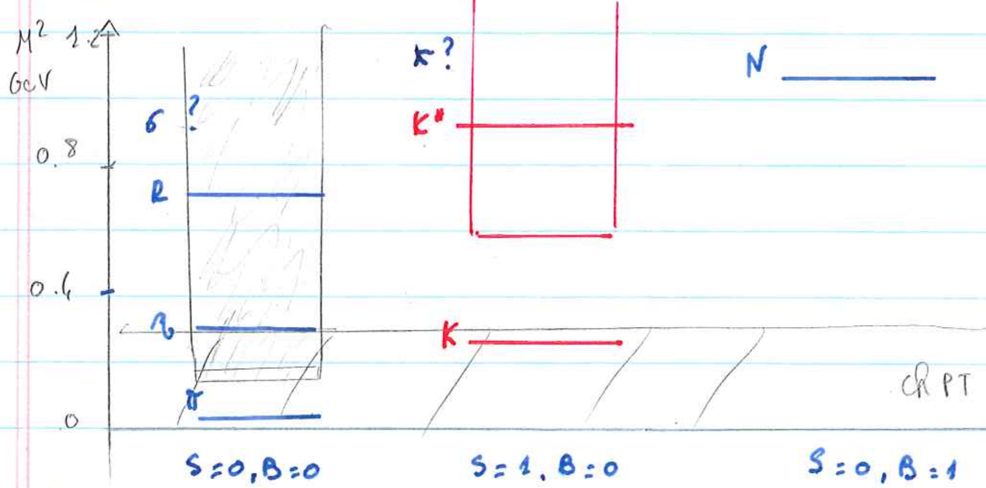
Chiral perturbation theory

I] Introduction

This is an EFT to describe strong interaction at low energy.
Instead of quarks and gluons
$\Rightarrow$ the observed particles are hadrons (due to confinement)

- mesons $q\bar{q}$: π^+ $u\bar{d}$, π^- : $\bar{u}d$
  - π^0 $u\bar{u}$ or $d\bar{d}$: $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
  - K^+ $u\bar{s}$, K^- : $s\bar{u}$ $\quad m_K \simeq 490$ MeV
  - K^0 : $d\bar{s}$ / $s\bar{d}$
  - η $\quad m_\eta = 547$ MeV
  - η' , ρ ... (770 MeV; 957 MeV)
- baryons : qqq : p uud
  - n ddu
  - Δ
  - Σ

QCD spectrum :



* chPT : energy scales $E << \Lambda_\chi \sim m_p = 1\,\mathrm{GeV}$

* d.o.f : hadron fields.

Build : $\mathcal{L}_{eff}$: most general $\mathcal{L}$ with d.o.f and compatible with the symmetries of the underlying theory i.e. QCD.

N.B: Matching the couplings of $\mathcal{L}_{eff}$ (Wilson coefficients or LECs) to QCD impossible!

⇒ not able to perform QCD computations at LE except with lattice QCD.

↳ does not look so good : not very predictive since the wilson coefficients not known.

⇒ However it turns out chiral symmetry severely constrains the interactions of light hadrons

⇒ EFT approach very useful to derive consequences of this approximate symmetry

→ predictivity for light mesons.

II] Chiral symmetry

The Lagrangian describing strong interactions: Build all the invariant under $SU(3)_c$ with the quarks

$$\boxed{\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + \sum_{k=1}^{N_f} \bar{q}_k (i\gamma^\mu D_\mu - m_k) q_k}$$

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \qquad D_\mu = \partial_\mu - i g_s \frac{\lambda_a}{2} A^a_\mu(x) \quad \left(\frac{\lambda_a}{2} = T_a\right)$$

$A^a_\mu(x)$ = gluon fields. $\quad - g_s [A^b_\mu, A^c_\nu]$ $\qquad N_f = 6 \quad u, d, s, c, b, t$

λ_a = Gell Mann matrices generator of $SU(3)_c$

N.B: Respect Lorentz invariance, P, C, T

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These matrices are traceless $Tr(\lambda_a) = 0$, hermitian : $\lambda^{a+} = \lambda^a$

$Tr[\lambda^a \lambda^b] = \delta^{ab}$

satisfy the SO(3) algebra : $\boxed{[\lambda^a, \lambda^b] = 2 i \underbrace{f^{abc}}_{\text{antisymmetric structure constants}} \lambda^c}$

gauge the theory $SU(3)_c \to$ local

↳ 8 different gauge fields : G^a_μ : the gluons g_s

Different parts of the Lagrangian :

- Kinetic terms : $G^a_{\mu\nu} G^{\mu\nu}_a$
- Interactions quark - gluon 
- Interactions gluon - gluon  
  - → Non abelian gauge group

≠ from QED : photon does not interact with itself !

One single universal coupling $\boxed{\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}}$ depends on energy

2 consequences : - asymptotic freedom
- confinement

7 unknowns in $\mathcal{L}_{QCD}$:
- 1: coupling constant
- 6 quark masses : m_k ⇒ No direct access because of confinement.

projectors
$P_L^2 = P_L$
$P_R^2 = P_R$
$P_L + P_R = 1$
$P_L P_R = P_R P_L = 0$

Chiral decomposition of fermion fields:

$$q = \frac{1}{2}(1-\gamma_5)q + \frac{1}{2}(1+\gamma_5)q \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= P_L q + P_R q$$
$$= q_L + q_R$$

$$\bar{q}\not{D}q$$
$$(q^+ P_L + q^+ P_R)\gamma_0 = q^+\gamma_0(P_R + P_L) = \bar{q}_R + \bar{q}_L$$

$$\mathcal{L}_{QCD} = \sum_k \Big[(\underbrace{\bar{q}_{L,k} + \bar{q}_{R,k}}_{(\bar{q}P_R + \bar{q}P_L)})\, i\not{D}(q_{L,k} + q_{R,k}) + m_k(\bar{q}_{L,k} + \bar{q}_{R,k})(q_{L,k} + q_{R,k}) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$$

$\boxed{\{\gamma_5, \gamma_\mu\} = 0}$, $\gamma_5^2 = 1$, $\gamma_5^+ = \gamma_5$

$$= \sum_k \bar{q}_{L,k}\, i\not{D}\, q_{L,k} + \bar{q}_{R,k}\, i\not{D}\, q_{R,k} - m_k(\bar{q}_{L,k} q_{R,k} + \bar{q}_{R,k} q_{L,k}) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a.$$

If $m_k \to 0$ $\mathcal{L}$ is invariant under a global symmetry
Consider only 3 flavours: $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$ ← flavour triplet

$q_L \to V_L q_L$ $\qquad \mathcal{L}^0_{QCD}$ invariant $\quad m_k \to 0$
$q_R \to V_R q_R$

V_L, V_R : 3×3 unitary matrices. = global rotation in flavour space.

$V_L^+ V_L = V_L V_L^+ = 1 \qquad V_L \in U(3)_L \qquad \psi_R$
$V_R^\dagger V_R = V_R V_R^+ = 1 \qquad V_R \in U(3)_R \qquad \psi_L$

N.B: helicity = chirality for massless fermions.
V_L, V_R = global rotation matrices in flavour space
$\frac{1}{2}\hat{h}\,\psi_{L,R} = \mp\frac{1}{2}\psi_{L,R}$, $\hat{h} = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$

while the gauge transformations act in colour space.

$$q_L \to V_L q_L$$
$$\bar{q}_L = q_L^+ \gamma^0 \to q_L^+ \gamma^0 V_L^+$$

$$\bar{q}_L i \not{\partial} q_L$$
$$\to \bar{q}_L V_L^+ i \not{\partial} V_L q_L = \bar{q}_L i \not{\partial} \underbrace{V_L^+ V_L}_{1} q_L$$

$$\bar{q}_R i \not{\partial} q_R \to \bar{q}_R V_R^+ i \not{\partial} V_R q_R = \bar{q}_R i \not{\partial} q_R$$

N.B: The mass term is not invariant! $\Rightarrow$ Mixes L and R

V_L and V_R can be parametrized as:

$$\boxed{V_{L,R} = \exp\left(i \alpha_{L,R} + \frac{i}{2} \sum_{a=1}^{8} \alpha^a_{L,R} \lambda^a\right)}$$

λ^a: Gell Mann matrices generators of SU(3)

$\to$ 2 flavours $\quad q \to \begin{pmatrix} u \\ d \end{pmatrix}$

σ^a: Pauli matrices generators of SU(2)

$$\boxed{V_{L,R} = \exp\left(i \alpha_{L,R} + i \sum_{a=1}^{3} \frac{\sigma^a}{2} \alpha^a_{L,R}\right)}$$

Rewrite $U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times \underbrace{U(1)_L \times U(1)_R}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U(1)_V \times U(1)_A$

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\bar{q}_L = \bar{q}^+ P_R$

Consider infinitesimal transformations $\Rightarrow$ Noether's theorem gives a classical conserved current for each transformation

$$\boxed{J_\mu \propto \frac{\delta \mathcal{L}}{\delta(\partial^\mu \psi)} \delta\psi}$$

$$\delta\psi = \psi' - \psi = (1 + i\delta\alpha_L)\psi - \psi = i\delta\alpha_L \psi$$

$L_\mu = \bar{q}_L \gamma_\mu q_L$, $L^a_\mu = \bar{q}_L \gamma_\mu \frac{\lambda^a}{2} q_L$

$R_\mu = \bar{q}_R \gamma_\mu q_R$, $R^a_\mu = \bar{q}_R \gamma_\mu \frac{\lambda_a}{2} q_R$

$\bar{q}_L = \bar{q}^+ P_R \gamma_\mu P_L q$

$$\left.\begin{aligned} \partial^\mu L^a_\mu &= 0 \\ \partial^\mu R^a_\mu &= 0 \end{aligned}\right\} \text{Conserved currents}$$

Instead of left and right currents, it is convenient to use axial and vector currents:

$V^\mu = L^\mu + R^\mu = \bar{q}\,\gamma^\mu q \Rightarrow U(1)_V$

$A^\mu = R^\mu - L^\mu = \bar{q}\,\gamma^\mu \gamma_5 q$

$\partial_\mu V^\mu = 0 \Rightarrow$ conservation of baryon number.

$\partial_\mu A^\mu = 0$ at the classical level

But at the quantum level $\partial_\mu A^\mu \neq 0$: A^μ is anomalous

$$\boxed{\partial_\mu A^\mu = \frac{N_c g_s^2}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} G^{a\mu\nu} G^{a\rho\sigma}} \quad (N_c = 3)$$

The remaining $SU(3)_L \times SU(3)_R \times U(1)_V$ transformations are symmetries of the quantum theory

we build also the vector and axial currents from L^a_μ and R^a_μ

$V^a_\mu = R^a_\mu + L^a_\mu = \bar{q}\,\gamma_\mu \frac{\lambda_a}{2} q$

$A^a_\mu = R^a_\mu - L^a_\mu = \bar{q}\,\gamma_\mu \gamma_5 \frac{\lambda_a}{2} q$

that are conserved $\partial^\mu V^a_\mu = 0$, $\partial^\mu A^a_\mu = 0$

We can build the associate conserved charges:

$$Q_V^a = \int d^3x \; V^{0a}(x) = \int d^3x \; \bar{q} \, \gamma^0 \frac{\lambda^a}{2} q$$

$$Q_A^a = \int d^3x \; A^{0a}(x) = \int d^3x \; \bar{q} \, \gamma^0 \gamma^5 \frac{\lambda^a}{2} q$$

$$Q_V = \int d^3x \; V^0(x) = \int d^3x \; \bar{q} \gamma^0 q = \int d^3x \; q^+ q$$

The $2 \times 8 + 1$ charges Q_V^a, Q_A^a and Q_V commute with the Hamiltonian H_0 of massless QCD.

$$[Q_V^a, H_0] = [Q_A^a, H_0] = [Q_V, H_0] = 0$$

$\Rightarrow$ Q is conserved $\Rightarrow$ commutes with H, annihilates the vacuum.

$$Q|0\rangle = 0 \Rightarrow \frac{dQ}{dt} = [Q, H] = 0$$

does not necessarily imply

But $[Q, H] = 0 \not\Rightarrow Q|0\rangle = 0$

2 situations:

1) $[Q, H] = 0$ and $Q|0\rangle = 0$: similar to QM

Wigner Weyl mode of symmetry realization

$H|\psi_p\rangle = E_p |\psi_p\rangle$

single particle state

$|\chi\rangle = Q|\psi\rangle$ $\quad H|\chi\rangle = H Q_V |\psi\rangle = Q_V H |\psi\rangle = E Q_V |\psi\rangle = E|\chi\rangle$

$H(e^{iQ}|\psi_p\rangle) = e^{iQ} H|\psi_p\rangle = E_p (e^{iQ}|\psi_p\rangle)$

$\Rightarrow$ Another state of same mass

Well known for $SU(3)_V$: eightfold way

$Q_V^a|0\rangle = 0$ shown by Vafa and Witten '84

2) $[Q, H] = 0$ and $Q|0\rangle \neq 0$

$\to$ the symmetry is spontaneously broken

The hamiltonian is invariant but not the vacuum

$\Rightarrow$ the symmetry is not visible in the spectrum!

The symmetry is realized according to Nambu goldstone mode.

Chiral symmetry is not manifest in QCD

$\Rightarrow$ It would tell us that the ρ meson (vector meson) must have the same mass as the a_1 meson (axial vector) but

$m_\rho = 0.770$ GeV , $m_{a_1} = 1.26$ GeV

$SU(3)_L \times SU(3)_R \xrightarrow{\text{spontaneously broken}} SU(3)_V$

Consider $|\psi\rangle$ with $H|\psi\rangle = E|\psi\rangle$

If axial symmetry exact $[H, Q_5^a] = 0$

$|\chi\rangle = Q_A^a|\psi\rangle$ has energy $H|\chi\rangle = H Q_A^a |\psi\rangle$

$= Q_A^a H|\psi\rangle$

$= Q_A^a E|\psi\rangle$

$= E Q_A^a|\psi\rangle$

$= E|\chi\rangle$ same energy.

$\Rightarrow$ Not the case!

what is happening?

The vacuum of QCD is not chirally invariant

$\to$ $Q_A^a|0\rangle \neq 0$ $\Rightarrow$ due to the condensation of $q\bar{q}$ *

This is spontaneous symmetry breaking

The Lagrangian has the symmetry but the vacuum breaks it.

$\Rightarrow$ goldstone bosons

* We need $\langle 0|[Q_A^a, O^b]|0\rangle \neq 0$

O_A = pseudoscalar operator $\Rightarrow$ simplest possibility $O^a = \bar{q}\gamma_5\lambda^a q$

$\langle 0|[Q_A^a, \bar{q}\gamma_5\lambda^b q]|0\rangle = -\frac{1}{2}\langle 0|\bar{q}\{\lambda^a, \lambda^b\}q|0\rangle = -\frac{2}{3}\delta_{ab}\langle 0|\bar{q}q|0\rangle$

Q Noether charge

$\Rightarrow$ there exists an operator $\mathcal{O}$ which satisfies:

$\langle 0 | [Q, \mathcal{O}] | 0 \rangle \neq 0$

$\Rightarrow$ Only possible if $Q|0\rangle \neq 0$

Goldstone's theorem: it exists a massless state $|G\rangle$ such that

$\langle 0 | J^0 | G \rangle \langle G | \mathcal{O} | 0 \rangle \neq 0$

$\Rightarrow$ quantum numbers of G are dictated by those of J^0 and $\mathcal{O}$

$\langle 0 | [Q, \mathcal{O}] | 0 \rangle$ = order parameter of the spontaneous symmetry breakdown.

$\mathcal{O}^a$ = pseudoscalar operator.

$\Rightarrow$ simplest are $\mathcal{O}^a = \bar{q} \gamma_5 \lambda^a q$

let us show that: $\langle 0 | [Q_A^a, \bar{q} \gamma_5 \lambda^b q] | 0 \rangle \propto \langle 0 | \bar{q} q | 0 \rangle$

$\hookrightarrow$ $[Q_L^a, q_L] = -\frac{\lambda^a}{2} q_L$ — Q_L^a generator of $SU(3)_L$

$[Q_R^a, q_R] = -\frac{\lambda^a}{2} q_R$ — Q_R^a " of $SU(3)_R$

$[Q_L^a, q_R] = [Q_R^a, q_L] = 0$

$Q_A^a = Q_R^a - Q_L^a$ $\Rightarrow$

$[Q_L^a, \bar{q}_L] = \bar{q}_L \frac{\lambda^a}{2}$

$[Q_R^a, \bar{q}_R] = \bar{q}_R \frac{\lambda^a}{2}$

The quark condensate $\langle 0|u\bar{u}|0\rangle = \langle 0|d\bar{d}|0\rangle = \langle 0|s\bar{s}|0\rangle \neq 0$ (ChPT 9)
is the order parameter of spontaneous symmetry breaking

goldstone theorem: There exists a massless

$$H\, Q_A^a\,|0\rangle = Q_A^a\, H|0\rangle = 0$$

since $Q_A^a|0\rangle \neq 0 \Rightarrow \exists$ 8 states with $E = 0$ massless states!
we end up with 8 states

$$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$$

the number of broken generators $N_f^2 - 1$ massless, parity odd, spin 0
states $\Rightarrow$ 8 pseudoscalars : the 8 lightest hadrons : $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$
$J^P = 0^-$

goldstone theorem: For every conserved charge associated to a continuous global symmetry which does not let leave the vacuum invariant it appears in the spectrum of the theory a massless particle called goldstone boson that has the same quantum numbers as the associated charge.

For the QCD vacuum

$$Q_V^a\,|0\rangle = 0 \quad a = 0, \dots 8$$
$$Q_A^a\,|0\rangle \neq 0 \quad a = 0, \dots 8$$

$\rightarrow$ However they have a mass! They are not massless.
Yes because the light quark masses break explicitely the chiral symmetry
$\Rightarrow$ pseudo goldstone bosons.

$$\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} + \mathcal{L}_{mass}$$

$$\mathcal{L}_{mass} = -\bar{q}\, M\, q \quad \text{with } M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$\hookrightarrow$ $\mathcal{L}$ mass small perturbation
Expand around $\mathcal{L}^{(0)}_{QCD}$ in powers of m_q
Chiral perturbation theory the low energy effective theory of QCD
$\Rightarrow$ is a simultaneous expansion in powers of momenta and quark masses.

where $h(\phi, g) \in H$: compensating transformation needed to return to a given choice of coset representative $\bar{\xi}$

In general h depends on ϕ and g

Since The same transformation $h(\phi, g)$ occurs in the left and right sectors (the two chiral sectors are related by a parity transformation which obviously leaves H invariant)

→ we can get rid of it by combining the two chiral relations into a simpler form

$U(\pi) \xrightarrow{G} g_R \, U(\pi) \, g_L^{+}$ with $U(\pi) = \xi_R(\pi) \, \xi_L^{+}(\pi)$

without loss of generality we can take a canonical choice of coset representative such that $\xi_R(\pi) = \xi_L^{+}(\pi) \equiv u(\pi)$

The 3×3 unitary matrix

$\boxed{U(\pi) = u(\pi)^2 = \exp\left\{\frac{i}{F} \lambda^a \pi^a\right\}}$ gives a convenient parametrization of the goldstone fields.

goldstone bosons do not interact at vanishing energy

They live in the coset space G/H

$G: SU(3)_L \otimes SU(3)_R \to H = SU(3)_V$

The transformation properties of the coordinates of G/H (goldstone coordinates) under the action of G are non linear. The changes of the goldstone coordinates under a chiral transformation $g = (g_L, g_R) \in G$ is given by

$$\xi_L(\pi) \xrightarrow{G} g_L\, \xi_L(\pi)\, h^\dagger(\,,g)$$
$$\xi_R(\pi) \to g_R\, \xi_R(\pi)\, h^\dagger(\,,g)$$

III] Effective Lagrangian

$m_q = 0$

Task: Construct an effective field theory for the goldstone bosons

d.o.f * 8 lightest hadrons: pseudoscalars $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

Symmetries * Incorporate all symmetry constraints of QCD

Applicability: $E \ll \Lambda \sim 1\,\mathrm{GeV}$

Proceed in 4 steps:

1. Collect $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$ in a common field U

$$\boxed{U = e^{\frac{i}{F}\lambda^a \underbrace{\pi^a}_{\phi^a}}}$$ for $SU(3)$ $\qquad \lambda^a$ = gell mann matrices generators of $SU(3)$

If we work only with π pions: $SU(2)$ $\qquad \sigma^a$ = Pauli matrices generator of $SU(2)$

$$\boxed{U = e^{i/F\, \sigma^a \underbrace{\pi^a}_{\phi^a}}}$$

To understand why the field is parametrized this way consider the quark mass term and coupling to photons.

$$\lambda^a \pi^a = \begin{pmatrix} \pi^3 + \frac{1}{\sqrt{3}}\pi^8 & \pi^1 - i\pi^2 & \pi^4 - i\pi^5 \\ \pi^1 + i\pi^2 & -\pi^3 + \frac{1}{\sqrt{3}}\pi^8 & \pi^6 - i\pi^7 \\ \pi^4 + i\pi^5 & \pi^6 + i\pi^7 & -\frac{2}{\sqrt{3}}\pi^8 \end{pmatrix}$$

The goldstones are identified with the octet of light mesons π, K and η.
Modulus a phase this correspondance is given by

$$|\pi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|\pi^1\rangle \mp i|\pi^2\rangle\right) , \quad |\pi^0\rangle = |\pi^3\rangle$$

$$|K^\pm\rangle = \frac{1}{\sqrt{2}}\left(|\pi^4\rangle \mp i|\pi^5\rangle\right) , \quad |K^0\rangle = \frac{1}{\sqrt{2}}\left(|\pi^6\rangle - i|\pi^7\rangle\right)$$

$$|\bar{K}^0\rangle = \frac{1}{\sqrt{2}}\left(|\pi^6\rangle + i|\pi^7\rangle\right) , \quad |\eta\rangle = |\pi^8\rangle$$

$$U(x) = \exp\left[\frac{i}{f}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right]$$

In SU(2) ChPT

$$U(x) = \exp\left[i\frac{\sigma^a\pi^a}{F}\right]$$

$$= \exp\left[\frac{i}{F}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right]$$

2) Specify the transformation properties of this field under

$G = SU(3)_L \times SU(3)_R$

$$U \xrightarrow{G} V_R \, U \, \underset{\substack{\shortparallel \\ V_L^{-1}}}{V_L^+}$$

3) Construct $\mathcal{L}_{eff}$ with the d.o.f = goldstone bosons invariant under chiral transformations G.

Since $U(x)$ is dimensionless the terms with higher powers of $U(x)$ are unsuppressed $\Rightarrow$ order in powers of momenta or derivatives.

$$\mathcal{L}_{eff} = \overset{O(1)}{f_0(U)} + f_1(U)\, Tr\,(U^\dagger \Box U) + f_2(U)\, Tr\,(\partial_\mu U \partial^\mu U^+) + \ldots$$

The invariance under $U \overset{G}{\to} U' = V_R U V_L^+$ implies

$f_{0,1,2}(U)$ does not depend on U.

why $\Rightarrow$ choose $V_R = 1$, $\overset{\leftarrow \text{unitary}}{V_L} = U$

$f_0(U) = f_0(V_R U V_L^+)$

$\Rightarrow f_0(U) = f_0(\underbrace{U U^+}_{1}) = \text{cst}$

$O(1)$ terms irrelevant since they are constant one can drop them.

Term proportional to $f_1(U) \Rightarrow$ absorb into f_2 by integration by part

$$\int d^4x\, f_1(U)\, Tr\,(\overset{u}{U^+} \overset{v'}{\Box U}) = -\int d^4x\, f_1(U)\, \partial_\mu U \partial^\mu U^+$$

$\partial_\mu \partial^\mu U)$

$u = U^+ \qquad u' = \partial^\mu U^+$

$v' = \partial_\mu \partial^\mu U \qquad v = \partial_\mu U$

$(uv)' = u'v + v'u$

$$\mathcal{L}_{eff} = \underbrace{c.\; Tr\,(\partial_\mu U \partial^\mu U^+)}_{\mathcal{L}_2} + \underbrace{\ldots}_{\mathcal{L}_4} + \underbrace{\ldots}_{\mathcal{L}_6}$$

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4}\, Tr\,(\partial_\mu U \partial^\mu U^+) + O(p^4)}$$

The prefactor chosen to get canonically normalized kinetic terms for the π's.

To see this, expand

$$U(x) = \exp\left(\frac{i}{F}\pi^a\lambda^a\right) = 1 + \frac{i}{F}\pi^a\lambda^a - \frac{1}{2F^2}\vec{\pi}^2 \cdot \mathbb{1} + \dots$$

$$\mathcal{L}_{eff} = \frac{F^2}{4}\,Tr\left(\partial_\mu U\,\partial^\mu U^\dagger\right) + O(p^4)$$

$$\partial_\mu U = \frac{i}{F}(\partial_\mu \pi^a)\lambda^a$$

$$= \frac{i}{F}\begin{pmatrix} \partial_\mu\pi^0 + \frac{1}{\sqrt{3}}\partial_\mu\eta & \sqrt{2}\,\partial_\mu\pi^+ & \sqrt{2}\,\partial_\mu K^+ \\ \sqrt{2}\,\partial_\mu\pi^- & -\partial_\mu\pi^0 + \frac{1}{\sqrt{3}}\partial_\mu\eta & \sqrt{2}\,\partial_\mu K^0 \\ \sqrt{2}\,\partial_\mu K^- & \sqrt{2}\,\partial_\mu \bar{K}^0 & -\frac{2}{\sqrt{3}}\partial_\mu\eta \end{pmatrix} + \dots$$

$$\partial^\mu U^\dagger = -\frac{i}{F}\left(\partial^\mu\pi^{a\dagger}\right)\underset{\lambda^a}{\overset{\shortparallel}{\lambda^{a\dagger}}}$$

$$= -\frac{i}{F}\begin{pmatrix} \partial^\mu\pi^0 + \frac{1}{\sqrt{3}}\partial^\mu\eta & \sqrt{2}\,\partial^\mu\pi^- & \sqrt{2}\,\partial^\mu K^- \\ \sqrt{2}\,\partial^\mu\pi^+ & -\partial^\mu\pi^0 + \frac{1}{\sqrt{3}}\partial^\mu\eta & \sqrt{2}\,\partial^\mu\bar{K}^0 \\ \sqrt{2}\,\partial^\mu K^+ & \sqrt{2}\,\partial^\mu K^0 & -\frac{2}{3}\partial^\mu\eta \end{pmatrix} + \dots$$

$$\mathcal{L}_{eff} = \frac{1}{4}\left(2\,\partial_\mu\pi^+\partial^\mu\pi^- + 2\,\partial_\mu K^+\partial^\mu K^- + \dots\right)$$
$$= \underline{\frac{1}{2}\partial_\mu\pi^+\partial^\mu\pi^- + \frac{1}{2}\partial_\mu K^+\partial^\mu K^-} + \dots$$

$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\,Tr\left(\partial_\mu U\,\partial^\mu U^\dagger\right)}$ is invariant under $G = SU(3)_L \times SU(3)_R$

$$Tr\left(\partial_\mu U\,\partial^\mu U^\dagger\right) \longrightarrow Tr\left[\partial_\mu\left(V_R U V_L^\dagger\right)\,\partial^\mu\left(V_R U V_L^\dagger\right)^\dagger\right]$$
$$= Tr\left[V_R(\partial_\mu U)V_L^\dagger\,V_L\,\partial^\mu U^\dagger\,V_R^\dagger\right]$$
$$= \underline{Tr\left[\partial_\mu U\,\partial^\mu U^\dagger\right]}$$

$$V_R V_R^\dagger = V_L V_L^\dagger = \mathbb{1}$$

Calculate the Noether currents L_a^μ, R_a^μ from $\mathcal{L}^{(2)}$ or vector and axial currents

To see this expand

$$U(x) = \exp\left(\frac{i}{F}\pi^a\lambda^a\right) = 1 + \frac{i}{F}\pi^a\lambda^a - \frac{1}{2F^2}\vec{\pi}^2\,\mathbb{1} + \dots$$

$$\mathcal{L}_{eff} = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U\,\partial^\mu U^+\right) + O(p^4)$$

$$= \underline{\frac{1}{2}\partial_\mu\pi^+\partial^\mu\pi^- + \frac{1}{2}\partial_\mu K^+\partial^\mu K^-} + \dots$$

Show in exercise → see χPT p 13'

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U\,\partial^\mu U^+\right)}$$ invariant under $SU(3)_L \times SU(3)_R = G$

$$\mathrm{Tr}\left(\partial_\mu U\,\partial^\mu U^\dagger\right) \xrightarrow{G} \mathrm{Tr}\left(\partial_\mu\left(V_R U V_L^+\right)\partial^\mu\left(V_R U V_L^+\right)^\dagger\right)$$

$$= \mathrm{Tr}\left(V_R(\partial_\mu U)V_L^+\,\partial^\mu\left(V_L U^+ V_R^+\right)\right)$$

$$= \mathrm{Tr}\left(V_R(\partial_\mu U)\,V_L^+ V_L\,(\partial^\mu U^\dagger)\,V_R^\dagger\right)$$

$$= \mathrm{Tr}\left(\partial_\mu U\,\partial^\mu U^\dagger\right)$$

$$V_R V_R^\dagger = 1 = V_L V_L^\dagger$$

Calculate the Noether currents: L_a^μ, R_a^μ from $\mathcal{L}^{(2)}$ or vector and axial currents

$$V_a^\mu = \underline{R_a^\mu + L_a^\mu} = \underline{i\frac{F^2}{4}\mathrm{Tr}\left(\lambda_a\left[\partial^\mu U, U^+\right]\right)}$$

$$A_a^\mu = \underline{R_a^\mu - L_a^\mu} = i\frac{F^2}{4}\mathrm{Tr}\left(\lambda_a\left\{\partial^\mu U, U^+\right\}\right)$$

Expand currents in power of π^a

$$A_a^\mu = -F\,\partial^\mu\pi_a + O(\pi_a^3)$$

$$\boxed{\langle 0 | A_a^\mu | \pi_b(p)\rangle = i p^\mu \delta_{ab} F}$$

⇒ F is the pion decay constant measured in $\pi^+ \to l^+\nu_l$

$F \simeq F_\pi = \underline{92.1\ \mathrm{MeV}}$

The effective Lagrangian has several remarquable properties : 1) One parameter F_π determines all π interactions. ChPT 14

2) Symmetry requires interactions with arbitrary many pions.

3) Derivative couplings : the interactions vanish if momentum go to 0.

IV] First prediction : $\pi\pi$ scattering

Isospin invariant amplitude :

$$M(\pi^a \pi^b \to \pi^c \pi^d) = \delta_{ab}\,\delta_{cd}\,A(s,t,u) + \delta_{ac}\,\delta_{bd}\,A(t,u,s) + \delta_{ad}\,\delta_{bc}\,A(u,s,t)$$

with
$$s = (p_a + p_b)^2 = (p_c + p_d)^2$$
$$t = (p_a - p_c)^2 = (p_d - p_b)^2$$
$$u = (p_a - p_d)^2 = (p_c - p_b)^2$$

Exercise : Using the effective Lagrangian show $A(s,t,u) = \frac{s}{F^2}$

V] ChPT and explicit symmetry breaking

Our effective lagrangian only valid in the limit $m_q = 0$
$\Rightarrow$ Implement the quark mass term that breaks the symmetry explicitely.

$$\boxed{\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} - \bar{q}\,M\,q}$$

The symmetry breaking term :

$$\bar{q}\,M\,q = (\bar{q}_L + \bar{q}_R)\,M\,(q_L\ q_R)$$
$$= \bar{q}_R\,M\,q_L + \bar{q}_L\,M\,q_R \quad \text{with } M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$\mathcal{L}_m$ becomes chiral invariant if M is transformed as :

$$M \to V_R\,M\,V_L^+$$

Use this property to build $\mathcal{L}_{eff}(U, M)$: treat M as an external source that transforms as: $M \to V_R M V_L^+$

$\mathcal{L}_{eff}$ must be invariant as well.

The lowest invariant term is:

$$\mathcal{L}_{s.b} = \frac{F^2 B_0}{2} Tr\left(M U^+ + M^+ U\right)$$

$$U \to V_R U V_L^+$$

$$M U^+ \to V_R M V_L^+ V_L U^+ V_R^+$$

$$\longrightarrow V_R M U^+ V_R^+$$

$$Tr(M U^+) \to Tr(M U^+)$$

$$Tr(M^+ U) \to Tr(M^+ U)$$

$$M^+ = M.$$

$$\underline{\mathcal{L}_{s.b} = F^2 B_0 \; Tr\left(M (U + U^+)\right)}$$

$$U = 1 + \frac{i}{F} \pi^a \lambda^a - \frac{1}{2F^2} \vec{\pi}^2 \mathbb{1}$$

$$U^+ = 1 - \frac{i}{F} \pi^a \lambda^a - \frac{1}{2F^2} \vec{\pi}^2$$

$$\boxed{\mathcal{L}_2 = \frac{F^2}{4}\left[Tr\left(\partial_\mu U^+ \partial^\mu U\right) + Tr\left(2 B_0 M (U + U^+)\right)\right]}$$

For $SU(2)$ $\mathcal{L}_{s.b} = F^2 B_0 \; Tr\left(\begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}\right)\left(-\frac{1}{2F^2} \vec{\pi}^2\right)$

$$= -\frac{B_0}{2}(m_u + m_d)\, \pi^2$$

$$= \underline{-\frac{M_\pi^2}{2} \vec{\pi}^2}$$

The masses of the pions are equal and proportional to the sum $\underline{m_u + m_d}$

B_0 is related to the $\bar{q} q$ condensate.

Gell Mann Oakes - Renner

$$\boxed{M_\pi^2 = \underset{\substack{\uparrow \\ \text{explicit} \\ O(p^2)}}{(m_u + m_d)} \Big(\underset{\substack{\uparrow \\ \text{spontaneous}}}{- \langle 0 | \bar{u} u | 0 \rangle} \frac{1}{F_\pi^2}\Big)} + O(m_q^2)$$

In the presence of quark mass the $\pi\pi$ scattering amplitude becomes:

$$\boxed{A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}}$$ Weinberg '66

In accordance with Goldstone theorem: $\pi\pi$ interaction vanishes at low energies : $s \to 0$, $m_q \to 0$

- Isospin amplitudes:

$$T^{I=0} = 3\,A(s,t,u) + A(t,u,s) + A(u,s,t)$$
$$T^{I=1} = A(t,u,s) - A(u,s,t)$$
$$T^{I=2} = A(t,u,s) + A(u,s,t)$$

- s wave scattering lengths : $\boxed{a_0^I = \frac{1}{32\pi} T^I (s = 4m_\pi^2, t = u = 0)}$

$$a_0^0 = \frac{7\,m_\pi^2}{32\pi F_\pi^2} = 0.16 \quad , \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045$$

Gell-Mann Oakes Renner formula 1968 Leutwyler's lecture

$$\boxed{M_\pi^2 = \underset{\text{explicit}}{(m_u + m_d)} \cdot \underset{\text{spontaneous.}}{\langle 0|\bar u u|0\rangle} \times \frac{1}{F_\pi^2}}$$

Coefficient: decay constant F_π

- why $M_\pi^2 \propto (m_u + m_d)$?

$\langle 0|\bar u(x)\gamma^\mu\gamma_5 d(x)|\pi^-\rangle = i\sqrt{2}\, F_\pi\, p^\mu e^{-ipx}$ → Axial current generates pion.

$\langle 0|\bar u(x)\, i\gamma_5\, d(x)|\pi^-\rangle = \sqrt{2}\, G_\pi\, e^{-ipx}$

Current conservation: Use Dirac equation → "see following page"

Exercise → $\partial_\mu(\bar u\gamma^\mu\gamma_5 d) = (m_u$

$$= \partial_\mu \bar u\, \gamma^\mu\gamma_5 d + \bar u\gamma^\mu\gamma_5\partial_\mu d$$

$$= i m_u\, \bar u\gamma^\mu\gamma_5 d - \underbrace{\bar u\gamma_5\gamma^\mu\partial_\mu d}_{+\, i\bar u\gamma_5 m_d d}$$

$$= (m_u + m_d)\, \bar u\, i\gamma_5 d$$

$\boxed{x = 0}$

$$-i^2\sqrt{2}\, F_\pi\, p^\mu p_\mu = (m_u + m_d)\sqrt{2}\, G_\pi$$

$$\sqrt{2}\, F_\pi\, \underbrace{p^2} = (m_u + m_d)\sqrt{2}\, G_\pi$$

$$F_\pi M_\pi^2 = (m_u + m_d)\, G_\pi$$

$\boxed{\gamma^5\gamma^\mu = -\gamma^\mu\gamma^3}$

$$\boxed{M_\pi^2 = (m_u + m_d)\frac{G_\pi}{F_\pi}}$$ exact

- Expansion in powers of m_u, m_d:

$$\frac{G_\pi}{F_\pi} = B + O(m)$$

$$\Rightarrow \boxed{M_\pi^2 = (m_u + m_d)\, B + O(m^2)}$$

~~$C = i\gamma^0$~~

$\psi \to -i\gamma_2 \psi^* = \psi_c$

$\bar{\psi} = \psi^\dagger \gamma^0$

$C \gamma^\mu C^{-1} = -\gamma^{\mu *}$

$(i\not{\partial} - m)\psi = 0$

$C: (i\not{\partial} - m)\psi = 0$

$C(i\not{\partial} - m)\psi = 0$

$C(i\not{\partial} - m)C^{-1}C\psi = 0$

$(Ci\not{\partial}C^{-1} - m)(-i\gamma_2\psi^*) = 0$

$(i(-\not{\partial}) - m)(-i\gamma_2\psi^*) = 0$

$-(i\not{\partial} - m)\psi_c = 0$

$i \to i$ oder c? $\Rightarrow (i\not{\partial} + m)\psi_c = 0$

$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$[(i\not{\partial} - m)\psi]^\dagger = 0$

$\psi^\dagger (i\not{\partial} - m)^\dagger = 0$

$\psi^\dagger(-i(\gamma^0\gamma^\mu\gamma^0)\partial - m) = 0$

$\psi^\dagger \gamma^0(-i\gamma^0\gamma^\mu\gamma^0\partial - \gamma^0\gamma^0 m) = 0$

$\psi^\dagger \gamma^0(-i\gamma^\mu\gamma^0\partial - \gamma^0 m) = 0$

$\bar{\psi}(-i\gamma^\mu\partial_\mu - m)\gamma^0 = 0$

$\bar{\psi}(i\gamma^\mu\partial_\mu + m) = 0$

$\Rightarrow \boxed{\bar{\psi}(i\not{\partial} + m) = 0}$