Lecture 3: The pion and dynamical chiral symmetry breaking

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The Pion – Nature’s strong messenger

- Hideki Yukawa in 1935 postulated a strongly interacting particle of mass $\sim 100$ MeV
  - Yukawa called this particle a “meson”
- Cecil Powell in 1947 discovered the $\pi$-meson from cosmic ray tracks in a photographic emulsion – a technique Cecil developed

- Cavendish Lab had said method is incapable of “reliable and reproducible precision measurements”
- The measured pion mass was: 130 – 150 MeV
- Both Yukawa & Powell received Nobel Prize – in 1949 and 1950 respectively
- Discovery of pion was beginning of particle physics; before long there was the particle zoo
The Pion in QCD

Today the pion is understood as both a bound state of a dressed-quark and a dressed-antiquark in QFT and the Goldstone mode associated with DCSB in QCD.

This dichotomous nature has numerous ramifications, e.g.:

\[ m_\rho/2 \sim M_N/3 \sim 350 \text{ MeV} \quad \text{however} \quad m_\pi/2 \simeq 0.2 \times 350 \text{ MeV} \]

The pion is unusually light, the key is dynamical chiral symmetry breaking.

The pion – \( \pi^+, \pi^0, \pi^- \) – is stable under the strong interaction, however does decay via the weak interaction: \( \tau_{\pi^\pm} \simeq 10^{-8}\text{s} \).

Pion properties are therefore difficult to measure but there have been numerous successes:

- masses; decay constants and rates; electromagnetic form factor and parton distribution function

Critical need for modern theory to examine the structure of the pion.

- the Dyson-Schwinger equations are the ideal tool
QCD’s Dyson-Schwinger Equations

- The equations of motion of QCD $\iff$ QCD’s Dyson–Schwinger equations
  - an infinite tower of coupled integral equations
  - tractability $\implies$ must implement a symmetry preserving truncation

- The most important DSE is QCD’s gap equation $\implies$ quark propagator

- ingredients – dressed gluon propagator & dressed quark-gluon vertex

\[ S(p) = \frac{Z(p^2)}{i \slashed{p} + M(p^2)} \]

- $S(p)$ has correct perturbative limit
- mass function, $M(p^2)$, exhibits dynamical mass generation
- complex conjugate poles
  - no real mass shell $\implies$ confinement

**Pion Elastic Form Factor**

- Form factors parametrize the interaction of the electromagnetic current with a hadron.
  - The pion form factor furnishes information on the distribution of charge inside the pion.

The interaction of the EM current with a pion has the form:

\[
\langle J_\mu^\alpha \rangle = (p'^\mu + p^\mu) F_\pi(Q^2)
\]

- In the impulse approximation the pion form factor is given by the (triangle) diagram:

**Ingredients:**
- dressed quark propagators
- homogeneous Bethe-Salpeter vertices
- dressed quark-photon vertex

**Measurement and computation of the pion form factor will shed light on DCSB in QCD**
Bethe-Salpeter Equation

- In QFT physical states appear as poles in \( n \)-point Green Functions
- That is, the full \( \bar{q}q \) scattering matrix or \( t \)-matrix, contains poles for all \( \bar{q}q \) bound states, that is, the physical mesons
- The \( \bar{q}q \) \( t \)-matrix is given by the Bethe-Salpeter equation (BSE):

\[
T = K + T K
\]

- In principle the kernel, \( K \), contains all possible 2PI diagrams

\[
K = \ldots + NJL \rightarrow \ldots + \ldots
\]

- The BSE is compact and correct – its solution in a particular channel gives the two-body (meson) propagator
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Bethe-Salpeter Vertex Functions

Near a bound state pole of mass $m$ a two-body $T$-matrix behaves as

$$
\mathcal{T}(p, k) \to \frac{i \Gamma(p, k) \bar{\Gamma}(p, k)}{p^2 - m^2}
$$

where $p = p_1 + p_2$, $k = p_1 - p_2$

* $\Gamma(p, k)$ is the homogeneous Bethe-Salpeter vertex & describes relative motion of the quark and anti-quark while they form the bound state

* Pion BSE vertex has the general form

$$
\Gamma_{\pi}(p, k) = \gamma_5 \left[ E_\pi(p, k) + \phi F_\pi(p, k) + k \cdot p \mathcal{G}(p, k) + \sigma^{\mu \nu} k_\mu p_\nu \mathcal{H}(p, k) \right]
$$

* the dominant amplitude is $E_\pi(p, k)$, $F_\pi(p, k)$ becomes important for large $Q^2$

* Bethe-Salpeter vertex needed for calculations e.g. $f_\pi$ or $F_\pi(Q^2)$
NJL BSE for the Pion

**NJL BSE for the Pion:**

\[ \mathcal{K}_\pi = -2i G_\pi (\gamma_5 \tau)_{\alpha\beta} (\gamma_5 \tau)_{\lambda\epsilon} \]

\[ \mathcal{T}(q) = \mathcal{K} + \int \frac{d^4k}{(2\pi)^4} \mathcal{K} S(q+k) S(k) \mathcal{T}(q) \]

Solving for the \( t \)-matrix and expanding about the pole:

\[ \mathcal{T} = \gamma_5 \tau_i \left( \frac{-2i G_\pi}{1 + 2 G_\pi \Pi_\pi (q^2)} \right) \gamma_5 \tau_i \rightarrow \frac{i Z_\pi}{q^2 - m^2_\pi} (\gamma_5 \tau_i)(\gamma_5 \tau_i) \implies \Gamma_\pi = \sqrt{Z_\pi} \gamma_5 \tau_i \]

- \( Z_\pi \) is effective pion-quark coupling constant & \( \Gamma_\pi \) the pion BS vertex

The pion mass is then given by

\[ -1 + 2 G_\pi \Pi_\pi (q^2) = m^2_\pi = 0 \]  - where

\[ \Pi_\pi (q^2) \delta_{ij} = 3i \int \frac{d^4k}{(2\pi)^4} \text{Tr} [\gamma_5 \tau_i S(k) \gamma_5 \tau_j S(k+q)] \]

- this result is straightforward to obtain
Is the pion a Goldstone boson? NJL gap equation gives:

\[ \Pi_\pi(q^2) = \frac{m}{2G_\pi M} - \frac{1}{2G_\pi} - q^2 I(q^2) \]

Pole condition – \( 1 + 2G_\pi \Pi_\pi(q^2 = m_\pi^2) = 0 \) – implies

\[ m_\pi^2 = \frac{m}{2G_\pi M I(m_\pi^2)} \]

Therefore as demanded by chiral symmetry we have: \( m_\pi^2 \propto m \) (GMOR)

also in the chiral limit – \( m \to 0 \) (\( M \neq 0 \)) – pion is massless

The NJL model also satisfies all another relations associated with chiral symmetry; for example

- \( f_\pi g_{\pi qq} = M g_{Aqq} \) Goldberger–Treiman (GT) relation
- \( f_\pi^2 m_\pi^2 = \frac{1}{2} (m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle \) Gell-Mann–Oakes–Renner (GMOR)
Chiral Partners

IF chiral symmetry was NOT dynamically broken in nature expect mass degenerate chiral partners; e.g. \( m_{\sigma} \approx m_{\pi} \) & \( m_{a_1} \approx m_{\rho} \)

The \( \rho \) and \( a_1 \) are the lowest lying vector (\( J^P = 1^- \)) and axial-vector (\( J^P = 1^+ \)) \( \bar{q}q \) bound states: \( m_{\rho}^{\text{exp't}} \approx 770 \text{ MeV} \) & \( m_{a_1}^{\text{exp't}} \approx 1260 \text{ MeV} \)

The associated NJL BSE pole conditions read:

\[
1 + 2 G_{\rho} \Pi_{\rho}(q^2 = m_{\rho}^2) = 0 \quad \& \quad 1 + 2 G_{\rho} \Pi_{a_1}(q^2 = m_{a_1}^2) = 0
\]

where \( \Pi_{a_1}(q^2) = M^2 I(q^2) + \Pi_{\rho}(q^2) \)

If \( m = 0 \) and there is NO DCSB (\( M = 0 \)) would have: \( m_{\rho} = m_{a_1} \)

In nature and NJL, DCSB splits chiral partner masses

NJL gives: \( m_{\rho} \equiv 770 \text{ MeV} \) & \( m_{a_1} \approx 1098 \text{ MeV} \)

agrees with the Weinberg relation: \( m_{a_1} \approx \sqrt{2} m_{\rho} \); \( [m_{\sigma}^2 \approx m_{\pi}^2 + 4 M^2] \)
Full DSE results uses the pion BSE vertex:

\[ \Gamma_\pi(p, k) = \gamma_5 \left[ E_\pi(p, k) + \not{\phi} F_\pi(p, k) + k \cdot p \not{G}(p, k) + \sigma^{\mu\nu} k_\mu p_\nu \not{H}(p, k) \right] \]

In gap equation use simple kernel \(\iff\) NJL model with \(\pi - a_1\) mixing

\[ g^2 D_{\mu\nu}(p - k) \Gamma^\nu(p, k) \rightarrow \frac{1}{m_G^2} g_{\mu\nu} \gamma^\nu \implies \Gamma_\pi(p, k) = \gamma_5 \left[ E_\pi + \not{\phi} F_\pi \right] \]

- quark no longer has a running mass

**Nature of interaction can have observable consequences for** \(Q^2 > 0\)


Measuring Pion Form Factor

Sullivan Process

- At low $Q^2$ pion form factor is measured by scattering a pion from the electron cloud of an atom \([t \equiv p^2]\)
- small mass of electron limits this to $Q^2 < 0.5$ GeV$^2$

- Higher $Q^2$ experiments have been performed at Jefferson Lab where a virtual photon scatters from a virtual pion that is part of the nucleon

- Initial pion is off its mass shell – $p^2 \leq 0$ – on mass shell $p^2 = m^2_\pi$
- need to extrapolate to the pion pole $p^2 = m^2_\pi$
Light-Front Wave Functions

- In equal-time quantization a hadron wave function is a frame dependent concept
- Boost operators are dynamical, that is, they are interaction dependent

- In high energy scattering experiments particles move at near speed of light
  - Natural to quantize a theory at equal light-front time: $\tau = (t + z)/\sqrt{2}$

- Light-front quantization $\implies$ light-front WFs; many remarkable properties:
  - Frame-independent; probability interpretation – as close as QFT gets to QM
  - Boosts are kinematical – *not dynamical*

- Parton distribution amplitudes (PDAs) are (almost) observables & are related to light-front wave functions

$$\varphi(x) = \int d^2k_\perp \psi(x, k_\perp)$$
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Parton distribution amplitudes (PDAs) are (almost) observables & are related to light-front wave functions

\[
\varphi(x) = \int d^2 \vec{k}_\perp \psi(x, \vec{k}_\perp)
\]
pion’s PDA – $\varphi_\pi(x)$: *is a probability amplitude that describes the momentum distribution of a quark and antiquark in the bound-state’s valence Fock state*

- it’s a function of the light-cone momentum fraction $x = \frac{k^+}{p^+}$ and the scale $Q^2$

\[ Q^2 F_\pi(Q^2) \rightarrow 16\pi f_\pi^2 \alpha_s(Q^2) \]

\[ Q^2 F_{\gamma^* \gamma \pi}(Q^2) \rightarrow 2 f_\pi \]

- PDAs enter numerous hard exclusive scattering processes

**GPDs**
Pion’s Parton Distribution Amplitude

- pion’s PDA – \( \varphi_\pi(x) \): is a probability amplitude that describes the momentum distribution of a quark and antiquark in the bound-state’s valence Fock state
- it’s a function of the light-cone momentum fraction \( x = \frac{k^+}{p^+} \) and the scale \( Q^2 \)
- The pion’s PDA is defined by

\[
f_\pi \varphi_\pi(x) = Z_2 \int \frac{d^4k}{(2\pi)^2} \delta(k^+ - x p^+) \text{Tr} \left[ \gamma^+ \gamma_5 S(k) \Gamma_\pi(k, p) S(k - p) \right]
\]

- \( S(k) \Gamma_\pi(k, p) S(k - p) \) is the pion’s Bethe-Salpeter wave function
- in the non-relativistic limit it corresponds to the Schrödinger wave function
- \( \varphi_\pi(x) \): is the axial-vector projection of the pion’s Bethe-Salpeter wave function onto the light-front [at twist-2 also pseudoscalar projection]

- Pion PDA is an essentially nonperturbative quantity whose asymptotic form is known; in this regime governs, e.g., \( Q^2 \) dependence of pion form factor

\[
Q^2 F_\pi(Q^2) \xrightarrow{Q^2 \to \infty} 16 \pi f_\pi^2 \alpha_s(Q^2) \quad \iff \quad \varphi_\pi^{\text{asy}}(x) = 6x(1-x)
\]
ERBL ($Q^2$) evolution for pion PDA [c.f. DGLAP equations for PDFs]

$$\mu \frac{d}{d\mu} \varphi(x, \mu) = \int_0^1 dy \ V(x, y) \varphi(y, \mu)$$

This evolution equation has a solution of the form

$$\varphi_\pi(x, Q^2) = 6x(1-x) \left[ 1 + \sum_{n=2, 4, \ldots} a_n^{3/2}(Q^2) C_n^{3/2}(2x-1) \right]$$

- $\alpha = 3/2$ because in $Q^2 \to \infty$ limit QCD is invariant under the collinear conformal group $SL(2; \mathbb{R})$
- Gegenbauer-$\alpha = 3/2$ polynomials are irreducible representations $SL(2; \mathbb{R})$

The coefficients of the Gegenbauer polynomials, $a_n^{3/2}(Q^2)$, evolve logarithmically to zero as $Q^2 \to \infty$: $\varphi_\pi(x) \to \varphi_\pi^\text{asy}(x) = 6x(1-x)$

At what scales is this a good approximation to the pion PDA?

E.g., AdS/QCD find $\varphi_\pi(x) \sim x^{1/2}(1-x)^{1/2}$ at $Q^2 = 1 \text{ GeV}^2$; expansion in terms of $C_n^{3/2}(2x-1)$ converges slowly: $a_{32}^{3/2}/a_2^{3/2} \sim 10\%$
Both DSE results, each using a different Bethe-Salpeter kernel, exhibit a pronounced broadening compared with the asymptotic pion PDA.

- Scale of calculation is given by renormalization point $\zeta = 2 \text{ GeV}$

A realization of DCSB on the light-front

As we shall see the dilation of pion’s PDA will influence the $Q^2$ evolution of the pion’s electromagnetic form factor.
Lattice QCD can only determine one non-trivial moment

\[
\int_0^1 dx \ (2x - 1)^2 \varphi_\pi(x) = 0.27 \pm 0.04
\]

[V. Braun et al., Phys. Rev. D 74, 074501 (2006)]

scale is \( Q^2 = 4 \text{ GeV}^2 \)

Standard practice to fit first coefficient of “asymptotic expansion” to moment

\[
\varphi_\pi(x, Q^2) = 6x (1-x) \left[ 1 + \sum_{n=2,4,...} a_n^{3/2}(Q^2) C_n^{3/2}(2x - 1) \right]
\]

however this expansion is guaranteed to converge rapidly only when \( Q^2 \to \infty \)

this procedure results in a double-humped pion PDA

Advocate using a generalized expansion

\[
\varphi_\pi(x, Q^2) = N_\alpha x^\alpha (1-x)^\alpha \left[ 1 + \sum_{n=2,4,...} a_n^{\alpha+1/2}(Q^2) C_n^{\alpha+1/2}(2x - 1) \right]
\]

Find \( \varphi_\pi \approx x^\alpha (1-x)^\alpha, \quad \alpha = 0.35^{+0.32}_{-0.24} \); good agreement with DSE: \( \alpha \approx 0.52 \)
Pion PDA from lattice QCD

- Lattice QCD can only determine one non-trivial moment
  \[
  \int_0^1 dx \ (2x - 1)^2 \varphi_\pi(x) = 0.27 \pm 0.04
  \]
  [V. Braun et al., Phys. Rev. D 74, 074501 (2006)]
- Scale is \( Q^2 = 4 \) GeV\(^2\)
- Standard practice to fit first coefficient of “asymptotic expansion” to moment
  \[
  \varphi_\pi(x, Q^2) = 6x (1 - x) \left[ 1 + \sum_{n=2, 4, \ldots} a_n^{3/2}(Q^2) C_n^{3/2}(2x - 1) \right]
  \]
  however this expansion is guaranteed to converge rapidly only when \( Q^2 \to \infty \)
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- Advocate using a generalized expansion
  \[
  \varphi_\pi(x, Q^2) = N_\alpha x^\alpha (1 - x)^\alpha \left[ 1 + \sum_{n=2, 4, \ldots} a_n^{\alpha+1/2}(Q^2) C_n^{\alpha+1/2}(2x - 1) \right]
  \]
  - Find \( \varphi_\pi \simeq x^\alpha (1 - x)^\alpha \), \( \alpha = 0.35^{+0.32}_{-0.24} \); good agreement with DSE: \( \alpha \sim 0.52 \)
Generalized expansion

\[ \varphi_\pi(x) = N_\alpha x^\alpha (1 - x)^\alpha \left[ 1 + \sum a_n^{\alpha+}(Q^2) C_n^{\alpha+}(2x - 1) \right] \]

Updated lattice QCD moment: [V. Braun et al., arXiv:1503.03656 [hep-lat]]

\[ \int_0^1 dx (2x - 1)^2 \varphi_\pi(x) = 0.2361 (41) (39) (?) \]

DSE prediction:

\[ \int_0^1 dx (2x - 1)^2 \varphi_\pi(x) = 0.251 \]
When is the Pion’s PDA Asymptotic

Under leading order $Q^2$ evolution the pion PDA remains broad to well above $Q^2 > 100 \text{ GeV}^2$, compared with $\varphi_{\pi}^{\text{asy}}(x) = 6x(1-x)$

Consequently, the asymptotic form of the pion PDA is a poor approximation at all energy scales that are either currently accessible or foreseeable in experiments on pion elastic and transition form factors

Importantly, $\varphi_{\pi}^{\text{asy}}(x)$ is only guaranteed be an accurate approximation to $\varphi_{\pi}(x)$ when pion valence quark PDF satisfies: $q_{v\pi}(x) \sim \delta(x)$

This is far from valid at forseeable energy scales
LO QCD evolution of momentum fraction carried by valence quarks

$$\langle x q_v(x) \rangle (Q^2) = \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{\gamma_{qq}^{(0)} / (2\beta_0)} \langle x q_v(x) \rangle (Q_0^2)$$

where \( \frac{\gamma_{qq}^{(0)} }{2\beta_0} > 0 \)

Therefore, as \( Q^2 \rightarrow \infty \) we have \( \langle x q_v(x) \rangle \rightarrow 0 \) implies \( q_v(x) \propto \delta(x) \)

At LHC energies valence quarks still carry 20% of pion momentum

- the gluon distribution saturates at \( \langle x g(x) \rangle \sim 55\% \)

**Asymptotia is a long way away!**
Pion Elastic Form Factor

- Direct, symmetry-preserving computation of pion form factor predicts maximum in $Q^2 F_\pi(Q^2)$ at $Q^2 \approx 6 \text{ GeV}^2$

- magnitude of this product is determined by strength of DCSB at all accessible scales

- The QCD prediction can be expressed as

\[ Q^2 F_\pi(Q^2) \sim Q^2 \gg \Lambda_{\text{QCD}}^2 \rightarrow 16 \pi f_\pi^2 \alpha_s(Q^2) w_\pi^2; \quad w_\pi = \frac{1}{3} \int_0^1 dx \frac{1}{x} \varphi_\pi(x) \]

- Within DSEs there is consistency between the direct pion form factor calculation and that obtained using the DSE pion PDA

- 15% disagreement explained by higher order/higher-twist corrections

- **We predict that QCD power law behaviour – with QCD’s scaling law violations – sets in at $Q^2 \sim 8 \text{ GeV}^2$**
At large $Q^2$ the hard gluon exchange in the $\gamma^* + \pi \rightarrow \pi$ form factor – needed to keep the pion intact – results in distinctly different behaviour to the pion transition form factor $\gamma^* + \pi \rightarrow \gamma$.

\[ Q^2 F_{\gamma^*\pi\gamma}(Q^2) \rightarrow 2 f_\pi \omega^2_\pi \quad \text{c.f.} \quad Q^2 F_\pi(Q^2) \rightarrow 16 \pi f_\pi^2 \alpha_s(Q^2) \omega^2_\pi \]

Therefore approach to asymptotic limit gives *inter alia* a unique window into quark-gluon dynamics in QCD.

In full DSE calculation of $\gamma^* \pi \rightarrow \gamma$ conformal limit approached from below.
To observe onset of perturbative power law behaviour – to differentiate from a monopole – optimistically need data at 8 GeV$^2$ but likely also at 10 GeV$^2$

this is a very challenging task experimentally

Scaling predictions are valid for both spacelike and timelike momenta

timelike data show promise as the means of verifying modern predictions