HUGS 2015 Lectures: Lattice QCD : W Detmold

Scope of lectures 1) motivation – why QCD and why lattice QCD?
2) basics of field theory with a spacetime lattice regulator
- scalar fields, gauge fields, fermion fields
3) introduce applications of LQCD

- Many good reference; eg Gattringer & Lang "Intro to lattice..."
- Today: motivation and review fundamentals of QCD

1.1 Hadrons, strong interactions & QCD : first a little bit of history

- Since the measurement of the magnetic moments of the proton and neutron, we have known that they are not fundamental particles but have substructure
- Elastic scattering experiments (Hofstadter 1950s) showed this clearly and mapped out the distribution of charge and currents in the proton and nuclei.
- In the late 1960s, higher energy deep inelastic scattering experiments at SLAC hinted at point-like constituents (partons) that behave almost free spin-½ particles
- A little earlier Gell-man - Ne'emann & Zweig realised that the zoo of hadrons that had been discovered by the 1960s could be organised by assuming baryons and mesons were composed of a set of more fundamental objects (quarks or aces) that came in different flavours (up, down, strange then, now also charm, bottom and top). Baryons were 3 quark objects and mesons were quark-anti-quark objects. Successfully predicted Ω (=sss) baryons mass
- In the early 1970s, Gross & Wilczek and Politzer showed that a gauge theory based upon the non-Abelian local symmetry SU(3) had the remarkable property of asymptotic freedom: as the energies involved in a physical process become large (or distances being probed get small), the strength with which they interact decreases logarithmically. Related calculations showed that QCD; quarks in the fundamental representation of SU(3) interacting via non-Abelian SU(3) adjoint-rep gauge bosons (gluons) was able to describe the DIS experiments at SLAC

- Indirect evidence for gluon degrees of freedom inside a proton was found through 3-jet events (PETRA expt @ DESY, mid 1970s)
- However quarks and gluons are not seen directly in experiment and the spectrum of states of QCD is very different from the particles of non-interacting QCD. QCD is fundamentally different from QED. Quarks and gluons are confined.
- Although QCD becomes asymptotically free at high energies, the same running of the coupling strength means that at low energies QCD interactions become very strong and the methods available in the 1970s were inapplicable
- In essence we had a theory that was beautifully able to describe experimental results at high energy, but we could only assume that the same theory produced the hadrons seen at low energies
- Although lattice QCD was invented by Wilson in 1974 (related ideas were around earlier - Wegner 1971) and did not rely on a small coupling strength the computational demands are such that it is only in the last decade that we have been able to calculate the masses and simple structure of simple hadrons in QCD
- Such calculations can be compared to experimental determinations finding agreement
- The significance of this milestone should not be overlooked - it is really of equivalent importance as the comparison to DIS experiments at high energies and shows that QCD is an effective description of the strong interactions at low energies as well as high energies.
- The story is far from complete as there are many quantities that we can only calculate at low precision (lack of computing or need to address systematic uncertainties) and many others that we do not know how to calculate at all (need a better theoretical understanding or algorithmic approach)
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2 The role of lattice QCD in nuclear & particle physics - some highlights

- Now LQCD plays an important role in many aspets of N & P physics

① Hadron spectroscopy: once the quark masses and lattice scale are set by enforcing agreement with experiment for a few quantities (typically meson masses) the spectrum of QCD for a large variety of quantum numbers (flavour, spin, parity, ...) can be post-dicted or predicted

- for simple states, serves as a test of the lattice method
- studies of excited mesons and baryons are an important counterpoint to experiments at JLab (GlueX) and BES III that are exploring these states - 1309.2608, also predictions for bottom baryons ($\Omega_b, \Xi_b', \ldots$)
- strong and EM isospin splittings between masses - 1406.4088
- baryons with bottom quarks and charm quarks can be predicted - 1409.0497

② Hadron structure: LQCD can also calculate many aspects of the structure of hadrons such as the proton and pion. These calculations, and related experiments give us a picture of the proton etc. Information on hadron structure is also important for various studies of neutrino physics and searches for physics beyond the SM.

- Electromagnetic and weak current form-factors - 1411.0078
- (Moments of) quark distribution functions, generalised parton distributions (GPDs) and transverse-momentum dependent parton distributions (TMDs)
- These quantities give a picture of which constituents are carrying the momentum and the spin of the proton - 1312.4816
- An important puzzle that LQCD could help resolve is the "proton radius puzzle" - 1502.05314
- Transition form factors

(3) Hadron interactions and nuclei : QCD should describe not only single hadrons, but also how they interact with one another and how they form nuclei. The intrinsic complexity of these topics makes them more difficult to address as does the Eulidean space nature of LQCD calculations

- Simple processes such as $\pi^+\pi^+$ scattering can be studied with high precision – 0706.3026
- Meson-baryon and baryon-baryon scattering are also studied at considerably less precision (1301.5790). Understanding these interactions as well as three body interactions is important in many contexts such as the physics of neutron stars
- Light nuclei are an active area of study with systems up to $A=4$ being resolved numerically – 1206.5219, 1502.04182
- First calculations of structure properties of nuclei, their magnetic moments, have recently appeared – 1409.3556
- Understanding nuclei from QCD will let us probe the many apparently accidental fine tunings in nuclear physics such as the triple-α process whereby ^{12}C forms.

(4) Hadronic matrix elements for electroweak decays : a topic that has driven the development of LQCD as a precision tool is the determination of hadronic contributions to various electroweak-induced decays of mesons and baryons. Such decays are sensitive to electroweak effects and potential physics beyond the Standard Model, but in order to extract this, the effects of QCD must be disentangled through lattice calculations

- Typical processes that are investigated involve changes of quark flavour, eg $B \to \pi l \nu$ to extract CKM matrix elt V_{ub}

- most studied processes involve only single hadron initial/final states and calculations reach few % level of accuracy where effects such as EM corrections must be dealt with - see FLAG review
- A few more complicated processes are tackled such as $K \to \pi\pi$ but calculations are much more difficult
- Beyond flavour physics, precision calculations of nucleon and even nuclear matrix elements will be required to make best use of ongoing and planned experiments over the next decade. Important examples are for dark matter detection experiments, $(g-2)_\mu$, neutrino-nucleus interaction

5. The phases of QCD : the study of QCD at extremes of temperature and density is another area of active development in LQCD
   - LQCD has unambiguously shown that QCD undergoes a crossover from a hadronic phase to a quark gluon plasma phase at a pseudocritical temperature $T_c \sim 155$ MeV - 1007.2580
   - The equation of state relating pressure and energy density has been determined over a wide range of temperatures - 1407.6387
   - Freezout parameters in heavy-ion collisions have been determined - 1404.6511
   - Investigations at non-zero density/chemical potential are very difficult in LQCD as currently formatted. The chemical potential makes the QCD action complex and not amenable to usual QCD tools. Finding alternate ways to tackle this problem is an open challenge

(6) <u>Strong dynamics in gauge theories beyond QCD</u> : there are strong theoretical motivations to study other strongly interacting gauge theories. It is possible that such a theory is responsible for electroweak symmetry-breaking (roughly technicolour theories). It is also possible that dark matter is comprised of strongly interacting matter. Finally it is important to understand strongly-interacting theories in their own right and see what kinds of physics they contain. LQCD methods can be, and are being, adapted to the study of such theories.

1.3 Review of QCD

- In these lectures we shall focus on QCD with two light (up/down) quarks and a strange quark as the relevant matter content. Heavier quarks (charm & bottom - top does not hadronise) will be discussed briefly

- The classical <u>QCD Lagrangian</u> is given by

$\not{D} = D_\mu \gamma^\mu$ ← repeated indices are summed

$$\mathcal{L}_{QCD} = -\frac{1}{4} tr[F^{\mu\nu}F_{\mu\nu}] + \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f + \Theta \epsilon_{\mu\nu\rho\sigma} tr[F^{\mu\nu}F^{\rho\sigma}]$$

quark masses

which contains <u>quark</u> degrees of freedom

$$\psi_{f,\alpha,c}(x) \quad \text{with} \quad \begin{aligned} f &= \text{flavour} = u,d,c,s,t,b \\ \alpha &= \text{Dirac} = 1,2,3,4 \\ c &= \text{colour} = 1,2,3 \end{aligned}$$

each flavour has $N_c N_s = 12$ components

and <u>gluon</u> degrees of freedom

$$A_{\mu,a}(x) \quad \text{with} \quad \begin{aligned} \mu &= \text{Lorentz index} = 0,1,2,3 \\ a &= \text{color} \in \{1,2,\dots,8\} \end{aligned}$$

which we can write in terms of a matrix valued field

$$A_\mu(x) = \underbrace{A_{\mu a}(x)\, T_a}_{\text{repeated indices summed}} \qquad \text{where } T_a \text{ are the generators of } su(3)$$

$$T_a = \frac{\lambda^a}{2} \leftarrow \text{Gell-Mann matrices}$$

<u>SU(3)</u> generators $T^a = \lambda^a/2 \qquad a = 1, \dots, 8$

$$\lambda_1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0&0&-i\\ 0&0&0\\ i&0&0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-2 \end{pmatrix}$$

satisfy su(3) algebra

$$[T^a, T^b] = i f^{abc} T^c \qquad f^{abc} = \text{antisymmetric structure constants}$$

have

$$T^{a\dagger} = T^a \quad , \; tr[T^a] = 0 \; , \; tr[T^a T^b] = \delta^{ab} \;\dots$$

see hep-ph/9507456 for many other identities

- The gauge field is used to build the covariant derivative

$$D^\mu = \partial^\mu - i g A^\mu \qquad \text{(matrix notation)}$$

and the gluon field strength

$$F^{\mu\nu} = \frac{2i}{g}[D^\mu, D^\nu] \equiv F^{\mu\nu}_a T_a \qquad \text{(matrix object)}$$

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^\mu_b A^\nu_c$$

- The fermion Lagrangian involves Dirac matrices γ^μ that are elements of a Clifford algebra, satisfying the anticommutator

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \; (\to 2\delta_{\mu\nu}) \quad \mu, \nu \in \{0,1,2,3\}$$

with the γ_μ being 4x4 matrices found in various bases in many textbooks

- $\mathcal{L}_{QCD}$ was constructed so that the quark field is invariant under local rotations in colour space

$$\psi(x) \to \psi'(x) = U(\vec{\theta})\psi(x)$$

with

$$U(\vec{\theta}) = \exp\left[-\frac{i}{2}\theta^a(x) T^a\right] \qquad \theta^a(x) \text{ parameterise transform}$$

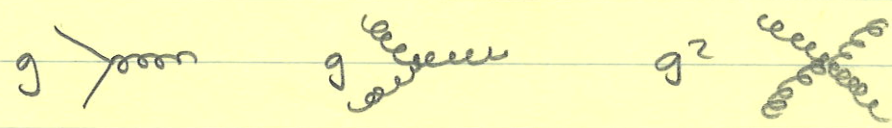
along with

$$A_\mu(x) \to A'_\mu(x) = U(\theta) A_\mu(x) U^{-1}(\theta) - \frac{2i}{g}[\partial_\mu U(\theta)] U^{-1}(\theta)$$

Additionally $\quad D^\mu \to U(\theta)\, D^\mu\, U^{-1}(\theta)$

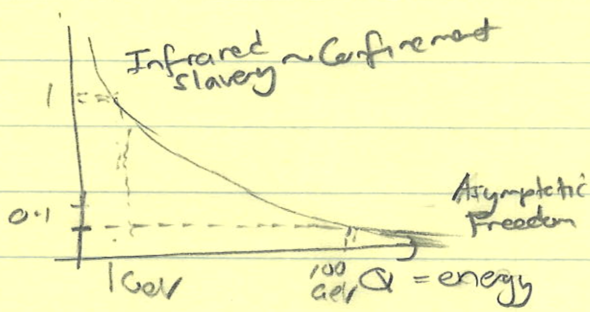
$\qquad\qquad\quad F^{\mu\nu} \to U(\theta)\, F^{\mu\nu}\, U^{-1}(\theta)$

- $\mathcal{L}_{QCD}$ contains interactions



and gluon self-interaction make QCD fundamentally different from QED. Two important consequences are asymptotic freedom and (presumably) confinement which occur through the running of the QCD coupling $\alpha_S = \frac{g^2}{4\pi}$ with the energy scale that results from quantum effects

$$\alpha_S(Q) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda_{QCD}^2}$$



where Λ_{QCD} is the QCD scale that emerges from the coupling

- Another important symmetry of QCD when quark masses are ignored is <u>chiral symmetry</u> which is a global invariance of the theory under axial transformations

Let us consider QCD with N_f flavours of massless quarks.
The quark Lagrangian is

$$\bar{\psi}\, i\not{D}\, \psi = (\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_{N_f}) \begin{pmatrix} i\not{D} & & \\ & i\not{D} & \\ & & \ddots \\ & & & i\not{D} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{N_f} \end{pmatrix}$$

we ignore colour as this is a global symmetry that we will discuss

This is invariant under global flavour rotations : $U(1) \times SU(N_f)$

vector symmetry
$$\begin{cases} \psi \to \exp(-\frac{i}{2}\alpha)\,\psi\ , & \bar{\psi} \to \bar{\psi}\exp(\frac{i}{2}\alpha) \qquad U(1) \\ \psi \to \exp(-\frac{i}{2}\alpha^a\tau^a)\,\psi\ , & \bar{\psi} \to \bar{\psi}\exp(+\frac{i}{2}\alpha^a\tau^a) \qquad SU(N_f) \end{cases}$$

where τ^a are the generators of $SU(N_f)$ in the fundamental representation [$N_f = 2$ they are the Pauli matrices. $N_f = 3$?]

- The theory has another global invariance under axial transformations.
  Define the Dirac matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ which satisfies
  $$\{\gamma_5, \gamma_\mu\} = 0 \quad , \quad \gamma_5^2 = 1 \quad , \quad \gamma_5^\dagger = \gamma_5$$
  Then the axial transformations
  $$\psi \to \exp(-\tfrac{i}{2}\alpha\gamma_5)\psi \qquad \bar\psi \to \bar\psi\exp(-\tfrac{i}{2}\alpha\gamma_5) \qquad U_1(A)$$
  $$\psi \to \exp(-\tfrac{i}{2}\alpha^a\tau^a\gamma_5)\psi \qquad \bar\psi \to \bar\psi\exp(-\tfrac{i}{2}\alpha^a\tau^a\gamma_5) \qquad SU(N_f)$$
  are also symmetries
  $$\bar\psi \not{D}\psi \to \bar\psi e^{-\frac{i}{2}\alpha^a\tau^a\gamma_5} D^\mu\gamma_\mu e^{-\frac{i}{2}\alpha^a\tau^a\gamma_5}\psi$$
  $$= \bar\psi D^\mu\gamma_5 e^{+\frac{i}{2}\alpha^a\tau^a\gamma_5} e^{-\frac{i}{2}\alpha^a\tau^a\gamma_5}\psi = \bar\psi\not{D}\psi \quad \checkmark$$

- Corresponding conserved vector and axial currents (Noether currents)
  $$V^{\mu a} = \tfrac{1}{2}\bar\psi\gamma_\mu\tau^a\psi \qquad A^{\mu a} = \tfrac{1}{2}\bar\psi\gamma_\mu\gamma_5\tau_a\psi \qquad a = 0,1,\dots N_f^2-1$$
  $$\tau_0 = 1$$
  and charges
  $$Q^a = \int d^3x\, V^{0a}(x) \qquad Q_5^a = \int d^3x\, A^{0a}(x)$$

- Recombine these symmetries in chiral symmetries generated by
  $$Q_L^a = \tfrac{1}{2}(Q^a - Q_5^a) \quad \text{and} \quad Q_R^a = \tfrac{1}{2}(Q^a + Q_5^a)$$
  $\Rightarrow$ global symmetry
  $$SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)$$

  [NB: these charges satisfy the chiral algebra
  $$[Q_L^a, Q_L^b] = if^{abc}Q_L^c \quad \text{and same for } L \to R$$
  $$[Q_L^a, Q_R^b] = 0$$
  and transform into each other under parity $PQ^a_{L,R}P^{-1} = Q^a_{R,L}$

- Defining projectors $P_L = \tfrac{1}{2}(1-\gamma_5) \qquad P_R = \tfrac{1}{2}(1+\gamma_5)$
  $$P_L^2 = P_L \quad P_R^2 = P_R \quad , \quad P_L + P_R = 1 \quad P_L P_R = P_R P_L = 0$$
  we can write
  $$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi$$

$$\Rightarrow \quad \bar{\psi} = \psi^\dagger \gamma_0 = (\psi^\dagger P_L + \psi^\dagger P_R)\gamma_0 = \psi^\dagger \gamma_0 (P_R + P_L) = \bar{\psi}_R + \bar{\psi}_L$$

So

$$\bar{\psi} \not{D} \psi = (\bar{\psi} P_L + \bar{\psi} P_R) \not{D} (P_L \psi + P_R \psi)$$
$$= \bar{\psi} P_L \not{D} P_L \psi + \bar{\psi} P_R \not{D} P_L \psi + \dots$$
$$= \cancel{\bar{\psi} \not{D} P_R P_L \psi} + \bar{\psi} \not{D} P_L P_L \psi + \dots$$

only survivors are $= \bar{\psi} P_R \not{D} P_L \psi + \bar{\psi} P_L \not{D} P_R \psi$
$$= \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R$$

the two different chiralities are independent

- Chiral symmetry is not manifest in QCD. It would tell us that the ρ meson (a vector meson) must have the same mass as the a_1 (axial vector) meson but

$$m_\rho = 0.770 \text{ GeV} \qquad m_{a_1} = 1.26 \text{ GeV}$$

[Consider state $|\psi\rangle$ with $H|\psi\rangle = E|\psi\rangle$

If axial symmetry is exact then $[H, Q_5^a] = 0$ so

$|\chi\rangle = Q_5^a|\psi\rangle$ has energy $H|\chi\rangle = HQ_5^a|\psi\rangle = Q_5^a H|\psi\rangle = E Q_5^a|\psi\rangle = E|\chi\rangle$]

- There is explicit symmetry breaking from quark masses

$$\bar{\psi} M \psi = \bar{\psi}_L M \psi_R + \bar{\psi}_R M \psi_L$$

but this is not sufficient for the large symmetry breaking that is observed

- We hypothesise (and lattice calculations now show unambiguously) that the vacuum of QCD is not chirally invariant $Q_5^a|0\rangle \neq 0$ because of the condensation of $\bar{q}q$ pairs. This is spontaneous symmetry breaking as the Lagrangian has the symmetry but the vacuum breaks it.
This symmetry breaking leads to massless Goldstone bosons (pions in QCD $N_f = 2$) and allows for many powerful predictions (chiral perturbation theory) regarding low energy hadronic physics

- The symmetry breaking pattern is down to the vector subgroup

$$SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f)$$

and Goldstone's theorem tells us there should be $N_f^2 - 1$ massless excitations (= # broken symmetry generators)

- The $U_A(1)$ symmetry is also not manifest and is broken by quantum effects — the axial anomaly

1.4 Quantum field theory and functional integrals

- One way to define the quantum theory is through the functional integral formalism
- The partition function (generating functional) contains all information about the theory

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}A_\mu \mathcal{D}\bar\Psi \mathcal{D}\Psi \exp\left[i\int d^4x \left(\mathcal{L}_{QCD} + J_\mu^a A^{\mu a} + \bar\eta \Psi + \bar\Psi \eta\right)\right]$$
$$= \exp\{i W[J,\eta,\bar\eta]\}$$

where $J, \eta, \bar\eta$ are external sources

Z corresponds to an integral over all possible gauge and fermion fields and generates correlation functions of the theory

$$\langle 0| A_\mu(x_1) \bar\Psi(y) \ldots \Psi(z)|0\rangle = \frac{1}{Z[0]} \frac{\delta}{\delta J_\mu(x)} \cdots \frac{\delta}{\delta \bar\eta(z)} Z[J,\eta,\bar\eta]\Big|_{J=\eta=\bar\eta=0}$$

- The fermion fields are anticommuting (Grassmann) number valued
- In order to construct Z, we must define the measure by regulating the theory. In perturbation theory, we expand Z and correlators in powers of couplings after fixing a gauge via the Faddeev-Popov-deWitt method or by requiring BRS invariance

- Instead we will define Z directly through a lattice regulator that is valid beyond perturbation theory. This provides mathematical definition of the theory and also results in a numerical method with which to evaluate correlation functions

2 Path Integrals in Quantum Mechanics and Field Theory

- In order to define the partition function, we start first with quantum mechanics

2.1 Quantum Mechanics

- We will start with a particle mass m in a 1D potential $U(x)$ defined in QM by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + U(\hat{x}) \qquad \hat{p} = -i\frac{d}{dx}, \quad [\hat{x}, \hat{p}] = i$$

- The Hamiltonian describes the time evolution of the particle and defines the partition function

$$Z_T = \mathrm{tr}\left[e^{-T\hat{H}}\right] \qquad T \in \mathbb{R}$$

and correlators

$$\langle \mathcal{O}_2(t)\,\mathcal{O}(0)\rangle = \frac{1}{Z_T}\mathrm{tr}\left[e^{-(T-t)\hat{H}}\,\mathcal{O}_2(t)\,e^{-t\hat{H}}\,\mathcal{O}_1(0)\right]$$

$\mathcal{O}_i$ might correspond to measuring position etc

- It should be clear we are considering evolution in Euclidean time (that is, $\tau \to it$) as this is mathematically more convenient [the operator $\hat{p}^2$ is Hermitian positive definite, so $e^{-\hat{p}^2}$ is also well defined]

- Consider an infinitesimal time step ε. We can write

$$\exp(-\varepsilon\hat{H}) = \underbrace{e^{-\frac{\varepsilon}{2}U(\hat{x})}\,e^{-\frac{\varepsilon}{2m}\hat{p}^2}\,e^{-\frac{\varepsilon}{2}U(\hat{x})}}_{\hat{W}_\varepsilon \;=\; \text{transfer matrix}} + \mathcal{O}(\varepsilon^2)$$

- Matrix elements between position eigenstates are easily evaluated

$$\langle x|\hat{W}_\varepsilon|y\rangle = \exp\left(-\frac{\varepsilon}{2}U(x)\right)\exp\left(-\frac{\varepsilon}{2}U(y)\right)\langle x|e^{-\frac{\varepsilon}{2m}\hat{p}^2}|y\rangle$$

$$= \exp\left(-\frac{\varepsilon}{2}(U(x)+U(y))\right)\int dp\,\langle x|e^{-\varepsilon\hat{p}^2/2m}|p\rangle\langle p|y\rangle$$

where we have used completeness of momentum eigenstates $\hat{p}|p\rangle = p|p\rangle$, $\int |p\rangle\langle p|\,dp = 1$

$$= \exp\left[-\frac{\varepsilon}{2}(U(x)+U(y))\right]\int dp\, e^{-\varepsilon p^2/2m}\langle x|p\rangle\langle p|y\rangle$$

$$= \exp\left[-\frac{\varepsilon}{2}(U(x)+U(y))\right]\int dp\, e^{-\varepsilon p^2/2m}\,e^{ip(x-y)}/2\pi$$

$$\langle p|y\rangle = \frac{e^{ipy}}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{m}{2\pi\varepsilon}}\exp\left[-\frac{\varepsilon}{2}(U(x)+U(y))\right]\exp\left[-\frac{(x-y)^2 m}{2\varepsilon}\right]$$

where we have completed the square to do the Gaussian integral

- Using the <u>Trotter formula</u> we can construct the finite time evolution

$$\exp(-TH) = \lim_{N_T \to \infty} \left(\hat{W}_\varepsilon\right)^{N_T}$$

where $\quad T = \varepsilon N_T$ is a fixed distance

- We can now write the partition function (or correlators) as

$$Z_T = \mathrm{Tr}[e^{-TH}] = \int dx_0 \langle x_0 | e^{-TH} | x_0 \rangle$$

(trace $\Rightarrow$ sum over complete basis)

$$= \lim_{N_T \to \infty} \int dx_0 \langle x_0 | \hat{W}_\varepsilon{}^{N_T} | x_0 \rangle$$

$$= \lim_{N_T \to \infty} \int dx_0\, dx_1 \dots dx_{N_T - 1} \langle x_0 | \hat{W}_\varepsilon | x_1 \rangle \langle x_1 | \hat{W}_\varepsilon \dots \langle x_{N_T-1} | \hat{W}_\varepsilon | x_0 \rangle$$

$$= \lim_{N_T \to \infty} \left(\frac{m}{2\pi\varepsilon}\right)^{N_T} \int dx_0 \dots dx_{N_T - 1} \exp\left[-\varepsilon \sum_{j=0}^{N_T - 1} \left\{\frac{m}{2}\frac{(x_j - x_{j+1})^2}{\varepsilon^2} + U(x_j)\right\}\right]$$

- Defining $t = \varepsilon j$ and noting $\frac{x_j - x_{j+1}}{\varepsilon}$ is a forward finite difference operator with $\lim_{\varepsilon \to 0} \frac{x_j - x_{j+1}}{\varepsilon} \to \frac{dx}{dt}$, we can write the exponent we can write

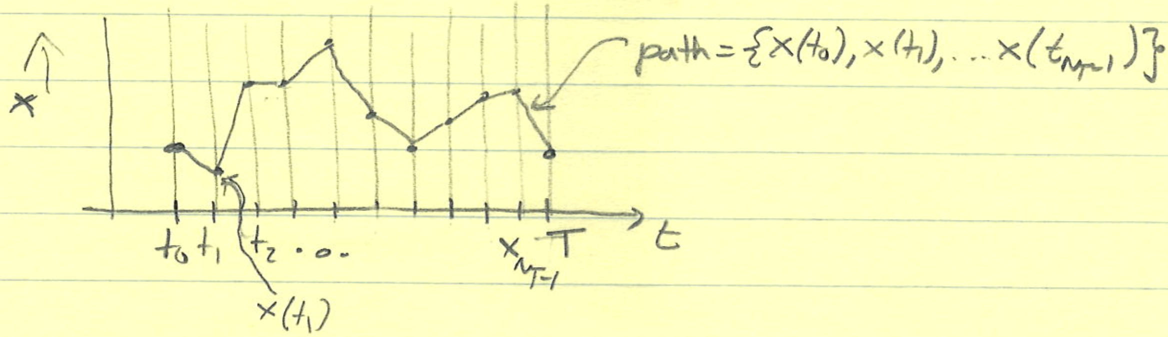
$$Z_T = \lim_{N_T \to \infty} \left(\frac{m}{2\pi\varepsilon}\right)^{N_T} \int dx_0 \dots dx_n \exp\left(-S_E(x, \dot{x})\right)$$

where the Euclidean action S_E is given by

$$S_E(x, \dot{x}) = \int_0^T dt \left\{\frac{m}{2} \dot{x}(t)^2 + U(x(t))\right\} + O(\varepsilon)$$

$$\varepsilon \sum_{j=0}^{N_T - 1} \to \int_0^T dt$$

- The result is that the partition function has been written as a multi-dimensional integral over the position of the particle at each (discretised) point in time weighted by the Euclidean action (energy functional). Paths with least energy dominate and others are exponentially damped. [in real time evolution, paths of large action are highly oscillatory contributions which average to zero]



- We can write the partition function as a path integral by taking the limit $N_T \to \infty$ (ie $\varepsilon \to 0$)

$$Z_T = \# \int [dx(t)]\, e^{-S_E(x,\dot{x})}$$

where there is an (infinite) normalisation and the integration measure $[dx(t)]$ is strictly defined through the limiting process

2.2 Scalar fields

- We consider a real scalar field of mass m with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 + V(\varphi) \qquad \varphi = \varphi(\vec{x},t)$$

and action

$$S = \int d^3x\, dt\, \mathcal{L}(\vec{x},t)$$

- We can quantise this theory by promoting fields to operators $\hat{\varphi}$ and introducing canonically conjugate momenta $\hat{\pi} = \dot{\hat{\varphi}}$ with $[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

$\Rightarrow$ Hamiltonian

$$\hat{H} = \int d^3x \left(\hat{\pi}\dot{\hat{\varphi}} - \mathcal{L}\right) \qquad \text{(Minkowski space)}$$

$$= \int d^3x \left(\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\vec{\nabla}\hat{\varphi})^2 + \frac{m^2}{2}\hat{\varphi}^2 + V(\hat{\varphi})\right)$$

- We shall treat time evolution as in the quantum mechanics case but also discretise space, defining a lattice of points separated by a shortest distance a, the lattice spacing

$$\vec{x} \to a\vec{n} \qquad n_i = 0, 1, \ldots N-1$$

- We denote the set of points as a 3D lattice

$$\Lambda_3 = \{ a\vec{n},\ n_i \in \{0,1,\dots,N-1\} \}$$

which has physical dimension $L = aN$ in each direction

- We will assume periodic boundary conditions (BCs), identifying $n_i = N$ with $n_i = 0$

- Derivatives and integrals are replaced by finite differences and sums, respectively

$$\partial_j \hat{\varphi}(\vec{x}) \to \frac{1}{2a}\left(\hat{\varphi}(\vec{n}+\hat{j}) - \hat{\varphi}(\vec{n}-\hat{j})\right) + \mathcal{O}(a^2) \quad \Big| \ \hat{j} = \text{unit vector in } j \text{ direction}$$

$$\int d^3x \to a^3 \sum_{\vec{n}\in\Lambda_3}$$

[NB: one could also use forward or backward finite differences or higher order differences for the derivative

- The Hamiltonian is now

$$\hat{H} = a^3 \sum_{\vec{n}\in\Lambda_3} \left\{ \frac{1}{2}\hat{\pi}(\vec{n})^2 + \frac{1}{8a^2}\sum_{j=1}^{3}\left(\hat{\varphi}(\vec{n}+\hat{j}) - \hat{\varphi}(\vec{n}-\hat{j})\right)^2 + \frac{m^2}{2}\hat{\varphi}(\vec{n})^2 + V(\varphi(\vec{n})) \right\}$$

and we are interested in the time evolution it generates

- We introduce a complete set of eigenstates of the field operators $\hat{\varphi}(\vec{n})$ (generalisations of position eigenstates)

$$\hat{\varphi}(\vec{n})|\varphi\rangle = \varphi(\vec{n})|\varphi\rangle$$

(NB $|\varphi\rangle$ contains information about field at every point

with

$$\langle\varphi|\varphi'\rangle = \delta(\varphi-\varphi') = \prod_{\vec{n}\in\Lambda_3} \delta(\varphi(\vec{n}) - \varphi'(\vec{n}))$$

and

$$1 = \int \mathcal{D}_3\varphi\, |\varphi\rangle\langle\varphi| \quad \text{where } \int \mathcal{D}_3\varphi = \prod_{\vec{n}\in\Lambda_3} \int_{-\infty}^{\infty} d\varphi(\vec{n})$$

integrate over the field at all points in Λ_3

and similarly a complete set of eigenstates of the conjugate momenta

$$\hat{\pi}(\vec{n})|\pi\rangle = \pi(\vec{n})|\pi\rangle \quad , \quad \langle\pi|\pi'\rangle = a^3\delta(\pi-\pi')$$

$$\int \mathcal{D}_3\pi\, |\pi\rangle\langle\pi| = 1 \qquad \mathcal{D}_3\pi = \prod_{\vec{n}\in\Lambda} \frac{a^3}{2\pi}\int_{-\infty}^{\infty} d\pi(\vec{n})$$

- Consistent with the commutators

$$\langle\varphi|\pi\rangle = \left(\frac{a^3}{2\pi}\right)^{N^3/2} \exp\left[i a^3 \sum_{\vec{n}\in\Lambda_3} \pi(\vec{n})\varphi(\vec{n})\right]$$

(plane waves in field configuration space)

- We can now consider time evolution as before

$$e^{-\hat{H}T} = \lim_{N_T\to\infty} (\hat{W}_\varepsilon)^{N_T}$$

with

$$\hat{W}_\varepsilon = \exp(-\varepsilon\hat{U}/2)\exp(-\varepsilon\hat{H}_0)\exp(-\varepsilon\hat{U}/2)$$

where we have separated the Hamiltonian in $\hat{H} = \hat{H}_0 + \hat{U}$

with $$\hat{H}_0 = \frac{a^3}{2}\sum_{\vec{n}\in\Lambda_3} \hat{\pi}(\vec{n})^2$$

$$U = a^3 \sum_{\vec{n}\in\Lambda_3} \left\{ \frac{1}{8a^2}\sum_{j=1}^{3} \left(\varphi(\vec{n}+\hat{j}) - \varphi(\vec{n}-\hat{j})\right)^2 + \frac{m^2}{2}\varphi(\vec{n})^2 + V(\varphi(\vec{n}))\right.$$

and write the infinitesimal time evolution of field eigenstates as

$$\langle\varphi_{i+1}|\hat{W}_\varepsilon|\varphi_i\rangle = \langle\varphi_{i+1}|e^{-\varepsilon\hat{U}/2}e^{-\varepsilon H_0}e^{-\varepsilon\hat{U}/2}|\varphi_i\rangle$$

time label

$$= e^{-\varepsilon U(\varphi_{i+1})/2}e^{-\varepsilon U(\varphi_i)/2}\langle\varphi_{i+1}|e^{-\varepsilon H_0}|\varphi_i\rangle$$

$$= \text{———} \int\mathcal{D}_3\pi\,\langle\varphi_{i+1}|\pi\rangle\langle\pi|e^{-\varepsilon H_0}|\varphi_i\rangle$$

where $|\varphi_j\rangle$ denotes the state at time $t=j\varepsilon$ and $\varphi_j(\vec{n})$ denote the eigenvalues

Now we can do the integral by completing the square

$$\int\mathcal{D}_3\pi\,\langle\varphi_{i+1}|\pi\rangle\langle\pi|e^{-\varepsilon\hat{H}_0}|\varphi_i\rangle = \int\mathcal{D}_3\pi\,\exp\left(-\frac{\varepsilon a^3}{2}\sum_{\vec{n}\in\Lambda}\pi(\vec{n})^2\right)\langle\varphi_{i+1}|\pi\rangle\langle\pi|\varphi_i\rangle$$

$$= \int\mathcal{D}_3\pi\,\exp a^3\sum_{\vec{n}\in\Lambda_3}\left[-\frac{\varepsilon}{2}\pi^2(\vec{n}) + i\pi(\vec{n})\left(\varphi_{i+1}(\vec{n}) - \varphi_i(\vec{n})\right)\right]$$

$$= \prod_{\vec{n}\in\Lambda_3}\frac{a^3}{2\pi}\int_{-\infty}^{\infty} d\pi(\vec{n})\,\exp\left[-\frac{\varepsilon a^3}{2}\pi^2(\vec{n}) + i\pi(\vec{n})\left(\varphi_{i+1}(\vec{n}) - \varphi_i(\vec{n})\right)\right]$$

$$= \prod_{\vec{n}\in\Lambda_3} \sqrt{\frac{a^3}{2\pi\varepsilon}} \exp\left[-\frac{a^3}{2\varepsilon}\left(\varphi_{l+1}(\vec{n}) - \varphi_l(\vec{n})\right)^2 \right]$$

Thus

$$\langle \varphi_{l+1} | \hat{W}_\varepsilon | \varphi_l \rangle = C^{N^3} \exp\left[-\frac{\varepsilon}{2}\left(U(\varphi_l) + U(\varphi_{l+1})\right) - \frac{a^3}{2\varepsilon} \sum_{\vec{n}\in\Lambda_3} \left(\varphi_{l+1}(\vec{n}) - \varphi_l(\vec{n})\right)^2 \right]$$

with

$$C = \sqrt{a^3/2\pi\varepsilon}$$

- Given this expression, we can construct the partition function and correlators (just drop the operators and normalisation to get Z_T)

$$X = \mathrm{tr}\left[e^{-(T-t)\hat{H}} \hat{O}_2(t)\, e^{-t\hat{H}} \hat{O}_1(0) \right]$$

$$= \lim_{N_T\to\infty} \mathrm{tr}\left[\hat{W}_\varepsilon^{\,N_T - n_t}\, \hat{O}_2\, \hat{W}_\varepsilon^{\,n_t}\, \hat{O}_1 \right] \qquad T = \varepsilon N_T\,,\ t = \varepsilon n_t$$

$$= \lim_{N_T\to\infty} \int \mathcal{D}_3\varphi_0\, \mathcal{D}_3\varphi_1 \ldots \mathcal{D}_3\varphi_{N_T-1}\, \mathcal{D}_3\tilde{\varphi}_1 \mathcal{D}_3\tilde{\varphi}_2 \langle \varphi_0 | \hat{W}_\varepsilon | \varphi_{N_T-1} \rangle \langle \varphi_{N_T-1} | \hat{W}_\varepsilon | \varphi_{N_T-2} \rangle \ldots$$

$$\ldots \langle \varphi_{n_t+1} | \hat{W}_\varepsilon | \tilde{\varphi}_2 \rangle \langle \tilde{\varphi}_2 | \hat{O}_2 | \varphi_{n_t} \rangle \langle \varphi_{n_t} | \hat{W}_\varepsilon | \varphi_{n_t-1} \rangle \ldots$$

$$\langle \varphi_1 | \hat{W}_\varepsilon | \tilde{\varphi}_1 \rangle \langle \tilde{\varphi}_1 | \hat{O}_1 | \varphi_0 \rangle$$

In most cases O_i will be a function of field operators $O_i(t_j) = O_i(\varphi_j)$

$$\langle \tilde{\varphi}_j | O | \varphi_k \rangle = O(\varphi_k) \langle \tilde{\varphi}_j | \varphi_k \rangle = O(\varphi_k)\, \delta(\tilde{\varphi}_j - \varphi_k)$$

So

$$X = \lim_{N_T\to\infty} C^{N^3 N_T} \int \mathcal{D}_3\varphi_0 \ldots \mathcal{D}_3\varphi_{N_T-1}\, O_2(\varphi_{n_t}) O_1(\varphi_0) \times$$

$$\times \exp\left[-\frac{\varepsilon}{2}\left(U(\varphi_0) + U(\varphi_{N_T-1})\right) - \frac{a^3}{2\varepsilon} \sum_{\vec{n}} \left(\varphi_0(\vec{n}) - \varphi_{N_T-1}(\vec{n})\right)^2 \right]$$

$$\times \ldots$$

$$\times \exp\left[-\frac{\varepsilon}{2}\left(U(\varphi_1) + U(\varphi_0)\right) - \frac{a^3}{2\varepsilon} \sum_{\vec{n}} \left(\varphi_1(\vec{n}) - \varphi_0(\vec{n})\right)^2 \right]$$

$$= \lim_{N_T\to\infty} C^{N^3 N_T} \int \mathcal{D}_3\varphi_0 \ldots \mathcal{D}_3\varphi_{N_T-1}\, O_2(\varphi_{n_t}) O_1(\varphi_0) \exp\left(-S_E[\varphi]\right)$$

with

$$S_E[\varphi] = \frac{a^3}{2\varepsilon} \sum_{j=0}^{N_T-1} \sum_{\vec{n}\in\Lambda_3} \left(\varphi_{j+1}(\vec{n}) - \varphi_j(\vec{n})\right)^2 + \varepsilon \sum_{j=0}^{N_T-1} U[\varphi_j] + O(\varepsilon^2, a^2)$$

where we assume $\varphi_{N_T} = \varphi_0$ (periodic BCs in time)

- We can define a 4D lattice, choosing the same length scale for the time step as the spatial lattice spacing, $\varepsilon = a$ (more complex schemes are possible)

$$\Lambda = \{ a\underbrace{(\vec{n}, n_4)}_{n_\mu} \mid \vec{n} \in \Lambda_3, \; n_4 \in \{0, 1, \dots, N_T - 1\}\}$$

$\Rightarrow$

$$S_E[\varphi] = a^4 \sum_{(\vec{n}, n_4) \in \Lambda} \Big\{ \frac{1}{2a^2} \left(\varphi_{n_4+1}(\vec{n}) - \varphi_{n_4}(\vec{n}) \right)^2 + \frac{1}{8a^2} \sum_{j=1}^{3} \left(\varphi_{n_4}(\vec{n}+\hat{j}) - \varphi_{n_4}(\vec{n}-\hat{j}) \right)^2$$
$$+ \frac{m^2}{2} \varphi_{n_4}(\vec{n})^2 \quad + V(\varphi_{n_4}(\vec{n})) \Big\}$$

- Defining $\mathcal{D}\varphi = \prod_{j=0}^{N_T-1} \mathcal{D}_3 \varphi_j = \prod_{j=0}^{N_T-1} \prod_{\vec{n} \in \Lambda_3} d\varphi_j(\vec{n})$ we can write

$$Z_T = \mathrm{tr}[e^{-T\hat{H}}] = \int \mathcal{D}\varphi \, e^{-S_E[\varphi]}$$

and

$$\langle \hat{O}_2(t) \hat{O}_1(0) \rangle = \frac{1}{Z_T} \mathrm{tr}[e^{-(T-t)\hat{H}} \hat{O}_2(t) e^{-t\hat{H}} O_1(0)]$$
$$= \frac{1}{Z_T} \int \mathcal{D}\varphi \, O_2[\varphi] O_1[\varphi] e^{-S_E[\varphi]}$$

- With a very small change, replacing the forward difference approximation of the time derivatives (arose from canonical quantisation) by a central difference, we restore the equality between space & time directions

$$S_E[\varphi] = a^4 \sum_{n_\mu \in \Lambda} \Big\{ \frac{1}{8a^2} \sum_{\nu=1}^{4} \left(\varphi(n_\mu + \hat{\nu}) - \varphi(n_\mu - \hat{\nu}) \right)^2 + \frac{m^2}{2} \varphi^2(n_\mu) + V(\varphi(n_\mu)) \Big\}$$

- Analogously to QM, we see that the field theory has been defined by integrals over all possible configurations of fields weighted by their Euclidean action
- The lattice procedure we went through provides a way to regulate the formally infinite functional integrals (at every non-zero a, every quantity is defined and finite
- The $a \to 0$ limit provides a definition of the theory beyond perturbation theory

2.3 Gauge fields

- There are multiple ways to introduce gauge fields, but a particularly clean geometric way is to associate them with links between sites of Λ where matter fields reside
- Consider the kinetic term for a set of complex scalar field

$$\tfrac{a}{2}\mathcal{L} \sim \mathrm{tr}[\partial_\mu \varphi^\dagger \partial_\mu \varphi] \qquad \text{trace over internal index}$$

If we demand invariance under local rotations in the internal index

$$\varphi(x) \to \Omega(\vec{\theta}(x))\,\varphi(x) = \exp\left(\tfrac{i}{2} T^a \theta^a(x)\right)\varphi(x)$$

then we must promote the derivatives to gauge covariant derivatives

$$\mathcal{L} \sim \mathrm{tr}[D_\mu \varphi^\dagger D^\mu \varphi]$$

- Starting from the free scalar theory and following our latticisation procedure we will arrive at

$$\mathcal{L}_{latt} \sim \sum_{\nu=1}^{4} \mathrm{tr}\left[(\varphi^\dagger(n_\mu+\hat{\nu}) - \varphi^\dagger(n_\mu))(\varphi(n_\mu+\nu) - \varphi(n_\mu))\right]$$

$$\sim \sum_{\nu=1}^{4} \mathrm{tr}\left[\varphi^\dagger(n+\hat{\nu})\varphi(n+\hat{\nu}) - \varphi^\dagger(n+\hat{\nu})\varphi(n) - \varphi^\dagger(n)\varphi(n+\hat{\nu}) + \varphi^\dagger(n)\varphi(n)\right]$$

Demanding $\quad \varphi(n) \to \Omega_n \varphi(n)\,, \quad \varphi(n+\hat{\mu}) \to \Omega_{n+\hat{\mu}}\varphi(n+\hat{\mu})$

we see this is not gauge invariant

- To fix this, we introduce link field $U_\mu(n)$ that transform as

$$U_\mu(n) \to U'_\mu(n) = \Omega(n)\,U_\mu(n)\,\Omega^\dagger(n+\hat{\nu})$$

and replace eg

$$\varphi^\dagger(n)\varphi(n+\hat{\nu}) \quad \text{by} \quad \varphi^\dagger(n)\,U_\nu(n)\,\varphi(n+\hat{\nu})$$

then we can maintain gauge invariance

$\Rightarrow$

$$\mathcal{L}_{latt} \sim \sum_{\nu=1}^{4} \mathrm{tr}\left[\varphi^\dagger(n+\hat{\nu})\varphi(n+\hat{\nu}) - \varphi^\dagger(n+\hat{\nu})\,U_{-\nu}(n+\hat{\nu})\,\varphi(n) + \varphi^\dagger(n)\varphi(n)\right]$$

with $\quad U_{-\nu}(n) = U_\nu(n-\hat{\nu})^\dagger$

$$\to \Omega(n)\,U_\nu(n-\hat{\nu})^\dagger\,\Omega^\dagger(n-\hat{\nu})$$

- Just as in continuum field theory, demanding a local symmetry has lead to the necessity of an additional vector field

- The link fields can be thought of as gauge-covariant transporters between the site n_μ and site $n_\mu + \nu$.
 They are specific cases of parallel transporters that map the coordinates in the internal symmetry space from one point to another, the general form of which are

$$E_\Gamma(x,y) = P\exp\{ i \int_\Gamma A \cdot ds \}$$

where Pexp is the "path-ordered exponential" along path Γ from $x \to y$

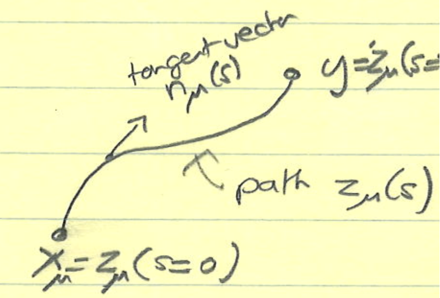
The path-ordered exponential is the solution to

$$D_s E_\Gamma(x,y) = 0$$

where

$$D_s = \frac{dz^\mu(s)}{ds} D_\mu = \frac{dz^\mu}{ds}\partial_\mu + i\frac{dz^\mu}{ds}A_\mu =$$

$$= \frac{d}{ds} + i\frac{dz^\mu}{ds} A_\mu(z)$$



is the covariant derivative along the path

Now

$$0 = D_s E_\Gamma(x,y) \Rightarrow \frac{dE_\Gamma}{ds} = \frac{i\,dz^\mu}{ds} A_\mu(z) E_\Gamma(x,y) \equiv -M(s) E_\Gamma(x,y)$$

Remembering A_μ is a matrix, this is solved by

$$E(x_\mu = z_\mu(0), y_\mu = z_\mu(t))$$

$$= \sum_{m=0}^{\infty} (-1)^m \int_0^t ds_m \int_0^{s_m} ds_{m-1} \dots \int_0^{s_2} ds_1 M(s_m) M(s_{m-1}) \dots M(s_1)$$

$$\equiv P\exp\left(i\int_0^t ds \frac{dz_\mu}{ds} A_\mu(z)\right) = P\exp\left(i\int_{x\,\Gamma}^{y} dz_\mu A^\mu(z)\right)$$

You should check that differentiating works and that

$$E_\Gamma(x,y) \longrightarrow \Omega(x) E_\Gamma(x,y) \Omega^\dagger(y)$$

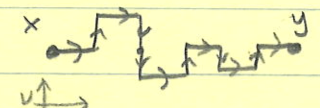
under a gauge transform

- Parallel transporters are sometimes called Wilson lines or, if along closed paths, Wilson loops

- For now it suffices to consider parallel transporters only between neighbouring sites and will assure that the lattice spacing is sufficiently small so that we can assume that $A_\nu(x)$ is constant over $x_\mu = n_\mu \to n_\mu + \hat{\nu}$ so the link variable

$$U_\nu(n) = E_\Gamma(n_\mu, n_\mu + \hat{\nu}) \quad \text{with } \Gamma \text{ a straight link path}$$
$$= \exp\left(i \int_n^{n+\hat{\nu}} dz_\mu A^\mu(z) \right)$$
$$= \exp\left(i A^\mu(n) \int_n^{n+\hat{\nu}} dz_\mu \right) = \exp\left(i a A_\nu \right) \quad \Big| \int_n^{n+\hat{\nu}} dz_\mu = a\,\delta_{\mu\nu}$$
$$\to 1 + i a A_\nu(n) + \mathcal{O}(a^2) \text{ for small } a$$

- An important property of link variables is that:

$E_\Gamma(x,y) =$  $= U_\mu(x)\, U_\nu(x+\hat{\mu}) \ldots U_\mu(y-\hat{\mu})$

transforms as $\quad E_\Gamma(x,y) \to \Omega(x) E_\Gamma(x,y) \Omega^\dagger(y) \quad$ [YOU SHOULD CHECK THIS]

for any path

$\Rightarrow \quad \varphi^\dagger(x) E_\Gamma(x,y) \varphi(y)$ is gauge invariant $\quad$ ⊛

as is $\quad E_\Gamma(x,x)$ for any closed path Γ

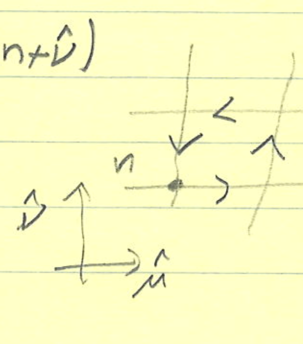
- Closed paths (Wilson loops) can be used to construct all gauge invariant quantities involving only the gauge field and quantities like ⊛ above span all gauge invariant quantities involving matter fields too. That is, the entirety of YM theory can be built from loops

$$L_\Gamma[U] = \mathrm{tr}\left[\prod_{n,\mu \in \Gamma} U_\mu(n) \right]$$

- The smallest loop is the 1×1 plaquette

$$U_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_{-\mu}(n+\hat{\mu}+\hat{\nu})\, U_{-\nu}(n+\hat{\nu})$$
$$= U_\mu(x)\, U_\nu(n+\hat{\mu})\, U^\dagger_\mu(n+\hat{\nu})\, U^\dagger_\nu(n)$$

in the $\hat{\mu}$-$\hat{\nu}$ plane



- Using the plaquettes, it is possible to construct a lattice gauge action as first done by Wilson in 1974 (<u>Wilson gauge action</u>)

$$S_G[U] = \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\left[1 - U_{\mu\nu}(n)\right]$$

which reduces to the continuum action $S_{cont}[A] = \frac{1}{2g^2}\int d^4x\, \mathrm{tr}\, F_{\mu\nu}^2(x)$ in the limit $a\to 0$

[To see this, we use the Baker-Campbell-Hausdorff formula to combine the product of the 4 exponentials in $U_{\mu\nu}(n)$: $e^A e^B = \exp\left(A+B+\frac{1}{2}[A,B]+\dots\right)$

$$U_{\mu\nu}(n) = \exp\Big[ia A_\mu(n) + ia A_\nu(n+\hat{\mu}) - \frac{a^2}{2}\left[A_\mu(n), A_\nu(n+\hat{\mu})\right] + \dots \text{ more terms } + \mathcal{O}(a^3)\Big]$$

We then expand

$$A_\nu(n+\hat{\mu}) = A_\nu(n) + a\partial_\mu A_\nu(n) + \mathcal{O}(a^2)$$

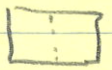
etc to give

$$U_{\mu\nu}(n) = \exp\left[ia^2\left(\partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) + i\left[A_\mu(n), A_\nu(n)\right]\right)\right] + \mathcal{O}(a^3)$$
$$= \exp\left[ia^2 F_{\mu\nu}(n)\right] + \mathcal{O}(a^3)$$
$$= 1 + ia^2 F_{\mu\nu}(n) - a^4 F_{\mu\nu}(n)^2 + \dots$$

$\Rightarrow$

$$S_G[U] = \frac{a^4}{2g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{tr}\left[F_{\mu\nu}^2(n)\right] + \mathcal{O}(a^2)$$
]

- This approximation to the continuum action becomes exact at the limit $a\to 0$ and can be modified to remove classical discretisation artifacts to $\mathcal{O}(a^4)$ or higher by making use of larger Wilson loops such as 2×1 rectangles or chairs

$S_{improved} = \alpha$  $+ \beta_1$  $+ \beta_2$  $+ \beta_2$ 

can be chosen to have only $\mathcal{O}(a^4)$ discretisation artifacts

[Technically improved actions can introduce unphysical poles in gluon propagators as they break reflection positivity [Luscher Weisz '84]]

- It is common to rewrite the Wilson action as

$$S_G[U] = \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(n)\right) \qquad \text{with} \quad \beta = \frac{2N_c}{g_0^2}$$

- With this we can now define correlation functions and the partition function of lattice Yang Mills theory in direct analogy to our discussion of the scalar field

$$\langle \hat{O}_a \hat{O}_b \rangle = \frac{1}{Z} \int \mathcal{D}U \, O_a[U]\, O_b[U]\, e^{-S_G[U]}$$

$$Z = \int \mathcal{D}U \, e^{-S_G[U]}$$

where the integration measure is

$$\mathcal{D}U = \prod_{n\in\Lambda} \prod_{\mu=1}^{4} dU_\mu(n)$$

- Since $U_\mu = e^{iaA_\mu}$ is a group-valued object, integration $dU_\mu(n)$ involves integration over all elements of the continuous (compact) Lie group G. Gauge invariance of Z and $S_G[U]$ demands that the measure is also invariant under gauge transforms and leads us to the <u>Haar measure</u>, well known in mathematical literature. Denoting group elements $U, V \in G$ this is defined by

$$dU = d(VU) = d(UV) \qquad \text{and} \qquad \int dU = 1$$

[We do not need the details, but the generic group element $U = U(\omega) = e^{iT^a\omega^a}$ with ω^a being parameters and T^a the generators of the Lie algebra.

- The metric in G is given by

$$ds^2 = g(\omega)_{mn}\, d\omega^n d\omega^m$$

where

$$g(\omega) = \mathrm{tr}\left[\frac{\partial U(\omega)}{\partial \omega^n} \frac{\partial U(\omega)^\dagger}{\partial \omega^m}\right]$$

- The measure is then

$$dU = c\sqrt{\det g(\omega)} \prod_a d\omega^a$$

in explicit form

- For a compact group G, the parameters ω^a take a finite range

- For a compact group, the integration at each spacetime point is over a compact region and so is finite. Using link variables, there is no need to gauge fix (although gauge fixing can be defined for lattice fields) - if we had used $A_\mu(n)$ we would need gauge fixing

- At a non-zero lattice spacing the functional integral is a finite (large) number of finite integrals and so is finite!!
- The lattice theory can be studied using weak coupling or strong coupling expansions
- A very important byproduct of the construction of the lattice gauge theory is that the regularised field theory (the theory at any fixed a) is now amenable to calculations on a computer

2.4 Renormalisation and the continuum limit

- The lattice theory at a fixed lattice spacing provides a particular way of regulating the underlying field theory. The lattice is unphysical just like any other regulator. It works by imposing a maximum allowed momentum component

$$p_{max} = \frac{\pi}{a}$$

at which momentum integrals are cut off. Allowed momenta are in Brillouin zone $\tilde{\Lambda} = \{ p_\mu \mid -\frac{\pi}{a} < p_\iota \leq \frac{\pi}{a}, \iota = 1,2,3,4 \}$
- In order to remove the dependence on the cutoff, the theory must be renormalised by adjustment of the coupling(s) of the theory as the cutoff is modified

- We want to hold a physical quantity constant by changing the coupling as we change the cutoff. This is the heart of Callan-Symanzik renormalisation group equation (RGE)

- Consider any physical quantity of dimension mass $H = H(g(a), a)$, a function of the coupling and cutoff. Here we consider pure Yang-Mills where there is a single coupling although the whole discussion can be extended to QCD with the quark masses being additional couplings.

We demand

$$0 = a\frac{d}{da}H(g(a),a) = a\frac{\partial H}{\partial a} + a\frac{dg}{da}\frac{\partial H}{\partial g} \qquad (*)$$

- Since the only dimensionful quantity we have is the cutoff and $[H] = \text{mass} \Rightarrow \quad H \sim a^{-1} f(g)$

So

$$a\frac{\partial H}{\partial a} = a\left(-\frac{1}{a^2}f(g)\right) = -H(g(a),a)$$

and we see that

$$a\frac{\partial H(g(a),a)}{\partial a} = -H(g(a),a)$$

Substituting this into $(*)$ gives

$$-\beta(g) \equiv a\frac{dg}{da} = \frac{H(g,a)}{\partial H/\partial g}$$

which is the β function that controls how the coupling g depends on the cutoff. Since cutoff dependence arises only at loop level, $\beta(g)$ starts at $\mathcal{O}(g^3)$ and has the form (allowing for N_F species of quarks)

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \ldots$$

with

$$\beta_0 = \frac{1}{16\pi^2}\left[\frac{11N_c}{3} - \frac{2N_F}{3}\right]$$

$$\beta_1 = \left(\frac{1}{16\pi^2}\right)^2\left[\frac{34N_c^2}{3} - \frac{10}{3}N_cN_F - N_F(N_c^2-1)/N_c\right]$$

$$\vdots$$

these two coefficients are universal, but higher order terms depend on the choice of H and on the details of the lattice cutoff

- The RGE can be solved

$$\frac{d(\ln a)}{dg} = \frac{1}{-\beta(g)} \Rightarrow \int_{g_0}^{g(a)} \frac{d\ln a}{dg'}dg' = \int_{g_0}^{g(a)} \frac{1}{\beta(g')}dg'$$

$$\Rightarrow \ln a = \int_{g_0}^{g(a)} dg' \frac{1}{\beta_0 g'^3} = \frac{-1}{2\beta_0}\left(\frac{1}{g^2(a)} - \frac{1}{g_0^2}\right)$$

$$\Rightarrow a(g) = \frac{1}{\Lambda_L} \exp\left(\frac{-1}{2\beta_0 g^2}\right)$$

where Λ_L is an integration constant arising from g_0

Including the g^5 term in $\beta(g)$ gives

$$a(g) = \frac{1}{\Lambda_L} (\beta_0 g^2)^{-\beta_1/2\beta_0^2} \exp\left(\frac{-1}{2\beta_0 g^2}\right) + \dots$$

Inverting gives

$$\frac{1}{g(a)^2} = -2\ln(a\Lambda_L) + \frac{\beta_1}{\beta_0}\ln\left(\ln(a^{-2}\Lambda_L^{-2})\right) + \dots$$

So taking $a \to 0$ requires $g(a) \to 0$ and the continuum limit is the limit of vanishing coupling with H held fixed

- Note that we should also be careful to take $N = L/a$ large as we take $a \to 0$ so that we are describing the same physical region of spacetime

3 Lattice Fermion

- Fermions present special challenges in lattice calculations that stem from the anticommuting nature of fermions and from chirality

3.1 - Fermions in field theory

- In the functional integral approach to field theory, the fermion fields are implemented as anticommuting Grassmann variables that satisfy

$$\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i \qquad \Rightarrow \quad \varepsilon_i^2 = 0$$

- Any function of a set of P Grassmann numbers $\{\varepsilon_i\}$ is a finite order polynomial $\quad f_i \in \mathbb{C}$ etc

$$f(\varepsilon_1, \varepsilon_2, \ldots) = f_0 + \sum_i f_i \varepsilon_i + \sum_{i<j} f_{ij} \varepsilon_i \varepsilon_j + \cdots + f_{12\ldots N} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_P$$

and these polynomials form a Grassmann algebra $\mathcal{G}$ that has multiplication and (commutative) addition and scalar multiplication.

- In the field theory context we will have an independent Grassmann variable for each point in spacetime: $\varepsilon_1 = \Psi(x_1)$ $\varepsilon_2 = \Psi(x_2) \ldots$ so the Grassmann algebra has an infinite dimensional basis, On a lattice, we have N^4 sites so $P = 2N^4$ (we also need to represent $\bar{\Psi}(x_1) \ldots$)
- We need to integrate over the fermion fields so need to define Grassmannian integration that maps $\mathcal{G} \to \mathbb{C}$ and is linear

normalisation here is a convention

$$\int d\varepsilon\, 1 = 0 \qquad \int d\varepsilon\, \varepsilon = \underline{1} \qquad \int d\varepsilon_i\, \varepsilon_j = \delta_{ij}$$

with the measure also a Grassmann valued object, Thus

$$\begin{aligned}\int d\varepsilon_1 d\varepsilon_2\, \varepsilon_1 \varepsilon_2 &= -\int d\varepsilon_2 \left(\int d\varepsilon_1\, \varepsilon_1\right) \varepsilon_2 \\ &= -\int d\varepsilon_1 \left(d\varepsilon_2\, \varepsilon_2\right) \varepsilon_1 \qquad = -1\end{aligned}$$

- Derivatives are defined in the obvious way

$$\frac{\partial}{\partial \varepsilon_1}\left(f_0 + f_1 \varepsilon_1 + f_{12} \varepsilon_1 \varepsilon_2\right) = f_1 + f_{12} \varepsilon_2$$

and are act identically to integration

Also

$$\int d\varepsilon_1 \frac{\delta}{\delta \varepsilon_1} f(\varepsilon_1, \varepsilon_2 \ldots) = 0$$

- We are particularly interested in Grassmannian functional integrals of functions that are Gaussians

$$I = \int \mathcal{D}\psi \mathcal{D}\bar\psi \exp\Big[\overbrace{-\int dx dy \bar\psi(x) M(x,y) \psi(y)}^{S[\psi,\bar\psi]}\Big]\, g[\psi,\bar\psi]$$

where the measure ← Grassmann

$$\mathcal{D}\psi = d\psi(x_1)\, d\psi(x_2) \ldots = \prod_{i=1}^{\infty} d\psi_i \qquad \text{(order must be consistent}$$

and $g[\psi,\bar\psi]$ is some functional of the Grassmann variables

- first take $g[\psi,\bar\psi] = 1$ and consider just 2 variables

$$\int d\xi d\bar\xi \exp[-m\bar\xi\xi] = \int d\xi d\bar\xi (1 - m\bar\xi\xi) =$$
$$= -m$$

- Now consider many variables we have

$$\int \mathcal{D}\psi \mathcal{D}\bar\psi \exp\left[-\int dx dy\, \bar\psi(x) M(x,y) \psi(y)\right]$$

which we think of as the limit $N \to \infty$ of (up to a sign from reordering the measure)

$$Z_N = \int d\varepsilon_n d\bar\varepsilon_n d\varepsilon_{n-1} d\bar\varepsilon_{n-1} \ldots d\varepsilon_1 d\bar\varepsilon_1 \qquad [\psi(x_i) = \varepsilon_i \text{ etc}$$
$$\times \exp\Big[-\sum_{ij} \bar\varepsilon_i M_{ij} \varepsilon_j\Big] \qquad \text{(absorbing in } M$$

which we evaluate by shifting variables ε_j to

$$\varepsilon_j' = \sum_k M_{jk} \varepsilon_k \quad \Rightarrow \quad \varepsilon_i = \sum_j (M^{-1})_{ij} \varepsilon_j'$$

under which the integration measure picks up a Jacobian

$$d\varepsilon_n d\bar\varepsilon_n \ldots d\varepsilon_1 d\bar\varepsilon_1 = \det M\, d\varepsilon_n' d\bar\varepsilon_n \ldots d\varepsilon_1' d\bar\varepsilon_1$$

[To see this; consider the normalisation integral

$$1 = \int d\varepsilon_n d\varepsilon_{n-1} \ldots d\varepsilon_1\, \varepsilon_1 \ldots \varepsilon_n = \int d\varepsilon_n' \ldots d\varepsilon_1' \varepsilon_1' \ldots \varepsilon_n'$$
$$= \int d\varepsilon_n' \ldots d\varepsilon_1' \sum_{k_1 \ldots k_n} M_{1k_1} \ldots M_{nk_n} \varepsilon_{k_1} \ldots \varepsilon_{k_n}$$
$$= \int d\varepsilon_n' \ldots d\varepsilon_1' \underbrace{\sum_{k_1 \ldots k_n} M_{1k_1} \ldots M_{nk_n} \varepsilon_{k_1 \ldots k_n}}_{\det M} \varepsilon_1 \ldots \varepsilon_n \qquad \text{(reorder them! using antisymmetry)}$$
$$= \det M \int d\varepsilon_n' \ldots d\varepsilon_1' \varepsilon_1 \ldots \varepsilon_n$$

So

$$Z_N = \det M \int d\varepsilon_n' d\bar\varepsilon_n \ldots d\varepsilon_1' d\bar\varepsilon_1 \exp\Big[+\sum_{ij} \bar\varepsilon_i M_{ij} \sum_k (M^{-1})_{jk} \varepsilon_k'\Big]$$
$$= \det M \int d\varepsilon_n' d\bar\varepsilon_n \ldots d\varepsilon_1' d\bar\varepsilon_1 \exp\Big[+\sum_i \bar\varepsilon_i \varepsilon_i'\Big]$$
$$= \det M \int d\varepsilon_n' d\bar\varepsilon_n \ldots d\varepsilon_1' d\bar\varepsilon_1 (1 + \bar\varepsilon_1 \varepsilon_1')(1 + \bar\varepsilon_2 \varepsilon_2') \ldots (1 + \bar\varepsilon_n \varepsilon_n')$$

$$= \overset{n}{=} \det M \int d\varepsilon_n' d\bar{\varepsilon}_n \ldots d\varepsilon_1' d\bar{\varepsilon}_1 \; \bar{\varepsilon}_1 \varepsilon_1' \bar{\varepsilon}_2 \varepsilon_2' \ldots \bar{\varepsilon}_n \varepsilon_n' \qquad \left(\begin{array}{l}\text{all other} \\ \text{terms } \int \text{ to zero}\end{array}\right.$$

$$= \det M$$

[This result is known as the Matthews-Salam formula.

- We can extend our Grassmann algebra to twice the size introducing source fields $\eta(x), \bar{\eta}(x) \Rightarrow \{\eta_1, \eta_2, \ldots \eta_n, \bar{\eta}_1 \ldots \bar{\eta}_n\}$ which anticommute with the fermion fields and define the generating functional

$$Z_N[\eta, \bar{\eta}] = \int d\varepsilon_1 d\bar{\varepsilon}_1 \ldots d\varepsilon_n d\bar{\varepsilon}_n \exp\left[\sum_{ij} \bar{\varepsilon}_i M_{ij} \varepsilon_j + \sum_k \bar{\eta}_k \varepsilon_k + \sum_k \bar{\varepsilon}_k \eta_k\right]$$

$$= \int d\varepsilon_1 d\bar{\varepsilon}_1 \ldots d\varepsilon_n d\bar{\varepsilon}_n \exp\left[\left(\bar{\varepsilon}_i + \bar{\eta}_k M^{-1}_{ki}\right) M_{ij} \left(\varepsilon_j + M^{-1}_{je} \eta_e\right) - \bar{\eta}_i M^{-1}_{ij} \eta_j\right]$$

now perform linear shifts $\varepsilon_i \to \varepsilon_i' = \varepsilon_i + M^{-1}_{ij} \eta_j$, $\bar{\varepsilon}_i \to \bar{\varepsilon}_i' = \bar{\varepsilon}_i + \bar{\eta}_j (M^{-1})_{ji}$ under which the integration measure is invariant (Jacobian is 1)

$$= \int d\varepsilon_1' d\bar{\varepsilon}_1' \ldots d\varepsilon_n' d\bar{\varepsilon}_n' \exp\left[\bar{\varepsilon}_i' M_{ij} \varepsilon_j' - \bar{\eta}_i M^{-1}_{ij} \eta_j\right]$$

$$= \det M \exp\left[-\bar{\eta}_i M^{-1}_{ij} \eta_j\right]$$

- Correlation functions arise as functional derivatives with respect to $\eta, \bar{\eta}$

$$\langle 0 | \psi(x) \bar{\psi}(y) \ldots | 0 \rangle = \frac{1}{Z_N[0]} \frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)} \ldots Z_N[\eta, \bar{\eta}] \Big|_{\eta = \bar{\eta} = 0} \qquad (N \to \infty)$$

but with the integration performed in Z, this is straightforward

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{\partial}{\partial \bar{\eta}_e} \frac{\partial}{\partial \eta_k} \ldots \exp\left[-\bar{\eta}_i (M^{-1})_{ij} \eta_j\right] \Big|_{\eta = \bar{\eta} = 0}$$

$$= \frac{\partial}{\partial \bar{\eta}_e} \frac{\partial}{\partial \eta_k} \ldots \left(1 - \bar{\eta}_1 M^{-1}_{11} \eta_1\right)\left(1 - \bar{\eta}_1 M^{-1}_{12} \eta_2\right) \ldots \left(1 - \bar{\eta}_n M^{-1}_{nn} \eta_n\right) \Big|_{\eta = \bar{\eta} = 0}$$

$$= (M^{-1})_{ek} .$$

or in general

$$\langle 0 | \psi_{i_1} \bar{\psi}_{j_1} \ldots \psi_{i_m} \bar{\psi}_{j_m} | 0 \rangle = (-1)^m \sum_{\substack{\text{perms} \\ P(1, \ldots, m)}} \operatorname{sign}(P) \, (M^{-1})_{i_1 j_{P(1)}} (M^{-1})_{i_2 j_{P(2)}} \ldots (M^{-1})_{i_m j_{P(m)}}$$

3.2 Naive fermions discretisation

- In a gauge theory the fermion matrix M depends on the gauge field (A_μ in continuum version, U_μ in lattice formulation) and all the fermion manipulation are performed under $\int$ over gauge degrees of freedom.

- As for the complex scalar field, the discretisation of the covariant derivative will involve parallel transporters in order to be gauge covariant

$$\int dx\, \bar\Psi \gamma_\mu D^\mu \psi \rightarrow a^4 \sum_{n\in\Lambda} \sum_{\mu=1}^{4} \bar\Psi(n)\gamma_\mu \left(\frac{U_\mu(n)\psi(n+\hat\mu) - U_{-\mu}(n)\psi(n-\hat\mu)}{2a} \right)$$

$$= a^4 \sum_{n,m\in\Lambda} \bar\Psi(n)\, D(n,m)\, \psi(m)$$

where $D(n,m)$ is a matrix in space-time as well as colour and spin. In component form

$$D(n,m)^{a\alpha}_{b\beta} = \sum_{\mu=1}^{4} (\gamma_\mu)_{\alpha\beta} \left(\frac{U_\mu(n)_{ab}\,\delta_{n+\hat\mu,m} - U_{-\mu}(n)_{ab}\,\delta_{n-\hat\mu,m}}{2a} \right) + M\,\delta_{ab}\,\delta_{\alpha\beta}\,\delta_{n,m}$$

where we have included the mass term as well

- NB: we had to use central difference as fwd/backward derivs not anti Hermitian (as $\partial_\mu \sim ik_\mu$ is)
- The above naive fermion action does not lead to the correct continuum limit

Consider the Fourier transform of the inverse free two point function (fermion propagator)

$$\tilde D(k,\ell) = \sum_{n,m} e^{-i(k\cdot n - \ell\cdot m)} \langle 0|\psi(n)\bar\Psi(m)|0\rangle^{-1}$$

$$= \sum_{n,m} e^{-i(k\cdot n - \ell\cdot m)}\, D(n,m)\big|_{U_\mu\to 1}$$

$$= \sum_{n,m} e^{-i(k\cdot n - \ell\cdot m)} \left(\sum_{\mu=1}^{4} \gamma_\mu (\delta_{n+\hat\mu,m} - \delta_{n-\hat\mu,m})/2a + M\,\delta_{nm} \right)$$

$$= \sum_{n} e^{-i(k-\ell)\cdot n} \left(\sum_{\mu=1}^{4} \gamma_\mu \left(e^{i\ell a} - e^{-i\ell a}\right)/2a + M\,\mathbb{I} \right)$$

$$= \delta^4(k-\ell)\left(m + \frac{i}{a}\sum_{\mu=1}^{4} \gamma_\mu \sin k_\mu a \right) = \delta^4(k-\ell)\,\tilde D(k)$$

Now

$$\tilde D(k)^{-1} = \frac{m - i/a \sum_\mu \gamma_\mu \sin k_\mu a}{(m - i/a \sum_\mu \gamma_\mu \sin k_\mu a)(m + i/a \sum_\mu \gamma_\mu \sin k_\mu a)} = \frac{m - i/a \sum_\mu \gamma_\mu \sin k_\mu a}{m^2 + \frac{1}{a^2}\sum_\mu \sin^2 k_\mu a}$$

- Now we can undo the Fourier transform and $D(n,m)^{-1}$ is the two point correlator (propagator)

- Already looking a $\tilde{D}(k)^{-1}$ we see that for $M=0$ the propagator will have a pole at $k^\mu = (0,0,0,0)$ about which it will behave correctly as the continuum limit is taken

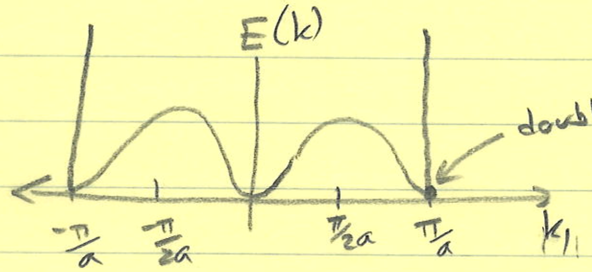
$$\tilde{D}(k)^{-1} = \frac{-i/a \sum_\mu \gamma_\mu \sin(k^\mu a)}{1/a^2 \sum_\mu \sin^2(k^\mu a)} \xrightarrow[a \to 0]{} \frac{-i \sum_\mu \gamma_\mu k^\mu}{\sum_\mu (k_\mu)^2}$$

However the propagator will also be singular at

$$k^\mu = (0,0,0,\pi/a), \ldots \left(\frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}\right)$$

These 15 unwanted <u>doubler</u> modes survive the continuum limit and the naive action describes a different theory

- The dispersion relation for this discretisation is



- It is easy to see that something like this has to occur as reality of the action and translational invariance mean that the momentum dependence of the propagator must be periodic over the Brillouin zone $\tilde{\Lambda}$

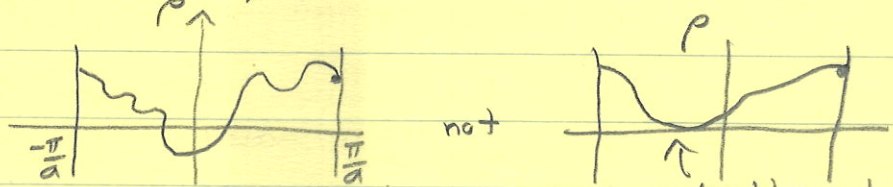
$$\tilde{\Lambda} = \{ p_\mu \mid -\tfrac{\pi}{a} < p_i \leq \tfrac{\pi}{a} \ \forall i \}$$

The general form for a chirally invariant fermion action is

$$S_F[\psi, \bar{\psi}] = a^4 \sum_{nm} \bar{\psi}(n) \gamma_\mu \rho^\mu(n-m) \psi(m)$$

Fourier transform

with $\tilde{\rho}_\mu(p)$ being regular (locality) and no other Dirac structures allowed (chirality). Given these constraints Nielsen & Ninomiya (also Karsten & Smit) showed that if there is a point where $\tilde{\rho}_\mu(p) = 0 \ \forall \mu$ then there must be at least one other point where $\tilde{p}_\mu(\bar{p})=0\ \forall\mu$ too. This is easy to see in 1D



3.3) Wilson Fermions

- To get around the Nielsen-Ninomiya theorem, Wilson proposed to break chiral symmetry explicitly even when $M=0$ by adding a second derivative term that acts as a large mass $\Big\{$ of order of the cutoff at the doubler poles but is irrelevant at zero momentum.

- The Wilson fermion action is

$$S_{\text{Wilson}}[\bar{\Psi},\Psi,U]=a^4\sum_{n,m\in\Lambda}\bar{\Psi}(n)\,D_{\text{Wilson}}(n,m)\,\Psi(m)$$

with

$$D_{\text{Wilson}}(n,m)=M\delta_{nm}+\sum_{\mu=1}^{4}\gamma_\mu\frac{U_\mu(n)\,\delta_{n+\hat{\mu},m}-U_{-\mu}(n)\,\delta_{n-\hat{\mu},m}}{2a}$$
$$+\frac{ar}{2}\sum_{\mu=1}^{4}\frac{U_\mu(n)\,\delta_{n+\hat{\mu},m}-2\delta_{n,m}+U_{-\mu}(n)\,\delta_{n-\hat{\mu},m}}{2a^2}$$

where the new Wilson term is a covariant discretised Laplacian and $0<r\leq 1$ is a free parameter (outside this range r would give complex dispersion relation)

- If we choose $r=1$ (usual convention) we can write

$$D_{\text{Wilson}}(n,m)=\mathcal{M}\,\delta_{n,m}+\frac{1}{2a}\sum_{\mu=1}^{4}\left(P_\mu^-U_\mu(n)\,\delta_{n+\hat{\mu},m}+P_\mu^+U_{-\mu}(n)\,\delta_{n-\hat{\mu},m}\right)$$

where

$$P_\mu^\pm=\frac{1}{2}(1\pm\gamma_\mu)\quad\text{and}\quad\mathcal{M}=M+4$$

- If we rescale the quark fields $\psi\to\mathcal{M}^{-1/2}\psi$, $\bar{\psi}\to\bar{\psi}\,\mathcal{M}^{-1/2}$

$$D_{\text{Wilson}}(n,m)=\delta_{n,m}+\kappa\sum_{\mu=1}^{4}\left(P_\mu^-U_\mu(n)\,\delta_{n+\hat{\mu},m}+P_\mu^+U_{-\mu}(n)\,\delta_{n-\hat{\mu},m}\right)$$

where the hopping parameter $\kappa=\frac{1}{2\mathcal{M}}$ is an input bare mass parameter which is renormalised in a similar way to the coupling $\Big\}$ for free fermions $m=0 \Rightarrow \kappa_c$

For completeness, the Nielsen-Ninomiya Theorem states that it is not possible to construct a Dirac operator $\tilde{D}(p)$ (here in momentum space) that simultaneously satisfies the following properties

a) $\tilde{D}(p)$ is regular (analytic) and periodic over the Brillouin zone

b) $\tilde{D}(p) = i\gamma_\mu p_\mu + \mathcal{O}(ap^2)$ for small momenta

c) $\tilde{D}(p)$ is invertible $\forall\, p$ that are non-vanishing mod $\frac{2\pi}{a}$

d) $\{\tilde{D}(p), \gamma_5\} = 0$ (chiral symmetry)

- The free propagator for Wilson fermions (zero gauge field) is

$$\tilde{D}^{-1}(k) = \left[m\mathbb{1} + \frac{i}{a}\sum_\mu \gamma_\mu \sin(k_\mu a) + \mathbb{1}\frac{1}{a}\sum_\mu (1-\cos k_\mu a)\right]^{-1}$$

for $k_\mu = (0,0,0,0)$ the new term vanishes leaving a pole, but for $k_\mu = (0, \frac{\pi}{a}, 0, -\frac{\pi}{a})$ etc the last term contributes an additional mass $m \to m + \frac{2\ell}{a}$ where $\ell = \#$ of $\frac{\pi}{a}$ components in k

As $a \to 0$, these modes decouple as their mass $\to \infty$

- Wilson's action achieves a single light fermion mode, but at the cost of introducing explicit chiral symmetry breaking as

$$\bar{\psi} D^2 \psi = \bar{\psi}_L D^2 \psi_R + \bar{\psi}_R D^2 \psi_L$$

is not chirally invariant.

- For heavy quarks this is not a big issue as chiral symmetry is broken by the mass anyway, but for physical mass u/d quarks, it is problematic. The absence of chiral symmetry also allows operators of different chirality to mix at nonzero lattice spacing which complicates many calculations (see later)

3.4 Chiral fermions

- In the 1990s, it was realised how to deal correctly with chiral fermions in lattice field theory

NNTh tells us we can not

- The key is to not preserve continuum chiral symmetry, but to preserve a "lattice chiral symmetry" that reduces to the usual symmetry in the chiral limit

- We consider a Dirac operator that satisfies the Ginsparg-Wilson relation

$$\gamma_5 D[U] + D[U]\gamma_5 = a\, D[U]\gamma_5 D[U]$$

(more complex forms also exist)

which reduces to a chiral invariance constraint for $a \to 0$

- Now consider a modified chiral transformation (Lüscher 1998)

$$\psi \to \psi' = \psi + \delta\psi = \left(1 + i\varepsilon^a T^a \gamma_5 \left(1 - \tfrac{a}{2} D[U]\right)\right)\psi$$

$$\bar\psi \to \bar\psi' = \bar\psi + \delta\bar\psi = \bar\psi\left(1 + i\varepsilon^a T^a \left(1 - \tfrac{a}{2} D[U]\right)\gamma_5\right)$$

which explicitly depends on the gluon field

- This is a symmetry of the lattice action for a GW preserving Dirac operator

$$\bar\psi' D[U]\psi' = \bar\psi\left(1 + i\varepsilon^a T^a \left(1 - \tfrac{a}{2} D[U]\right)\gamma_5\right) D[U] \times \left(1 + i\varepsilon^b T^b \gamma_5 \left(1 - \tfrac{a}{2} D[U]\right)\right)\psi$$

$$= \bar\psi D[U]\psi + \bar\psi\, i\varepsilon^a T^a \left(\gamma_5 D + D\gamma_5 - \tfrac{2a}{2} D\gamma_5 D\right)\psi + \mathcal{O}(\varepsilon^2)$$

$$= \bar\psi D[U]\psi \quad \text{by GWR}$$

(here we include $T^0 = \mathbb{1}$)

- Very interestingly the fermion integration measure is not invariant

$$\mathcal{D}\bar\psi'\mathcal{D}\psi' = \mathcal{D}\bar\psi\left(1 + i\varepsilon^a T^a \left(1 - \tfrac{a}{2} D\right)\gamma_5\right)\left(1 + i\varepsilon^a T^a \gamma_5 \left(1 - \tfrac{a}{2} D\right)\right)\mathcal{D}\psi$$

$$= \mathcal{D}\bar\psi\mathcal{D}\psi\left(1 + i\varepsilon^a \mathrm{Tr}\left[T^a \gamma_5 (2 - aD)\right]\right) + \mathcal{O}(\varepsilon^2)$$

$$= \mathcal{D}\bar\psi\mathcal{D}\psi\left(1 - i\varepsilon^0\, \mathrm{tr}_0\left[\gamma_5 D\right]\right)$$

$T^0 = \mathbb{1}$ ie $U_A(1)$ rotation

using $\mathrm{tr}[T^a] = 0$ $a = 1 \dots N_f^2 - 1$ and $\mathrm{tr}[\gamma_5] = 0$

So just as in the continuum the axial $U_A(1)$ anomaly arises from the non-invariance of the measure [This should be contrasted to Wilson fermions where the action is not invariant but the integration measure is]

$\mathrm{tr}[\gamma_5 D]$ counts ~~chiral~~ zero modes

- The modified symmetry is enough to protect quantities usually protected by chiral symmetry

- There are 3 known constructions of lattice Dirac operators that satisfy the GW relation
    - domain wall fermions (DB Kaplan; Shamir)
    - overlap fermions (Narayanan & Neuberger)
    - perfect actions (Hasenfratz et al.)
      a.k.a. fixed point fermions

- We will not have time to go into these (see Chandrasekharan & Wiese review) but they all avoid the Nielsen-Ninomiya theorem by introducing a minimal breaking of continuum chiral symmetry but the GW relation and symmetry above is enough to give all the benefits of chiral symmetry anyway.