Hadron Physics
& QCD's Dyson-Schwinger Equations

Lecture 2: The Dyson-Schwinger Equations

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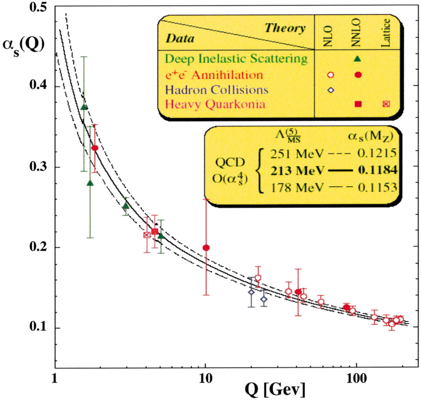
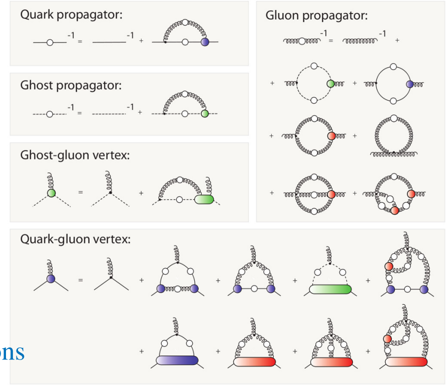
A Toolkit for QCD

- QCD is innately non-perturbative and is characterized by two emergent phenomena: Confinement & *DCSB*
  - both these phenomena are *NOT* apparent from the QCD Lagrangian
- The QCD interaction is non-perturbative over 98% of the proton's volume
- Critical need for modern theory to guide modern experiment. *Desired attributes:*



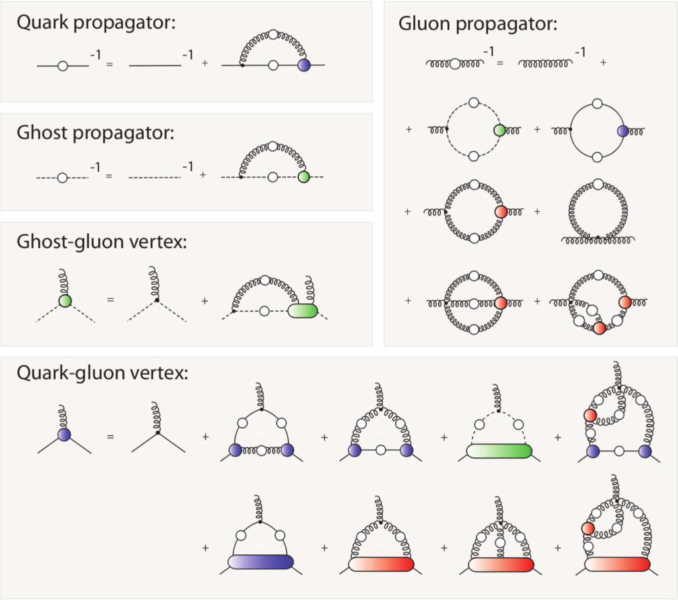
  - must possess a direct link to QCD, so that connection with established predictions of (perturbative) QCD can be established
  - must be capable of calculating *hadron wave functions*
  - capable of connecting wave functions with Wigner distributions $\Longrightarrow$ *generalized parton distribution* and *transverse momentum dependent distribution* functions
  - must be able to unify meson & baryon properties
- Both lattice QCD and the DSEs provide such a framework

QCD's Dyson-Schwinger Equations



- The equations of motion of QCD $\Longleftrightarrow$ QCD's Dyson–Schwinger equations
  - an infinite tower of coupled integral equations
  - must implement a symmetry preserving truncation
- Some key features of the DSEs:
  - provides a non-perturbative, continuum approach to QCD
  - hadrons are composites of quarks & gluons
  - Poincaré covariant and renormalizable
  - encapsulates dynamical chiral symmetry breaking $\Longleftrightarrow$ the generation of mass from nothing
  - coloured objects are confined $\Longleftrightarrow$ exhibits colour confinement
  - its elements have a direct connection with QCD
- Computationally inexpensive so can therefore provide rapid feedback and guidance to experiment. *Physics is an empirical science* $\Longleftrightarrow$ *experiment*

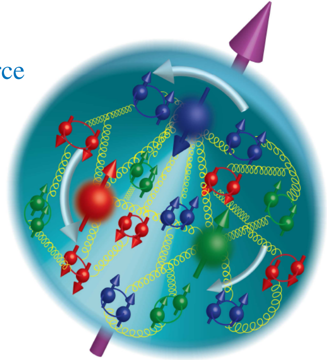
QCDs Dyson–Schwinger Equations



ETC!

Roads to Discovery

- Meson and baryon spectroscopy
  - the discovery of exotic or hybrid hadrons would force a dramatic reassessment of the distinction between the notions of matter fields and force fields

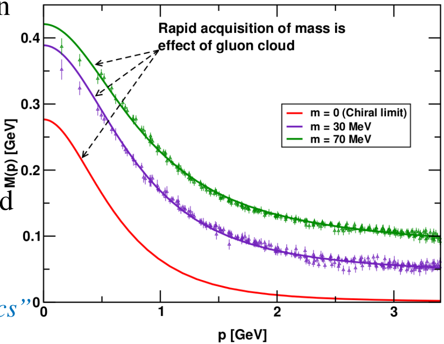


- Exploit opportunities provided by new data on nucleon elastic and transition form factors
  - chart infrared evolution of QCD's coupling and dressed-masses
  - reveal correlations that are key to nucleon structure
  - expose the facts or fallacies in modern descriptions of nucleon structure
- Precision experimental study of valence region, together with theoretical computation of distribution functions and distribution amplitudes
  - computation is critical – without computation an endless amount of data can only reveal a limited amount about the theory underlying strong interaction physics
- *The DSEs are an ideal tool with which to address these challenges!*

Significant Progress using DSEs

- A novel understanding of quark & gluon confinement – and its consequences – is beginning to emerge
- Provides a compelling picture that connects the perturbative domain of QCD's Green functions with the infrared
  - a prominent example is for the quark propagator; soon to be a textbook result: *"Foundations of Nuclear & Particle Physics"*

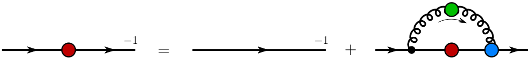
[M. S. Bhagwat *et al.*, Phys. Rev. **C 68**, 015203 (2003)]



- Arriving at a clear picture of how hadron masses emerge dynamically in a universe with light quarks through DCSB
- Detailed understanding of the Goldstone nature of the pion and its internal structure
- Performed realistic calculations of ground and excited state hadron wave functions whose structure reflects that of QCD
  - illuminated the important quark-quark correlations inside baryons

QCD's Gap Equation

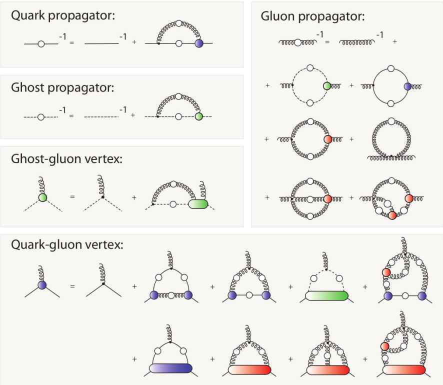
- Most important DSE is QCD's gap equation $\Longrightarrow$ *dressed quark propagator*



  - ingredients – *dressed gluon propagator & dressed quark-gluon vertex*

$$S(p)^{-1} = Z_2\left(i\not{p} + m_0\right) + Z_1 \int \frac{d^4k}{(2\pi)^4}\, g^2\, D_{\mu\nu}(p-k)\, \frac{\lambda^a}{2}\, \gamma^\mu\, S(k)\, \Gamma^{a,\nu}(p,k)$$

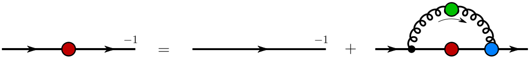
  - $S(p)$ dressed quark propagator
  - $D_{\mu\nu}(p-k)$ dressed gluon propagator
  - $\Gamma^{a,\nu}(p,k)$ dressed quark-gluon vertex
  - m_0 bare current quark mass
  - $Z_1,\, Z_2$ vertex and quark wave function renormalization constants



- Gap equation is exact – yet deceptively simply
  - sums a countable infinity of diagrams
  - impossible in perturbation theory

QCD's Gap Equation

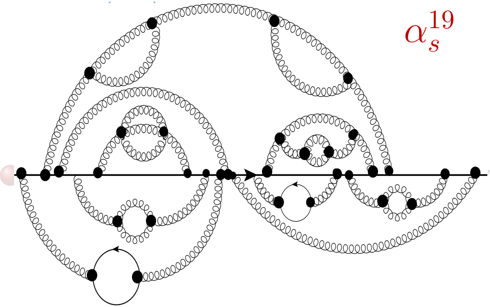
- Most important DSE is QCD's gap equation $\implies$ *dressed quark propagator*

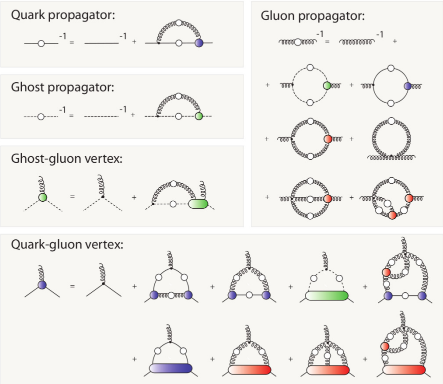


- ingredients – *dressed gluon propagator & dressed quark-gluon vertex*

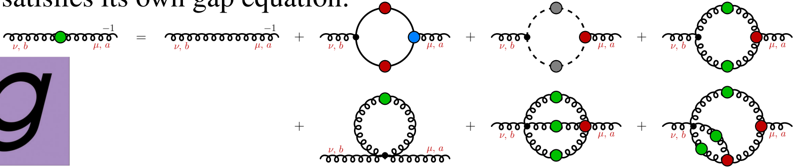
$$S(p)^{-1} = Z_2\left(i\not p + m_0\right) + Z_1 \int \frac{d^4k}{(2\pi)^4}\, g^2\, D_{\mu\nu}(p-k)\, \frac{\lambda^a}{2}\, \gamma^\mu\, S(k)\, \Gamma^{a,\nu}(p,k)$$

- $S(p)$ dressed quark propagator
- $D_{\mu\nu}(p-k)$ dressed gluon propagator





- Gluon satisfies its own gap equation:

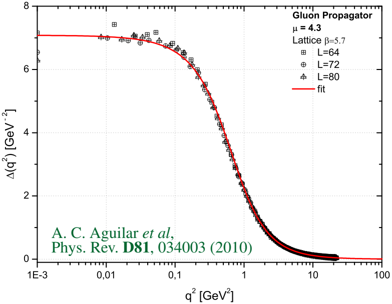


  - additional ingredients: ghost propagator; ghost-gluon vertex; 3-, 4-gluon vertices

- In covariant gauge gluon propagator has one dressing function

$$D^{\mu\nu}(q) = \left(\delta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\Delta(q^2) + \xi\,\frac{q^\mu q^\nu}{q^4}$$

  - usually choose Landau gauge $\xi = 0$; fixed point of the RGEs



- Gluons also possess a dynamically generated mass

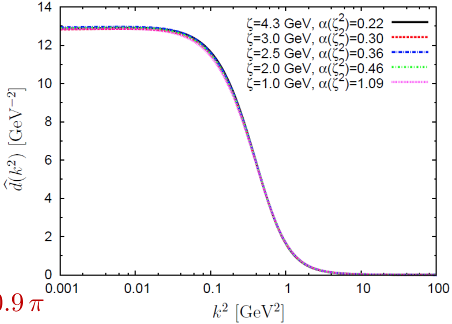
- *Dynamically generated masses for quarks and gluons means that QCD dynamically generates its own infrared cutoffs*

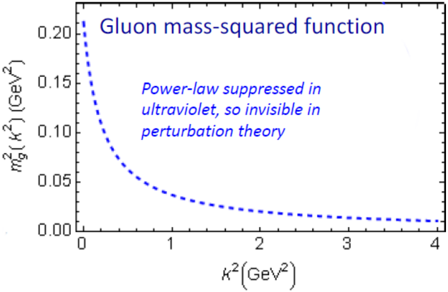
- Careful analysis of the complex of quark, gluon and ghost gap equations yields the RGI function:

$$\hat{d}(k^2) = \frac{\alpha_s(\zeta)}{k^2 + m_g^2(k^2,\zeta)}$$

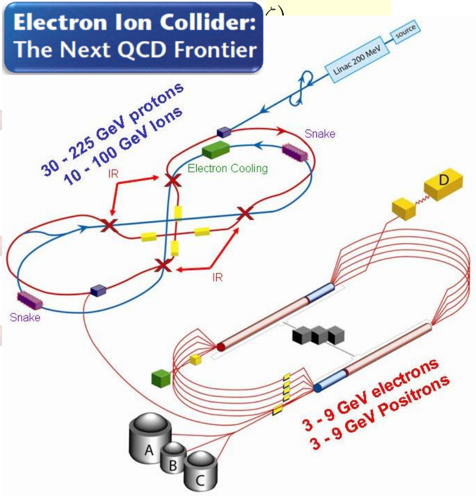
- Can identify a gluon mass function with the infrared scales: $m_g^2(0) = (0.46\,\text{GeV})^2;\ \alpha_s(0) = 2.77 \simeq 0.9\,\pi$
- Role of gluons with wavelength larger than $1/m_g(0)$ are greatly suppressed
- Hadron structure at low Bjorken-x is dominated by gluons
  - features in this regime must reflect infrared properties of gluon dressing function; e.g. gluon saturation @ an EIC

[D. Binosi *et al.*, Phys. Lett. B **742**, 183 (2015)]





- Careful analysis of the complex of quark, gluon and ghost gap equations yields the RGI function:



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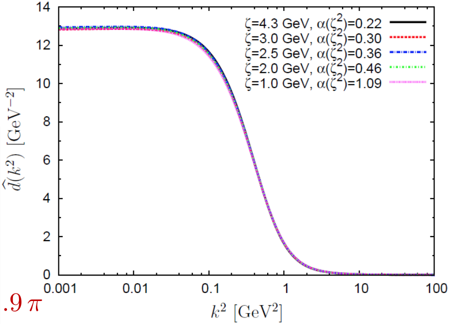
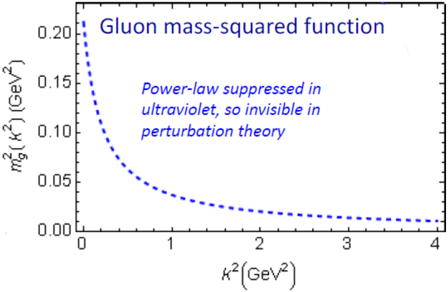
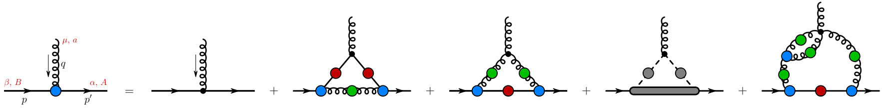
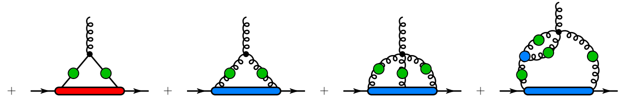


.9 π





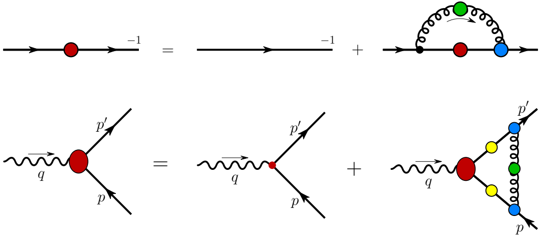


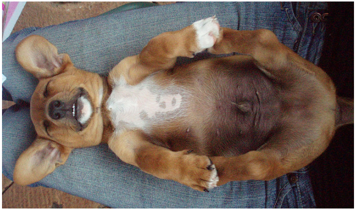
- The quark-gluon vertex presents the biggest challenge and necessitates a truncation
- In general the quark-gluon vertex has the form

$$\Gamma^{a,\mu}_{gqq}(p',p) = \frac{\lambda^a}{2} \sum_{i=1}^{12} \Lambda_i^\mu \; f_i(p'^2,p^2,q^2) = \frac{\lambda^a}{2} \left[\Gamma^\mu_L(p',p) + \Gamma^\mu_T(p',p)\right]$$

- Truncation scheme that must maintain symmetries of theory
  - we will define the truncation in Landau gauge ($\xi = 0$), so $SU(3)_c$ gauge invariance becomes a moot point
  - in principle could use the Landau-Khalatnikov-Fradkin transformations (LKFT) to tranform Green functions from one gauge to another

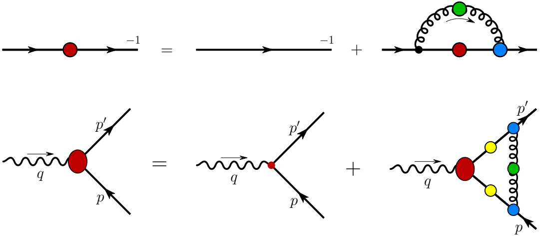
Preserving Symmetries





- The truncation must preserve the global symmetries of QCD, as well as exhibit DCSB and colour confinement
- Confinement can be checked by investigation of the analytic structure of the propagtors of coloured states
- To guarantee that the consequences of DCSB are respected need to be determine the properties of bound states
  - for relativistic two-body bound states these properties are given by the Bethe-Salpeter equation
  - *kernels of the gap and Bethe-Salpeter equations must be intimately related*

Ward–Takahashi identities



- Consider the vector and axial-vector Ward–Takahashi identities (WTIs)

$$q_\mu \, \Gamma^\mu_{\gamma qq}(p',p) = \hat{Q}_q \left[S_q^{-1}(p') - S_q^{-1}(p) \right]$$

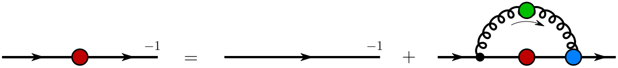
$$q_\mu \, \Gamma_5^{\mu,i}(p',p) = S^{-1}(p') \, \gamma_5 \, t_i + t_i \, \gamma_5 \, S^{-1}(p) + 2 \, m \, \Gamma_\pi^i(p',p)$$

  - relates quark-photon vertex and the inhomogeneous axial-vector & pseudoscalar vertices with quark propagator

- Satisfing these WTIs will guarantee, for example, electromagnetic current conservation and a robust realization of DCSB

- *Therefore feedback from experiment can help constrain the elements of QCD within the framework provided by the DSEs*

Rainbow Ladder Truncation



$$S(p)^{-1} = Z_2\left(i\not{p} + m_0\right) + Z_1 \int \frac{d^4k}{(2\pi)^4}\, g^2\, D_{\mu\nu}(p-k)\, \frac{\lambda^a}{2}\, \gamma^\mu\, S(k)\, \Gamma^{a,\nu}(p,k)$$

- A leading symmetry preserving truncation to the DSEs is rainbow-ladder:

$$\tfrac{1}{4\pi}\, g^2\, D_{\mu\nu}(p-k)\, \Gamma_\nu(p,k) \quad\longrightarrow\quad \alpha_{\text{eff}}(p-k)\, D^{\text{free}}_{\mu\nu}(p-k)\, \gamma_\nu$$

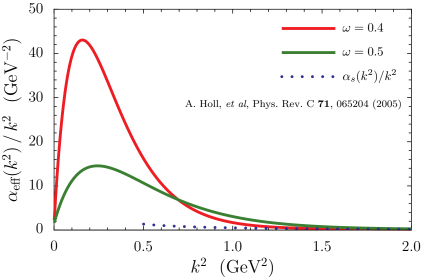
- Need model for $\alpha_{\text{eff}}(k^2)$ – must agree with perturbative QCD for large k^2
  - Maris–Tandy model is historically the most successful example [PRC 60, 055214 (1999)]

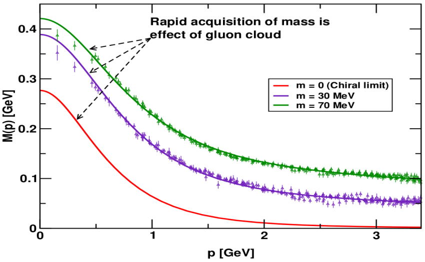
$$\alpha_{\text{eff}}(k^2) = \frac{\pi\, D}{\omega^6}\, k^4\, e^{-k^2/\omega^2} + \frac{24\,\pi}{25}\left(1 - e^{-k^2/\mu^2}\right) \ln^{-1}\!\left[e^2 - 1 + \left(1 + k^2/\Lambda^2_{\text{QCD}}\right)^2\right]$$

  - $\mu = 1\,\text{GeV}, \quad \Lambda_{QCD} = \Lambda^{(4)}_{\overline{\text{MS}}} = 0.234\,\text{GeV}, \quad \omega\, D = (0.72\,\text{GeV})^3$

- Correct LO perturbative limit is build in: $\alpha_{\text{eff}}(k^2) \overset{k^2\to\infty}{\longrightarrow} \frac{12}{25}\, \frac{\pi}{\ln\left[k^2/\Lambda^2_{\text{QCD}}\right]}$
  - one parameter model for QCDs infra-red behaviour

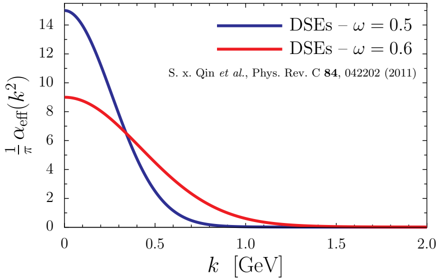
QCD's Quark Propagator

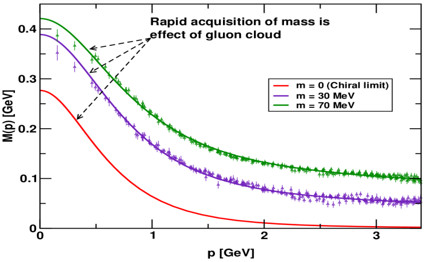




- Quark propagator: $$S(p) = \frac{Z(p^2)}{i\not{p} + M(p^2)} = \frac{1}{i\not{p}\,A(p^2) + B(p^2)}$$
- Dynamical mass generation, $M \propto \langle\bar{q}q\rangle \iff \langle\bar{q}q\rangle \neq 0 \iff$ DCSB
  - Higgs mechanism is almost irrelevant for light quarks
- DCSB generates 98% of the mass in the visible universe
- In perturbative QCD: $$B(p^2) = m\left[1 - \frac{\alpha}{\pi}\ln\left(\frac{p^2}{m^2}\right) + \ldots\right] \overset{m\to 0}{\to} 0$$
- QCD is an innately non-perturbative theory! The only example in nature

QCD's Quark Propagator





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Solving the QCD Gap Equation

$$S(p,\mu^2)^{-1} = Z_2(\mu^2,\Lambda^2)\,S_0(p) + \tfrac{4}{3}\,Z_1(\mu^2,\Lambda^2)\int g^2\,D_{\mu\nu}(p-k)\,\gamma^\mu\,S(k,\mu^2)\,\Gamma^\nu(p,k)$$

- Use quark propagator: $S^{-1}(p,\mu^2) = i\not{p}\,A(p^2,\mu^2) + B(p^2,\mu^2)$
- Rainbow ladder truncation:

$$\frac{g^2}{4\pi}\,\Gamma^\nu(p,k) \to \alpha_{\text{eff}}(k^2)\,\gamma^\mu, \qquad D_{\mu\nu}(k) \to D^{\text{free}}_{\mu\nu}(k)$$

- Use *off-shell subtraction scheme* for renormalization:

$$S(p)^{-1}\Big|_{p^2=\mu^2} = i\,\not{p} + m(\mu^2)$$

  - $m(\mu^2)$ is the renormalized current quark mass: $m(\mu^2) = \frac{m_0(\Lambda^2)}{Z_m(\mu^2,\Lambda^2)}$

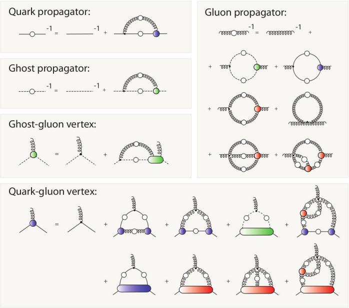
- Gap equation becomes set of coupled integral eqs. for $A(p^2)$ & $B(p^2)$:

$$A(p^2,\mu^2) = Z_2(\mu^2,\Lambda^2)\,A_0(p^2,\Lambda^2) \quad \& \quad B(p^2,\mu^2) = Z_2(\mu^2,\Lambda^2)\,B_0(p^2,\Lambda^2)$$

- Then solve the two coupled equations by iteration

Charting Interaction between light quarks

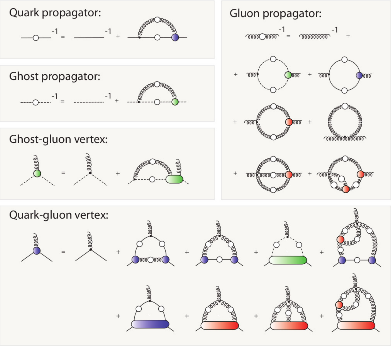
- Formally, hadronic observables are related to QCD's Schwinger functions
- E.g., the quark propagator is a Schwinger function and the gap equation relates this to:
  - the gluon propagator: $D^{\mu\nu}(k)$
  - the quark-gluon vertex $\Gamma^{a,\mu}_{gqq}(p,p')$
- The quark propagator is the building block of hadrons in the DSEs



- The DSEs are therefore a tool that can relate QCD's Schwinger Functions to hadronic observables
- Measurements of, for example, the hadron mass spectrum, elastic and transition form factors, PDFs, etc must provide information on the long-range interaction between light quarks and gluons
- Interplay between DSEs & experiment provides a framework to extract infrared behaviour of QCD's Schwinger functions

The Simplest Truncation

- The full machinery of the DSEs provides a connection between QCD & experiment
  - there remains much to be explored, notably baryon PDFs, TMDs & GPDs
  - however DSEs calculations are time & resource intensive – useful to have some physics intuition before embarking upon DSE studies

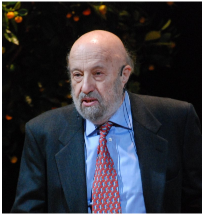


- To build intuition and understanding is it good to explore hadron structure with a simplified quark-gluon interaction
- Replace gluon propagator with a δ-function in configuration space:

$$g^2\, D_{\mu\nu}(p-k)\Gamma^{\nu}(p,k) \;\to\; \frac{1}{m_g^2}\, g_{\mu\nu}\,\gamma^{\nu}$$

- This *contact interaction* framework is basically equivalent to the Nambu–Jona Lasinio (NJL) model
- The NJL model is a proven and powerful tool with which to explore hadron structure and guide experiment

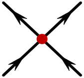
The Nambu–Jona-Lasinio Model



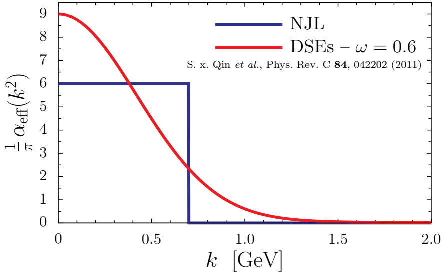
- The Nambu–Jona-Lasinio (NJL) Model was invented in 1961 by *Yoichiro Nambu* and *Giovanni Jona-Lasinio* while at The University of Chicago
  - was inspired by the BCS theory of superconductivity
  - was originally a theory of elementary nucleons
  - rediscovered in the 80s as an effective quark theory
- It is a relativistic quantum field theory, that is relatively easy to work with, and is very successful in the description of hadrons, nuclear matter, etc
- Nambu won half the 2008 Nobel prize in physics in part for the NJL model:

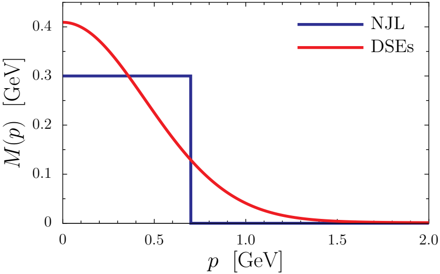
  "*for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics*" [Nobel Committee]

Continuum QCD

"integrate out gluons" →  $\frac{1}{m_g^2}\ \Theta(\Lambda^2 - k^2)$

- this is just a modern interpretation of the Nambu–Jona-Lasinio (NJL) model
- model is a Lagrangian based covariant QFT, exhibits dynamical chiral symmetry breaking & quark confinement; elements can be QCD motivated via the DSEs



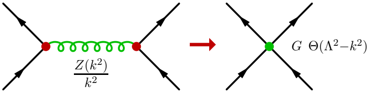


- Proper-time regularization: Λ_{IR} & $\Lambda_{UV} \Longrightarrow$ Confinement
- Quark propagator: $[\not{p} - m + i\varepsilon]^{-1} \rightarrow Z(p^2)[\not{p} - M + i\varepsilon]^{-1}$
  - wave function renormalization vanishes at quark mass-shell: $Z(p^2 = M^2) = 0$

Constructing the Lagrangian

- In general the NJL Lagrangian has the form

$$\mathcal{L} = \overline{\psi}\left(i\,\partial\!\!\!/ - m\right)\psi + \sum_{\alpha} G_{\alpha}\left(\overline{\psi}\,\Gamma_{\alpha}\,\psi\right)^{2}$$



  - Γ_α represents a product of Dirac, colour and flavour matrices

- What about $\mathcal{L}_I$? – effective theories should maintain symmetries of QCD
- In chiral limit QCD Lagrangian has symmetries

$$\mathcal{S}_{QCD} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U(1)_A \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

  - $SU(N_f)_A$ is broken dynamically – DCSB
  - $U(1)_A$ is broken in the anomalous mode – $U(1)$ problem – massive η'

- NJL interaction Lagrangian must respect the symmetries

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

  - in NJL $SU(3)_c$ will be considered a global gauge symmetry
  - $U(1)_A$ is often broken explicitly $\Longrightarrow m_{\eta'} \neq 0$

NJL Symmetries

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

- The NJL Lagrangian should be symmetric under the transformations

$$\begin{aligned}
SU(N_f)_V: &\quad \psi \longrightarrow e^{-i\,\boldsymbol{t}\cdot\boldsymbol{\theta_V}}\,\psi && \bar\psi \longrightarrow \bar\psi\,e^{i\,\boldsymbol{t}\cdot\boldsymbol{\theta_V}} \\
SU(N_f)_A: &\quad \psi \longrightarrow e^{-i\,\gamma_5\,\boldsymbol{t}\cdot\boldsymbol{\theta_A}}\,\psi && \bar\psi \longrightarrow \bar\psi\,e^{-i\,\gamma_5\,\boldsymbol{t}\cdot\boldsymbol{\theta_A}} \\
U(1)_V: &\quad \psi \longrightarrow e^{-i\,\theta}\,\psi && \bar\psi \longrightarrow \bar\psi\,e^{i\,\theta} \\
U(1)_A: &\quad \psi \longrightarrow e^{-i\,\gamma_5\,\theta}\,\psi && \bar\psi \longrightarrow \bar\psi\,e^{-i\,\gamma_5\,\theta}
\end{aligned}$$

- Nambu and Jona-Lasinio chose the Lagrangian

$$\mathcal{L} = \bar\psi\,(i\partial\!\!\!/ - m)\,\psi + G_\pi\left[(\bar\psi\psi)^2 - (\bar\psi\,\gamma_5\boldsymbol{\tau}\,\psi)^2\right]$$

- Can choose any combination of these $4-$fermion interactions

$$(\bar\psi\psi)^2, \quad (\bar\psi\,\gamma_5\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\,\psi)^2 \quad (\bar\psi\,\gamma^\mu\gamma_5\,\psi)^2, \quad (\bar\psi\,i\sigma^{\mu\nu}\,\psi)^2,$$
$$(\bar\psi\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma_5\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\gamma_5\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,i\sigma^{\mu\nu}\,\boldsymbol{t}\,\psi)^2.$$

NJL Lagrangian ($N_f = 2$)

- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + G_\pi\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\,\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right] + G_\omega\left(\bar{\psi}\,\gamma^\mu\,\psi\right)^2 + G_\rho\left[\left(\bar{\psi}\,\gamma^\mu\boldsymbol{\tau}\,\psi\right)^2 + \left(\bar{\psi}\,\gamma^\mu\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right]$$
$$+ G_h\left(\bar{\psi}\,\gamma^\mu\gamma_5\,\psi\right)^2 + G_\eta\left[\left(\bar{\psi}\,\gamma_5\,\psi\right)^2 - \left(\bar{\psi}\,\boldsymbol{\tau}\,\psi\right)^2\right] + G_T\left[\left(\bar{\psi}\,i\sigma^{\mu\nu}\psi\right)^2 - \left(\bar{\psi}\,i\sigma^{\mu\nu}\boldsymbol{\tau}\,\psi\right)^2\right]$$

  - $\mathcal{L}_I$ is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

$\bar{\psi}\psi$	$\longleftrightarrow$	σ	$\longleftrightarrow$	$(J^P, T) = (0^+, 0)$
$\bar{\psi}\,\gamma_5\boldsymbol{\tau}\,\psi$	$\longleftrightarrow$	π	$\longleftrightarrow$	$(J^P, T) = (0^-, 1)$
$\bar{\psi}\,\gamma^\mu\,\psi$	$\longleftrightarrow$	ω	$\longleftrightarrow$	$(J^P, T) = (1^-, 0)$
$\bar{\psi}\,\gamma^\mu\boldsymbol{\tau}\,\psi$	$\longleftrightarrow$	ρ	$\longleftrightarrow$	$(J^P, T) = (1^-, 1)$
$\bar{\psi}\,\gamma^\mu\gamma_5\,\psi$	$\longleftrightarrow$	f_1, h_1	$\longleftrightarrow$	$(J^P, T) = (1^+, 0)$
$\bar{\psi}\,\gamma^\mu\gamma_5\boldsymbol{\tau}\,\psi$	$\longleftrightarrow$	a_1	$\longleftrightarrow$	$(J^P, T) = (1^+, 1)$
$\bar{\psi}\boldsymbol{\tau}\psi$	$\longleftrightarrow$	a_0	$\longleftrightarrow$	$(J^P, T) = (0^+, 1)$
$\bar{\psi}\,\gamma_5\,\psi$	$\longleftrightarrow$	η, η'	$\longleftrightarrow$	$(J^P, T) = (0^-, 0)$

NJL Lagrangian ($N_f = 3$)

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- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + G_\pi\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right] + G_\omega\left(\bar{\psi}\gamma^\mu\,\psi\right)^2 + G_\rho\left[\left(\bar{\psi}\gamma^\mu\boldsymbol{\tau}\,\psi\right)^2 + \left(\bar{\psi}\gamma^\mu\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right]$$
$$+ G_h\left(\bar{\psi}\gamma^\mu\gamma_5\,\psi\right)^2 + G_\eta\left[\left(\bar{\psi}\gamma_5\,\psi\right)^2 - \left(\bar{\psi}\boldsymbol{\tau}\,\psi\right)^2\right] + G_T\left[\left(\bar{\psi}\,i\sigma^{\mu\nu}\psi\right)^2 - \left(\bar{\psi}\,i\sigma^{\mu\nu}\boldsymbol{\tau}\,\psi\right)^2\right]$$

  - $\mathcal{L}_I$ is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

- The most general $N_f = 3$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L}_I = G_\pi\left[\tfrac{1}{6}\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\,\boldsymbol{t}\,\psi\right)^2 - \tfrac{1}{6}\left(\bar{\psi}\gamma_5\,\psi\right)^2 - \left(\bar{\psi}\gamma_5\,\boldsymbol{t}\,\psi\right)^2\right]$$
$$-\frac{1}{2}G_\rho\left[\left(\bar{\psi}\gamma^\mu\,\boldsymbol{t}\,\psi\right)^2 + \left(\bar{\psi}\gamma^\mu\gamma_5\,\boldsymbol{t}\,\psi\right)^2\right] - \frac{1}{2}G_\omega\left(\bar{\psi}\gamma^\mu\,\psi\right)^2 - \frac{1}{2}G_f\left(\bar{\psi}\gamma^\mu\gamma_5\,\psi\right)^2$$

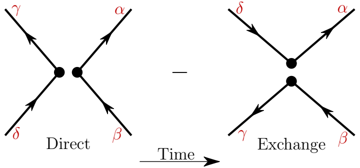
- Enlarging the $SU(N_f)_V \otimes SU(N_f)_A$ chiral group from $N_f = 2$ to $N_f = 3$ reduces the number of coupling from six to four

- The $N_f = 3$ Lagrangian is automatically $U(1)_A$ invariant
  - $U(1)_A$ is then often broken by the 't Hooft term – a 6-quark interaction

$$\mathcal{L}_I^{(6)} = K\left[\det\left(\bar{\psi}(1+\gamma_5)\psi\right) + \det\left(\bar{\psi}(1-\gamma_5)\psi\right)\right]$$

NJL Interaction Kernel

- Using Wick's theorem and the NJL Lagrangian there are 2 diagrams for the interaction between a quark and an anti-quark



$$2i\,G\left[\Omega^i_{\alpha\beta}\overline{\Omega}^i_{\gamma\delta} - \Omega^i_{\alpha\delta}\overline{\Omega}^i_{\gamma\beta}\right]$$

- Using Fierz transformations can express each *exchange term* as a sum of *direct terms*

- The $SU(2)$ NJL interaction kernel then takes the form

$$K_{\alpha\beta,\gamma\delta} = 2i\,G_\pi\left[(\mathbb{1})_{\alpha\beta}(\mathbb{1})_{\gamma\delta} - (\gamma_5\boldsymbol{\tau})_{\alpha\beta}(\gamma_5\boldsymbol{\tau})_{\gamma\delta}\right] - 2i\,G_\omega\,(\gamma_\mu)_{\alpha\beta}(\gamma^\mu)_{\gamma\delta}$$
$$- 2i\,G_\rho\left[(\gamma_\mu\boldsymbol{\tau})_{\alpha\beta}(\gamma^\mu\boldsymbol{\tau})_{\gamma\delta} + (\gamma_\mu\gamma_5\boldsymbol{\tau})_{\alpha\beta}(\gamma^\mu\gamma_5\boldsymbol{\tau})_{\gamma\delta}\right] + \ldots$$

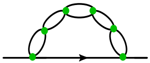
- This kernel enters the NJL gap and meson Bethe-Salpeter equations

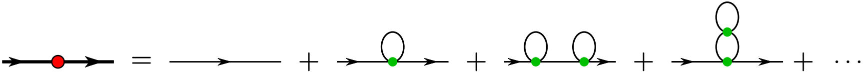
Regularization Schemes

- The NJL model is non-renormalizable $\Longrightarrow$ cannot remove regularization
  - regularization parameter(s) play a dynamical role
- Popular choices are:
  - 3-momentum cutoff: $\vec{p}^{\,2} < \Lambda^2$
  - 4-momentum cutoff $p_E^2 < \Lambda^2$
  - Pauli-Villars
- We will use the proper-time regularization scheme

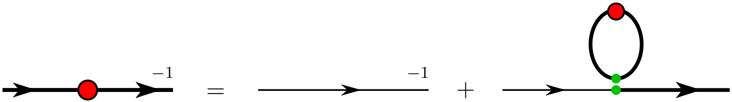
$$\frac{1}{X^n} = \frac{1}{(n-1)!}\int_0^\infty d\tau\, \tau^{n-1}\, e^{-\tau X} \;\to\; \frac{1}{(n-1)!}\int_{1/\Lambda_{UV}^2}^{1/\Lambda_{IR}^2} d\tau\, \tau^{n-1}\, e^{-\tau X}$$

  - only Λ_{UV} is needed to render the theory finite
  - however, as we shall see, Λ_{IR} plays a very important role; it prevents quarks going on their mass shell and hence *simulates quark confinement*

- Complete expression for the quark propagator cannot be obtained
  - need a truncation
- Do not in include diagrams like: 
  - would give a momentum dependent mass function
- Include all diagrams of the form:

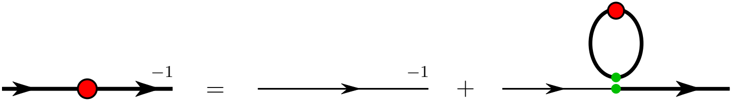


- All these diagrams can be summed via an integral equation



- The most general quark propagator has the form

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p)} = \frac{Z(p^2)}{\not{p} - M(p^2)}$$



- The NJL gap equation has the form

$$S^{-1}(k) = S_0^{-1}(k) - \Sigma(k) = [\not{k} - m] - \sum_j \int \frac{d^4\ell}{(2\pi)^4} \text{Tr}\left[S(\ell)\,\overline{\Omega}^j\right]\Omega^j$$

- The only piece of the interaction kernel that contributes is:

$$K^{\sigma}_{\alpha\beta,\gamma\delta} = 2i\,G_\pi\,(\mathbb{1})_{\gamma\delta}\,(\mathbb{1})_{\alpha\beta}$$

- Solving this equation give a quark propagator of the form

$$S^{-1}(k) = \not{k} - M + i\varepsilon$$

- The constituent quark mass satisfies the equation

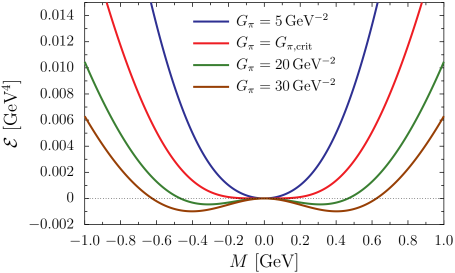
$$M = m + 48i\,G_\pi\,M \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2 - M^2 + i\varepsilon} = m + M\,\frac{3\,G_\pi}{\pi^2}\int d\tau\,\frac{e^{-\tau\,M^2}}{\tau^2}$$

The True Ground State

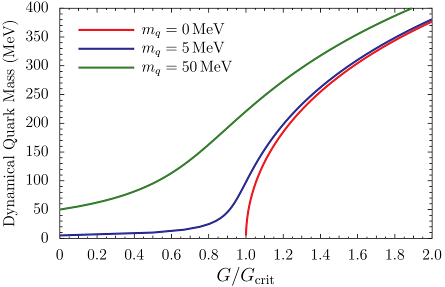
$$M = m + M \frac{3\,G_\pi}{\pi^2} \int d\tau \, \frac{e^{-\tau\, M^2}}{\tau^2}$$

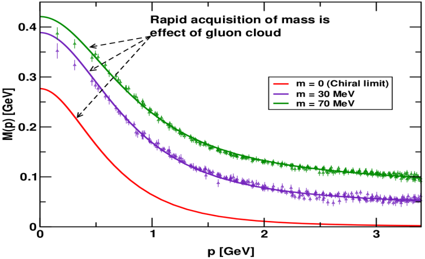
- For the case $m = 0$ the gap equation has two solutions:
  - trivial solution: $M = 0$ & non-trivial solution: $M \neq 0$
- Which solution does nature choose, that is, which solution minimizes the energy. Compare vacuum energy density, $\mathcal{E}$, for each case

$$\mathcal{E}(M) - \mathcal{E}(M=0) = -\frac{3}{4\pi^2} \int d\tau \frac{1}{\tau^3} \left(e^{-\tau M^2} - 1 \right) + \frac{M^2}{4\,G_\pi}$$



- For $G_\pi > G_{\pi,\mathrm{crit}}$ the lowest energy solution has $M \neq 0$
- Therefore for $G_\pi > G_{\pi,\mathrm{crit}}$ NJL has DCSB
- DCSB $\Longleftrightarrow$ generates mass from nothing





- NJL constituent mass is given by: $M = m - 2\,G_\pi\,\langle\bar{\psi}\psi\rangle$
- Chiral condensate is defined by

$$\langle\bar{\psi}\psi\rangle \equiv \lim_{x\to y} \mathrm{Tr}\left[-iS(x-y)\right] = -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[i\,S(k)\right]$$

- Mass is generated via interaction with vacuum
- Dynamically generated quark masses $\iff$ $\langle\bar{\psi}\psi\rangle \neq 0$
- $\langle\bar{\psi}\psi\rangle = \langle\bar{u}u + \bar{d}d\rangle$ is an order parameter for DSCB

Confinement in NJL model

- In general the NJL model is not confining; quark propagator is simply

$$S(k) = \frac{1}{\not{k} - M + i\varepsilon} = \frac{\not{k} + M}{k^2 - M^2 + i\varepsilon}$$

  - quark propagator has a pole $\Longrightarrow$ quarks are part of physical spectrum

- However the proper-time scheme is unique

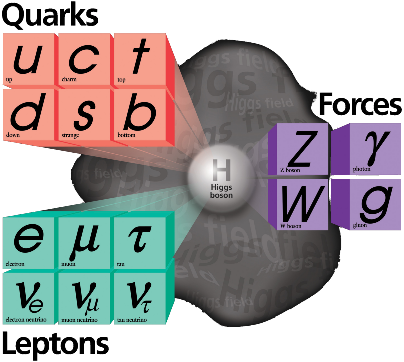
$$S(k) = \int_0^\infty d\tau\, (\not{k} + M)\, e^{-\tau\left(k^2 - M^2\right)} \to \underbrace{\frac{\left[e^{-(k^2-M^2)/\Lambda_{UV}^2} - e^{-(k^2-M^2)/\Lambda_{IR}^2}\right]}{k^2 - M^2}}_{\equiv Z(k^2)} [\not{k} + M]$$

  - quark propagator does not have a pole: $Z(k^2) \overset{k^2 \to M^2}{=} \frac{1}{\Lambda_{IR}^2} - \frac{1}{\Lambda_{UV}^2} \neq \infty$

- Are confinement and DCSB related?
  - NJL model is proof that DCSB can exist without confinement
  - *however commonly believed cannot have dynamically generated confinement without DCSB*

Summary

- QCD and therefore Hadron Physics is unique:
  - must confront a fundamental theory in which the elementary degrees-of-freedom are intangible (confined) and only composites (hadrons) reach detectors
- QCD will only be solved by deploying a diverse array of experimental and theoretical methods:
  - must define and solve the problems of confinement and its relationship with DCSB



- These are two of the most important challenges in fundamental Science
- The DSEs are an important tool with which to meet these challenges
  - the NJL model, while not sophisticated enough to directly address these issues within QCD, has and will continue to provide critical guidance