QCD and hadron structure

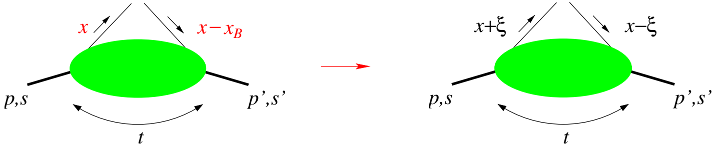
Lecture 2: properties and physics of GPDs

M. Diehl

Deutsches Elektronen-Synchroton DESY

Jefferson Lab, June 2016

GPDs: definition and properties

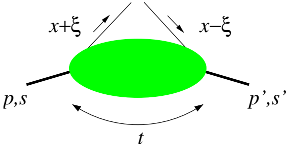


$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

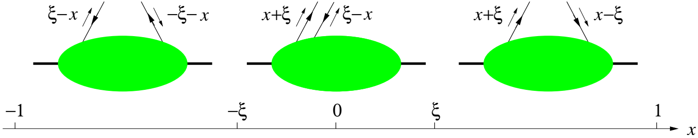
- use now symmetric parameteriz'n of momentum fractions
- kinematic variables:
  x, ξ momentum fractions w.r.t. $P = \frac{1}{2}(p+p')$
  $\xi = (p-p')^+/(p+p')^+$ plus-momentum transfer
  in DVCS: $\xi = x_B/(2-x_B)$, x integrated over
  t can trade for transverse momentum transfer $\boldsymbol{\Delta} = \boldsymbol{p}' - \boldsymbol{p}$

$$t = -\frac{4\xi^2 m^2 + \boldsymbol{\Delta}^2}{1-\xi^2}$$

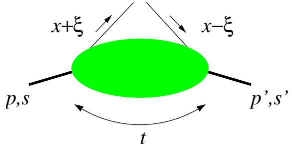
GPDs: definition and properties



- nonzero for $-1 \le x \le 1$
- $|x| > \xi$ similar to parton densities
  correlation $\psi^*_{x-\xi}\,\psi_{x+\xi}$ instead of probability $|\psi_x|^2$
  $|x| < \xi$ coherent emission of $q\bar{q}$ pair
- regions related by Lorentz invariance
  spacelike partons incoming in some frames, outgoing in others



GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$$= H^q\; \bar{u}(p', s')\gamma^+ u(p, s) + E^q\; \bar{u}(p', s')\, \frac{i}{2m_p} \sigma^{+\alpha}(p'-p)_\alpha\, u(p, s)$$

- proton spin structure:

  $H^q \leftrightarrow s = s'$ for $p = p'$ recover usual densities:

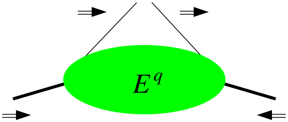
  $$H^q(x, \xi = 0, t = 0) = \begin{cases} q(x) & x > 0 \\ -\bar{q}(-x) & x < 0 \end{cases}$$

  $E^q \leftrightarrow s \neq s'$ decouples for $p = p'$

- similar definitions for polarized quarks $\tilde{H}^q, \tilde{E}^q$ and for gluons

  $H^g(x, \xi = 0, t = 0) = xg(x)$ for $x > 0$

GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$$= H^q\; \bar{u}(p', s')\gamma^+ u(p, s) + E^q\; \bar{u}(p', s')\, \frac{i}{2m_p}\sigma^{+\alpha}(p' - p)_\alpha\, u(p, s)$$

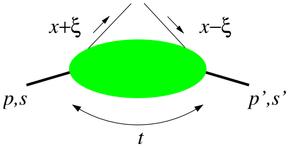
- more precisely: for proton helicities (λ', λ)

$$F^q_{\lambda'=\lambda} \;\propto\; H^q + \frac{\xi^2}{1-\xi^2}\, E^q$$

$$F^q_{\lambda'\neq\lambda} \;\propto\; e^{\pm i\varphi}\, \frac{|\boldsymbol{\Delta}|}{2m_p}\, E^q \qquad \varphi = \text{azimuthal angle of } \boldsymbol{\Delta}$$

- $E^q \neq 0$ needs orbital angular momentum between partons
  $\Delta L^3 = \pm 1$ from helicity imbalance M. Burkardt, G. Schnell '05

GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi} \, e^{ixP^+z^-} \big\langle p', s' \big| \, \bar{q}(-\tfrac{1}{2}z) \, W[-\tfrac{1}{2}z, \tfrac{1}{2}z] \gamma^+ q(\tfrac{1}{2}z) \, \big| \, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$$= H^q \; \bar{u}(p', s')\gamma^+ u(p, s) + E^q \; \bar{u}(p', s') \, \frac{i}{2m_p} \sigma^{+\alpha}(p' - p)_\alpha \, u(p, s)$$

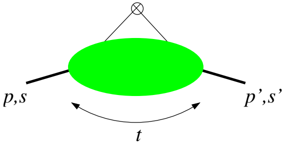
- time reversal invariance $\to$

$$H^q(x, \xi, t) = H^q(x, -\xi, t)$$

same for other distrib's

easy to see with var's x, ξ ... more complicated with x, x_B

GPDs: definition and properties



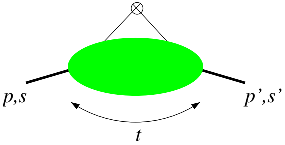
$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$
$$= H^q\ \bar{u}(p',s')\gamma^+ u(p,s) + E^q\ \bar{u}(p',s')\, \frac{i}{2m_p}\sigma^{+\alpha}(p'-p)_\alpha\, u(p,s)$$

- Mellin moments: $\int dx\, x^n \ \to$ local operator $\to$ form factors
- can be calculated in lattice QCD
- $\int dx \ \to$ vector current $\bar{q}(0)\gamma^+ q(0)$

$\sum_q e_q \int dx\, H^q(x,\xi,t) = F_1(t)$ Dirac f.f.

$\sum_q e_q \int dx\, E^q(x,\xi,t) = F_2(t)$ Pauli f.f.

GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \big\langle p', s' \big|\, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z] \gamma^+ q(\tfrac{1}{2}z) \,\big|\, p, s \big\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

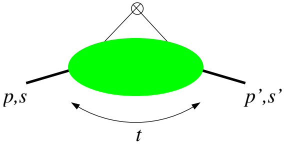
$$= H^q\ \bar{u}(p', s')\gamma^+ u(p, s) + E^q\ \bar{u}(p', s')\, \frac{i}{2m_p} \sigma^{+\alpha}(p' - p)_\alpha\, u(p, s)$$

- Mellin moments: $\int dx\, x^n \ \rightarrow$ local operator $\rightarrow$ form factors
- $\int dx\, x^n\, e^{ixP^+z^-} \ \rightsquigarrow\ \delta^{(n)}(z^-)\ \rightsquigarrow$ operators with derivatives ∂^+
- Lorentz invariance $\rightarrow$ polynomiality property

$$\int dx\, x^{n-1} H^q(x, \xi, t) = \sum_{k=0}^{n} (2\xi)^k\, A^q_{n,k}(t)$$

$$\Delta^+ = -2\xi P^+$$

GPDs: definition and properties



$$F^q = \int \frac{dz^-}{4\pi}\, e^{ixP^+z^-} \langle p', s' \big| \, \bar{q}(-\tfrac{1}{2}z)\, W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\gamma^+ q(\tfrac{1}{2}z) \, \big| \, p, s\rangle_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$$= H^q \, \bar{u}(p', s')\gamma^+ u(p, s) + E^q \, \bar{u}(p', s') \, \frac{i}{2m_p}\sigma^{+\alpha}(p' - p)_\alpha \, u(p, s)$$

- $\int dx\, x \;\to\;$ energy-momentum tensor

  Ji's sum rule $\quad \frac{1}{2}\int_{-1}^{1} dx\, x(H^q + E^q) = J^q(t)$

  $J^q(0) =$ total angular momentum carried by quark flavor q (helicity and orbital part)

  recall: E^q needs orbital angular momentum

  for gluons: $\int_{-1}^{1} dx\, (H^g + E^g) = J^g(t)$

Localizing partons: impact parameter

- states with definite light-cone momentum p^+
  and transverse position (impact parameter):

$$|p^+, \boldsymbol{b}\rangle = \frac{1}{(2\pi)^2} \int d^2\boldsymbol{p}\, e^{-i\boldsymbol{b}\,\boldsymbol{p}}\, |p^+, \boldsymbol{p}\rangle$$

  formal: eigenstates of 2 dim. position operator

- can exactly localize proton in 2 dimensions
  no limitation by Compton wavelength
- and stay in frame where proton moves fast
  $\rightsquigarrow$ parton interpretation
- different from localization in 3 spatial dimensions
  well-known for form factors; also for GPDs

Belitsky, Ji, Yuan '03; Brodsky et al. '06

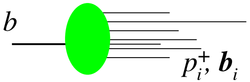
Localizing partons: impact parameter

- states with definite light-cone momentum p^+ and transverse position (impact parameter):

$$|p^+, \boldsymbol{b}\rangle = \frac{1}{(2\pi)^2} \int d^2\boldsymbol{p}\, e^{-i\boldsymbol{b}\,\boldsymbol{p}}\, |p^+, \boldsymbol{p}\rangle$$

formal: eigenstates of 2 dim. position operator

- $\boldsymbol{b}$ is center of momentum of the partons in proton



$$\boldsymbol{b} = \frac{\sum_i p_i^+ \boldsymbol{b}_i}{\sum_i p_i^+} \qquad (i = q, \bar{q}, g)$$

consequence of Lorentz invariance: transverse boosts

$$k^+ \to k^+ \qquad \boldsymbol{k} \to \boldsymbol{k} - k^+ \boldsymbol{v}$$

nonrelativistic analog: Galilei invariance $\xrightarrow{\text{Noether}}$ center of mass

Impact parameter GPDs

for simplicity take $\xi = 0$ ($\xi \neq 0$ and $s \neq s'$ later)

$\rightarrow$ blackboard

Impact parameter GPDs

for simplicity take $\xi = 0$ ($\xi \neq 0$ and $s \neq s'$ later)

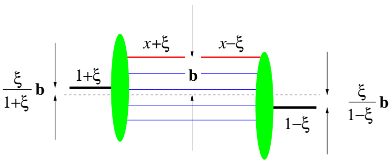
- $q(x, b^2) = (2\pi)^{-2} \int d^2\mathbf{\Delta}\, e^{-i\mathbf{b}\mathbf{\Delta}}\, H^q(x, \xi = 0, t = -\mathbf{\Delta}^2)$
  gives distribution of quarks with
  - longitudinal momentum fraction x
  - transverse distance b from proton center
- average impact parameter

$$\langle b^2 \rangle_x = \frac{\int d^2b\, b^2\, q(x, b^2)}{\int d^2b\, q(x, b^2)} = 4\, \frac{\partial}{\partial t} \log H(x, \xi = 0, t) \Big|_{t=0}$$

- integrated over x $\rightsquigarrow$ form factor

$$\langle b^2 \rangle = \frac{\int dx \int d^2b\, b^2\, q(x, b^2)}{\int dx \int d^2b\, q(x, b^2)} = 4\, \frac{\partial}{\partial t} \log F_1(t) \Big|_{t=0}$$

Impact parameter GPDs: $\xi \neq 0$



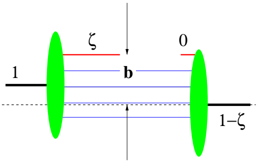
DGLAP region   ERBL region

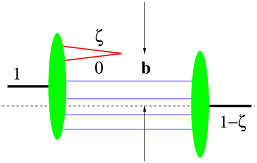
- Fourier transf. w.r.t. $\mathbf{\Delta}$

$t = -\frac{4\xi^2 m^2 + \mathbf{\Delta}^2}{1-\xi^2}$

- hadron center of momentum shifts because of plus-momentum transfer
- key observable: t dependence of cross sections at given ξ

Impact parameter GPDs: $\xi \neq 0$





- especially simple for $x = \xi$

  change to asymmetric variables: $\xi = \frac{\zeta}{2-\zeta}$ and $t = -\frac{\zeta^2 m_p^2 + \mathbf{\Delta}^2}{1-\zeta}$
- Fourier transf. w.r.t. $\mathbf{\Delta}$

  $\rightsquigarrow$ distance of struck parton from spectator system

in following concentrate on $\xi = 0$

Apples, oranges, and other fruit

form factor	distribution	$\langle b^2 \rangle$
F_1^p	$\sum_q e_q\,(q - \bar{q})$	$(0.66\,\text{fm})^2$
G_E^p		$(0.71\,\text{fm})^2 = (0.66\,\text{fm})^2 + \dfrac{\kappa_p}{m_p^2}$
G_A	$\Delta u + \Delta\bar{u} - (\Delta d + \Delta\bar{d}\,)$	$(0.52$ to $0.54\,\text{fm})^2$

- in form factor integral parton distributions have average $x \sim 0.2$
- generalized gluon dist. at $x = 10^{-3}$ $\rightsquigarrow$ $\langle b^2 \rangle = (0.57$ to $0.60\,\text{fm})^2$
  from J/Ψ photoproduction at HERA

note:

$4\,\frac{\partial}{\partial t} \log G(t)\Big|_{t=0} =$ squared impact parameter

$6\,\frac{\partial}{\partial t} \log G(t)\Big|_{t=0} =$ squared radius

numbers: G_E and F_1 from Particle Data Group; G_A from Bernard et al. '01

Interlude: lattice QCD in a tiny nutshell

- represent amplitudes as path integral:

$$\int \mathcal{D}[A]\,\mathcal{D}[\bar{\psi}\psi]\;\psi(x_1)\,\bar{\psi}(x_2)\ldots\bar{\psi}(x_n)\;\exp\!\big(i\int d^4x\,\mathcal{L}\big)$$

- action quadratic in $\bar{\psi}, \psi$ $\rightsquigarrow$ integrate over fermion fields analytically remaining $\int \mathcal{D}[A]$ not suited for numerical evaluation

Interlude: lattice QCD in a tiny nutshell

- represent amplitudes as path integral:

$$\int \mathcal{D}[A]\, \mathcal{D}[\bar{\psi}\psi]\; \psi(x_1)\, \bar{\psi}(x_2) \ldots \bar{\psi}(x_n)\; \exp\big(i \int d^4x\, \mathcal{L}\big)$$

- action quadratic in $\bar{\psi}, \psi$ $\rightsquigarrow$ integrate over fermion fields analytically
  remaining $\int \mathcal{D}[A]$ not suited for numerical evaluation
- discretize space-time $\rightsquigarrow$ integrate over fields on space-time lattice
  later take limit $a \to 0$ for lattice spacing — typical $a \sim 0.1\,\text{fm}$
- take lattice of finite size $(Na)^4$
  later take limit $Na \to \infty$ — typical $N \sim 16 \ldots 64$
- go to Euclidean time $\tau = it$, find $\mathcal{L}(x) \to -\mathcal{L}_E(x_E) < 0$

$$\exp\big(i \int d^4x\, \mathcal{L}\big) \to \exp\big(-\int d^4x_E\, \mathcal{L}_E\big)$$

  $\rightsquigarrow$ evaluate highly multidimensional integral by Monte-Carlo methods
- computational effort bigger for smaller m_q
  $\rightsquigarrow$ mostly take unphysically large m_q, later extrapolate to physical value

Interlude: lattice QCD in a tiny nutshell

- represent amplitudes as path integral:

$$\int \mathcal{D}[A]\, \mathcal{D}[\bar{\psi}\psi]\; \psi(x_1)\, \bar{\psi}(x_2) \ldots \bar{\psi}(x_n)\; \exp\!\big(i \int d^4x\, \mathcal{L}\big)$$

- action quadratic in $\bar{\psi}, \psi$ $\rightsquigarrow$ integrate over fermion fields analytically
  remaining $\int \mathcal{D}[A]$ not suited for numerical evaluation
- discretize space-time $\rightsquigarrow$ integrate over fields on space-time lattice
  later take limit $a \to 0$ for lattice spacing   typical $a \sim 0.1\,\text{fm}$
- take lattice of finite size $(Na)^4$
  later take limit $Na \to \infty$ typical $N \sim 16 \ldots 64$
- go to Euclidean time $\tau = it$, find $\mathcal{L}(x) \to -\mathcal{L}_E(x_E) < 0$

$$\exp\!\big(i \int d^4x\, \mathcal{L}\big) \to \exp\!\big(-\int d^4x_E\, \mathcal{L}_E\big)$$

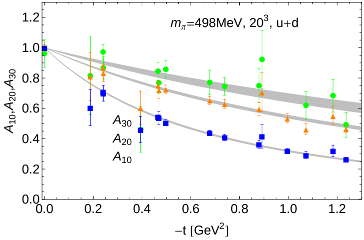
  $\rightsquigarrow$ evaluate highly multidimensional integral by Monte-Carlo methods
- computational effort bigger for smaller m_q
  $\rightsquigarrow$ mostly take unphysically large m_q, later extrapolate to physical value
- suited to calculate e.g. hadron masses and local matrix elements like

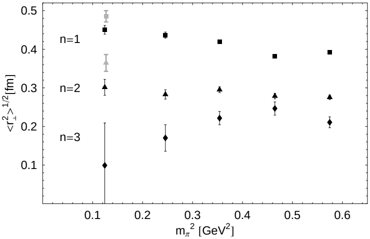
$$\langle h'(p')|\, \bar{\psi}(0)\Gamma\psi(0)\, |h(p)\rangle$$

Lattice calculations

- results for GPD moments

$$A_{n,0}(t) = \int dx\, x^{n-1} H(x, \xi = 0, t) = \int d^2\boldsymbol{b}\; e^{i\boldsymbol{b}\boldsymbol{\Delta}} \int dx\, x^{n-1} q(x, b^2)$$





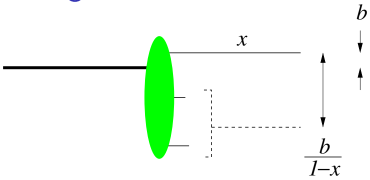
black: $L^3 = 28^3$, grey: $L^3 = 20^3$

LHPC Collaboration, arXiv:0705.4295

- steeper t slope for larger n

  $\rightsquigarrow$ decrease of $\langle b^2 \rangle_x$ with x

Large x



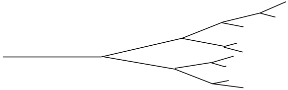
- for $x \to 1$ get $b \to 0$
  nonrel. analog:
  center of mass of atom
- $\Leftrightarrow \quad t$ dependence becomes flat

- $d = b/(1-x)$
  $=$ distance of selected parton from spectator system
  gives lower bound on overall size of proton
- finite size of configurations with $x \to 1$ implies

$$\langle b^2 \rangle_x \sim (1-x)^2$$

Small x

- partons with smaller $x \to$ broader in b
- Gribov diffusion: parton branching as random walk in b space



$$\to \langle b^2 \rangle \propto \alpha' \log(1/x)$$

- Regge phenomenology: simplest ansatz

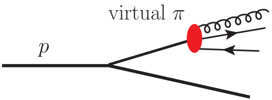
$$H(x, \xi = 0, t) \sim e^{tB} \left(\frac{1}{x}\right)^{\alpha_0 + \alpha' t} = x^{-\alpha_0} e^{t\alpha' \log(1/x) + tB}$$

  - is effective power-law in limited range of x and t at given μ^2
  - works well in fits of forward parton distributions
  - used in GPD models with further ansatz to generate ξ dep'ce
- for gluons $\alpha' \sim 0.15\,\text{GeV}^{-2}$ from HERA J/Ψ production
  barely known: value for valence and sea quarks, interplay with gluons

Large b

- prediction from chiral dynamics
  $\langle b^2 \rangle \sim e^{-\kappa b_T}/b_T$ with $\kappa \sim 2m_\pi = (0.7\,\mathrm{fm})^{-1}$

sets in for $x \lesssim m_\pi/m_p$
for larger x pion virtuality $\gg m_\pi^2$



Ch. Weiss et al

Now add spin

- $E \leftrightarrow$ nucleon helicity flip $\langle\downarrow|\mathcal{O}|\uparrow\rangle$

  $\leftrightarrow$ transverse pol. difference $|X\pm\rangle = \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle \pm |\downarrow\rangle\,\big)$

  $\langle X{+}|\mathcal{O}|X{+}\rangle - \langle X{-}|\mathcal{O}|X{-}\rangle = \langle\uparrow|\mathcal{O}|\downarrow\rangle + \langle\downarrow|\mathcal{O}|\uparrow\rangle$

- quark density in proton state $|X{+}\rangle$ shifted in y direction:

$$q^{\Uparrow}(x,\boldsymbol{b}) = q(x,\boldsymbol{b}^2) - \frac{b^y}{m}\frac{\partial}{\partial \boldsymbol{b}^2}\, e^q(x,\boldsymbol{b}^2)$$

  $e^q(x,b)$ is Fourier transform of $E^q(x,\xi=0,t)$

  derivative from Fourier trf. of

$$\frac{i\boldsymbol{\Delta}^y}{2m}\, E^q(x,\xi=0,t=-\boldsymbol{\Delta}^2)$$

Now add spin

- $E \leftrightarrow$ nucleon helicity flip $\langle\downarrow|\mathcal{O}|\uparrow\rangle$

  $\leftrightarrow$ transverse pol. difference $|X\pm\rangle = \frac{1}{\sqrt{2}}(\,|\uparrow\rangle \pm |\downarrow\rangle\,)$

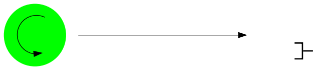
  $\langle X{+}|\mathcal{O}|X{+}\rangle - \langle X{-}|\mathcal{O}|X{-}\rangle = \langle\uparrow|\mathcal{O}|\downarrow\rangle + \langle\downarrow|\mathcal{O}|\uparrow\rangle$

- quark density in proton state $|X{+}\rangle$ shifted in y direction:

$$q^{\Uparrow}(x,\boldsymbol{b}) = q(x,\boldsymbol{b}^2) - \frac{b^y}{m}\frac{\partial}{\partial \boldsymbol{b}^2}\, e^q(x,\boldsymbol{b}^2)$$

  $e^q(x,b)$ is Fourier transform of $E^q(x,\xi=0,t)$

- semi-classical picture: rotating matter distribution

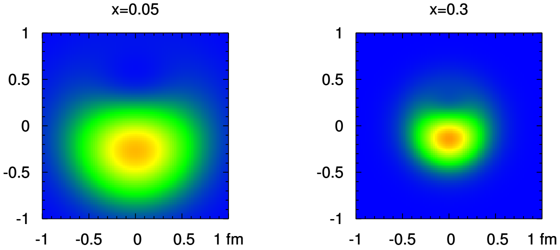


- gives alternative view on Ji's sum rule $L^x = b^y\, p^z$ M. Burkardt '05 quark density in proton state $|X+\rangle$

$$q^{\Uparrow}(x, \boldsymbol{b}) = q(x, \boldsymbol{b}^2) - \frac{b^y}{m} \frac{\partial}{\partial \boldsymbol{b}^2} e^q(x, \boldsymbol{b}^2)$$

is shifted

$(d - \bar{d})$ density in transverse plane



GPD model from MD, Th Feldmann, R Jakob, P Kroll '04

- from p and n magnetic moments $\kappa_p = \frac{2}{3}\kappa_u - \frac{1}{3}\kappa_d, \;\; \kappa_n = \frac{2}{3}\kappa_d - \frac{1}{3}\kappa_u$

$$\int dx\, E^u(x,0,0) = F_2^u(0) = \kappa^u \approx 1.67$$
$$\int dx\, E^d(x,0,0) = F_2^d(0) = \kappa^d \approx -2.03$$

$\rightsquigarrow$ large spin-orbit correlations for $q - \bar{q}$

- size of effect for sea quarks and gluons $\rightsquigarrow$ wait for EIC

- density representation

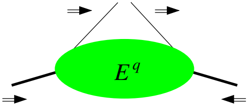
$$q^{\Uparrow}(x,\boldsymbol{b}) = q(x,\boldsymbol{b}^2) - \frac{b^y}{m}\frac{\partial}{\partial \boldsymbol{b}^2}\, e^q(x,\boldsymbol{b}^2)$$

gives positivity bound M. Burkardt '03

$$\Big| E^q(x,\xi=0,t=0) \Big| \quad \le \quad q(x)\, m \sqrt{\langle \boldsymbol{b}^2 \rangle_x}$$

have more restrictive bounds involving polarized distributions

$\Rightarrow$ E^q must fall faster than H^q at large x



- E $\leftrightarrow$ orbital angular momentum
  $\Rightarrow$ not carried by partons with large x

The proton spin budget

- Ji's sum rule: $\frac{1}{2} = J^g + \sum_q J^q$ with

$$J^q = \frac{1}{2} \int dx\, x \left(H^q + E^q\right)\Big|_{\substack{t=0\\ \xi=0}} \qquad J^g = \frac{1}{2} \int dx \left(H^g + E^g\right)\Big|_{\substack{t=0\\ \xi=0}}$$

- further decompose $J^q = L^q + \frac{1}{2}\Sigma$ and $J^g = L^g + \Delta g$
  with Σ and Δg from ordinary parton densities $\qquad \Sigma_{\overline{\mathrm{MS}}} \approx 25\%$
  operator interpretation of L^g nontrivial
- alternative decomp. $\frac{1}{2} = \mathcal{J}^g + \sum_q \mathcal{J}^q$ Bashinski, Jaffe '98

  with $\mathcal{J}^g = \mathcal{L}^g + \Delta g$ and $\mathcal{J}^q = \mathcal{L}^q + \frac{1}{2}\Sigma$
  - $J^q \neq \mathcal{J}^q$, $L^q \neq \mathcal{L}^q$ and $J^g \neq \mathcal{J}^g$

The proton spin budget

- Ji's sum rule: $\frac{1}{2} = J^g + \sum_q J^q$ with

$$J^q = \frac{1}{2} \int dx\, x \,(H^q + E^q)\Big|_{\substack{t=0 \\ \xi=0}} \qquad J^g = \frac{1}{2} \int dx \,(H^g + E^g)\Big|_{\substack{t=0 \\ \xi=0}}$$

- further decompose $J^q = L^q + \frac{1}{2}\Sigma$ and $J^g = L^g + \Delta g$
  with Σ and Δg from ordinary parton densities $\Sigma_{\overline{\mathrm{MS}}} \approx 25\%$
  operator interpretation of L^g nontrivial
- alternative decomp. $\frac{1}{2} = \mathcal{J}^g + \sum_q \mathcal{J}^q$ Bashinski, Jaffe '98
- ambiguities in decomposition reflect difficulty to separate
  - "quark" from "gluon" contrib's in presence of interactions
    gluon field contains physical and unphysical (gauge) d.o.f.
  - "intrinsic" from orbital angular momentum for spin-1 particles

  similar issues already in QED

  there need not be a unique choice

recent discussion: K-F Liu, C Lorcé 2015

The proton spin budget

- Ji's sum rule: $\frac{1}{2} = J^g + \sum_q J^q$ with

$$J^q = \frac{1}{2} \int dx\, x \left(H^q + E^q\right)\Big|_{\substack{t=0 \\ \xi=0}} \qquad J^g = \frac{1}{2} \int dx \left(H^g + E^g\right)\Big|_{\substack{t=0 \\ \xi=0}}$$

- further decompose $J^q = L^q + \frac{1}{2}\Sigma$ and $J^g = L^g + \Delta g$
  with Σ and Δg from ordinary parton densities $\Sigma_{\overline{\text{MS}}} \approx 25\%$
  operator interpretation of L^g nontrivial
- lattice $\rightsquigarrow$ Σ and J^q J^g difficult, Δg impossible
  - directly get integrals over x at $\xi = 0$
- exclusive processes $\rightsquigarrow$ GPDs $\rightsquigarrow$ J^q and (more difficult) J^g
  - exclusive (and inclusive) processes: $\int dx$ difficult
  - measure at $\xi \neq 0$
  - but direct access to x dependence of $E^{q,g}(x,x,t)$

Some lattice estimates at scale $\mu = 2\,\mathrm{GeV}$

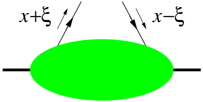
- QCDSF, M. Ohtani et al. '07
  $J^u = 0.230(8) \qquad J^d = -0.004(8)$
  $L^{u+d} = 0.025(27)$
- LHPC, J.D. Bratt, Ph. Hägler et al. '10
  $J^u = 0.236(6) \qquad J^d = 0.0018(37)$
  $L^{u+d} = 0.056(11)$ or $0.030(12)$
- C. Alexandrou et al. '13
  $J^u = 0.214(27) \qquad J^d = -0.003(17)$
  $L^{u+d} = -0.092(41)$
- still important systematic uncertainties
- trend: $J^u > 0, \;\; J^d \approx 0$
  $L^u < 0$, $L^d > 0$ and $L^u + L^d \approx 0$
- small L^{u+d} does not mean absence of orbital angular momentum

Evolution

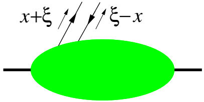
- for non-singlet combinations (e.g. $q - \bar{q}$ or $u - d$)

$$\mu^2 \frac{d}{d\mu^2} H^{\mathrm{NS}}(x,\xi,t) = \int dx' \, V^{\mathrm{NS}}(x,x',\xi) \, H^{\mathrm{NS}}(x',\xi,t)$$

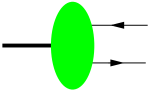
- for singlet $\sum_q (q + \bar{q})$: matrix equation for mixing with gluon GPD
- same evolution for E (independent of proton spin)



generalization of DGLAP
evolution to $\xi \neq 0$
recover usual DGLAP for $\xi = 0$



ERBL evolution as for
meson distribution amplitudes



Evolution

- for non-singlet combinations (e.g. $q - \bar{q}$ or $u - d$)

$$\mu^2 \frac{d}{d\mu^2} H^{\mathrm{NS}}(x,\xi,t) = \int dx' \, V^{\mathrm{NS}}(x,x',\xi) \, H^{\mathrm{NS}}(x',\xi,t)$$

- for singlet $\sum_q (q + \bar{q})$: matrix equation for mixing with gluon GPD
- same evolution for E (independent of proton spin)
- evolution local in t (take $-t \ll \mu^2$ to be safe)
  Fourier trf $\rightsquigarrow$ evolution local in b (take $1/\mu \ll b$ to be safe)
- for $\xi = 0$: $q(x, b^2)$ fulfills usual DGLAP evolution equation

$$\mu^2 \frac{d}{d\mu^2} q_{\mathrm{NS}}(x, b^2) = \int_x^1 \frac{dz}{z} \, P_{\mathrm{NS}}\Big(\frac{x}{z}\Big) q_{\mathrm{NS}}(z, b^2)$$

Evolution

- for non-singlet combinations (e.g. $q - \bar{q}$ or $u - d$)

$$\mu^2 \frac{d}{d\mu^2} H^{\mathrm{NS}}(x,\xi,t) = \int dx'\, V^{\mathrm{NS}}(x,x',\xi)\, H^{\mathrm{NS}}(x',\xi,t)$$

- for singlet $\sum_q (q+\bar{q})$: matrix equation for mixing with gluon GPD
- same evolution for E (independent of proton spin)
- evolution local in t (take $-t \ll \mu^2$ to be safe)
  Fourier trf $\rightsquigarrow$ evolution local in b (take $1/\mu \ll b$ to be safe)
- average impact parameter

$$\langle b^2 \rangle_x = \frac{\int d^2b\; b^2\, q(x,b^2)}{\int d^2b\; q(x,b^2)}$$

evolves as

$$\mu^2 \frac{d}{d\mu^2} \langle b^2 \rangle_x = -\,\frac{1}{q_{\mathrm{NS}}(x)} \int_x^1 \frac{dz}{z}\, P_{\mathrm{NS}}\Big(\frac{x}{z}\Big)\, q_{\mathrm{NS}}(z) \Big[\langle b^2 \rangle_x - \langle b^2 \rangle_z \Big]$$

Model ansätze for GPDs (only a sketch)

general strategy

- use standard parton densities at input (or boundary cond.)
- ansatz for t dependence
  ensure that sum rules for $H \leftrightarrow F_1$ and $E \leftrightarrow F_2$ satisfied
  factorized ansätze like $H(x, \xi, t) = h(x, \xi)\, F_1(t)$
  disfavored by theory and lattice results
- generate ξ dependence so as to satisfy polynomiality
  requires special constructions, e.g. double distributions
- check that positivity bounds satisfied
  not always done, often only possible numerically

Double distributions and the D term

$$H(x,\xi,t) = \int d\beta\, d\alpha\, \delta(x-\beta-\xi\alpha)\, f(\beta,\alpha,t)$$

- $f(\beta,\alpha,t) =$ double distribution
- forward limit: $\int d\alpha\, f(\beta,\alpha,0) = q(\beta)$
- ensures polyomiality:

$$\int dx\, x^{n-1} H(x,\xi,t) = \int d\beta\, d\alpha\, (\beta+\alpha\xi)^{n-1}\, f(\beta,\alpha,t) = \sum_{k=0}^{n-1} (2\xi)^k\, A_{n,k}(t)$$

- misses power ξ^n for H and E

  for $H+E$ and $\tilde{H}$, $\tilde{E}$ highest allowed power is ξ^{n-1}
- add Polyakov-Weiss $/D$ term

$$H_{DD}(x,\xi,t) + D(x/\xi,t) \qquad E_{DD}(x,\xi,t) - D(x/\xi,t)$$

gives power $\int dx\, x^{n-1} D(x/\xi,t) = \xi^n \int d\alpha\, \alpha^{n-1} D(\alpha,t)$

Double distributions for modeling

I Musatov, V Radyushkin '99, ...

$$H(x,\xi,t) = \int d\beta\, d\alpha\, \delta(x-\beta-\xi\alpha)\, f(\beta,\alpha,t)$$

ansatz:

$$f(\beta,\alpha,t) = h^{(b)}(\beta,\alpha)\, H(\beta,0,t)$$

with "profile function"

$$h^{(b)}(\beta,\alpha) \propto \frac{[(1-|\beta|)^2-\alpha^2]^b}{(1-|\beta|)^{2b+1}}$$

- forward limit: satisfies $\int d\alpha\, f(\beta,\alpha,0) = H(\beta,0,0) = f_1(\beta)$
- profile parameter b free
  often separate b for valence $(q-\bar{q})$ and sea $(\bar{q})$
- add Polyakov-Weiss term $D^q(x/\xi,t)$ for full polynomiality