Why the proton radius is smaller in Virginia

(Part I – Fits of G_E)

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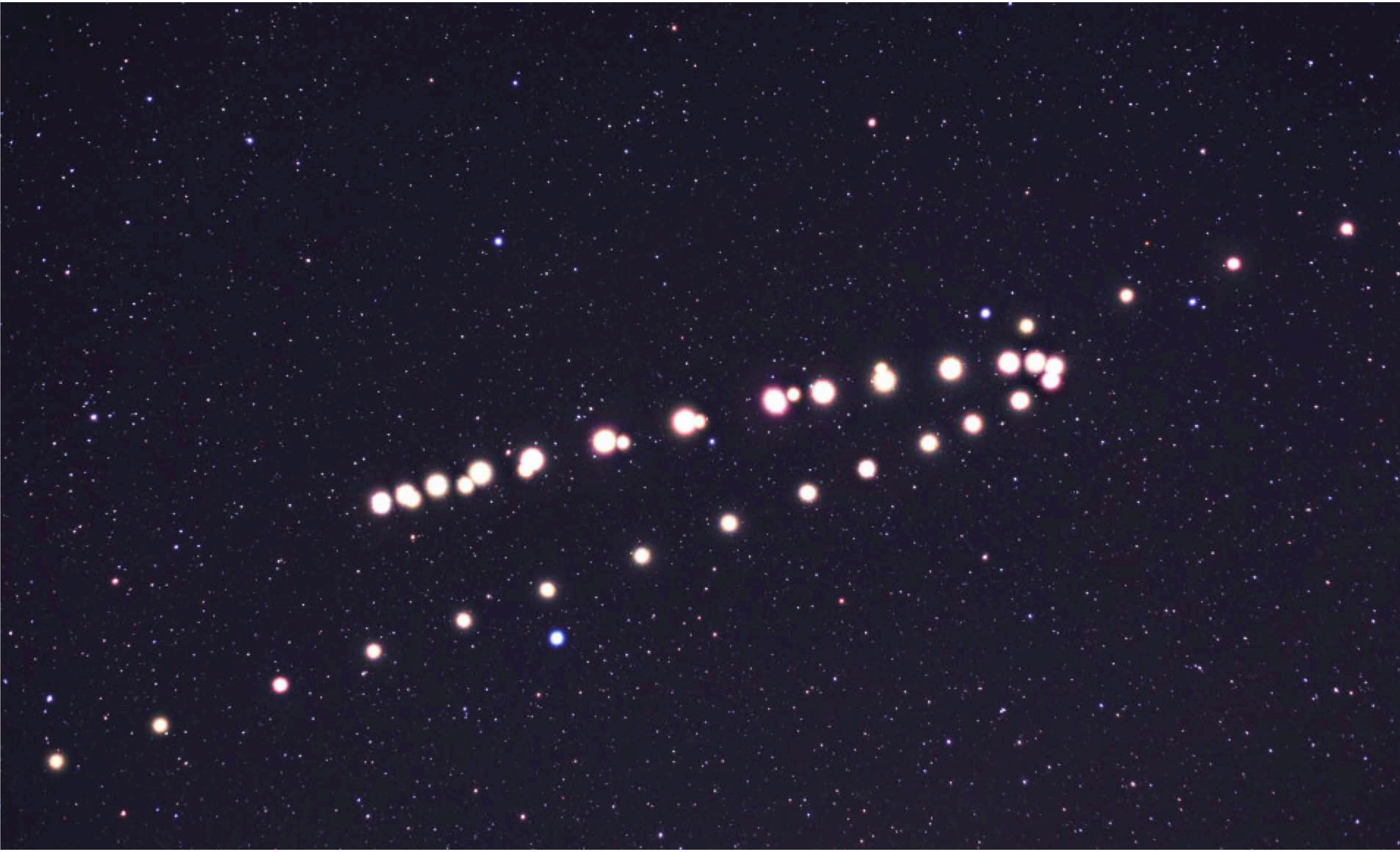
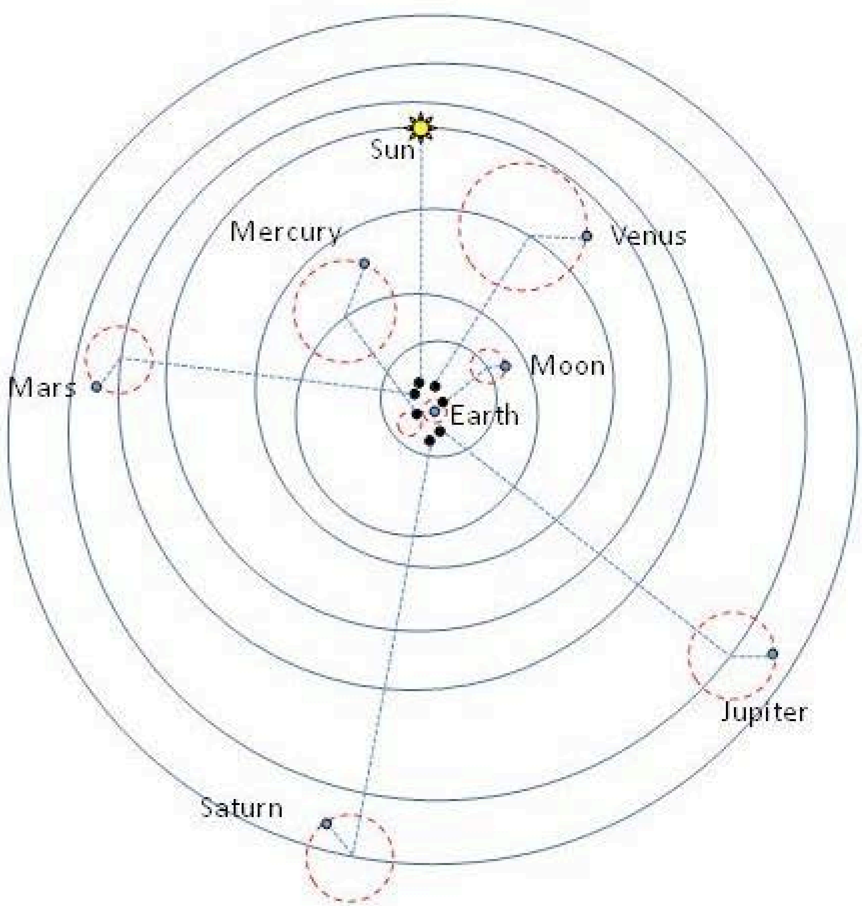
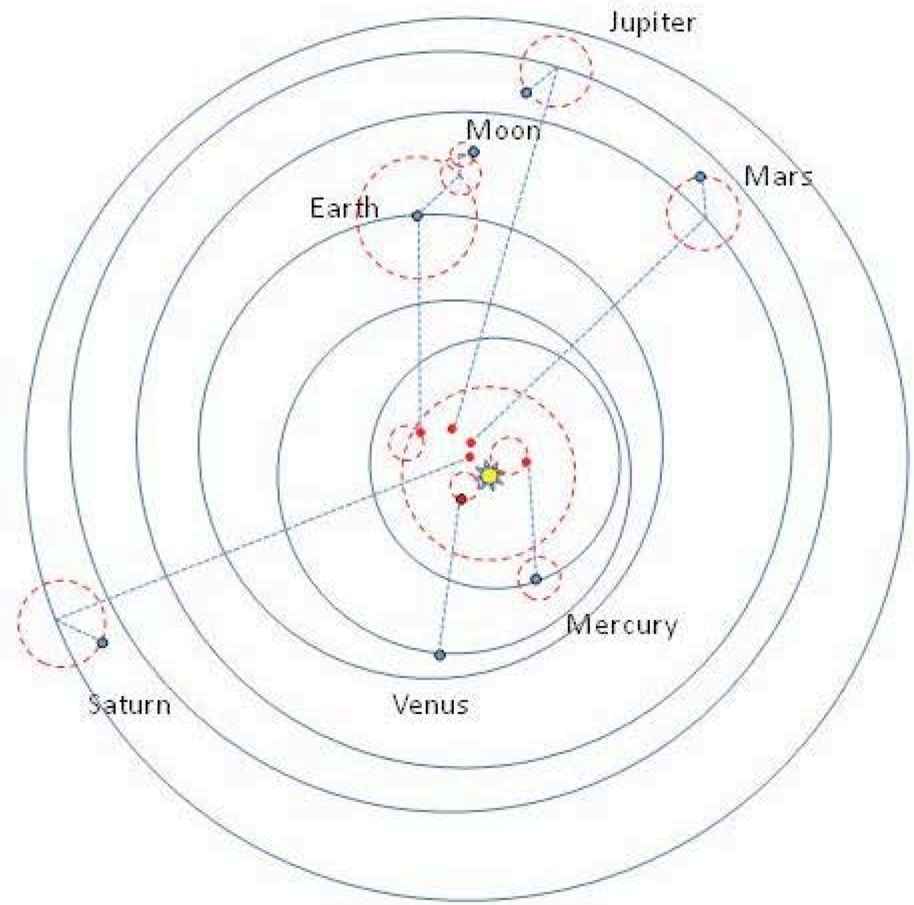
Retrograde Motion of Mars As Seen From Earth



Earth vs. Sun Centered Models



Ptolemaic Model



Copernican Model

Phases & Elliptical Orbits

- At first, with orbits as perfect circles, Ptolemaic models were better at predicting the orbits of the planets then Copernican models.
- It was the phases of the Venus (Galileo 1610) along with the elliptical orbits of Kepler (1609) [fitting the "naked eye" data of Brahe (1574)] that proved to be the downfall of the Ptolemaic model.

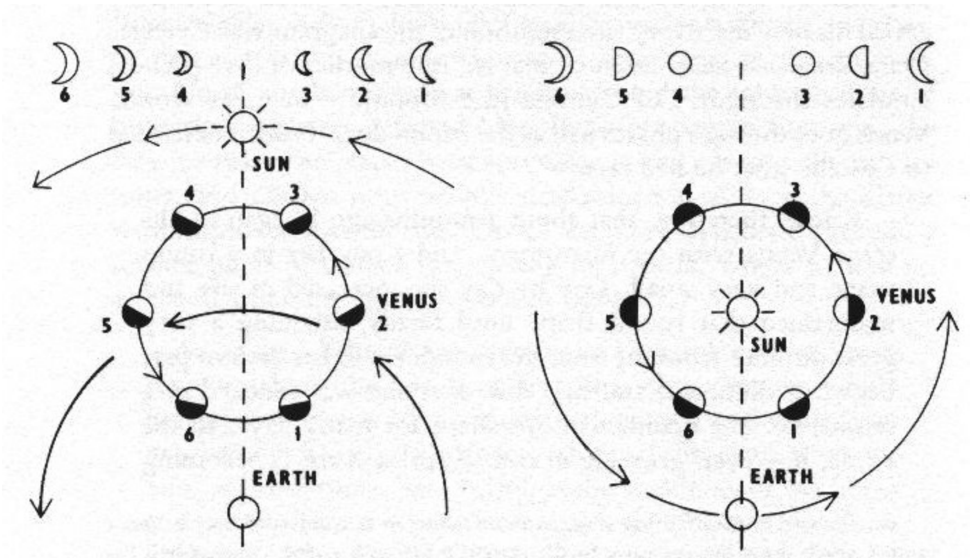
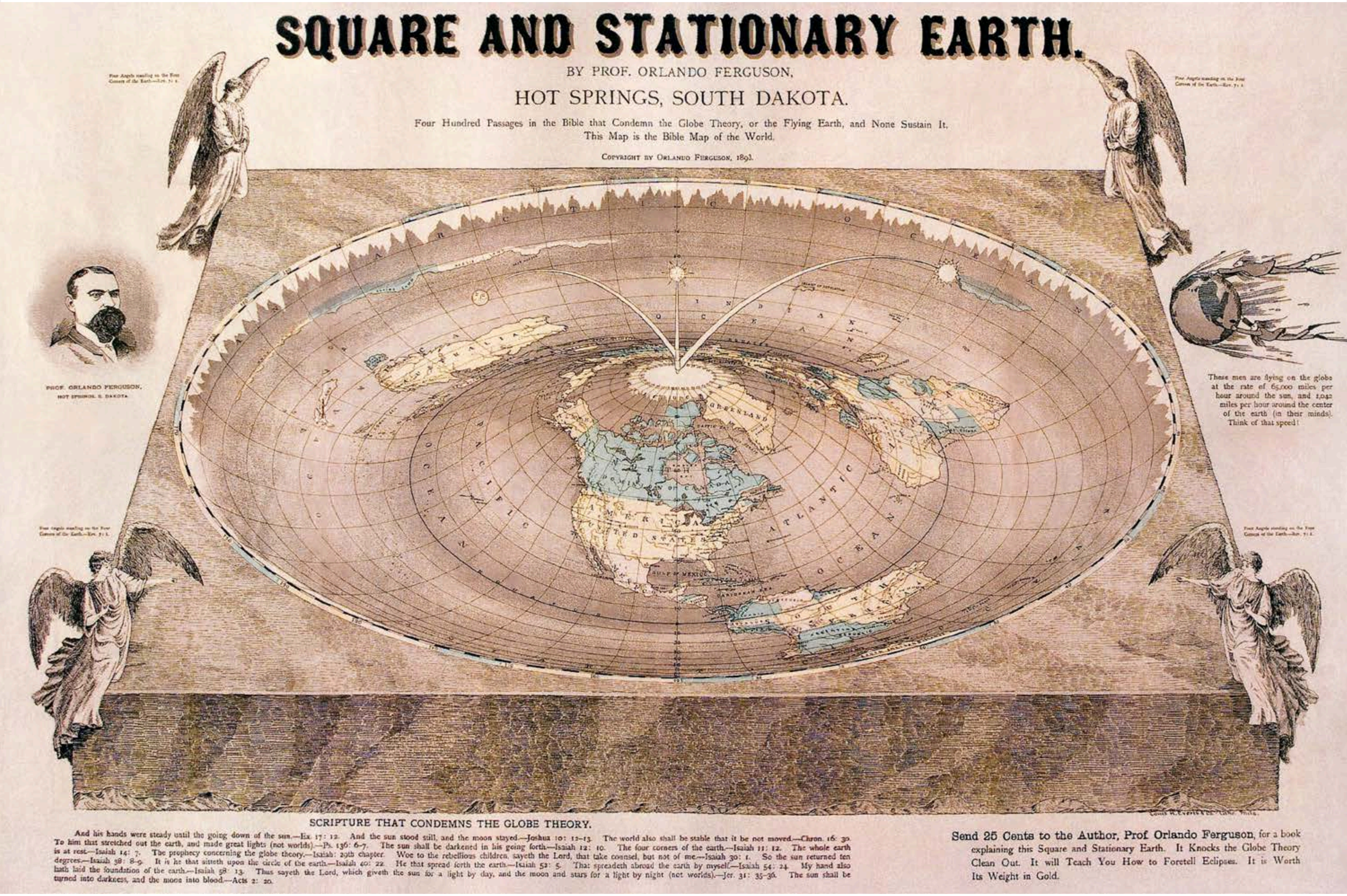


Illustration by Galileo Galilei in *Sidereus Nuncius (Starry Messenger) 1610.*


SQUARE AND STATIONARY EARTH.
BY PROF. ORLANDO FERGUSON,
HOT SPRINGS, SOUTH DAKOTA.
Four Hundred Passages in the Bible that Condemn the Globe Theory, or the Flying Earth, and None Sustain It.
This Map is the Bible Map of the World.
Copyright by Orlando Ferguson, 1893.
PROF. ORLANDO FERGUSON,
HOT SPRINGS, S. DAKOTA.
These men are flying on the globe at the rate of 65,000 miles per hour around the sun, and 1,042 miles per hour around the center of the earth (in their minds). Think of that speed!
PACIFIC OCEAN
ATLANTIC OCEAN
INDIAN OCEAN
GREENLAND
AMERICA
UNITED STATES
GULF OF MEXICO
SCRIPTURE THAT CONDEMNS THE GLOBE THEORY.
And his hands were steady until the going down of the sun.—Ex. 17: 12. And the sun stood still, and the moon stayed.—Joshua 10: 12–13. The world also shall be stable that it be not moved.—Chron. 16: 30. To him that stretched out the earth, and made great lights (not worlds).—Ps. 136: 6–7. The sun shall be darkened in his going forth.—Isaiah 12: 10. The four corners of the earth.—Isaiah 11: 12. The whole earth is at rest.—Isaiah 14: 7. The prophecy concerning the globe theory.—Isaiah: 29th chapter. Woe to the rebellious children, sayeth the Lord, that take counsel, but not of me.—Isaiah 30: 1. So the sun returned ten degrees.—Isaiah 38: 8–9. It is he that sitteth upon the circle of the earth.—Isaiah 40: 22. He that spread forth the earth.—Isaiah 52: 5. That spreadeth abroad the earth by myself.—Isaiah 54: 24. My hand also hath laid the foundation of the earth.—Isaiah 58: 13. Thus sayeth the Lord, which giveth the sun for a light by day, and the moon and stars for a light by night (not worlds).—Jer. 31: 35–36. The sun shall be turned into darkness, and the moon into blood.—Acts 2: 20.
Send 25 Cents to the Author, Prof. Orlando Ferguson, for a book explaining this Square and Stationary Earth. It Knocks the Globe Theory Clean Out. It will Teach You How to Foretell Eclipses. It is Worth Its Weight in Gold.

Occam's Razor

- William Occam (1287 – 1347)
- One can always explain failing explanations with an ad hoc hypothesis, thus in Science, simpler theories are preferable to more complex ones. (e.g. the Sun centered vs. Earth centered)
- Layman's version of Occam's Razor is "the simplest explanation is usually the correct one" (i.e. KISS)
- In statistical versions of Occam's Razor, one uses a rigorous formulation instead of a philosophical argument. In particular, one must provide a specific definition of simple:
  - F test, Akaike information criterion, Bayesian information criterion, etc.
  - In statistical modeling of data too simple is under-fitting and too complicated is over-fitting.

How many ways can **YOU** determine the radius of a perfect sphere?!

Image of the sphere created to test theory of relativity on the Gravity Probe B spacecraft.

Some Answers

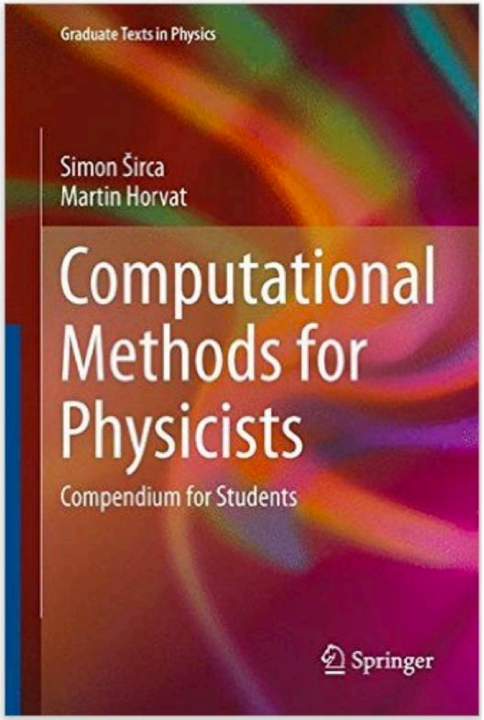
- Diameter = 2 r
- Area = πr^2
- Volume = $4/3 \pi r^3$ (displacement of water)
- Momentum of Inertia
  - $2/5\ m r^2$ (solid sphere)
  - $2/3\ m r^2$ (hollow sphere)

All Models Are Wrong

"The most that can be expected from any model is that it can supply a useful approximation to reality: All models are wrong; some models are useful." - George Box (1919 – 2013)

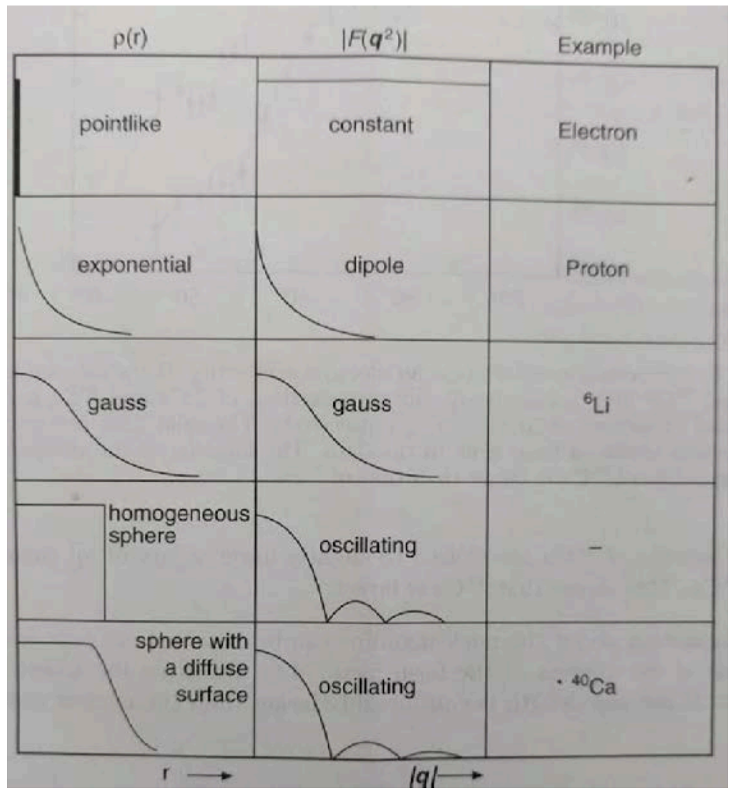
"An ever increasing amount of computational work is being relegated to computers, and often we almost blindly assume that the obtained results are correct."

- Simon Širca & Martin Horvat



Charge Radii from Electron Scattering

- For heavy nuclei, one typically measures the charge form factor, $G_E(Q^2)$, and with a Fourier transformation finds the charge radius.
- Diffractive minima also help determine radius and for a perfectly homogeneous sphere the minima would determine the radius exactly.

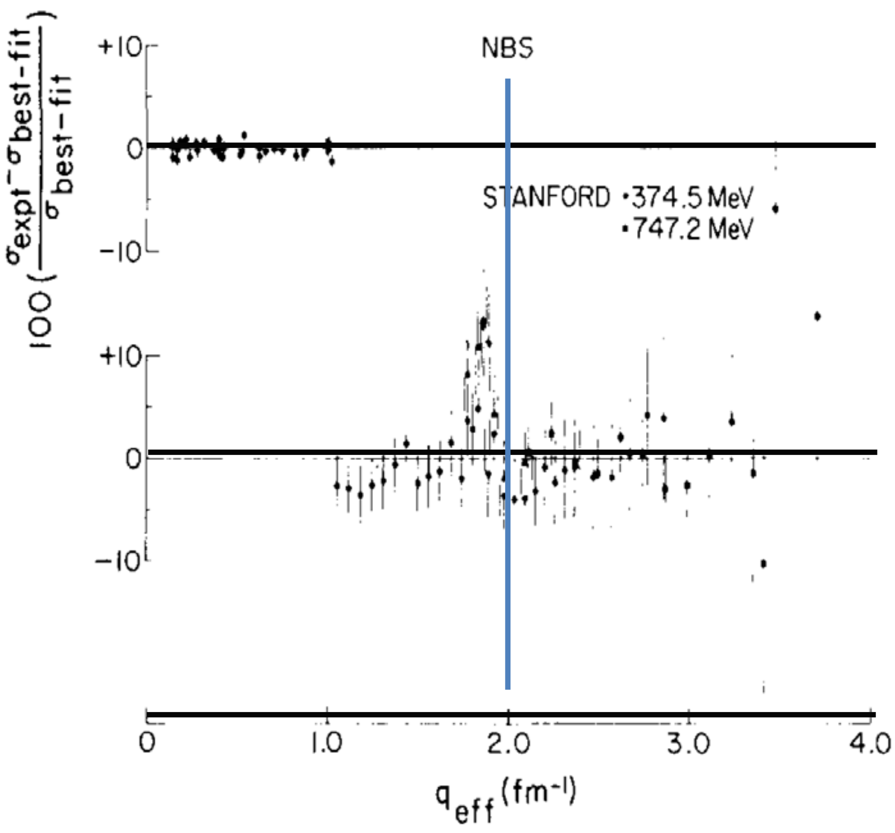


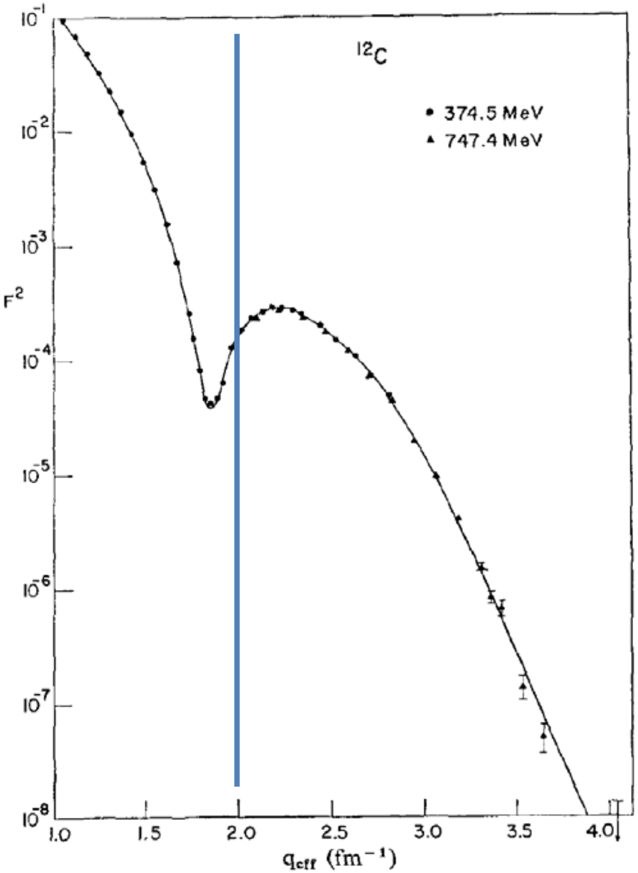
Textbook example from Povh, Rith, Scholz, Zetsche, Particles and Nuclei 2nd Edition (1999) Springer.

Determining the Charge Radius of Carbon

Stanford high Q^2 data from I. Sick and J.S. McCarthy, Nucl. Phys. **A150** (1970) 631.

National Bureau of Standards (NBS) low Q^2 data from L. Cardman et. al., Phys. Lett. **B91** (1980) 203.





See the L. Cardman's paper for details of the carbon radius (2.46 fm) analysis.

Proton Radius Puzzle

- There are currently only a few ways to determine the radius of the proton:
  - Atomic Hydrogen Lamb Shift (~ 0.88 fm)
  - Muonic Hydrogen Lamb Shift (~ 0.84 fm)
  - And of course elastic electron scattering!
- New measurements are coming!
  - Prad: electron scattering (going on right now)
  - NIST *& other labs*: Atomic Hydrogen Lamb Shift
- My focus today will be on the electron scattering data.

Elastic Scattering on a Proton

From relativistic quantum mechanics one can derive the the formula electron-proton scattering where one has assumed the exchange of a single virtual photon.

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \cdot \frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]$$

where G_E and G_M form factors take into account the finite size of the proton.

$G_E = G_E(Q^2)$, $G_M = G_M(Q^2)$; $G_E(0)=1$, $G_M(0) = \mu_p$

$Q^2 = 4\, E\, E' \sin^2(\theta/2)$ and $\tau = Q^2/4m_p^2$

Elastic cross sections at small angles and small Q^2's are dominated by G_E (Prad Hall B)

Elastic cross Sections at large angles and large Q^2's are dominated by G_M (GMP Hall A)

For moderate $Q^{2'}$s one can separate G_E and G_M with Rosenbluth technique.

Charge Radius of the Proton

- Proton G_E has no measured minima and it is too light for the Fourier transformation to work in a model independent way.
- Thus for the proton we make use of the fact that as Q^2 goes to zero the charge radius is proportional to the slope of G_E

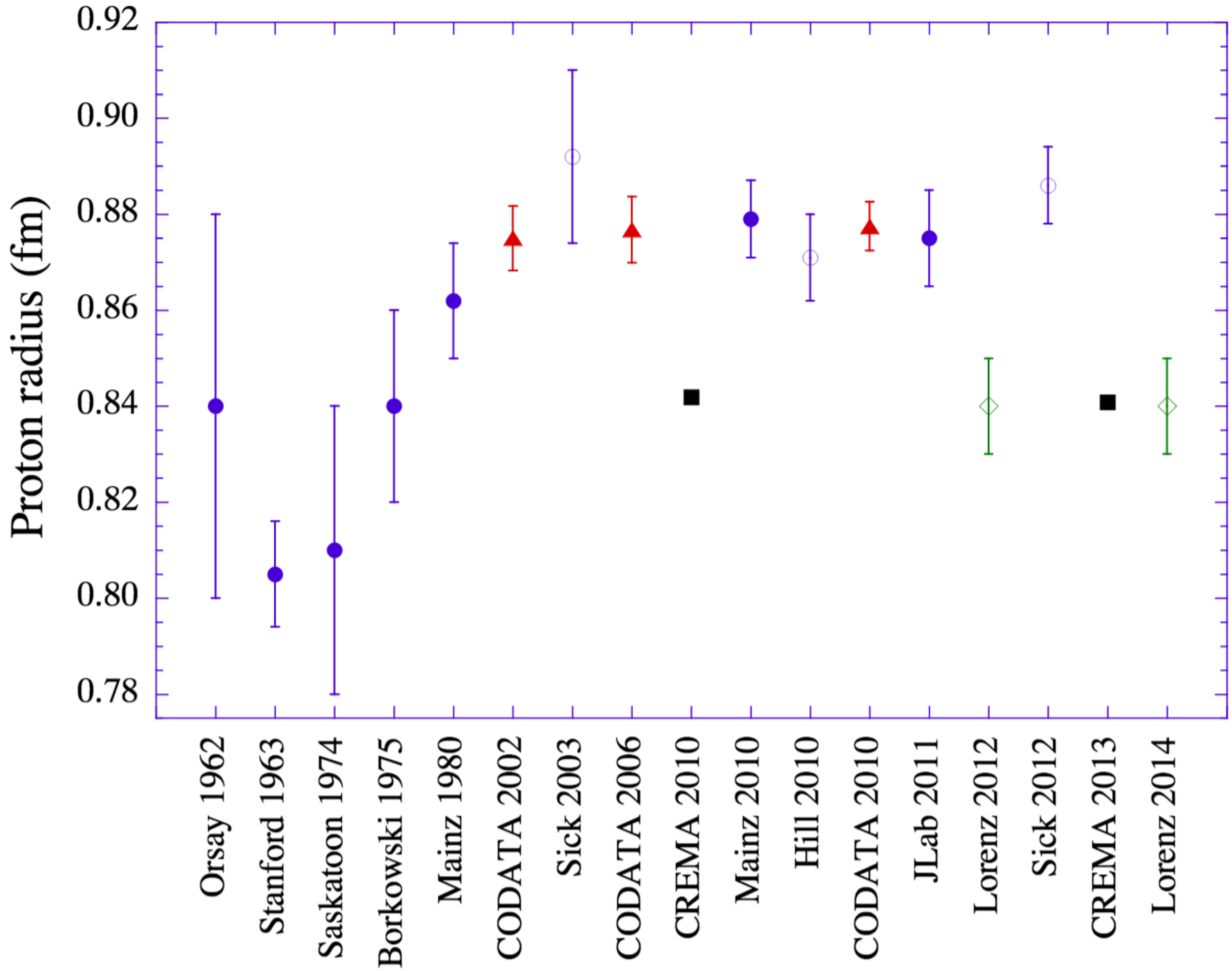
$$G_E(Q^2) = 1 + \sum_{n \geq 1} \frac{(-1)^n}{(2n+1)!} \left\langle r^{2n} \right\rangle Q^{2n}$$

$$r_p \equiv \sqrt{\langle r^2 \rangle} = \left(-6 \left. \frac{\mathrm{d}G_E(Q^2)}{\mathrm{d}Q^2} \right|_{Q^2=0} \right)^{1/2}$$

We don't measure to Q^2 of zero, so this is going to be an extrapolation problem.

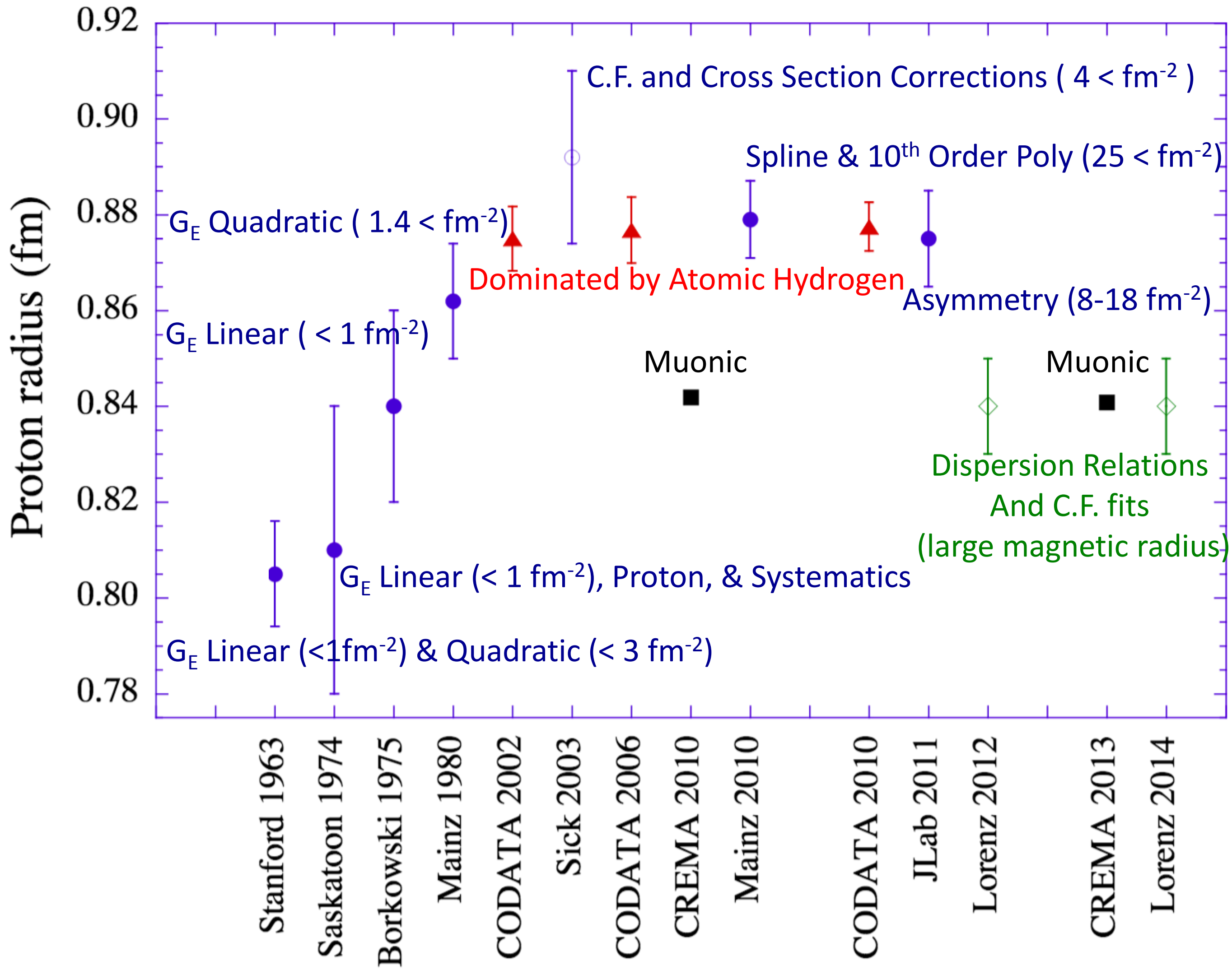
Proton Radius vs. Time

V. Punjabi et al., Eur. Phys. J. **A51** (2015) 79.

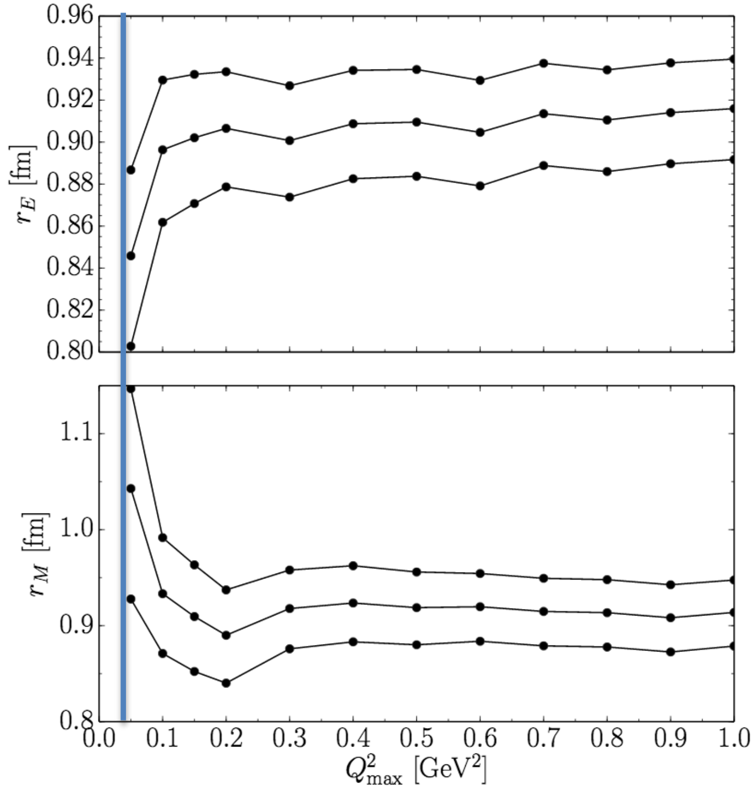


Proton Radius vs. Time

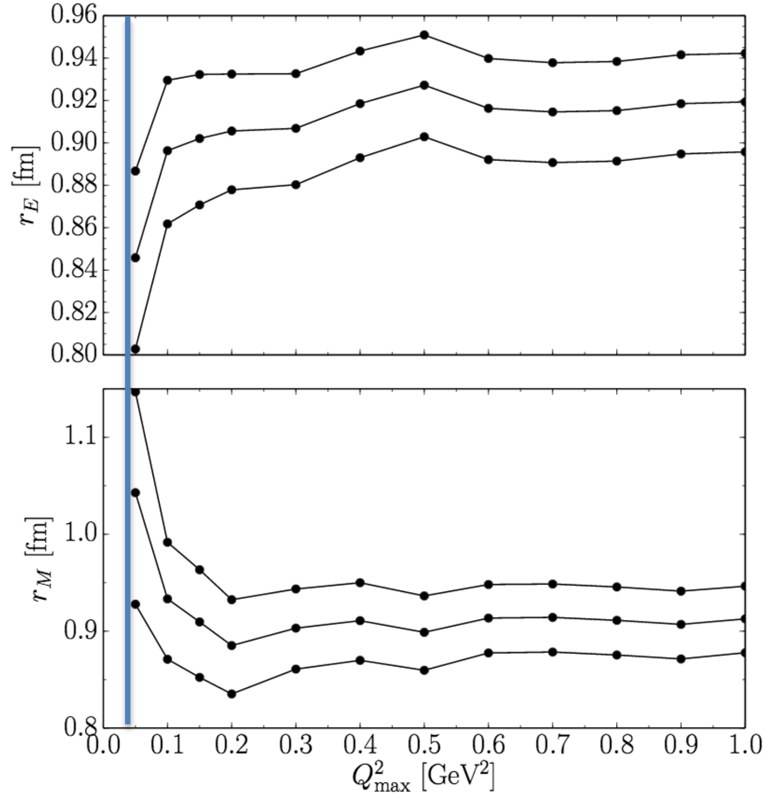
V. Punjabi et al., Eur. Phys. J. **A51** (2015) 79.



G. Lee, J. Arrington and R. Hill, Phys. Rev. **D92** (2015) 013013.



World Cross Section Data



Adding Polarization Data

At very Low Q^2 G_E dominates the cross section & very high Q^2 G_M dominates the cross section.

$1\ \mathrm{fm}^{-2} \cong 0.04\ \mathrm{GeV}^2$

Arrington and I look at this same plot and see things that support our points of view….

Warning: Danger of Confirmation Bias

In psychology and cognitive science, confirmation bias is a tendency to search for or interpret information in a way that confirms one's preconceptions, leading to statistical errors.

"Proton Radius Puzzle" in 1975 !?

F. Borkowski, G.G. Simon, V. H. Walther, and R. D. Wendling, Nucl. Phys. **B93** (1975) 461.

$$G_{E,M}(q^2) = 1 - \tfrac{1}{6}\langle r^2_{E,M}\rangle |\boldsymbol{q}|^2 + \tfrac{1}{120}\langle r^4_{E,M}\rangle |\boldsymbol{q}|^4 - + \ldots , \qquad (6)$$

For $q^2 < 0.9\ \mathrm{fm}^{-2}$ the contributions of the higher terms in the expansion (6) are negligable and the series can be truncated to give $G_E(q^2) = \delta + \beta q^2$. From fitting this expression to the form factors of fig. 5, the solid line of fig. 5 has been obtained. The best fit parameters were $\delta = 0.994 \pm 0.002$ and $\beta = -0.118 \pm 0.004\ \mathrm{fm}^2$. The reduced χ^2 was 0.5. The result of the fit did not depend significantly on the fitted q^2 range. This was checked by fitting additionally the G_E values of table 2 up to $1.2\ \mathrm{fm}^{-2}$. The addition of a q^4 term to the fit formula did not improve the fit, moreoever the error of the additional parameter turned out to be larger than its value. The best fit value of the parameter δ is well within the normalization error of the G_E values. The best fit value of the parameter β gives a proton r.m.s. radius of $\langle r^2_E\rangle^{\frac{1}{2}} = 0.84 \pm 0.02$ fm. This value is higher than the dipole value of 0.81 fm, but within the error limits it is compatible with the result (0.81 ± 0.04 fm) of a similar experiment carried out at Saskatoon [7].

And then a model dependent correction is made . . .

Test of Additional Term

A textbook statistics problem is to quantify when to stop adding terms to a nested statistical model (e.g. a Maclaurin series).

One way to do this is with an F-distribution test.

$$F = \frac{\chi^2(j-1) - \chi^2(j)}{\chi^2(j)}(N - j - 1)$$

where j is the order of the fit and N the number points being fit.

(see James 2nd edition page 282 or Bevington 3rd edition page 207)

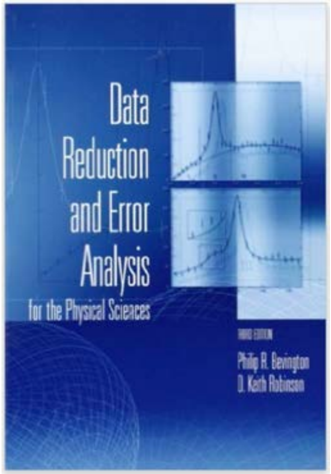
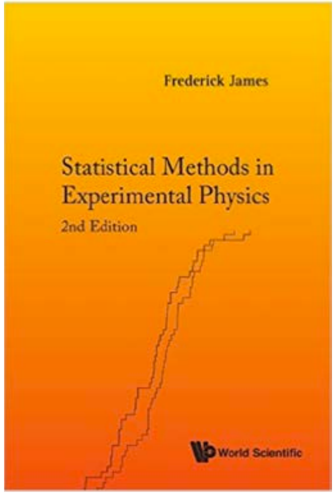


Table 10.2. Maximum degree needed in polynomial approximation.

$N-j-1$	2	3	4	6	8	12	20	60	120
Reject j^{th} order to 95% confidence level if F is smaller than	18.5	10.1	7.7	6	5.3	4.7	4.3	4	3.9



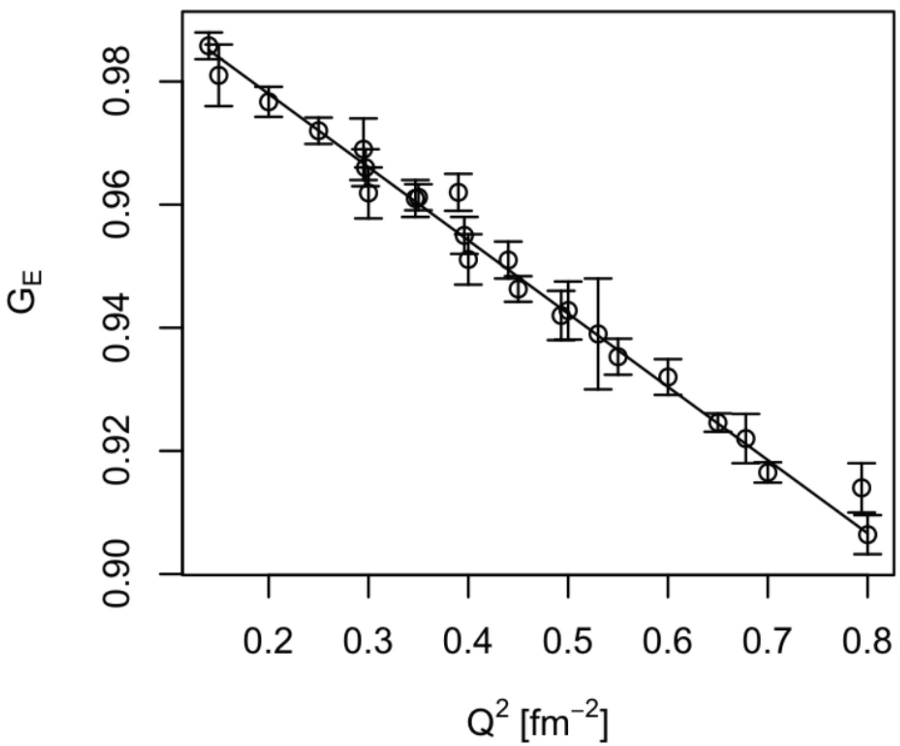
Quantifies the statement “doesn’t significantly improve the fit” from Borkoski *et al.*(1975).

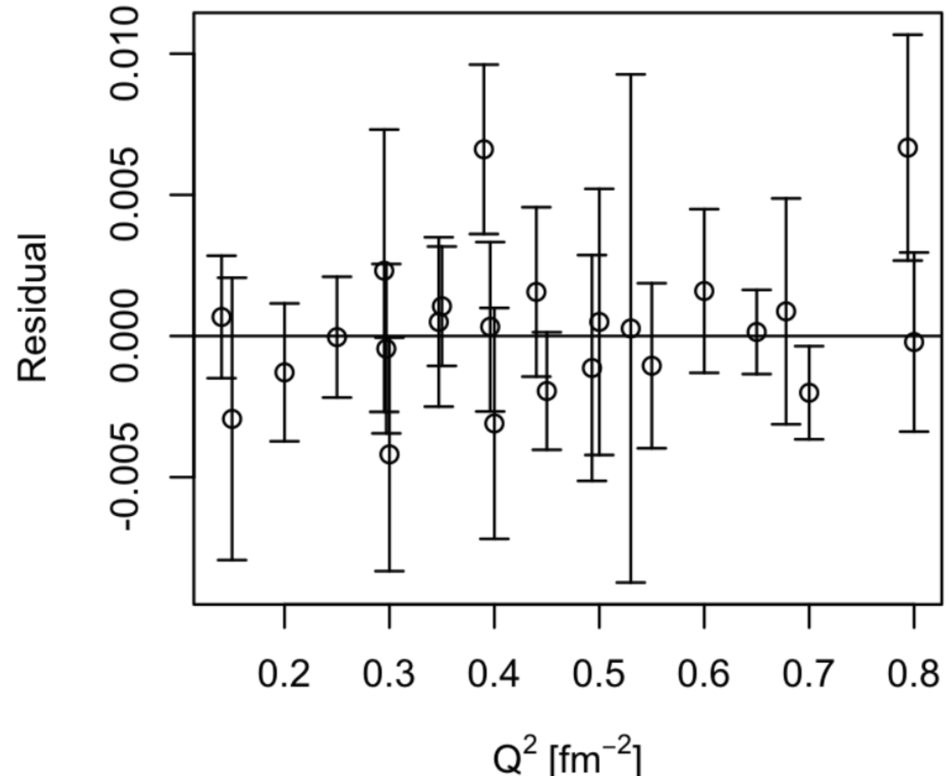
Saskatoon '74 and Mainz '80

G. G. Simon, C. Schmitt, F. Borkowski, and V. H. Walther, Nucl. Phys. **A333** (1980) 381.
J. J. Murphy, Y. M. Shin, and D. M. Skopik, Phys. Rev. **C9** (1974) 2125.

$$f(Q^2) = n_0 G_E(Q^2) \approx n_0 \left(1 + \sum_{i=1}^{m} a_i Q^{2i}\right)$$

$$f_1(Q^2) = n_0 \left(1 + a_1 Q^2\right) \ ,$$
$$f_2(Q^2) = n_0 \left(1 + a_1 Q^2 + a_2 Q^4\right) \ ,$$





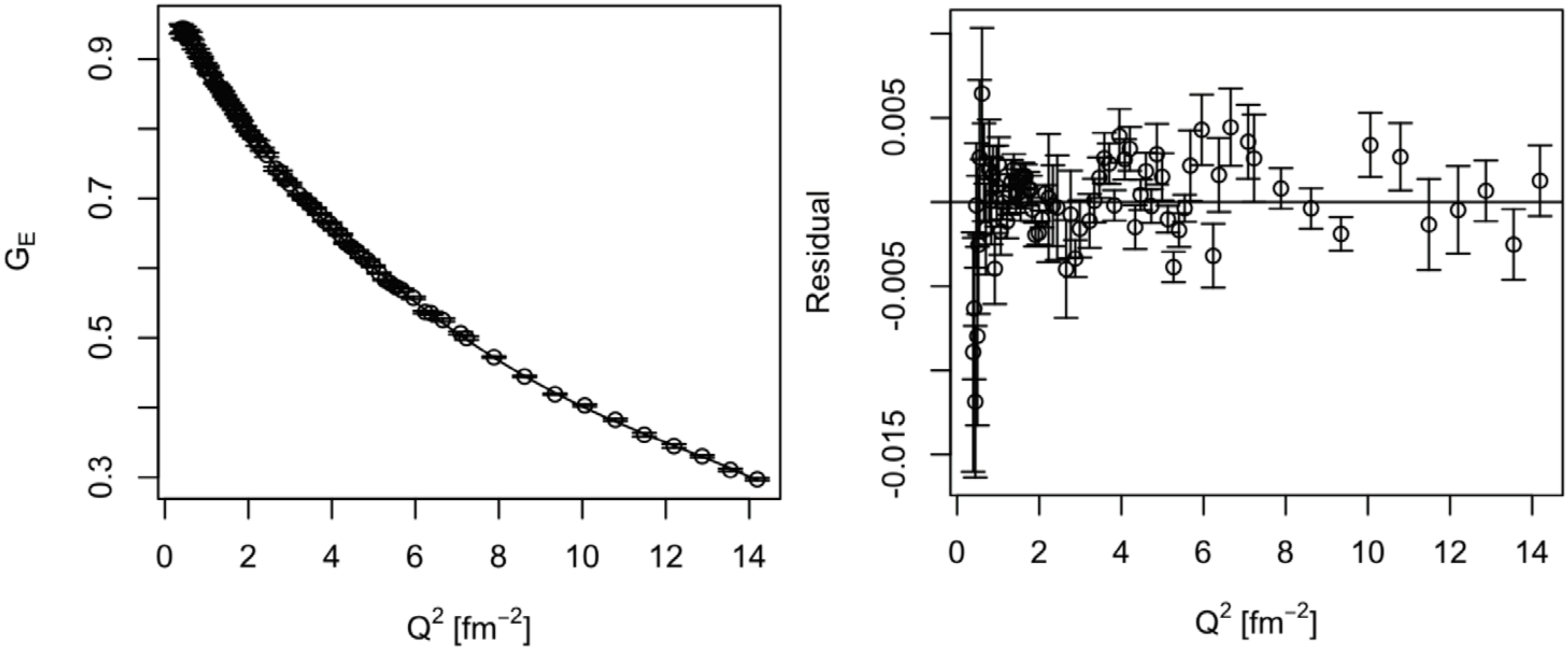
F-test rejects the $f_2(Q^2)$ statistical model.
For $f_1(Q^2)$, i.e. a linear extrapolation, we find a 0.84(1) fm radius.

Mainz 2014 G_E Rosenbluth Data

J. Bernauer *et al.*, Phys Rev. **C90** (2014) 015206 supplemental material.

$$f(Q^2) = n_0 G_E(Q^2) \approx n_0 \left(1 + \sum_{i=1}^{m} a_i Q^{2i}\right)$$

Using AIC, one rejects the 6th order polynomial (j=7). F-test gives the same result.



BUT one should be very wary of using a high order polynomial to extrapolate beyond the data.

Fixed Radius Fits

- Again using the Mainz 2014 Rosenbluth results.
- Fit the Maclaurin series with radius fixed to the two competing hypotheses
  - 0.84 fm from Muonic hydrogen
  - 0.88 fm from Atomic hydrogen

Fixed Radius	χ^2	χ^2/ν	n_0	a_2	a_3	a_4	a_5
0.84 fm	56.34	0.783	0.994(1)	$1.12(1) \cdot 10^{-2}$	$-0.93(2) \cdot 10^{-3}$	$5.0(1) \cdot 10^{-5}$	$1.20(5) \cdot 10^{-6}$
0.88 fm	142.1	1.97	1.003(1)	$1.62(1) \cdot 10^{-2}$	$-1.78(1) \cdot 10^{-3}$	$1.14(1) \cdot 10^{-4}$	$-2.90(7) \cdot 10^{-6}$

Padé Approximant & Continued Fractions

Pade' Approximant

When it exists, the Pade' approximant (N,M) of a Tayler series is unique.

$$f(x) = \frac{a_0 + a_1 x^1 + a_2 x^2 \ldots + a^M * x^M}{1 + b_1 x^1 + b_2 x^2 \ldots + b^N * x^N}$$

In our case we want $f(x) = n_0\, G_E(Q^2)$, so

$$f(x) = n_0 \frac{1 + a_1 Q_2 + a_2 Q^4 \ldots + a^{M*2} * Q^{M*2}}{1 + b_1 Q_2 + b_2 Q^4 \ldots + b^{N*2} * x^{N*2}}$$

(Henri Padé ~ 1860)

Continued Fraction

$$f(Q^2) = \cfrac{c_1}{1 + \cfrac{c_2 Q^2}{1 + \cfrac{c_3 Q^2}{1 + \cfrac{c_4 Q^2}{1 + \ldots}}}}$$

(Ancient Greeks)

Further reading: **Extrapolation algorithms and Padé approximations: a historical survey**
C. Brezinski, Applied Numerical Mathematics 20 (1996) 299.

Maclaurin, Padé Approximant & Dipole Fits

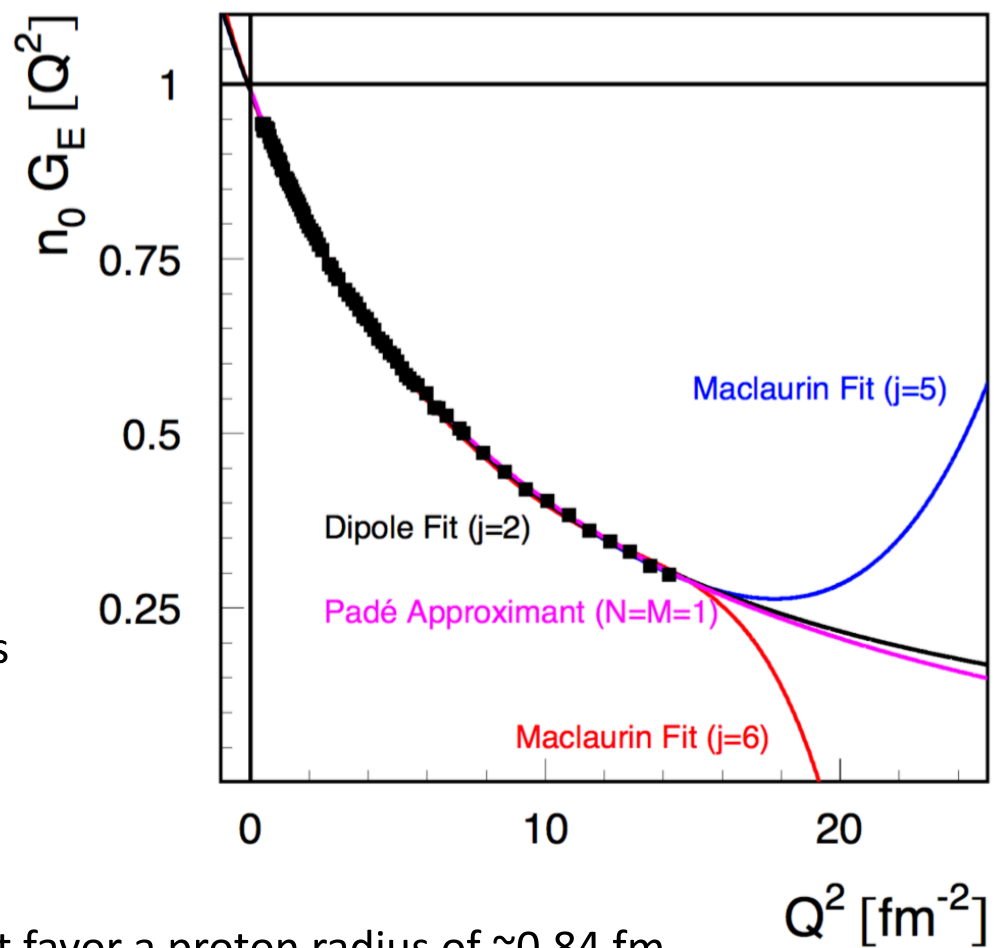
Using the Mainz14 "Rosenluth" Results (**where G_E & G_M well constrained by the data**).

$$f(Q^2) = n_0 G_E(Q^2) \approx n_0\left(1 + \sum_{i=1}^{m} a_i Q^{2i}\right)$$

Used f test to rule out j=7 (i = 6 + n_0 term)

WARNING: F test can reject functions, but It doesn't tell you which of the remaining is "best" or most appropriate.

(i.e. inspect the results)

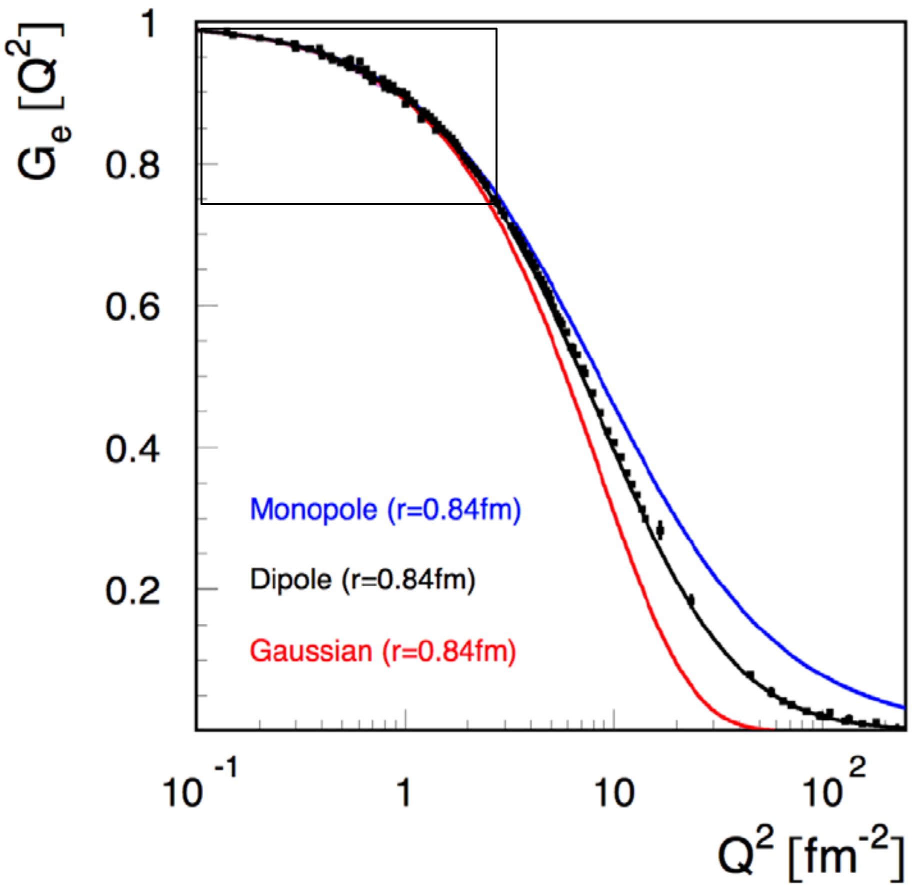


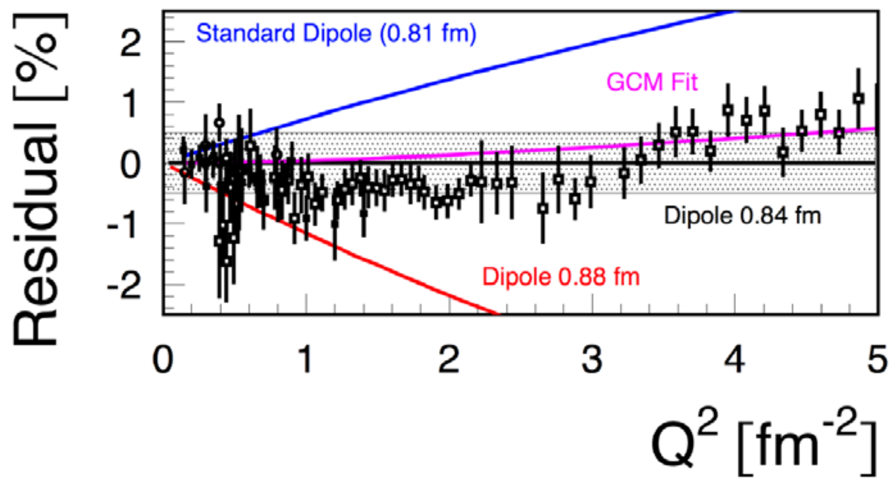
These fits all give results that favor a proton radius of ~0.84 fm.
Note how Padé and dipole fits extrapolate nicely, while the Maclaurin quickly diverge.

Fitting with Textbook Functions

Using the "old" Stanford, Jlab, Mainz, Saskatoon data along with the Mainz 2014 "Rosenbluth" G_E Results

Functions straight out of Povh, Rith, Scholz, and Zetsche, Particles and Nuclei 2nd Edition (1999) Springer.



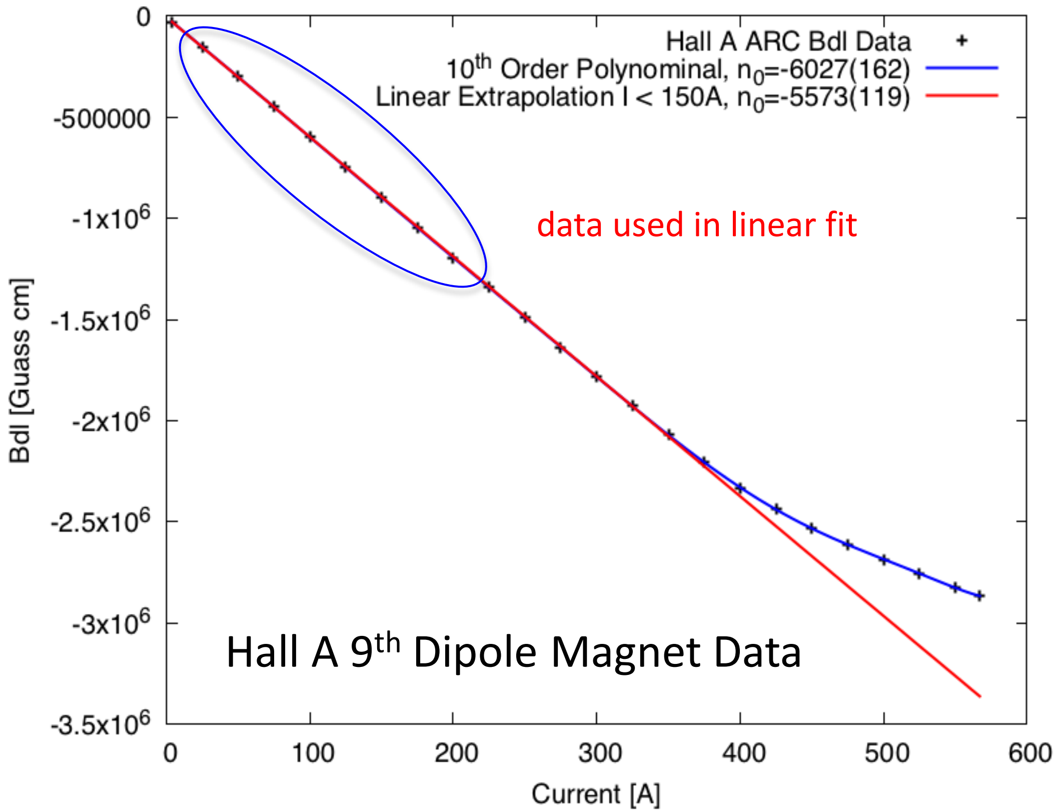


Data shown with 1/sqrt(N) errors only.
Gray error is a 0.5% systematic error band.

"Every Model Is Wrong", but the dipole function with the 0.84 fm radius is pretty amazing.

Precise Fitting vs. Accurate Extrapolation

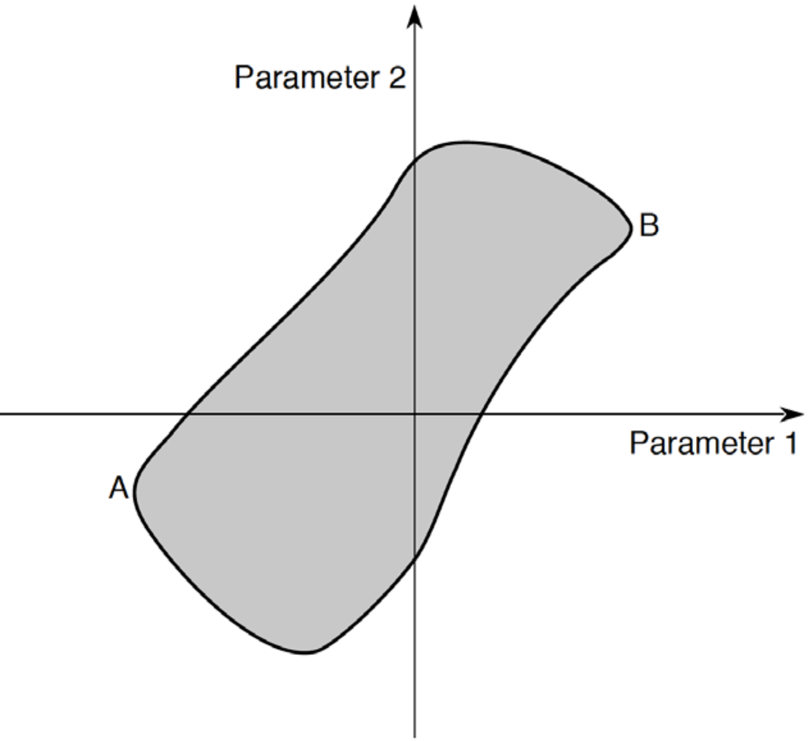
Warning!! The result below are shown with only standard errors **estimates** which are only valid over the range of the fit.



The 10th Order Polynomial Fit Precisely Describes The Data But Doesn't Accurately Extrapolate

Celina Pearson (Virginia Governor's School Senior going to VT) was given this data without the first point. Including even Pade' and C.F. fits, her n_0 was closest to truth with a linear extrapolation of last two points…

Multivariate Errors



As per the particle data handbook, one should be using a co-variance matrix and calculating the probably content of the hyper-contour of the fit.   Default setting of Minuit of “up”(often call $\Delta\chi^2$ is one.

Also note standard Errors often underestimate true uncertainties. (manual of gnuplot fitting has an explicate warning about this)

Number of Parameters	Confidence level (probability contents desired inside hypercontour of $\chi^2 = \chi^2_{min} + up$)				
	50%	70%	90%	95%	99%
1	0.46	1.07	2.70	3.84	6.63
2	1.39	2.41	4.61	5.99	9.21
3	2.37	3.67	6.25	7.82	11.36
4	3.36	4.88	7.78	9.49	13.28
5	4.35	6.06	9.24	11.07	15.09
6	5.35	7.23	10.65	12.59	16.81
7	6.35	8.38	12.02	14.07	18.49
8	7.34	9.52	13.36	15.51	20.09
9	8.34	10.66	14.68	16.92	21.67
10	9.34	11.78	15.99	18.31	23.21
11	10.34	12.88	17.29	19.68	24.71
	If FCN is $-\log(\text{likelihood})$ instead of χ^2, all values of up should be divided by 2.				

The Interpretation of Errors in Minuit (2004 by James)

seal.cern.ch/documents/minuit/mnerror.pdf

In ROOT: **SetDefaultErrorDef(X.X)**

Default is 1 and doesn’t change unless you change it!

Summary (part I)

- Occam's Razor - *Among competing hypotheses, the one with the fewest assumptions should be selected.*
- Confirmation Bias - *Tendency to search for or interpret information in a way that confirms one's preconceptions.*
- To avoid confirmation bias, one can apply statistical modeling techniques such as F-tests, AIC, Stepwise Regression, etc. to determine the function to fit a given set of data.
  - R based Stepwise Regression Code Posted Along With Example Data Sets
  - http://jeffersonlab.github.io/model-selection/
- With this technique, one finds radii consistent with the Muonic hydrogen data (0.84 fm)
  - With the lowest Q^2 data ($< 1 fm^2$), statistical modeling of the data indicates one should use a linear extrapolations as one would expect from the Maclaurin expansion of $G_E(Q^2)$.
  - If one wants to try to fit large Q^2 ranges, functions such as the Pade' approximant & C.F. should likely be used though even Maclaurin fits favor the Muonic results.
  - **Warning:** One should keep in mind that a function that gives a precise fit may not be appropriate for accurately extrapolating. (a fundamental math problem)
- The Hand Paper Challenge (Hand *et al.*, Rev. of Modern Phys. 35 (1963) 342.)
  - In the review article by Hand the author claims a consistent 0.805 fm radius.
    - The paper has a single paragraph on the radius fit, yet this paper is the radius of standard dipole.
  - What do you get!?
    - (use anything from a ruler to a Gaussian process regression)
    - Try to follow what Hand et al. did OR use your own cut-offs
  - **We will discuss your results on Thursday!**