

③

• General properties of S-matrix

Definition of the S-matrix

$$S = I + iT, \quad S_{ab} = \delta_{ab} + iT_{ab}$$

↙ reaction matrix and takes care for the interaction

$$S_{ab} S_{ab}^\dagger = I \leftarrow \text{unitarity}, \quad \underline{\text{explain what unitarity is}}$$

Example

$$\langle f | S_{ei} | i \rangle = \begin{bmatrix} \langle \pi N | S | \pi N \rangle & \langle \pi N | S | \gamma N \rangle \\ \langle \gamma N | S | \pi N \rangle & \langle \gamma N | S | \gamma N \rangle \end{bmatrix}$$

Let's work out a bit the reaction matrix T_{ab}

$$T_{ab} = (2\pi)^4 \delta^{(4)} \left(\sum_a p_i - \sum_b k_j \right) \prod_a \frac{1}{\sqrt{2p_{0i}}} \prod_b \frac{1}{\sqrt{2k_{0j}}} M_{ab}$$

defined in a way that M_{ab} is Lorentz invariant
the typical process is $2 \rightarrow N$

If we square T_{ab} we get the probability

$$[(2\pi)^4 \delta^{(4)}(0) = VT]$$

we get the measurable probability density

$$d\Gamma_j = \left(\frac{1}{\sqrt{2k_{0j}}} \right)^2 \left(\frac{d^3 k_j}{(2\pi)^3} \right) = \frac{d^3 k_j}{2(2\pi)^3 k_{0j}} = 2\pi \delta_+(k_j^2 - m_j^2) \frac{d^4 k_j}{(2\pi)^4}$$

(2)

$$d\sigma(a \rightarrow b) = \frac{1}{J} |M_{ab}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_b k_j) \frac{1}{[m!]} \prod_b d^3(k_j)$$

and J is the flux [number of particles per unit area and unit time]

$$J = 4 p_0 (s) \sqrt{s}$$

• unitarity

let's work out a bit unitarity

$$S S^\dagger = 1 \Rightarrow T_{ab} - T_{ab}^\dagger = i (T T^\dagger)_{cb}$$

$$\Leftrightarrow \frac{1}{i} (T_{ab} - T_{ba}^*) = \sum_c T_{ac} T_{cb}^*$$

If the interaction is time reversal invariant
(OK for strong interaction)

$$T_{ba} = T_{ab}, \text{ then } \frac{1}{i} (T_{ab} - T_{ab}^*) = 2 \text{Im} T_{ab}$$

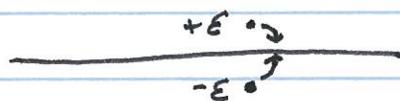
You can work out $\frac{M_{ab} - M_{ab}^*}{i}$

and if $a=b$ $2 \text{Im} M_{aa} = J \sigma_{\text{tot}}^a$ [optical theorem]

[Remind weak interaction]

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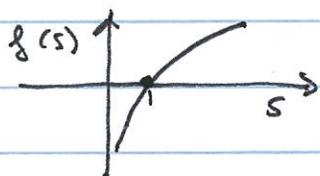
Let's think a bit about unitarity.

 and they do not reach the same value.

The function ~~is~~ does not seem to be continuous

→ let's do an example:

$$f(s) = \log s \quad \text{if } s \in \mathbb{R} \quad \text{and } f(s) \in \mathbb{R}$$



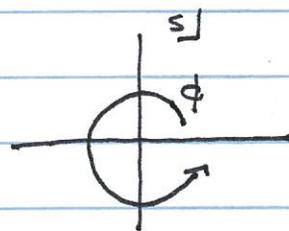
$$\mathbb{R} \xrightarrow{\log} s \in \mathbb{R} \Rightarrow s \in \mathbb{C}$$

$s \rightarrow |s| e^{i\phi}$ Let's apply the logarithm

$$\log s = \log[|s| e^{i\phi}] = \log |s| + i\phi \log e = \log |s| + i\phi$$

$$\log(s + i\epsilon) = \log |s| + i(\epsilon + \phi)$$

$$\log(s - i\epsilon) = \log |s| + i(2\pi - \epsilon)$$



$$\log(s + i\epsilon) - \log(s - i\epsilon) = -2\pi i$$

so when $\epsilon \rightarrow 0$ $\text{Im } \log s = -\pi$

we have a cut [Riemann sheets]

↳ Careful with programming

④

• Mandelstam plane for $2 \rightarrow 2$ scattering

M_{ab} is a Lorentz invariant function \rightarrow will depend on Lorentz invariant variables

For a $2 \rightarrow 2$ process $1+2 \rightarrow 3+4$

How many Lorentz invariant variables do we have?

4 momentum four-vectors

$$4 \times 4 = 16$$

- 4 four-momenta due to momentum conservation

$$16 - 4 = 12$$

- 4 on-mass-shell conditions

$$12 - 4 = 8$$

- 6 parameters [3 rotations + 3 Lorentz boosts]

$$8 - 6 = 2$$

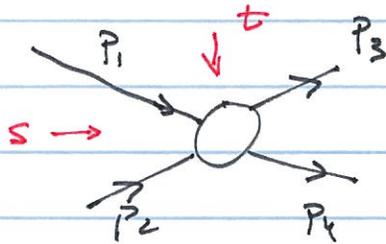
the general case is $n_1 \rightarrow n_2$

\downarrow Lorentz + Rotations

$$\underbrace{4(n_1 + n_2 - 1)}_{\text{momentum}} - \underbrace{(n_1 + n_2) - 6}_{\text{on-mass-shell}} = 3(n_1 + n_2) - 10$$

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For a $2 \rightarrow 2$ process we have 2 independent variables. We define



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

which are connected by $s + t + u = \sum_{i=1}^4 m_i^2$

let's set the masses equal to m

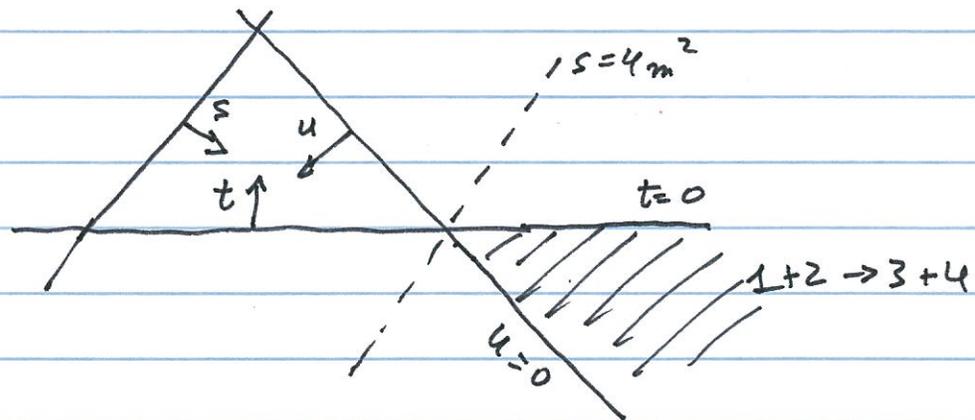
in c.o.m. $s = (p_1 + p_2)^2 = \underline{\underline{E_c^2}}$

The minimum quantity for s is $4m^2$, so

$$s \geq 4m^2 \Rightarrow t \leq 0 \text{ and } u \leq 0$$

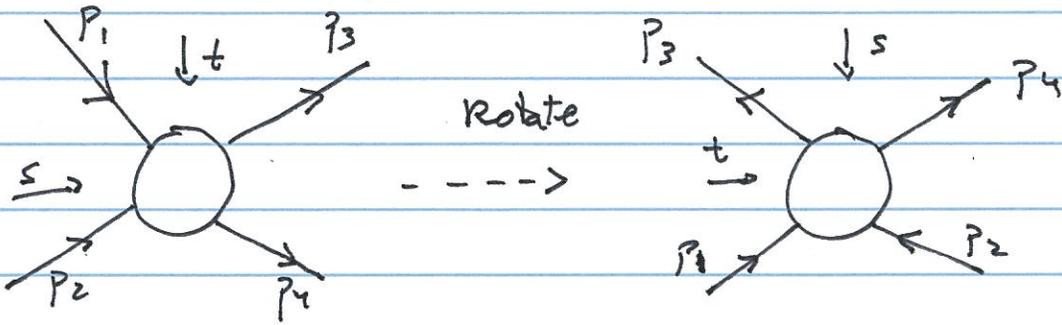
\uparrow
 $s + u + t = 4m^2$

let's draw a Mandelstam triangle



⑥

Crossing symmetry



$$1+2 \rightarrow 3+4$$

$$1+\bar{3} \rightarrow \bar{2}+4$$

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_3 - p_1)^2 \end{aligned} \quad \left\{ \begin{array}{l} s \geq 4m^2 \\ t \leq 0 \\ u \leq 0 \end{array} \right.$$

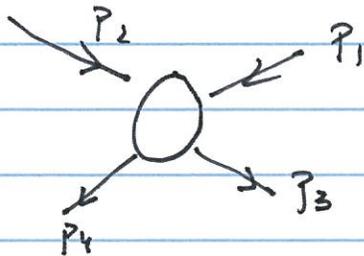
s-channel

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1 - \bar{p}_2)^2 \leq 0 \\ t &= (p_3 - p_1)^2 = (-\bar{p}_3 - p_1)^2 \\ &= (\bar{p}_3 + p_1)^2 \geq 4m^2 \end{aligned}$$

t-channel \rightarrow

$$\left\{ \begin{array}{l} t \geq 4m^2 \\ s \leq 0 \\ u \leq 0 \end{array} \right.$$

u-channel



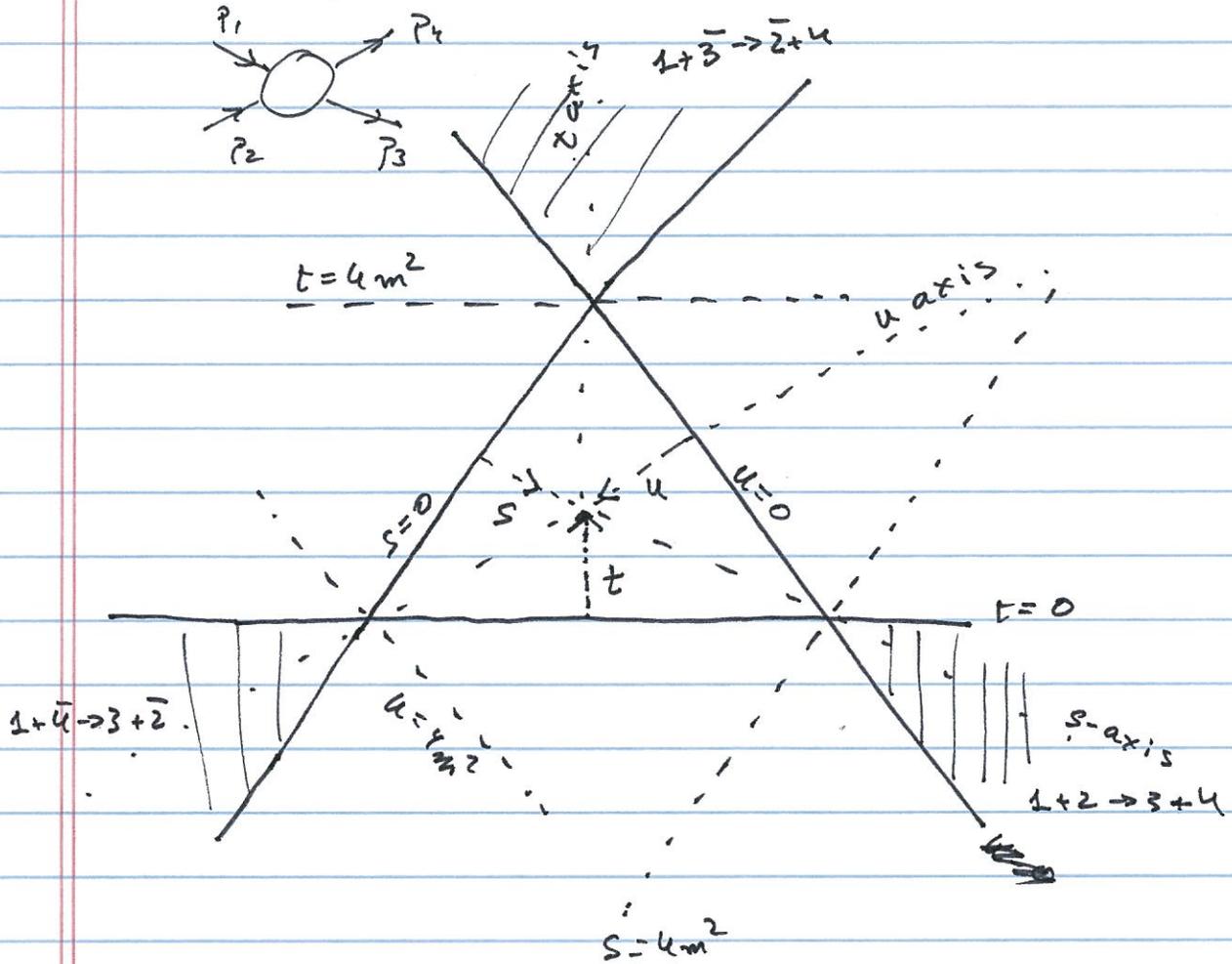
$$\bar{2}+4 \rightarrow 1+\bar{3}$$

Let's expand the Mandelstam plane

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2 → 2 process

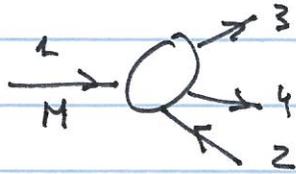
$$s + t + u = 4m^2$$



Mandelstam Plane

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Decay channel

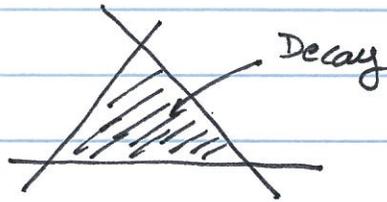


$$s = (p_1 + p_2)^2 = (p_1 - \bar{p}_2)^2 \geq (\mu - m)^2 > 0$$

$$t = (p_1 - p_3)^2 = (p_1 - p_2)^2 \geq (\mu - m)^2 > 0$$

$$u = (p_2 - p_4)^2 = (-\bar{p}_2 - p_4)^2 = (\bar{p}_2 + p_4)^2 \geq 4m^2 > 0$$

so, $s, u, t \geq 0$



Unitarity SERIOUSLY restricts the scattering amplitudes. These restrictions are different in each of the three crossing channels.

IF we can analytically continue from one region to another satisfying in each region ~~the~~ the corresponding unitarity condition, we only need one function for the three processes.

Causality ensures Analyticity

and this can be done.

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Moreover:

"Analytic properties of the exact amplitude coincide with those of the corresponding perturbation theory diagrams, even if perturbation theory cannot be applied"

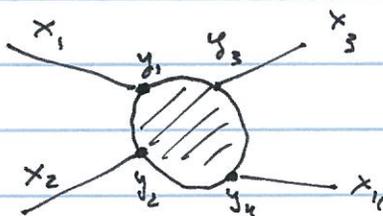
This statement only depends on QFT

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Analyticity and unitarity

Causality \Rightarrow Analyticity

• Four-point Green-function

$$A(x_1, x_2; x_3, x_4) =$$


$$= \int f(y_1, y_2, y_3, y_4) \left\{ \prod_{i=1}^4 D(y_i - x_i) d^4 y_i \right\}$$

where $D(y-x)$ describes the propagation of a free particle (scalar)

$$D(y_\mu - x_\mu) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \int \frac{dp_0}{2\pi i} \frac{\exp[-i p^\mu (y-x)_\mu]}{m^2 - p^2 - i\epsilon}$$

Time ordering

Initial states, we close the integration contour in energy around the pole at $p_0 = \sqrt{m^2 + \vec{p}^2}$

$$D(y_\mu - x_\mu) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \psi_{\vec{p}}(y) \psi_{\vec{p}}^\dagger(x) \quad y_0 > x_0$$

For final states

$$D(y_\mu - x_\mu) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \psi_{\vec{p}}(x) \cdot \psi_{\vec{p}}^\dagger(y) \quad x_0 > y_0$$

(11)

so we get

$$\mathcal{M}(p_i) = \int f(y_1, y_2, y_3, y_4) e^{-i(p_1 y_1 + p_2 y_2) + i(p_3 y_3 + p_4 y_4)} \pi d^4 y_i$$

we put energy-momentum conservation in the mix and
let's do *forward scattering*

$$\mathcal{M} \Rightarrow (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \int e^{i p_1 (y_2 - y_1)} f(y_{13}; p_2) d^4 y_{13}$$

$p_1 \approx p_3, p_2 \approx p_4$

Causality means:

$$f(y) = \theta(y_0) \theta(y_0^2) \cdot f_s(y) + f_0(y)$$

where $f_0(y)$ does not contribute to the amplitude

$$\int d^4 y f_0(y) \exp[i p \cdot y] = 0$$

it does not imply $f_0 = 0$

we know the amplitude is going to depend only on s

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2 m_2 E_1$$

in the rest frame of 2 E_1 is the energy of the
incident particle.

and now we write:

(12)

$$\begin{aligned}M(\vec{E}_1) &= \int d^4y f_1(y) \theta(y_0) \theta(y_1^2) e^{i p_1 y} \\ &= \int d^3\vec{y} \int_{-\infty}^{\infty} dt e^{i E_1 (t - v_1 z)} f_1(y)\end{aligned}$$

if we put the momentum in the z direction.

$$p_1 y \equiv E_1 t - \vec{p}_1 \cdot \vec{y} = E_1 (t - v_1 z)$$

because of the θ functions

$$t > 0 \quad \text{and} \quad (t - v_1 z) > 0$$

$$t > \sqrt{z^2 + y_1^2} \geq |z| > |v_1 z| \Rightarrow (t - v_1 z) > 0$$

the phase is positively definite

As a consequence $M(\vec{E}_1) \equiv M(s)$

is a regular analytic function in the upper half-plane of complex energies E_1 .

Actually, if the integral converges, it gets better

convergence in the complex plane due to $\exp[-\text{Im} E_1 (t - v_1 z)]$.

So

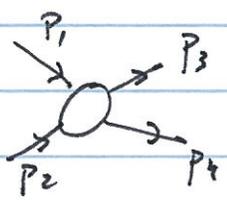
Causality \Rightarrow Analyticity

(13)

Cross-channel singularities of Born diagrams from s-channel point of view

QFT model scalar particles with $\lambda\phi^3$ interaction

This is the simplest example of a renormalizable theory
No ground state (stable vacuum) but is well suited for a qualitative analysis of the singularities of scattering amplitudes.



$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

$$s + t + u = 4m^2$$

Born approximation

$$\text{Diagram 1} = \frac{\lambda^2}{m^2 - s}$$

$$\text{Diagram 2} = \frac{\lambda^2}{m^2 - t}$$

$$\text{Diagram 3} = \frac{\lambda^2}{m^2 - u}$$

↑

Pole in energy
scattering via an
intermediate state

↑

Pole in t
channel is
related to
the interacting
radius

↑

Pole related to
exchange potential

Any problem with unitarity?

where does the complexity come from?

(135)

let's look at the unitarity condition

$$2 \operatorname{Im} H_{aa} = \sum_c M_{ac} M_{ac}^*$$



$$2 \operatorname{Im} \text{Diagram} = \text{Diagram} + \text{Diagram}$$

$$\left\{ \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right\} \begin{matrix} \xrightarrow{*} \\ \xrightarrow{*} \end{matrix} \left\{ \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right\}$$

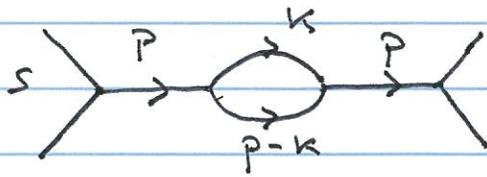
$$2 \operatorname{Im} T = \text{Diagram} + \dots$$

$\lambda \quad \lambda \quad \lambda \quad \lambda$
 $O(\lambda^4)$

so the scattering amplitude must be complex above the 2-particle threshold $s > 4m^2$

(136)

Two-particle thresholds *



We are trying to look for the origin of the complexity
we forget about the real pole factors and concentrate
on the loop integral

$$\bar{\Sigma}(s) = \frac{1}{2!} \int \frac{d^4 k}{(2\pi)^4 i} \frac{\lambda^2}{(m^2 - k^2 - i\epsilon)(m^2 - (p-k)^2 - i\epsilon)}$$

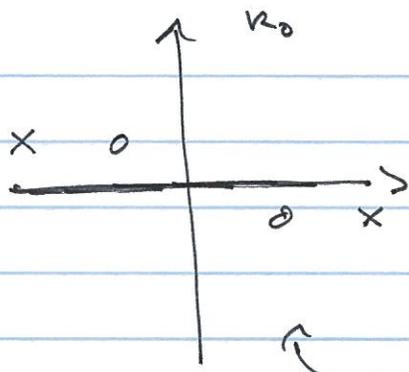
↑
symmetry factor

Singularities are at:

$$k_0 = \pm \sqrt{m^2 + \vec{k}^2} \mp i\epsilon$$
$$k_0 = p_0 \pm \sqrt{m^2 + (\vec{p} - \vec{k})^2} \mp i\epsilon$$

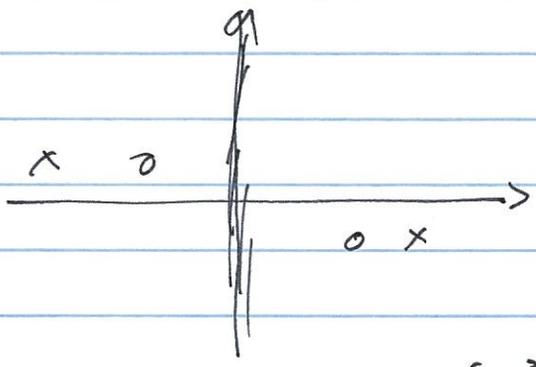
13d

$$s < 0$$



We can make $p_0 = 0$ and poles are symmetric

We rotate $k_0 \rightarrow ix$



and
$$\Sigma(-\vec{p}) = \frac{1}{2!} \int \frac{d^3 \vec{k}}{(2\pi)^3} \int \frac{d^2 x}{2\pi} \frac{\lambda^2}{(m^2 + x^2 + \vec{k}^2)^2 (m^2 + x^2 + (\vec{p} - \vec{k})^2)}$$

The answer is real valued \rightarrow so, no complexity here

13e

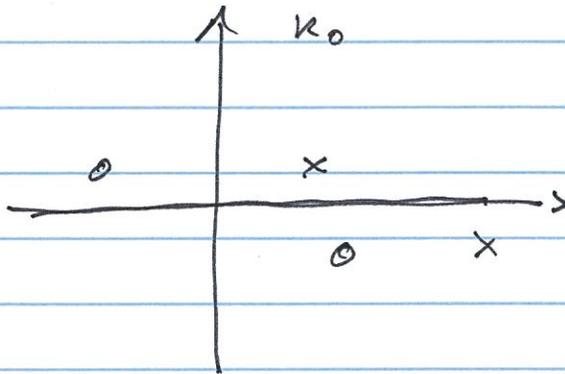
$s > 0$

we pick the frame $\vec{p}' = \vec{0}$

$$k_0 = \pm \sqrt{m^2 + \vec{k}^2} \mp i\epsilon$$

$$k_0 = \pm \sqrt{m^2 + \vec{k}^2} + p_0 \mp i\epsilon$$

with $p_0 > 0$ and increasing



and there is a moment when

$$-\sqrt{m^2 + \vec{k}^2} + p_0 = \sqrt{m^2 + \vec{k}^2}$$

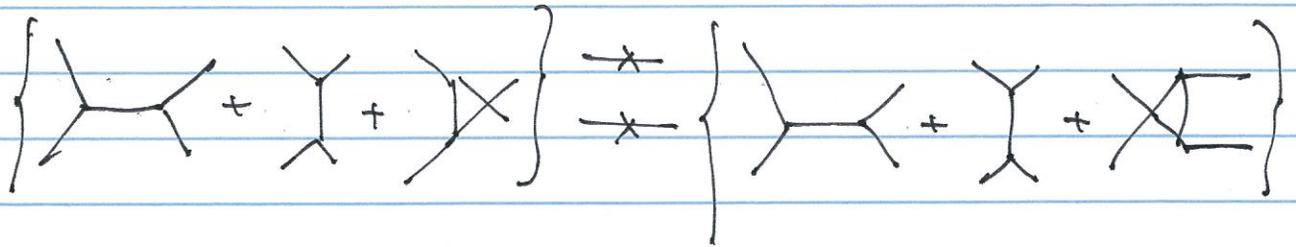
They collide and the integral becomes complex!

we are integrating in \vec{k} , and this happens for the first time at

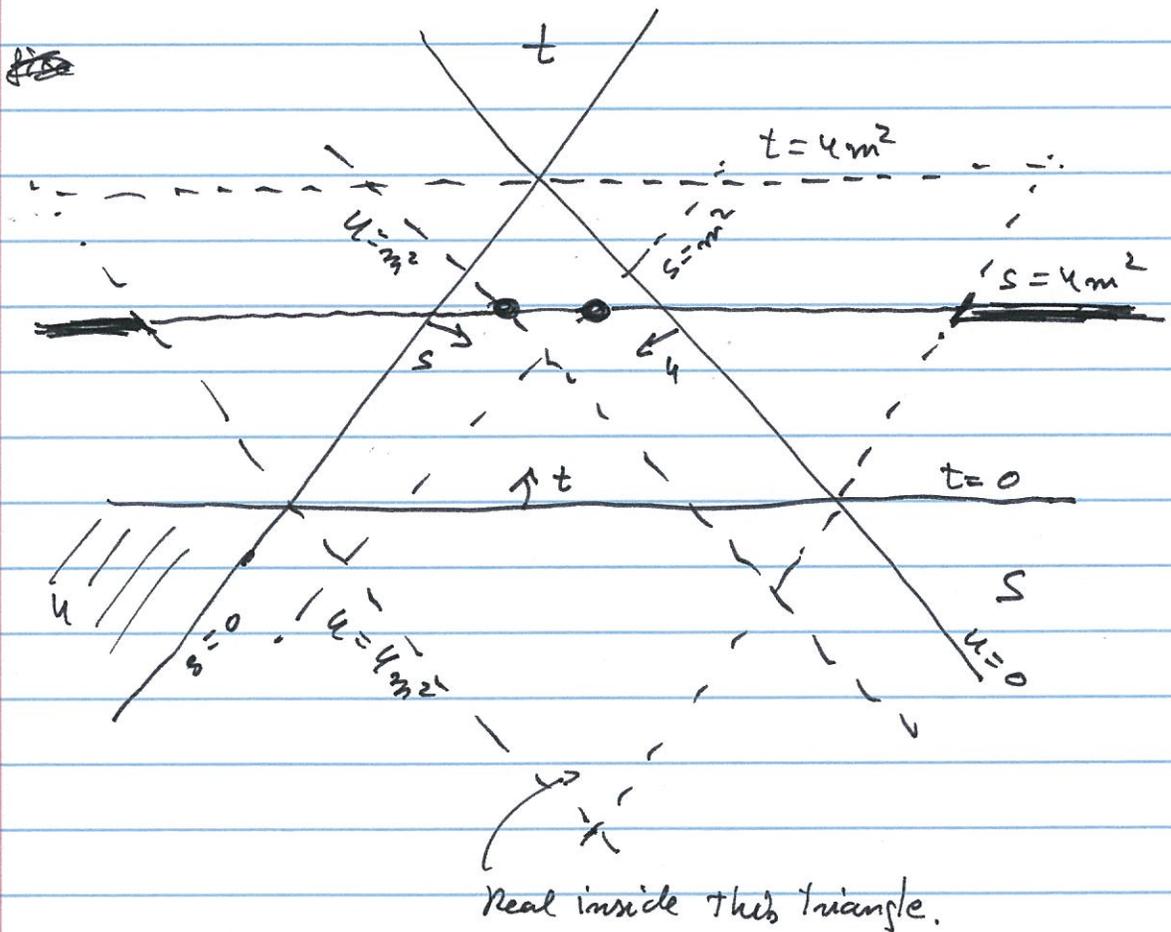
$$s = 4m^2 \leq \left(2\sqrt{m^2 + \vec{k}^2} \right)^2$$

and that is the 2-particle energy threshold

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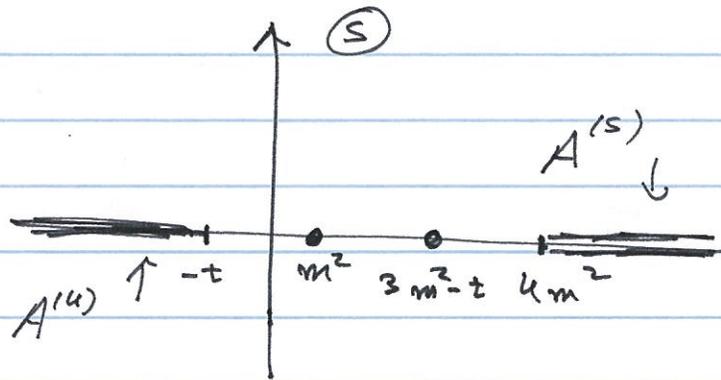


9 cut diagrams describe the imaginary part of the full amplitude in the order λ^4



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fixed t



Analytic structure of the amplitude

We proved that the upper part is analytical
Reflectivity principle allows us to continue to the
lower plane

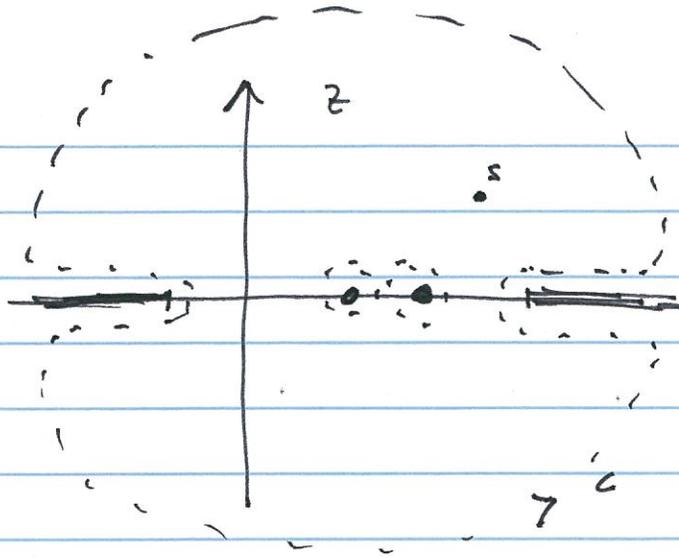
• Dispersion relations

Once the imaginary part of the analytic function
is known, we can restore its real part using the
dispersion relation

Cauchy integral around a point s in the complex plane

$$\int_C \frac{dz}{2\pi i} \frac{A(z)}{z-s} = A(s)$$

(16)



and cuts:

$$\text{Im}_s A = \frac{1}{2i} [A(s+i\epsilon, t) - A(s-i\epsilon, t)] \quad s > 4m^2$$

$$\text{Im}_u A = \frac{1}{2i} [A(u+i\epsilon, t) - A(u-i\epsilon, t)] \quad u > 4m^2$$

We can do this if the amplitude falls when $s \rightarrow \infty$

$\frac{\lambda^2}{m^2 - t}$ does not fall with s

we apply one subtraction

$$A(s) - A(0) = \int_C \frac{dz}{2\pi i} \left[\frac{A(z)}{z-s} - \frac{A(z)}{z} \right] = \frac{s}{\pi} \int_C \frac{dz}{2i} \frac{A(z)}{z(z-s)}$$

$$A(s) = A(0, t) + \frac{s}{\pi} \int_C \frac{dz}{2i} \frac{A(z)}{z(z-s)}$$

(17)

Twice subtracted dispersion relation

$$A(s) = \alpha + \alpha' s + \frac{s^2}{\pi} \int \frac{dz}{z^2} \frac{A(z)}{z(z-s)}$$

and so on if necessary. But we need input

to determine α and α'

- One can systematically, with the help of "semesters" Feynman diagrams, study the pattern of singularities of the interaction amplitude.

For each singularity

- (1) its position is determined by masses of real hadrons
- (2) its character derives from the topology of the interaction process
- (3) the coefficient in front of a singularity is expressed in terms of the physical on-mass-shell amplitudes