Extraction of Hadron Resonances in Lattice QCD

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Motivation

- use lattice methods to explore the low-lying stationary states of QCD in a finite volume
- extract hadron resonance properties: masses, decay widths
- explore the possibilities for exotic hadrons: *e.g.* tetraquarks, glueballs
- understand the nature of controversial resonances: *e.g.* $\Lambda(1405)$

Content

1. Introduction
   - two important equations
   - QCD action
   - Operator smearing and displacements
2. hadron operator construction
   - single hadron operator
   - projection onto O_h^D Irreps
   - multi-hadron operators
3. calculating correlation functions
   - stochastic estimation of quark propagator
   - estimating quark lines using stochastic LapH
4. extracting energies from correlation functions
   - a toy model
   - analysis of T_{1u}^+ channel energies
5. **work in progress**: Effective Hamiltonian methods

Path Integral on Lattice

$$C_{ij}(t) = \langle 0| O_i(t+t_0)\, \overline{O}_j(t_0) |0\rangle = \sum_n \langle 0| O_i |n\rangle \langle n| \overline{O}_j |0\rangle e^{-E_n t}$$

- RHS to extract matrix elements of the operators and spectrum of the theory

$$\lim_{T\to\infty} \frac{1}{Z_t} tr[e^{-(T-t)\hat{H}} \hat{O}_2 e^{-t\hat{H}} \hat{O}_1] = \frac{1}{Z_t} \int D[\phi] e^{-S_E[\phi]} O_2[\phi(.,t)] O_1[\phi(.,0)]$$

- The integral is over all possible configurations of the field ϕ
- Two operators $\hat{O}_i$'s are translated to functionals
- RHS evaluated numerically on the lattice

Lattice QCD Basics

- define QCD on set of discrete points in Euclidean spacetime in a finite volume
- introduce simplest, gauge-invariant, discretized action that reduces to the continuum QCD action in continuum and thermodynamic limit
- quark field: Dirac 4-spinors $\psi^{f}_{\alpha c}(x)$, $\bar{\psi}^{f}_{\alpha c}(x)$; gauge field $A^{\mu}_{cd}(x)$

QCD action

- The Fermionic Part

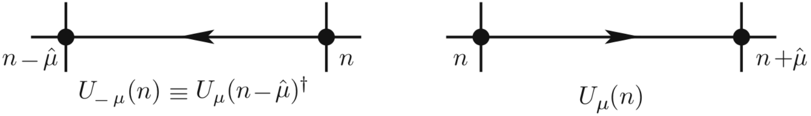
$$S_F[\psi,\bar{\psi},A] = \sum_{n=1}^{N_f} \int d^4x \bar{\psi}^{(f)}(x)(\gamma_\mu(\partial_\mu + iA_\mu(x)) + m^{(f)})\psi^{(f)}(x)$$

$$= \sum_{n=1}^{N_f} \int d^4x \bar{\psi}^{(f)}(x)_{\alpha c}((\gamma_\mu)_{\alpha\beta}(\delta_{cd}\partial_\mu + iA_\mu(x)_{cd}) + m^{(f)}\delta_{\alpha\beta}\delta_{cd})\psi^{(f)}(x)_{\alpha c}$$

- The Gluonic Part

$$S_G[A] = \frac{1}{2g^2}\int d^4x tr[F_{\mu\nu}(x)F_{\mu\nu}(x)]$$

Introduction of Gauge-link Variables



$$\psi'(n) = \Omega(n)\psi(n); \bar{\psi}'(n) = \bar{\psi}(n)\Omega^\dagger(n)$$

$$U'_\mu(n) = \Omega(n)U_\mu\Omega^\dagger(n+\mu)$$

- The lattice version for free Fermion action reads

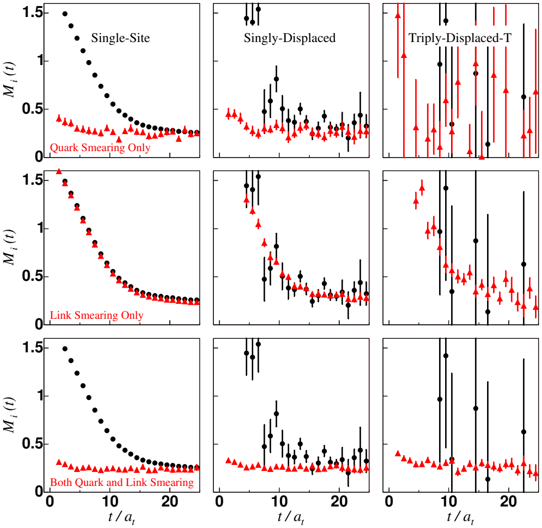
$$S_F[\psi,\bar{\psi}] = a^4 \sum_{n\in\Lambda} \bar{\psi}(n)(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu\psi(n+\hat{\mu}) - U_{-\mu}\psi(n-\hat{\mu})}{2a} + m\psi(n))$$

- gauge fields introduced through gauge-link variables

$$U_\mu(x) = \mathcal{P}\exp\left\{ig\int_x^{x+\hat{\mu}} d\eta \cdot A(\eta)\right\} \approx \exp\left\{igaA_\mu(x)\right\}$$

Operator Smearing and Displacements

- Smearing quark fields reduces the excited state contamination
- Smearing gauge-link fields reduces the error for displaced operators
- Displacement in different directions: orbital structure; Displacement by different distances: radial structure



Constructing Single-hadron Operators

- building blocks: covariantly-displaced LapH-smeared quark fields
  - stout-smeared links $\widetilde{U}_j(x)$
  - 3d gauge-covariant Laplacian $\widetilde{\Delta}$ in terms of $\widetilde{U}$
  - Laplacian-Heaviside (LapH) smeared quark fields

$$\widetilde{\psi}_{a\alpha}(x) = \mathcal{S}_{ab}(x,y)\ \psi_{b\alpha}(y), \qquad \mathcal{S} = \Theta\left(\sigma_s^2 + \widetilde{\Delta}\right)$$

  - displaced quark fields:

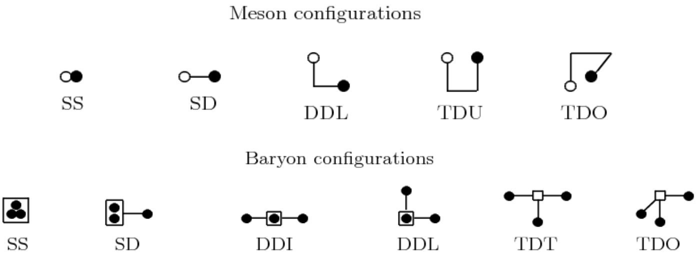
$$q^A_{a\alpha j} = D^{(j)}\widetilde{\psi}^{(A)}_{a\alpha}, \qquad \overline{q}^A_{a\alpha j} = \widetilde{\overline{\psi}}^{(A)}_{a\alpha}\,\gamma_4\, D^{(j)\dagger}$$

  - displacement $D^{(j)}$ is product of smeared links:

$$D^{(j)}(x,x') = \widetilde{U}_{j_1}(x)\ \widetilde{U}_{j_2}(x{+}d_2)\ \widetilde{U}_{j_3}(x{+}d_3)\ldots\widetilde{U}_{j_p}(x{+}d_p)\delta_{x',\ x+d_{p+1}}$$

Extended Operators for Single Hadrons

- quark displacements build up orbital, radial structure



- eg $\rightarrow$ single-site: $\epsilon_{abc}\widetilde{\psi}^{A}_{a\alpha}\widetilde{\psi}^{B}_{b\beta}\widetilde{\psi}^{C}_{c\gamma}$

  singly-displaced: $\epsilon_{abc}\widetilde{\psi}^{A}_{a\alpha}\widetilde{\psi}^{B}_{b\beta}(\widetilde{D}^{(p)}_{j}\widetilde{\psi})^{C}_{c\gamma}$

  doubly-displaced L: $\epsilon_{abc}\widetilde{\psi}^{A}_{a\alpha}(\widetilde{D}^{(p)}_{j}\widetilde{\psi})^{B}_{b\beta}(\widetilde{D}^{(p)}_{k}\widetilde{\psi})^{C}_{c\gamma},\ j \neq k$

Why an Anisotropic Lattice?

$$C_{ij}(t) = \langle 0| O_i(t+t_0)\, \overline{O}_j(t_0) |0\rangle = \sum_n \langle 0| O_i |n\rangle \langle n| \overline{O}_j |0\rangle e^{-E_n t}$$

- temporal correlation function: errors generally increase as t increases
  - to increase 'good' data points, we need a fine temporal spacing
- coarser spatial directions reduce computational needs
  - as a trade-off, this raises the need for improved actions

Quantum Numbers in Toroidal Box

- periodic boundary conditions in cubic box
  - not all directions equivalent $\Rightarrow$ using J^{PC} is wrong!!
- Which J-irreps of $SO(3)$ map into which irreps of octahedral group O? $\rightarrow$ Subduction
- label stationary states of QCD in a periodic box using irreps of cubic space group even in continuum limit
  - spatial inversion: double point group O_h^D

$$A_{1a}, A_{2a}, E_a, T_{1a}, T_{2a}, \quad G_{1a}, G_{2a}, H_a, \qquad a = g, u$$

Spin Content of Cubic Box Irreps

- $n_J^\Gamma = \frac{1}{g_O} \sum_p N_p \chi^J_{\downarrow O}(C_p) \chi^\Gamma(C_p)^*$
- numbers of occurrences of Λ irreps in subduced reps of $SO(3)$ restricted to O

J	A_1	A_2	E	T_1	T_2	J	G_1	G_2	H
0	1	0	0	0	0	$\frac{1}{2}$	1	0	0
1	0	0	0	1	0	$\frac{3}{2}$	0	0	1
2	0	0	1	0	1	$\frac{5}{2}$	0	1	1
3	0	1	0	1	1	$\frac{7}{2}$	1	1	1
4	1	0	1	1	1	$\frac{9}{2}$	1	0	2
5	0	0	1	2	1	$\frac{11}{2}$	1	1	2
6	1	1	1	1	2	$\frac{13}{2}$	1	2	2
7	0	1	1	2	2	$\frac{15}{2}$	1	1	3

Construction of Elemental Operators and Projection onto O_h^D Irreps

- Meson: $\overline{\Phi}^{AB}_{\alpha\beta}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot(\mathbf{x}+\frac{1}{2}(\boldsymbol{d}_\alpha+\boldsymbol{d}_\beta))}\delta_{ab}\ \overline{q}^B_{b\beta}(\boldsymbol{x},t)\ q^A_{a\alpha}(\boldsymbol{x},t)$
- Baryon: $\overline{\Phi}^{ABC}_{\alpha\beta\gamma}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\mathbf{x}}\varepsilon_{abc}\ \overline{q}^C_{c\gamma}(\boldsymbol{x},t)\ \overline{q}^B_{b\beta}(\boldsymbol{x},t)\ \overline{q}^A_{a\alpha}(\boldsymbol{x},t)$
- group-theory projections onto irreps of lattice symmetry group
  $\overline{M}_l(t) = c^{(l)*}_{\alpha\beta}\ \overline{\Phi}^{AB}_{\alpha\beta}(t)$ $\qquad$ $\overline{B}_l(t) = c^{(l)*}_{\alpha\beta\gamma}\ \overline{\Phi}^{ABC}_{\alpha\beta\gamma}(t)$

  $$B^{\Lambda\lambda F}_{Pi}(t) = \frac{d_\Lambda}{g_{O_h^D}}\sum_{R\in O_h^D}\Gamma^\Lambda_{\lambda\mu}(R)U_R B^F_i(t)U^\dagger_R$$
- Partner operators in other rows are obtained using transfer operation of $B^{\Lambda\lambda F}_i(t)$
- definite momentum $\boldsymbol{p}$, irreps of little group of $\boldsymbol{p}$

Two-hadron Operators

- our approach: superposition of products of single-hadron operators of definite momenta

$$c^{I_{3a} I_{3b}}_{\boldsymbol{p}_a \lambda_a;\ \boldsymbol{p}_b \lambda_b} \ B^{I_a I_{3a} S_a}_{\boldsymbol{p}_a \Lambda_a \lambda_a i_a} \ B^{I_b I_{3b} S_b}_{\boldsymbol{p}_b \Lambda_b \lambda_b i_b}$$

- fixed total momentum $\boldsymbol{p} = \boldsymbol{p}_a + \boldsymbol{p}_b$, fixed $\Lambda_a, i_a, \Lambda_b, i_b$
- group-theory projections onto little group of $\boldsymbol{p}$ and isospin irreps
- efficient creating large numbers of two-hadron operators
- generalizes to three, four, . . . hadron operators

Monte Carlo Estimate of Path Integrals

- Remember: the QCD action

$$S[\overline{\psi}, \psi, U] = \overline{\psi}\ K[U]\ \psi + S_G[U]$$

- after quark integration,

$$C_{ij}(t) = \frac{\int D[U]\ \det K[U]\, W[K^{-1}(U)]\ \exp(-S_G[U])}{\int D[U]\ \det K\ \exp(-S_G[U])}$$

- We can use Monte Carlo method now. BUT:
  - inclusion of $\det K[U]$ and evaluation of $K^{-1}[U]$ are computationally expensive!!
    - $N_{\text{tot}} = N_{\text{site}} N_{\text{spin}} N_{\text{color}}$
    - for $32^3 \times 256$ lattice, $N_{\text{tot}} \sim 101$ million

Stochastic Estimation of Quark Propagator

- Need an approximation on the inverse of the Dirac matrix $K[U]$
- Use noise vectors η such that $E(\eta_i) = 0$ and $E(\eta_i \eta_j^*) = \delta_{ij}$
  - $Z_4 = \{+1, -1, +i, -i\}$ noise
- Generate N_R noise vectors $\eta^{(r)}$ and solve $K[U]X^{(r)} = \eta^{(r)}$
- Then $E(X_i \eta_j^*) = E(K_{ik}^{-1} \eta_k \eta_j^*) = K_{ik}^{-1} E(\eta_k \eta_j^*) = K_{ij}^{-1}$

$$\Longrightarrow K_{ij}^{-1} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} X_i^{(r)} \eta_j^{(r)*}$$

Can We Make this Exact?

- For any noise vector $\eta^{(r)}$,

  Suppose $\eta_j^{(r)} = \sum_{s=1}^{N} \eta_j^{(r)[s]}$, where $\eta_j^{(r)[s]} = \eta_j^{(r)} \delta_{js}$ (no sum over j), then

  $$\sum_{s=1}^{N} X_i^{(r)[s]} \eta_j^{(r)[s]*} = K_{ij}^{-1} \eta_j^{(r)} \eta_j^{(r)*} = K_{ij}^{-1},$$

  $\rightarrow$ Exact for Z_4 noise

- But N solution vectors $X_i^{(r)[s]}$ had to be computed. Not feasible!

Variance Reduction through Noise Dilution

- Introduce a complete set of projection operators $P^{(a)}$:

$$P^{(a)}P^{(b)} = \delta^{ab}P^{(a)}, \quad \sum_a P^{(a)} = 1, \quad P^{(a)\dagger} = P^{(a)}$$

$$\eta_k^{[a]} = P_{kk'}^{(a)}\eta_{k'}, \qquad \eta_j^{[a]*} = P_{jj'}^{(a)*}\eta_{j'}^*$$

- Define $X^{[a]}$ to be the solution of $K_{ik}X_k^{[a]} = \eta_i^{[a]}$, then

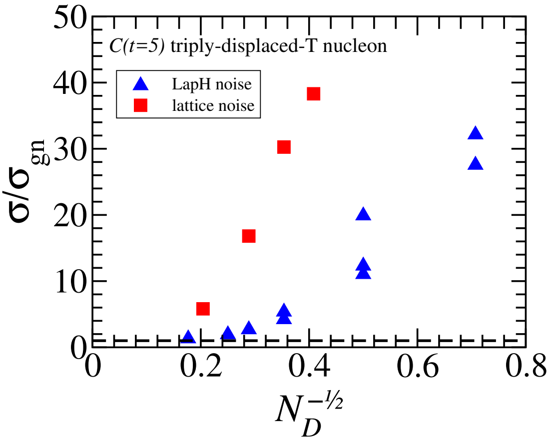
$$\sum_a E(X_i^{[a]}\eta_j^{[a]*}) = K_{ik}^{-1}\sum_a E(\eta_k^{[a]}\eta_j^{[a]*}) = K_{ik}^{-1}\sum_a P_{kj}^{(a)} = K_{ik}^{-1}$$

$$\Longrightarrow \ K_{ij}^{-1} \approx \frac{1}{N_R}\sum_{r=1}^{N_R}\sum_a X^{(r)[a]}\ \eta_j^{(r)[a]*}$$

- **An improvement because** $Var(\sum_a \eta_k^{[a]}\eta_j^{[a]*}) < Var(\eta_k\eta_j^*)$

Laplacian Heaviside (LapH) Smearing

- why bother finding propagator to/from high energy modes?
- use the N_v lowest eigenvectors of the covariant Laplacian to define the LapH subspace

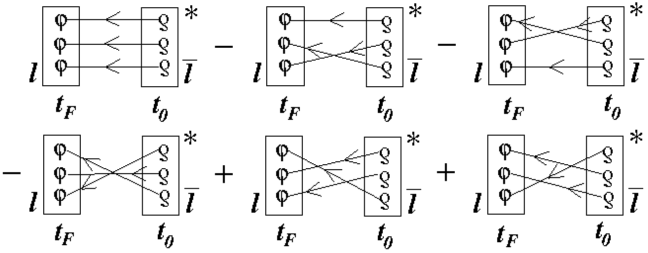


Correlators and quark line diagrams

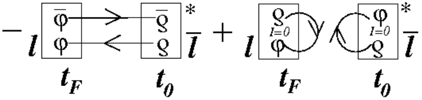
- baryon correlator

$$C_{l\bar{l}} \approx \frac{1}{N_R}\sum_{r}\sum_{d_A d_B d_C} \mathcal{B}_l^{(r)[d_A d_B d_C]}(\varphi^A,\varphi^B,\varphi^C)\,\mathcal{B}_{\bar{l}}^{(r)[d_A d_B d_C]}(\varrho^A,\varrho^B,\varrho^C)^*$$

- express diagrammatically



- meson correlator



Excited States from Correlation Matrices

- in finite volume, energies are discrete

$$C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)*} \ e^{-E_n t}, \qquad Z_j^{(n)} = \langle 0 | \, O_j \, | n \rangle$$

- not practical to do fits using above form
- define new correlation matrix $\widetilde{C}(t)$ using a single rotation

$$\widetilde{C}(t) = U^\dagger \ C(\tau_0)^{-1/2} \ C(t) \ C(\tau_0)^{-1/2} \ U$$

- columns of U are eigenvectors of $C(\tau_0)^{-1/2} \, C(\tau_D) \, C(\tau_0)^{-1/2}$
- choose τ_0 and τ_D large enough so $\widetilde{C}(t)$ diagonal for $t > \tau_D$
- effective energies

$$\widetilde{m}_\alpha^{\rm eff}(t) = \frac{1}{\Delta t} \ln \left(\frac{\widetilde{C}_{\alpha\alpha}(t)}{\widetilde{C}_{\alpha\alpha}(t+\Delta t)} \right)$$

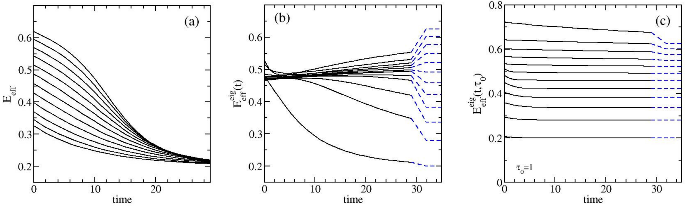
tend to N lowest-lying stationary state energies in a channel

Correlator matrix toy model

- **Theorem:** For every $t \geq 0$, let $\lambda_n(t)$ be the eigenvalues of an $N \times N$ Hermitian correlation matrix $C(t)$ ordered such that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{N-1}$, then
$$\lim_{t\to\infty} \lambda_n(t) = b_n e^{-E_n t}\Big[1 + O(e^{-t\Delta_n})\Big],$$
$$b_n > 0, \quad \Delta_n = \min_{m\neq n} |E_n - E_m|.$$
- Example: $N_e = 200$ eigenstates with energies
$$E_0 = 0.20, \qquad E_n = E_{n-1} + \frac{0.08}{\sqrt{n}}, \qquad n = 1, 2, \ldots, N_e - 1.$$
for $N \times N$ correlator matrix, $N = 12$, overlaps
$$Z_j^{(n)} = \frac{(-1)^{j+n}}{1 + 0.05(j-n)^2}.$$

Correlator matrix toy model (con't)

- toy model $N_e = 200$ with 12×12 correlator matrix $C(t)$



- left: effective energies of diagonal elements of correlator matrix
- middle: effective energies of eigenvalues of $C(t)$
- right: effective energies of eigenvalues of $C(\tau_0)^{-1/2}\ C(t)\ C(\tau_0)^{-1/2}$ for $\tau_0 = 1$

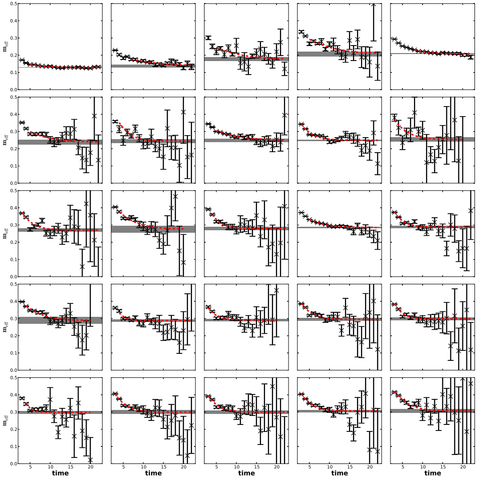
Ensembles and run parameters

- focusing on two Monte Carlo ensembles
  - $(32^3|240)$: 412 configs $32^3 \times 256$, $m_\pi \approx 240$ MeV, $m_\pi L \sim 4.4$
  - $(24^3|390)$: 551 configs $24^3 \times 128$, $m_\pi \approx 390$ MeV, $m_\pi L \sim 5.7$
- anisotropic improved gluon action, clover quarks (stout links)
- QCD coupling $\beta = 1.5$ such that $a_s \sim 0.12$ fm, $a_t \sim 0.035$ fm
- strange quark mass $m_s = -0.0743$ nearly physical (using kaon)
- work in $m_u = m_d$ limit so $SU(2)$ isospin exact
- generated using RHMC, configs separated by 20 trajectories

- stout-link smearing in operators $\xi = 0.10$ and $n_\xi = 10$
- LapH smearing cutoff $\sigma_s^2 = 0.33$ such that
  - $N_v = 112$ for 24^3 lattices
  - $N_v = 264$ for 32^3 lattices
- source times:
  - 4 widely-separated t_0 values on 24^3
  - 8 t_0 values used on 32^3 lattice

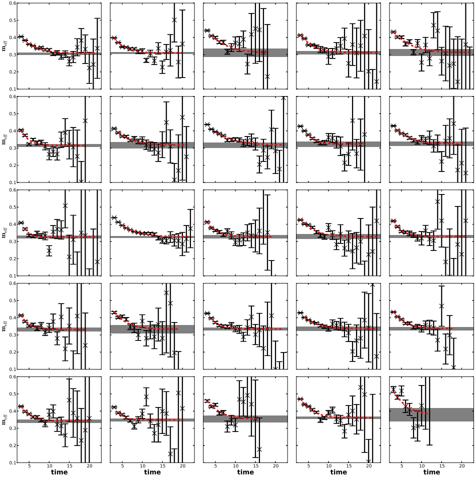
$I = 1, S = 0, T_{1u}^{+}$ Channel Energies

- Effective energies for first 25 levels (B. Fahy, PhD thesis)
- Energies obtained using a two-exponential fit



$I = 1, S = 0, T_{1u}^{+}$ Channel Energies, Continued

- Effective energies for next 25 levels
- Energies obtained using a two-exponential fit



Level identification

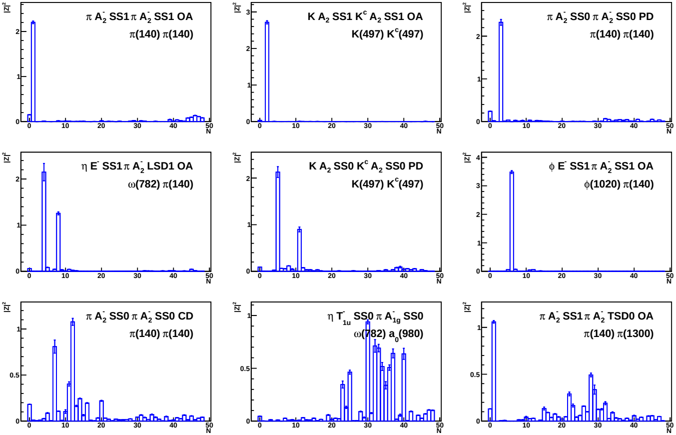
- level identification inferred from Z overlaps with probe operators
- keep in mind:
  - probe operators $\overline{O}_j$ act on vacuum, create a "probe state" $|\Phi_j\rangle$, Z's are overlaps of probe state with each eigenstate

$$|\Phi_j\rangle \equiv \overline{O}_i|0\rangle, \qquad Z_j^{(n)} = \langle\Phi_j|n\rangle$$

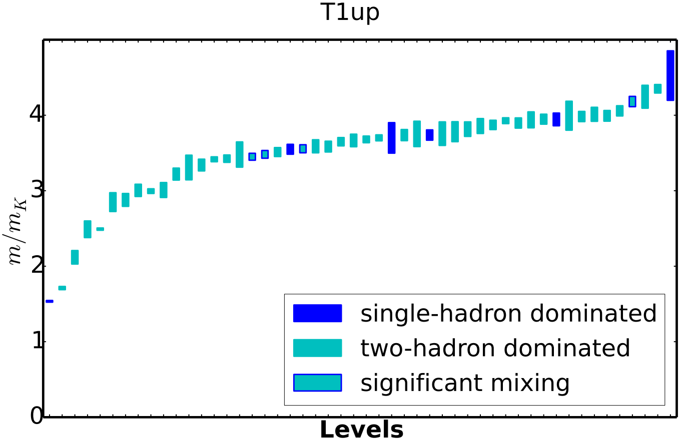
  - identify by dominant probe state(s) whenever possible

Level identification

- overlaps for various operators



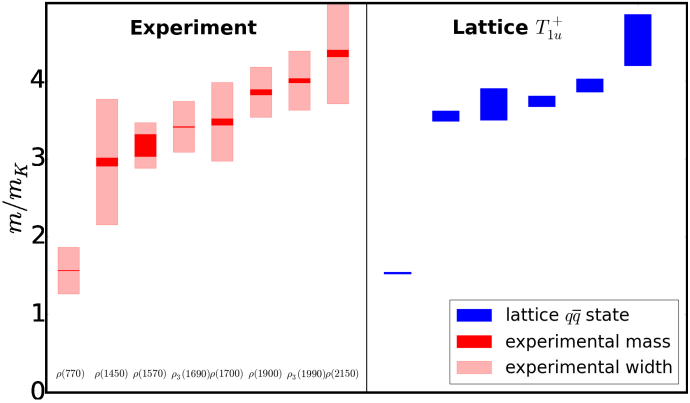
$I = 1, S = 0, T_{1u}^+$ Channel Staircase Plot



Lowest 50 energies in the $I = 1, S = 0, T_{1u}^+$ channel extracted from a 63×63 correlator matrix. Uses 412 configs on a clover-improved anisotropic $32^3 \times 256$ lattice with $m_\pi \approx 240\ MeV$. Total $P = 0$.

Single Hadron T_{1u}^{+} Spectrum Compared to Experiment

- right: energies of $\bar{q}q$-dominant states as ratios over m_K for $(32^3|240)$ ensemble (resonance precursor states)
- left: experiment



Effective Hamiltonian method

- We use "Lüscher method" to extract resonance information from finite volume energies: complicated
- Working on possible alternative: Effective Hamiltonian (Wu et al)
- kinetic terms
$$|\,\sigma\rangle\, m_\sigma\,\langle\,\sigma| \;+\; \left|\,\alpha_1\alpha_2\,\vec{k}\right\rangle\left(\sqrt{m_{\alpha_1}^2+\vec{k}^2}+\sqrt{m_{\alpha_2}^2+\vec{k}^2}\right)\left\langle\,\alpha_1\alpha_2\,\vec{k}\right|$$
- interaction $1-2$ and $2-2$ terms
$$|\,\sigma\rangle\, g(k)\left\langle\,\alpha_1\alpha_2\vec{k}\right| + \left|\,\alpha_1\alpha_2\vec{k_1}\right\rangle g(k_1,k_2)\left\langle\,\beta_1\beta_2\vec{k_2}\right|$$
- momentum dependence for various interactions to have form $\frac{a}{1+b\vec{k}^2}$ for S-wave and $k(\frac{a}{1+b\vec{k}^2})^{\frac{3}{2}}$ for P-wave
- parameters of Hamiltonian determined from fits to finite-volume spectra
- Lippmann-Schwinger (or other methods) to extract infinite-volume resonances

Applying to Lattice QCD

- We can analyze the spectra in different symmetry channels with definite $I, I_3, S, \eta,$ and G- parity
- consider zero total momentum first
- An important formula: $|\,\Lambda_P \lambda\rangle = \sum \Gamma^{\Lambda_P *}_{\lambda\lambda}(R)\hat{U}_R \,|\, J^P m_J\rangle$
- We can retain J as it is invariant under rotational and parity transformation: $|\,\Lambda_P \lambda J n \xi\rangle = \sum c^{\Lambda \lambda n}_{J m_J} \,|\, J^P m_J \xi\rangle$
- Prob 1. States with different J's can give non zero matrix elemnts if they the same sector?
- Prob 2. Same J could show up in different lattice irreps?

The Effective Hamiltonian in Lattice

- To understand the questions, let's write the hamiltonian explicitly in JLS basis:
- $\hat{H} = \hat{H}_0 + \hat{H}_I$
- $\hat{H}_0 = \sum \left| \sigma\Lambda_P^G\lambda Jni \right\rangle M_{\sigma Ji}^{\Lambda_P^G} \left\langle \sigma\Lambda_P^G\lambda Jni \right|$

  $+ \sum \left| \alpha\Lambda_P^G\lambda JL = J - S, Ss_1s_2, \eta_1\eta_2 : \eta_1\eta_2(-1)^{J-S} = P; nij \right\rangle N \left\langle ... \right|$
- $\hat{H}_I(1 \to 2) =$
  $\sum \left| \sigma\Lambda_P^G\lambda Jni \right\rangle g \left\langle \alpha\Lambda_P^G\lambda JL = J - S, Ss_1s_2, \eta_1\eta_2 : \eta_1\eta_2(-1)^{J-S} = P; njk \right.$
- $\hat{H}_I(2 \to 2) =$
  $\sum \left| \alpha\Lambda_P^G\lambda JL = J - S, Ss_1s_2, \eta_1\eta_2 : \eta_1\eta_2(-1)^{J-S} = P; nij \right\rangle v$
  $\left\langle \beta\Lambda_P^G\lambda JL = J - S', S's_3s_4, \eta_3\eta_4 : \eta_3\eta_4(-1)^{J-S'} = P; mkl \right|$

Comments:

- Different $'M'$ parameters are not independent. If we try to relate the states between continuous and cubic symmetries:
  $| \Lambda_p \lambda J n \xi \rangle = \sum_{m_J} c_{J m_J}^{\Lambda_p \lambda n} | J^P m_J \xi \rangle$
  we see that, in fact, $M_i = \sum c_i c_i^* m$. This will reduce the number of independent mass parameters.
- But, for example, we can construct three single particle states in A_{1u}^- channel: $J = 0 \rightarrow 2$ states, $J = 4 \rightarrow 1$ state; all have independent M
- So no $'g'$ term between different J, but E, T_1 and T_2 occur in $J = 4$ irreps
- if we construct states that don't retain J? $\rightarrow$ more non-zero g's than before (??)

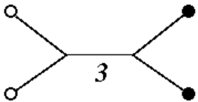
Working with a weird basis

- for $1 \to 2$ interaction: we have $|\,\sigma\Lambda\lambda\rangle\, g_{\Lambda\lambda}^{SM_S}(\vec{k})\left\langle\,\alpha SM_S\vec{k}\right|$ where
  $g_{\Lambda\lambda}^{SM_S}(\vec{k}) = \sum d_{JM_J}^{\Lambda\lambda} g_{SM_S}^{JM_J}(\vec{k}) = \sum d_{JM_J}^{\Lambda\lambda} \sum c_{LM_L SM_S}^{JM_J} g^{LM_L}(\vec{k})$
- The term $g^{LM_L}(\vec{k})$ transforms like spherical harmonics:
  $g^{LM_L}(\vec{k}) = Y^{LM_L}(\vec{k})u(|\vec{k}|)$
- Matches with the form of the g-term mentioned before
- We can also work it out for non-zero total constituent momenta

Summary

- comprehensive survey of energy spectrum of QCD stationary states in a finite volume
- Effective Hamiltonian method to extract resonance information from the finite volume spectra

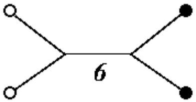
Tetraquark Color Structure



$$\overline{\Phi}^{(3)ABCD}_{\alpha\beta\gamma\delta}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \left[\overline{q}^{A}_{a\alpha}(\boldsymbol{x},t)\ \overline{q}^{B}_{b\beta}(\boldsymbol{x},t)\ \varepsilon_{abe}\right]$$

$(3_A \otimes \overline{3}_A)$

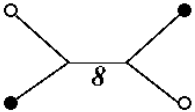
$$\times \left[D^{(3)}_{ef}\ \varepsilon_{cdf}\ q^{C}_{c\gamma}\ q^{D}_{d\delta}\right](\boldsymbol{x},t)$$



$$\overline{\Phi}^{(6)ABCD}_{\alpha\beta\gamma\delta}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \left[\overline{q}^{A}_{a\alpha}(\boldsymbol{x},t)\ \overline{q}^{B}_{b\beta}(\boldsymbol{x},t)\ C^{l*}_{ab}\right]$$

$(\overline{6}_S \otimes 6_S)$

$$\times \left[D^{(6)}_{lm}\ C^{m}_{cd}\ q^{C}_{c\gamma}\ q^{D}_{d\delta}\right](\boldsymbol{x},t)$$



$$\overline{\Phi}^{(8)ABCD}_{\alpha\beta\gamma\delta}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \left[\overline{q}^{A}_{a\alpha}(\boldsymbol{x},t)\ q^{B}_{b\beta}(\boldsymbol{x},t)\ \lambda^{i}_{ab}\right]$$

$(8 \otimes 8)$

$$\times \left[D^{(8)}_{ij}\ \lambda^{j}_{cd}\ \overline{q}^{C}_{c\gamma}\ q^{D}_{d\delta}\right](\boldsymbol{x},t)$$

Transformation of Diquarks

- the quark fields transform in the 3-dimensional irrep

$$q_a \to V^{(3)}_{ab} q_b, \qquad \overline{q}_a \to \overline{q}_b V^{(3)\dagger}_{ba}$$

- need the diquarks to transform properly.
  - *e.g.* the 6-dimensional irrep

$$\begin{aligned} C^m_{cd}\, q_c\, q_d \quad &\to C^m_{cd}\, V^{(3)}_{cc'}\, q_{c'}\, V^{(3)}_{dd'}\, q_{d'} \\ &= V^{(6)}_{mn}\, C^n_{c'd'}\, q_{c'}\, q_{d'} \end{aligned}$$

  - this implies the following transformation of C^m_{cd}

$$C^m_{cd} V^{(3)}_{cc'} V^{(3)}_{dd'} = V^{(6)}_{mn} C^n_{c'd'}$$

- similar for other irreps

Diquark Displacements

- we also need the gauge link variables in the 6 and 8 irreps in order to displace the diquarks
- consider the following definition for $D_{ij}^{(8)}(x, y)$

$$D_{ij}^{(8)}(x, y) \equiv \lambda_i^{ab} D_{bc}^{(3)}(x, y)\ \lambda_j^{cd} D_{da}^{(3)}(x, y)$$

- then, one can show it transforms as

$$D_{ij}^{(8)}(x, y) \to \Omega_{ii'}^{(8)}(x) D_{i'j'}^{(8)}(x, y) \Omega_{j'j}^{(8)\dagger}(y)$$

by using

$$\Omega_{ik}^{(8)} \lambda_k^{ab} = \Omega_{aa'}^{(3)\dagger} \lambda_i^{a'b'} \Omega_{b'b}^{(3)}$$

- similar for the 6 irrep

Tetraquark Correlators

- Must project the tetraquark operators onto the irreps of the lattice symmetry group
$$\overline{T}_l(t) = c^{(l)*}_{\alpha\beta\gamma\delta}\,\overline{\Phi}^{ABCD}_{\alpha\beta\gamma\delta}(t)$$
- recall how stochastic LapH estimates quark lines
$$\mathcal{Q}^{(A)}_{a\alpha;b\beta}(\boldsymbol{x},t;\boldsymbol{x}_0,t_0|U) \approx \frac{1}{N_r}\sum_{r=1}^{N_r}\sum_{A_d}\varphi^{(A,r)[A_d]}_{a\alpha}(\boldsymbol{x},t|U)\,\varrho^{(A,r)[A_d]}_{b\beta}(\boldsymbol{x}_0,t_0|U)^*$$
- next define a tetraquark function as
$$\mathcal{T}^{[b_1b_2b_3b_4]}_{l,(3)}(\varrho_1,\varrho_2,\varphi_3,\varphi_4;t) = c^{(l)}_{\alpha\beta\gamma\delta}\sum_{\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\,\varrho^{[b_1]}_{a\alpha\boldsymbol{x}t}(\rho_1)^*\,\varrho^{[b_2]}_{b\beta\boldsymbol{x}t}(\rho_2)^*$$
$$\times\ \varepsilon_{abe}\ \varepsilon_{cde}\ \varphi^{[b_3]}_{c\gamma\boldsymbol{x}t}(\rho_3)\varphi^{d\delta\boldsymbol{x}t}(\rho_4)$$
- correlator is sum over products of these tetraquark functions

Excited States from Correlation Matrices

- in finite volume, energies are discrete

$$C_{ij}(t) = \sum_{n}^{\infty} Z_i^{(n)} Z_j^{(n)*} \ e^{-E_n t}, \qquad Z_j^{(n)} = \langle 0| \, O_j \, |n\rangle$$

- not practical to do fits using above form
- define new correlation matrix $G(t)$

$$G(t) = C(\tau_0)^{-1/2} \ C(t) \ C(\tau_0)^{-1/2}$$

- let columns of $V(t)$ be the eigenvectors and $\lambda_n(t)$ be the eigenvalues of $G(t)$
- for large t, $\lambda_n(t) \to |Z'_n|^2 e^{-E_n t}$ and

$$Z_j^{(n)} \approx C_{jk}(\tau_0)^{1/2} V_{kn}(t) Z'_n$$

- does not assume $C_{ij}(t)$ is approximated by first N terms

Correlator matrix toy model

- Example: $N_e = 200$ eigenstates with energies

$$E_0 = 0.20, \qquad E_n = E_{n-1} + \frac{0.08}{\sqrt{n}}, \qquad n = 1, 2, \ldots, N_e - 1.$$

  for $N \times N$ correlator matrix, $N = 12$, overlaps

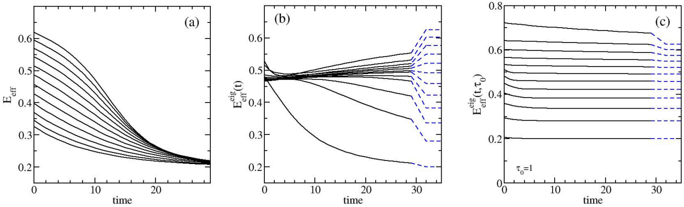
$$Z_j^{(n)} = \frac{(-1)^{j+n}}{1 + 0.05(j-n)^2}.$$

- recall definition of effective energies

$$m_\alpha^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left(\frac{C_{\alpha\alpha}(t)}{C_{\alpha\alpha}(t + \Delta t)} \right)$$

Correlator matrix toy model (con't)

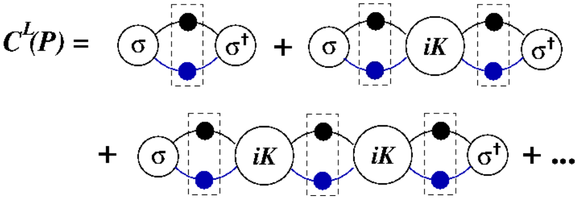
- toy model $N_e = 200$ with 12×12 correlator matrix $C(t)$



- left: effective energies of diagonal elements of correlator matrix
- middle: effective energies of eigenvalues of $C(t)$
- right: effective energies of eigenvalues of $C(\tau_0)^{-1/2}\, C(t)\, C(\tau_0)^{-1/2}$ for $\tau_0 = 1$

Scattering Phase Shifts from Finite-volume Energies

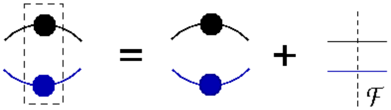
- correlator of two-particle operator σ in finite volume



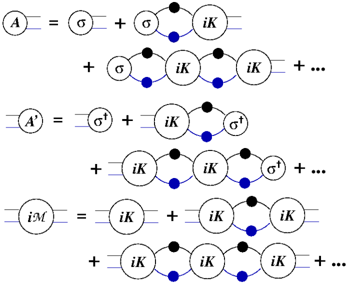
- $C^{\infty}(P)$ has branch cuts where two-particle thresholds begin
- momentum quantization in finite volume: cuts $\rightarrow$ series of poles
- C^{L} poles: two-particle energy spectrum of finite volume theory

Phase Shift from Finite-volume Energies (con't)

- finite-volume momentum sum is infinite-volume integral plus correction $\mathcal{F}$

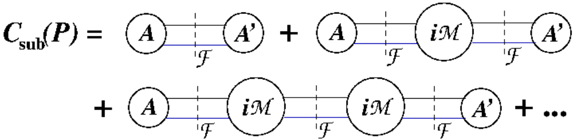


- define the following quantities: A, A', invariant scattering amplitude $i\mathcal{M}$



Phase Shifts from Finite-volume Energies (con't)

- subtracted correlator $C_{\rm sub}(P) = C^L(P) - C^\infty(P)$ given by



- sum geometric series

$$C_{\rm sub}(P) = A\ \mathcal{F}(1 - i\mathcal{M}\mathcal{F})^{-1}\ A'.$$

- poles of $C_{\rm sub}(P)$ are poles of $C^L(P)$ from $\det(1 - i\mathcal{M}\mathcal{F}) = 0$

Phase Shifts from Finite-volume Energies (con't)

- work in spatial L^3 volume with periodic b.c.
- total momentum $\boldsymbol{P} = (2\pi/L)\boldsymbol{d}$, where $\boldsymbol{d}$ vector of integers
- masses m_1 and m_2 of particle 1 and 2
- calculate lab-frame energy E of two-particle interacting state in lattice QCD
- boost to center-of-mass frame by defining:

$$\begin{aligned} E_{\rm cm} &= \sqrt{E^2 - \boldsymbol{P}^2}, \qquad \gamma = \frac{E}{E_{\rm cm}}, \\ \boldsymbol{q}_{\rm cm}^2 &= \frac{1}{4}E_{\rm cm}^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4E_{\rm cm}^2}, \\ u^2 &= \frac{L^2 \boldsymbol{q}_{\rm cm}^2}{(2\pi)^2}, \qquad \boldsymbol{s} = \left(1 + \frac{(m_1^2 - m_2^2)}{E_{\rm cm}^2}\right)\boldsymbol{d} \end{aligned}$$

- E related to S matrix (and phase shifts) by

$$\det[1 + F^{(\boldsymbol{s},\gamma,u)}(S - 1)] = 0.$$

Phase shifts from finite-volume energies (con't)

- F matrix in JLS basis states given by

$$F^{(\boldsymbol{s},\gamma,u)}_{J'm_{J'}L'S'a';\; Jm_JLSa} = \frac{\rho_a}{2}\delta_{a'a}\delta_{S'S}\Big\{\delta_{J'J}\delta_{m_{J'}m_J}\delta_{L'L}$$
$$+W^{(\boldsymbol{s},\gamma,u)}_{L'm_{L'};\; Lm_L}\langle J'm_{J'}|L'm_{L'},Sm_S\rangle\langle Lm_L,Sm_S|Jm_J\rangle\Big\},$$

- total angular mom J, J', orbital mom L, L', intrinsic spin S, S'
- a, a' channel labels
- $\rho_a = 1$ distinguishable particles, $\rho_a = \frac{1}{2}$ identical

$$W^{(\boldsymbol{s},\gamma,u)}_{L'm_{L'};\; Lm_L} = \frac{2i}{\pi\gamma u^{l+1}}\mathcal{Z}_{lm}(\boldsymbol{s},\gamma,u^2)\int d^2\Omega\, Y^*_{L'm_{L'}}(\Omega)Y^*_{lm}(\Omega)Y_{Lm_L}(\Omega)$$

- Rummukainen-Gottlieb-Lüscher (RGL) shifted zeta functions $\mathcal{Z}_{lm}$
- $F^{(\boldsymbol{s},\gamma,u)}$ diagonal in channel space, mixes different J, J'
- recall S diagonal in angular momentum, but off-diagonal in channel space

P-wave $I = 1$ $\pi\pi$ scattering

- for P-wave phase shift $\delta_1(E_{\mathrm{cm}})$ for $\pi\pi$ $I = 1$ scattering
- define

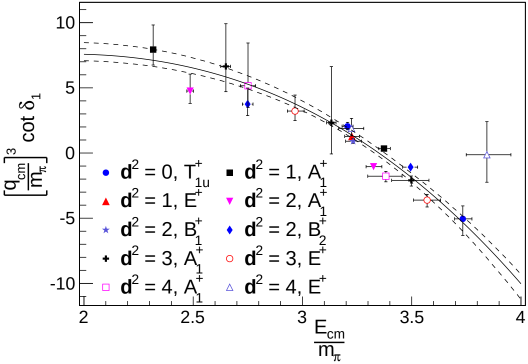
$$w_{lm} = \frac{\mathcal{Z}_{lm}(\boldsymbol{s}, \gamma, u^2)}{\gamma \pi^{3/2} u^{l+1}}$$

$\boldsymbol{d}$	Λ	$\cot \delta_1$
(0,0,0)	T_{1u}^{+}	$\mathrm{Re}\ w_{0,0}$
(0,0,1)	A_1^{+}	$\mathrm{Re}\ w_{0,0} + \frac{2}{\sqrt{5}}\mathrm{Re}\ w_{2,0}$
	E^{+}	$\mathrm{Re}\ w_{0,0} - \frac{1}{\sqrt{5}}\mathrm{Re}\ w_{2,0}$
(0,1,1)	A_1^{+}	$\mathrm{Re}\ w_{0,0} + \frac{1}{2\sqrt{5}}\mathrm{Re}\ w_{2,0} - \sqrt{\frac{6}{5}}\mathrm{Im}\ w_{2,1} - \sqrt{\frac{3}{10}}\mathrm{Re}\ w_{2,2},$
	B_1^{+}	$\mathrm{Re}\ w_{0,0} - \frac{1}{\sqrt{5}}\mathrm{Re}\ w_{2,0} + \sqrt{\frac{6}{5}}\mathrm{Re}\ w_{2,2},$
	B_2^{+}	$\mathrm{Re}\ w_{0,0} + \frac{1}{2\sqrt{5}}\mathrm{Re}\ w_{2,0} + \sqrt{\frac{6}{5}}\mathrm{Im} w_{2,1} - \sqrt{\frac{3}{10}}\mathrm{Re}\ w_{2,2}$
(1,1,1)	A_1^{+}	$\mathrm{Re}\ w_{0,0} + 2\sqrt{\frac{6}{5}}\mathrm{Im}\ w_{2,2}$
	E^{+}	$\mathrm{Re}\ w_{0,0} - \sqrt{\frac{6}{5}}\mathrm{Im}\ w_{2,2}$

$I = 1$ $\pi\pi$ scattering phase shift and width of the ρ

- preliminary results $32^3 \times 256$, $m_\pi \approx 240$ MeV (J. Bulava et al.):

$$g_{\rho\pi\pi} = 5.99(26),\ m_\rho/m_\pi = 3.350(24),\ \chi^2/\text{dof} = 1.04$$



- fit $\quad g_{\rho\pi\pi}^2 q_{\text{cm}}^3 \cot(\delta_1) = 6\pi E_{\text{cm}}(m_\rho^2 - E_{\text{cm}}^2)$

Effective Hamiltonian method

- relating finite-volume energies to resonance parameters via “Lüscher method” very complicated
- alternative: use an effective hadron Hamiltonian matrix
  - Wu et al, PRC **90**, 055206 (2014)
- use single and two-particle states as basis states
- interaction terms from symmetry, with assumed form with respect to momenta
- parameters of Hamiltonian determined from fits to finite-volume spectra
- Lippmann-Schwinger (or other methods) to extract infinite-volume resonances

Hamiltonian in total zero momentum sector

- non-interacting part

$$H_0 = \sum_{i=1,n} |\sigma_i\rangle m_i^0 \langle\sigma_i| + \sum_{\alpha} \int d\vec{k} |\alpha(\vec{k})\rangle [\sqrt{m_{\alpha_1}^2 + \vec{k}_{\alpha_1}^2} + \sqrt{m_{\alpha_2}^2 + \vec{k}_{\alpha_2}^2}] \langle\alpha(\vec{k})|$$

- interaction term

$$H_I = g + v$$

- $2-1$ interaction

$$g = \sum_{\alpha} \int d\vec{k} \sum_{i=1,n} \{|\alpha(\vec{k})\rangle g_{i,\alpha}^{\dagger}(k)\langle i| + |i\rangle g_{i,\alpha}(k)\langle\alpha(\vec{k})|\}$$

- $2-2$ interaction

$$v = \sum_{\alpha,\beta} \int d\vec{k} d\vec{k}' \, |\alpha(\vec{k})\rangle v_{\alpha,\beta}(k,k')\langle\beta(\vec{k}')|$$

- the S matrix is given by

$$S_{\alpha,\beta}(E) = 1 + 2iT_{\alpha,\beta}(k_{0\alpha}, k_{0\beta}; E)$$

Reference

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- S. Basak et al., *Lattice QCD determination of patterns of excited baryon states*, Phys. Rev. D **76**, 074504 (2007).
- Wu et al., *Finite-volume Hamiltonian method for coupled-channels interactions in lattice QCD*, Phys. Rev. C **90**, 055206 (2014)