Fragmentation Functions and Global QCD Fits
The 32nd Annual Hampton University Graduate Studies Program (HUGS2017)

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   ▶ the basics: recap on factorisation and evolution
   ▶ single inclusive annihilation as a case study
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2 Lecture 2: Methodological aspects of global QCD fits
   ▶ determining the probability density in a space of functions
   ▶ goodness of fit
   ▶ parton parametrisation, uncertainty representation, parameter optimisation
   ▶ fit validation

3 Lecture 3: Phenomenology of Fragmentation Functions
   ▶ overview of recent determinations of Fragmentation Functions
   ▶ higher-order corrections, heavy quark mass effects
   ▶ applications of Fragmentation Functions

DISCLAIMER

These lectures contain a personal selection of topics and are certainly not exhaustive
Bibliography

1. General textbooks on perturbative QCD

2. Reviews on Fragmentation Functions
   ▶ S. Albino et al. [arXiv:0804:2021]

3. Reviews on Parton Distribution Functions

4. Specific topics not addressed above
   ▶ more journal references along the way as we proceed

**DISCLAIMER**

These lectures will focus on collinear leading-twist Fragmentation Functions for single-inclusive unpolarised hadron production. Transverse-momentum-dependent/multi-hadron fragmentation not covered here.
Fragmentation Functions and Global QCD Fits

Lecture 1: Theoretical framework
Outline

1 Foreword: why Fragmentation Functions?
   from partons in the initial state to partons in the final state

2 The basics
   definition of Fragmentation Functions, factorisation, evolution
   properties of splitting functions, theoretical constraints

3 SIA as a case study
   SIA cross sections/multiplicities: theoretical expectations and data sets
   flavours schemes: changing the number of active flavours
   higher-order QCD corrections

4 Other processes
   hadron production in SIDIS: multiplicities and data sets
   hadron production in $pp$ collisions: cross section and data sets

Focus the attention on single-inclusive annihilation
as a paradigm for the illustration of features/issues of fragmentation
in the cleanest theoretical framework

Try to minimise overlap with J. W. Qiu’s lectures
1.1 Foreword: why Fragmentation Functions?
Hadron physics, or the quest for the nucleon structure

Nucleons make up all nuclei, and hence most of the visible matter in the Universe. They are bound states with internal structure and dynamics.
The QCD picture of the nucleon

naive picture

three non-relativistic quarks

realistic picture

indefinite number of relativistic quarks and gluons

QCD
factorization, evolution
QCD, a powerful theoretical framework

1 What are the general features of QCD?
   KEYWORDS: renormalisation; asymptotic freedom; infrared safety; factorisation
   → addressed in Jianwei Qiu’s lecture on the first week

2 How reliable is a theoretical QCD calculation?
   KEYWORDS: scale dependence; higher-order corrections; resummation(s)
   → addressed in Jianwei Qiu’s lecture on the first week

3 How can we relate QCD to experiment?
   KEYWORDS: factorisation; evolution; distribution functions; global analysis
   → addressed in these lectures (with a focus on Fragmentation Functions)

4 QCD has proven to be successful in the last 35+ years

GOALS

Qualitative QCD: discovery physics
Set up and check the framework

Quantitative QCD: precision physics
Understand and describe a large class of hard-scattering processes and phenomena

Precision QCD: new physics
Investigate possible deviations from the Standard Model
Lepton-hadron facilities
Hadron-hadron facilities

**Fermilab TeVatron** [1987 → 2011]
- $p\bar{p}$ collisions 0.63, 1.8, 1.96 TeV
- top discovery, jet physics, ...
- further established QCD

**CERN SPPS** [1981 → 1990]
- $p\bar{p}$ collisions 540, 630 GeV
- $W, Z$ discovery, jets, ...
- early successes of QCD

**CERN LHC** [operating]
- $p\bar{p}$ collisions 7, 14 TeV
- a QCD machine, discoveries?
- also PbPb and pPb program

**BNL RHIC** [2000 → ...]
- $p\bar{p}$ collisions up to 500 GeV
- the World’s first and only polarized collider
- spin dep. phenomena, spin strct. of the nucleon
- also versatile heavy ion program
The densities of partons \( f = q, \bar{q}, g \) with momentum fraction \( x \)

\[
f(x) \equiv f^{\uparrow}(x) + f^{\downarrow}(x) \quad \Delta f(x) \equiv f^{\uparrow}(x) - f^{\downarrow}(x)
\]

\[
q(x) = \begin{array}{c}
\text{parton} \\
xP_i \\
\text{proton}
\end{array} 
\]

\[
\Delta q(x) = \begin{array}{c}
\text{collinear transition of a massless proton } h \\
i \rightarrow \text{a massless parton } i \\
\text{with fractional momentum } x \\
\text{local OPE} \implies \text{lattice formulation}
\end{array}
\]

\[
q(x) = \frac{1}{4\pi} \int dy^- e^{-iy^-xP^+} \langle P|\bar{\psi}(0, y^-, 0_\perp)\gamma^+\psi(0)|P \rangle
\]

\[
\Delta q(x) = \frac{1}{4\pi} \int dy^- e^{-iy^-xP^+} \langle P, S|\bar{\psi}(0, y^-, 0_\perp)\gamma^+\gamma^5\psi(0)|P, S \rangle
\]

\[
y = (y^+, y^-, y_\perp), \quad y^+ = (y^0 + y^z)/\sqrt{2}, \quad y^- = (y^0 - y^z)/\sqrt{2}, \quad y_\perp = (v^x, v^y)
\]

The definitions can be generalised to include a transverse-momentum dependence.
### Nucleons in the initial state: Parton Distribution Functions

<table>
<thead>
<tr>
<th>Process</th>
<th>Reaction</th>
<th>Subprocess</th>
<th>PDFs probed</th>
<th>$x$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell^\pm {p, n} \rightarrow \ell^\pm + X$</td>
<td>$\gamma^* q \rightarrow q$</td>
<td>$q, \bar{q}, g$</td>
<td>$x \gtrsim 0.01$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm n/p \rightarrow \ell^\pm + X$</td>
<td>$\gamma^* d/u \rightarrow d/u$</td>
<td>$d/u$</td>
<td>$x \gtrsim 0.01$</td>
<td></td>
</tr>
<tr>
<td>$\nu(\bar{\nu})N \rightarrow \mu^- (\mu^+) + X$</td>
<td>$W^* q \rightarrow q'$</td>
<td>$q, \bar{q}$</td>
<td>$0.01 \lesssim x \lesssim 0.5$</td>
<td></td>
</tr>
<tr>
<td>$\nu N \rightarrow \mu^- \mu^+ + X$</td>
<td>$W^* s \rightarrow c$</td>
<td>$s$</td>
<td>$0.01 \lesssim x \lesssim 0.2$</td>
<td></td>
</tr>
<tr>
<td>$\bar{\nu} N \rightarrow \mu^+ \mu^- + X$</td>
<td>$W^* \bar{s} \rightarrow \bar{c}$</td>
<td>$\bar{s}$</td>
<td>$0.01 \lesssim x \lesssim 0.2$</td>
<td></td>
</tr>
<tr>
<td>$e^\pm p \rightarrow e^\pm + X$</td>
<td>$\gamma^* q \rightarrow q$</td>
<td>$g, q, \bar{q}$</td>
<td>$0.0001 \lesssim x \lesssim 0.1$</td>
<td></td>
</tr>
<tr>
<td>$e^\pm p \rightarrow e^\pm c\bar{c} + X$</td>
<td>$\gamma^* c \rightarrow c, \gamma^* g \rightarrow c\bar{c}$</td>
<td>$c, g$</td>
<td>$0.0001 \lesssim x \lesssim 0.1$</td>
<td></td>
</tr>
<tr>
<td>$e^\pm p \rightarrow jet(s) + X$</td>
<td>$\gamma^* g \rightarrow q\bar{q}$</td>
<td>$g$</td>
<td>$0.01 \lesssim x \lesssim 0.1$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm {\bar{p}', \bar{d}', \bar{n}'} \rightarrow \ell^\pm + X$</td>
<td>$\gamma^* q \rightarrow q$</td>
<td>$\Delta q + \Delta q, \Delta g$</td>
<td>$0.003 \lesssim x \lesssim 0.8$</td>
<td></td>
</tr>
<tr>
<td>$pp \rightarrow \mu^+ \mu^- + X$</td>
<td>$u\bar{u}, d\bar{d} \rightarrow \gamma^*$</td>
<td>$\bar{q}$</td>
<td>$0.015 \lesssim x \lesssim 0.35$</td>
<td></td>
</tr>
<tr>
<td>$pn/pp \rightarrow \mu^+ \mu^- + X$</td>
<td>$\frac{(u\bar{d})}{(u\bar{u})} \rightarrow \gamma^*$</td>
<td>$d/\bar{u}$</td>
<td>$0.015 \lesssim x \lesssim 0.35$</td>
<td></td>
</tr>
<tr>
<td>$p\bar{p}(pp) \rightarrow jet(s) + X$</td>
<td>$gg, gq, qq \rightarrow 2 jets$</td>
<td>$g, q$</td>
<td>$0.005 \lesssim x \lesssim 0.5$</td>
<td></td>
</tr>
<tr>
<td>$p\bar{p} \rightarrow (W^\pm \rightarrow \ell^\pm \nu) + X$</td>
<td>$ud \rightarrow W^+, \bar{u}d \rightarrow W^-$</td>
<td>$u, d, \bar{u}, \bar{d}$</td>
<td>$x \gtrsim 0.05$</td>
<td></td>
</tr>
<tr>
<td>$pp \rightarrow (W^\pm \rightarrow \ell^\pm \nu) + X$</td>
<td>$ud \rightarrow W^+, \bar{d}u \rightarrow W^-$</td>
<td>$u, d, \bar{u}, \bar{d}, (g)$</td>
<td>$x \gtrsim 0.001$</td>
<td></td>
</tr>
<tr>
<td>$p\bar{p}(pp) \rightarrow (Z \rightarrow \ell^+ \ell^-) + X$</td>
<td>$uu, dd(u\bar{u}, d\bar{d}) \rightarrow Z$</td>
<td>$u, d(g)$</td>
<td>$x \gtrsim 0.001$</td>
<td></td>
</tr>
<tr>
<td>$pp \rightarrow (W^+ + c) + X$</td>
<td>$gs \rightarrow W^- c, g\bar{s} \rightarrow W^+ \bar{c}$</td>
<td>$s, \bar{s}$</td>
<td>$x \sim 0.01$</td>
<td></td>
</tr>
<tr>
<td>$pp \rightarrow t\bar{t} + X$</td>
<td>$gg \rightarrow t\bar{t}$</td>
<td>$g$</td>
<td>$x \sim 0.01$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm {\bar{p}, \bar{d}} \rightarrow \ell^\pm h + X$</td>
<td>$\gamma^* q \rightarrow q$</td>
<td>$\Delta u, \Delta \bar{u}, \Delta d, \Delta \bar{d}$</td>
<td>$0.05 \lesssim x \lesssim 0.4$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm {\bar{p}, \bar{d}} \rightarrow \ell^\pm D + X$</td>
<td>$gg \rightarrow qg, qg \rightarrow qg$</td>
<td>$\Delta g$</td>
<td>$0.05 \lesssim x \lesssim 0.4$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm {\bar{p}, \bar{d}} \rightarrow \ell^\pm h + X$</td>
<td>$\gamma^* g \rightarrow c\bar{c}$</td>
<td>$\Delta u, \Delta \bar{u}, \Delta d, \Delta \bar{d}$</td>
<td>$0.005 \lesssim x \lesssim 0.5$</td>
<td></td>
</tr>
<tr>
<td>$\ell^\pm {\bar{p}, \bar{d}} \rightarrow \ell^\pm h + X$</td>
<td>$\gamma^* g \rightarrow c\bar{c}$</td>
<td>$\Delta g$</td>
<td>$0.06 \lesssim x \lesssim 0.2$</td>
<td></td>
</tr>
</tbody>
</table>
Accessed kinematic coverage

\( \mathcal{O}(4000) \) data points after cuts

\[ Q_{\text{cut}}^2 = 1 \ \text{GeV}^2 \quad W_{\text{cut}}^2 = 3 - 12.5 \ \text{GeV}^2 \]

kinematic cuts: \( Q^2 \geq Q_{\text{cut}}^2 \) (pQCD) and \( W^2 = m_p^2 + \frac{1-x}{x} Q^2 \geq W_{\text{cut}}^2 \) (no HT)
# Unpolarised and polarised PDFs

## Unpolarised PDFs

<table>
<thead>
<tr>
<th></th>
<th>CT14</th>
<th>MMHT14</th>
<th>NNPDF3.1</th>
<th>ABM16</th>
<th>HERAPDF2.0</th>
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<tr>
<td>fixed-target DIS</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
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<td>HERA</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>fixed-target DY</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>Tevatron ($W, Z$)</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
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<tr>
<td>Tevatron (jets)</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>LHC</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
</tbody>
</table>


## Polarised PDFs

<table>
<thead>
<tr>
<th></th>
<th>DSSV</th>
<th>NNPDFpol11.1</th>
<th>JAM</th>
<th>LSS</th>
<th>BB</th>
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</thead>
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<tr>
<td>DIS</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>SIDIS</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>$pp$</td>
<td>(jets, $\pi^0$)</td>
<td>✔ (jets, $W^\pm$)</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
</tbody>
</table>


Most of them are publicly available through the LHAPDF interface [EPJC 75 (2015) 132]
https://lhapdf.hepforge.org/
Polarised and unpolarised PDFs

Largely benefit from HERA and LHC programs
Polarised and unpolarised PDFs

Are starting to benefit from JLAB and RHIC programs
1.2 Fragmentation Functions: the basics
Hadrons in the final state: Fragmentation Functions

FFs allow for a proper field-theoretic definition as matrix elements of bilocal operators

\[ D_i^h(z) = \frac{1}{12\pi} \sum_X \int dy^- e^{-i\frac{P^+}{z}y^-} \text{Tr} \left[ \gamma^+ \langle 0 | \psi(y) \mathcal{P} | h(P) X \rangle \langle h(P) X | \mathcal{P}' \bar{\psi}(0) | 0 \rangle \right] \]

with light-cone coordinates and appropriate gauge links \( \mathcal{P}, \mathcal{P}' \)

\[ y = (y^+, y^-, y_\perp), \quad y^+ = (y^0 + y^z)/\sqrt{2}, \quad y^- = (y^0 - y^z)/\sqrt{2}, \quad y_\perp = (v^x, v^y) \]

All these definitions have ultraviolet divergences which must be renormalized in order to define finite PDFs to be used in the factorization formulas (PDF/FFs are scheme dependent)

All these definitions can be generalised to include longitudinal/transverse polarizations

Factorisation of physical observables

1. A variety of sufficiently inclusive processes allow for a factorised description

   short-distance part
   hard interaction of partons
   process-dependent kernels

   factorisation
   scheme & scale $\mu$

   long-distance part
   nucleon structure
   universal parton distributions

2. Physical observables are written as a convolution of coefficient functions and FFs

   $O_I = \sum_{i=q,\bar{q},g} C_{Ii}(y, \alpha_s(\mu^2)) \otimes D_i(y, \mu^2) + \text{p.s. corrections}$

   $f \otimes g = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y)$

3. Coefficient functions allow for a perturbative expansion

   $C_{If}(y, \alpha_s) = \sum_{k=0} a_s^k C_{If}^{(k)}(y), \quad a_s = \alpha_s/(4\pi)$

4. After factorization, all quantities (including FFs/PDFs) depend on $\mu$
Evolution of FFs: DGLAP equations

1. A set of \((2n_f + 1)\) integro-differential equations, \(n_f\) is the number of active flavors

\[
\frac{\partial}{\partial \ln \mu^2} D_i(x, \mu^2) = \sum_{j}^{n_f} \int_{x}^{1} \frac{dz}{z} P_{ji}(z, \alpha_s(\mu^2)) \ D_j \left( \frac{x}{z}, \mu^2 \right)
\]

2. Often written in a convenient basis of PDFs

\[
D_{NS;\pm} = (D_q \pm D_{\bar{q}}) - (D_{q'} \pm D_{\bar{q}'}) \quad D_{NS;v} = \sum_{q}^{n_f} (D_q - D_{\bar{q}}) \quad D_{\Sigma} = \sum_{q}^{n_f} (D_q + D_{\bar{q}})
\]

\[
\frac{\partial}{\partial \ln \mu^2} D_{NS;\pm,v}(x, \mu^2) = P_{\pm,v}(x, \mu_F^2) \otimes D_{NS;\pm,v}(x, \mu^2)
\]

\[
\frac{\partial}{\partial \ln \mu^2} \left( \begin{array}{c} D_{\Sigma}(x, \mu^2) \\ D_g(x, \mu^2) \end{array} \right) = \left( \begin{array}{cc} P_{qq} & 2n_f P_{qg} \\ \frac{1}{2n_f} P_{qg} & P_{gg} \end{array} \right) \otimes \left( \begin{array}{c} D_{\Sigma}(x, \mu^2) \\ D_g(x, \mu^2) \end{array} \right)
\]

3. With perturbative computable (time-like) splitting functions

\[
P_{ji}(z, \alpha_s) = \sum_{k=0}^{k+1} a_s^{k+1} P_{ji}^{(k)}(z), \quad a_s = \alpha_s/(4\pi)
\]

\[
P_{qq}^{(0)} \quad P_{gg}^{(0)} \quad P_{qg}^{(0)} \quad P_{gg}^{(0)}
\]
Splitting functions: LO and NLO

**LO: 1973**

\[
P_{n^2}(x) = C_F \left(2p_{qq}(x) + 3\delta(1-x)\right)
\]

\[
P_{ps}(x) = 0
\]

\[
P_{s^2}(x) = 2n_f p_{s^2}(x)
\]

\[
P_{gs}(x) = 2C_F p_{gs}(x)
\]

\[
P_{gg}(x) = C_A \left(4p_{gg}(x) + \frac{11}{3}\delta(1-x)\right) - \frac{2}{3}n_f \delta(1-x)
\]

\[
P_{s^2}(1-x) = P_{n^2}(x) = 0
\]

\[
P_{ps}(1-x) = P_{s^2}(x) = 0
\]

\[
P_{gs}(1-x) = P_{gg}(x) = 2C_F p_{gs}(x)
\]

\[
P_{gg}(1-x) = C_A \left(4p_{gg}(x) + \frac{11}{3}\delta(1-x)\right) - \frac{2}{3}n_f \delta(1-x)
\]

**NLO: 1980**

\[
P_{n^2}(1-x) = 4C_F C_F \left(p_{qq}(x) \left[\frac{67}{18} - \xi_2 + \frac{11}{6}H_0 + H_{0,0}\right] + p_{qq}(-x) \left[\xi_2 + 2H_{-1,0} - H_{0,0}\right]
\]

\[
+ \frac{14}{3}(1-x) + 8(1-x)\right) + \frac{17}{24}(1-x) - 3\xi_3\right)\right) - 4C_F n_f p_{qq}(x) \left[p_{qq}(x) \left[\frac{5}{9} + \frac{1}{3}H_0\right] + \frac{2}{3}(1-x)
\]

\[
+ 8(1-x) - \frac{3}{12} + \frac{2}{3}\xi_3\right)\right) + 4C_F^2 \left[2p_{qq}(x) \left[H_{1,0} - \frac{3}{4}H_0 + H_2\right] - 2p_{qq}(-x) \left[\xi_2 + 2H_{-1,0}
\]

\[
- H_{0,0}\right] - (1-x) \left[1 - \frac{3}{2}H_0\right] - H_0 - (1+x)H_{0,0} + 8(1-x) \left[\frac{3}{8} - 3\xi_3 + 6\xi_4\right]
\]

\[
P_{n^2}(1-x) = P_{n^2}(1-x) + 16C_F C_F \left(p_{qq}(x) \left[\xi_2 + 2H_{-1,0} - H_{0,0}\right] - 2(1-x)
\]

\[
- (1+x)H_0\right)
\]

\[
P_{ps}(1-x) = 4C_F n_f p_{qq}(x) \left[p_{qq}(x) \left[\frac{20}{9} - 2 + 6x - 4H_0 + x^2 \left[\frac{8}{3}H_0 - \frac{56}{9}\right] + (1+x) \left[5H_0 - 2H_{0,0}\right]\right]
\]

\[
P_{gs}(1-x) = 4C_F n_f p_{gs}(x) \left[p_{gs}(x) \left[H_{1,0} - 2H_{-1,0} - 2p_{gs}(x)H_{1,1} + x^2 \left[\frac{44}{3}H_0 - \frac{218}{9}\right]
\]

\[
+ 4(1-x) \left[H_{0,0} - 2H_0 + xH_1\right] - 4\xi_2 + 6H_{0,0} + 9H_0\right] + 4C_F n_f p_{gs}(x) \left[H_{1,0} + H_{1,1} + H_2
\]

\[
- \xi_2\right] + 4x^2 \left[H_0 + H_{0,0} + \frac{5}{2}\right] + 2(1-x) \left[H_0 + H_{0,0} - 2xH_1 + \frac{29}{4}\right] - \frac{15}{2} - H_0 - \frac{1}{2}H_0\right)
\]

\[
P_{gg}(1-x) = 4C_A C_F \left[\frac{1}{x} + 2p_{gg}(x) \left[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6}H_1\right] - x^2 \left[\frac{8}{3}H_0 - \frac{44}{9}\right] + 4\xi_2 - 2
\]

\[
- 7H_0 + 2H_{0,0} - 2H_1 x + (1+x) \left[2H_{0,0} + 6H_0 + \frac{37}{9}\right] - 2p_{gg}(-x)H_{-1,0}\right] - 4C_F n_f p_{gg}(x) \left[\frac{2}{3}H_1 - \frac{10}{9}\right]\right) + 4C_F^2 \left(p_{gg}(x) \left[3H_1 - 2H_{1,1}\right] + (1+x) \left[H_{0,0} - \frac{7}{2} + H_0\right] - 3H_{0,0}
\]

\[
+ 1 - \frac{3}{2}H_0 + 2H_1 x\right)
\]

\[
P_{gg}(1-x) = 4C_A n_f \left(1 - x - \frac{10}{9}p_{gg}(x) - \frac{13}{9} \left(1 - x^2\right) - \frac{2}{3}(1+x)H_0 - \frac{2}{3} \delta(1-x)\right) + 4C_A^2 \left[27
\]

\[
+ (1+x) \left[\frac{11}{3}H_0 + 8H_{0,0} - \frac{27}{2}\right] + 2p_{gg}(-x) \left[H_{0,0} - 2H_{-1,0} - \xi_2\right] - \frac{67}{9} \left(1 - x^2\right) - 12H_0
\]

\[
- \frac{44}{3} \left(3\right)H_0 + 2p_{gg}(x) \left[\frac{67}{18} - \xi_2 + H_{0,0} + H_{2,1,0} + H_2\right] + 8(1-x) \left[\frac{8}{3} + 3\xi_3\right]\right) + 4C_F n_f \left[2H_0
\]

\[
+ \frac{21}{3} + 10 \left(x^2 - 12 + (1+x) \left[4 - 5H_0 - 2H_{0,0}\right]\right)
\]

\[
+ \frac{21}{3} + 10 \left(x^2 - 12 + (1+x) \left[4 - 5H_0 - 2H_{0,0}\right]\right)
\]

Emanuele R. Nocera (Oxford)  FFs and Global QCD Fits  June 11 2017  21 / 48
Splitting functions: NNLO
Properties of splitting functions

1. At LO:
   time-like and space-like splitting functions are equal, provided \( P_{qq}^{S,(0)} \leftrightarrow P_{qq}^{T,(0)} \)

   time-like and space-like splitting functions are related by analytic continuation

   an uncertainty still remains on the exact form of \( P_{qq}^{(2)} \) (it does not affect its log behavior)
Properties of splitting functions

1. At LO:
   time-like and space-like splitting functions are equal, provided $P_{qg}^S(0) \leftrightarrow P_{gq}^T(0)$

   time-like and space-like splitting functions are related by analytic continuation

   an uncertainty still remains on the exact form of $P_{qg}^{(2)}$ (it does not affect its log behavior)
Properties of splitting functions

Must be careful with fixed-order splitting functions as \( z \to 0 \) \((m = 1, \ldots, 2k + 1)\)

**SPACE-LIKE CASE**

\[
P_{ji} \propto \frac{a_s^{k+1}}{x} \log^{k+1-m} \frac{1}{x}
\]

**TIME-LIKE CASE**

\[
P_{ji} \propto \frac{a_s^{k+1}}{z} \log^{2(k+1)-m-1} z
\]

Soft gluon logarithms diverge more rapidly in the TL case than in the SL case: as \( z \) decreases, the unresummed SGLs spoil the convergence of the FO series for \( P(z, a_s) \) if \( \log \frac{1}{z} \geq O \left( a_s^{-1/2} \right) \)
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Soft gluon logarithms diverge more rapidly in the TL case than in the SL case: as $z$ decreases, the unresummed SGLs spoil the convergence of the FO series for $P(z, a_s)$ if $\log \frac{1}{z} \geq O \left( a_s^{-1/2} \right)$
Solving DGLAP equations

1. Transform to Mellin space (convolutions become ordinary products)

\[
\frac{\partial}{\partial \ln \mu^2} D_i(N, \mu^2) = \sum_j n_f \gamma_{ji} (N, \alpha_s(\mu^2)) D_j (N, \mu^2) 
\]

\[
\gamma_{ji}(N) = \int_0^1 z z^{N-1} P_{ji}(z)
\]

2. Solve evolution equations in Mellin space

\[
D_i(N, \mu^2) = \sum_j \Gamma_{ij} (N, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) D_j (N, \mu_0^2)
\]

3. The evolution kernels \( \Gamma_{ij} (N, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) \) satisfy the evolution equations

\[
\frac{\partial}{\partial \ln \mu^2} \Gamma_{ij}(N, \mu^2) = \sum_k n_f \gamma_{ik} (N, \alpha_s(\mu^2)) \Gamma_{kj} (N, \mu^2) 
\]

\[
\frac{d\alpha_s}{d \ln Q^2} = \beta(\alpha_s) = - \sum_{n=0}^{\infty} \alpha_s^{n+2} \beta_n
\]

\[
\frac{\partial}{\partial \alpha_s} \Gamma_{ij} (N, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) = - \sum_k R_{ik}(N, \alpha_s) \Gamma_{kj} (N, \alpha_s(\mu^2), \alpha_s(\mu_0^2))
\]

4. The perturbative solution of the matrix \( R_{ij} = R = R^{(0)} + \alpha_s R^{(1)} + \ldots \) is

\[
R^{(0)} = \frac{\gamma^{(0)}}{\beta_0} \quad R^{(k)} = \frac{\gamma^{(k)}}{\beta_0} - \sum_{i=1}^{k} \frac{\beta_i}{\beta_0} R^{(k-i)} 
\]

\[
\gamma = \sum_{n=0}^{\infty} \alpha_s^{n+1} \gamma^{(0)}
\]
Solving DGLAP equations: LO example

1. For the flavour nonsinglet and valence quark FFs

\[ \Gamma_{\text{NS,LO}} \left( N, \alpha_s(\mu^2), \alpha_s(\mu_0^2) \right) = \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} R_{\text{NS}}^{(0)} \]

2. For the flavour singlet, diagonalise the $2 \times 2$ $\mathbf{R}$ matrix

\[ \Gamma_{\text{S,LO}}(N, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) = e_+(N) \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\lambda_+(N)} + e_-(N) \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\lambda_-(N)} \]

\[ \lambda_{\pm}(N) = \frac{1}{2\beta_0} \left[ \gamma_{qq}^{(0)}(N) + \gamma_{gg}^{(0)}(N) \pm \sqrt{\left( \gamma_{qq}^{(0)}(N) - \gamma_{gg}^{(0)}(N) \right)^2 + 4\gamma_{gg}^{(0)}(N)\gamma_{qq}^{(0)}(N)} \right] \]

\[ e_{\pm}(N) = \pm \frac{1}{\lambda_+(N) - \lambda_-(N)} \left( R_{\text{S}}^{(0)}(N) - \lambda_{\mp}(N)I \right) \]

3. Determine the evolution factors $\Gamma_{ij}(z, \alpha_s(\mu^2), \alpha_s(\mu_0^2))$ by inverse Mellin transform

\[ \Gamma_{ij}(z, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) = \int_C \frac{dN}{2\pi i} x^{-N} \Gamma_{\text{NS}}(N, \alpha_s(\mu^2), \alpha_s(\mu_0^2)) \]

4. Use public codes which solve the evolution equations numerically

Theoretical constraints

1. Momentum sum rule

\[ \sum_h \int_0^1 dz z D_i^h(z, \mu^2) = 1 \quad \forall \text{ parton } i \]

2. Charge sum rule

\[ \sum_h \int_0^1 dz e_h D_i^h(z, \mu^2) = e_i \quad \forall \text{ parton } i \]

where \( e_{h(i)} \) is the electric charge of the hadron \( h \) (parton \( i \)).

3. Positivity of cross sections

implies that FFs should be positive-definite at LO.

4. Charge conjugation symmetry

\[ D_{q(q)}^{h^+} = D_{\bar{q}(q)}^{h^-} \quad D_g^{h^+} = D_g^{h^-} \]

5. Isospin symmetry of the strong interaction

\[ D_{u(u)}^{\pi^+} = D_{d(d)}^{\pi^-} \quad D_{d(d)}^{\pi^+} = D_{u(u)}^{\pi^-} \]

approximate, as \( m_u \sim m_d \), but no phenomenological evidences of violation.
1.3 SIA as a case study
SIA cross section: general structure

\[ e^+(k_1) + e^-(k_2) \xrightarrow{\gamma,Z^0} h(P_h) + X \]

\[ q = k_1 + k_2 \quad q^2 = Q^2 > 0 \quad z = \frac{2P_h \cdot q}{Q^2} \]

\[ \frac{d\sigma^h}{dz} = \mathcal{F}^h_T(z, Q^2) + \mathcal{F}^h_L(z, Q^2) = \mathcal{F}^h_2(x, Q^2) \]

\[ \mathcal{F}^h_{k=T,L,2} = \frac{4\pi \alpha^2_{\text{em}}}{Q^2} \langle e^2 \rangle \left\{ D^h_{\Sigma} \otimes C^{\Sigma}_{k,q} + n_f D^h_g \otimes C^{\Sigma}_{k,g} + D^h_{\text{NS}} \otimes C^{\text{NS}}_{k,q} \right\} \]

\[ \langle e^2 \rangle = \frac{1}{n_f} \sum_{p=1}^{n_f} \hat{e}^2_p \quad D^h_{\Sigma} = \sum_{p=1}^{n_f} \left( D^h_p + D^h_{\bar{p}} \right) \quad D^h_{\text{NS}} = \sum_{p=1}^{n_f} \left( \frac{\hat{e}^2_p}{\langle e^2 \rangle} - 1 \right) \left( D^h_p + D^h_{\bar{p}} \right) \]

\[ \hat{e}^2_p = e^2_p - 2e_p \chi_1(Q^2) v_e v_p + \chi_2(Q^2)(1 + v^2_e)(1 + v^2_p) \]

\[ \chi_1(s) = \frac{1}{16 \sin^2 \theta_W \cos^2 \theta_W} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + \Gamma^2_Z M_Z^2} \]

\[ \chi_2(s) = \frac{1}{256 \sin^4 \theta_W \cos^4 \theta_W} \frac{s^2}{(s - M_Z^2)^2 + \Gamma^2_Z M_Z^2} \]
1. Quark and antiquark FFs always appear through the combination $D_{q^+} = D_q + D_{\bar{q}}$
   $\rightarrow$ no separation between quark and antiquark FFs

2. The leading contribution to $C_{S_{2,g}}^g$ is $\mathcal{O}(\alpha_s)$
   $\rightarrow$ direct sensitivity to $D_g$ only beyond LO
   $\rightarrow$ indirect sensitivity to $D_g$ via DGLAP evolution

3. The effective electroweak charges $\hat{e}_q$ depend on the energy scale
   $\rightarrow$ $\hat{e}_q^2 / \langle e^2 \rangle (M_Z) \sim 1$; $\hat{e}_q^2 / \langle e^2 \rangle (10 \text{ GeV}) \gg 1$
   $\rightarrow$ partial flavour flavour separation

4. Measurements are often multiplicities, i.e. cross sections normalised to $\sigma_{tot}$

   \[
   \sigma_{tot}(Q) = \frac{4\pi \alpha_s(Q)}{Q^2} \left( \sum_{q} \hat{e}_q^2(Q) \right) \sum_{n=1} \left( \frac{\alpha_s(Q)}{\pi} C_n \left( \frac{s}{\mu^2} \right) \right)^n
   \]

   with coefficients

   \[ C_1(1) = 1 \]

   \[ C_2(1) \approx 1.986 - 0.115n_f \]

   \[ C_3(1) \approx -6.637 - 1.200n_f - 0.005n_f^2 - 1.240 \left( \sum_{q} \hat{e}_q^2 \right)^2 / \sum_{q} \hat{e}_q^2 \]
Experimental data

\begin{align*}
\frac{d\sigma^h}{dz} &= \frac{4\pi\alpha^2(Q^2)}{Q^2} F_2^h(z, Q^2) \quad h = \pi^+ + \pi^-, K^+ + K^-; \quad \text{possibly normalised to } \sigma_{\text{tot}}
\end{align*}
Scaling violations

As the scale increases, multiplicities are shifted towards lower values.
Flavour schemes

1. Assumption: the final-state hadron has no intrinsic heavy quark component

2. Fixed-flavour number scheme (FFNS)
   the only quarks treated as partons are the $n_L$ light quarks
   \[
   \mathcal{F}_2^{(n_L)}(z, Q^2, m_h^2) = \mathcal{F}_2^{L,(n_L)}(z, Q^2) + \mathcal{F}_2^{H,(n_L)}(z, Q^2, m_h^2)
   \]
   \[
   \mathcal{F}_2^{L,(n_L)}(z, Q^2) = \sum_{i} C_{2,i}^{L,(n_L)} \left(z, \frac{Q^2}{\mu^2} \right) \otimes D_{i}^{(n_L)}(z, \mu^2)
   \]
   \[
   \mathcal{F}_2^{H,(n_L)}(z, Q^2, m_h^2) = \sum_{i} C_{2,i}^{H,(n_L)} \left(z, \frac{Q^2}{m_h^2}, \frac{m_h^2}{m_h^2} \frac{Q^2}{\mu^2} \right) \otimes D_{i}^{(n_L)}(z, \mu^2)
   \]
   is accurate in the quark mass threshold region and below
suffers from unresummed $\ln(Q^2/m_h^2)$ in the Wilson coefficients, large for $Q^2 \gg m_h^2$

3. Zero-mass variable flavour number scheme (ZM-VFNS)
   treat heavy quarks as massless partons below their threshold
   introduce a heavy quark FF
   the renormalisation of the FF resums logs due to parton splitting via DGLAP
   \[
   \mathcal{F}_2^{(n_L+1)}(z, Q^2) = \sum_{i} C_{2,i}^{(n_L+1)} \left(z, \frac{Q^2}{\mu^2} \right) \otimes D_{i}^{(n_L+1)}(z, \mu^2)
   \]
   the reliability of the ZM-VFNS is reduced where powers of $m_h^2/Q^2$ are significant
Changing the number of active flavours

1. All partons with mass $m < \mu = \sqrt{s}$ must be assigned as active → even though they are not active at $\mu = \mu_0$ → the number of active partons depend on the scale of the process

2. Determine the matching conditions between a scheme with $n_f$ and $n_f + 1$ flavours

3. Consider the production of a light hadron $h$

$$
\frac{d\sigma}{dz} = \sum_{i \in \mathbb{I}_{n_L}} D^{(n_L)}_i(z, \mu^2) \otimes \frac{d\hat{\sigma}^{(n_L)}_i(z, \mu^2)}{dz} + D^{(n_L)}_g(z, \mu^2) \otimes \frac{d\sigma_{h\bar{h}g}(z)}{dz}
$$

→ the first term involves the FF and the $\hat{\sigma}_i$ for all partons $i$ excluding heavy flavours
→ the second term is the $O(\alpha_s)$ heavy-flavour contribution

$$
\frac{d\sigma_{h\bar{h}g}(z)}{dz} = \sigma_{h\bar{h}} \frac{\alpha_s}{2\pi} C_F 2 \frac{1 + (1 - y)^2}{y} \left\{ \ln \frac{s}{m_h^2} + \ln(1 - y) - 1 \right\}
$$

4. In the $\overline{\text{MS}}$ scheme, treating all flavour as massless

$$
\frac{d\sigma}{dz} = \sum_{i \in \mathbb{I}_n} D^{(n)}_i(z, \mu^2) \otimes \frac{d\hat{\sigma}^{(n)}_i(z, \mu^2)}{dz} \quad \mathbb{I}_n = \mathbb{I}_{n_L} \cup \{h, \bar{h}\}$$
Changing the number of active flavours

The difference between the two cross sections should vanish

\[ 0 = \sum_{i \in \mathbb{1}, i \neq g} \left[ D^{(n)}_i(z, \mu^2) - D^{(nL)}_i(z, \mu^2) \right] \otimes \frac{d\hat{\sigma}_i(z, \mu^2)}{dx} \]

\[ + \left[ D^{(n)}_g(z, \mu^2) \otimes \frac{d\hat{\sigma}^{(n)}(z, \mu^2)}{dz} - D^{(nL)}_g(z, \mu^2) \otimes \frac{d\hat{\sigma}^{(nL)}(z, \mu^2)}{dz} \right] \]

\[ + \sum_{i \in \{h, \bar{h}\}} D^{(n)}_i(z, \mu^2) \otimes \frac{d\hat{\sigma}(z, \mu^2)}{dz} - D^{(nL)}_i(z, \mu^2) \otimes \frac{d\sigma_{h\bar{h}g}(z)}{dz} \]

\( \hat{\sigma}^{(n)}_i \) and \( \hat{\sigma}^{(nL)}_i \) replaced with \( \hat{\sigma}_i \), since these cross sections differ by terms \( \mathcal{O}(\alpha_s^2) \)

\( \hat{\sigma}_g \) is \( \mathcal{O}(\alpha_s) \), hence \( D^{(nL)}_g = D^{(n)}_g \) (they differ by \( \mathcal{O}(\alpha_s) \))

\( d\hat{\sigma}^{(n)}_g(z, \mu^2)/dz - d\hat{\sigma}^{(nL)}_g(z, \mu^2)/dz = d\hat{\sigma}_{h\bar{h}g}(z, \mu^2)/dz \) (massless \( \overline{\text{MS}} \))

\( D^{(n)}_{h/\bar{h}} \) are \( \mathcal{O}(\alpha_s) \), so that only the Born hard cross section is needed

\[ \frac{d\hat{\sigma}_{h/\bar{h}}(z, \mu^2)}{dz} - \sigma_{h\bar{h}} \delta(1 - z) + \mathcal{O}(\alpha_s) \]

\[ \frac{d\hat{\sigma}_{h\bar{h}g}(z, \mu^2)}{dz} = \sigma_{h\bar{h}} \frac{\alpha_s}{2\pi} CF \frac{1 + (1 - z)^2}{z} \left\{ 2 \ln z + \ln(1 - z) + \ln \frac{Q^2}{\mu^2} \right\} \]
The matching conditions for the light and heavy quarks are

\[ D^{(n)}_i(z, \mu^2) = D^{(nL)}_i(z, \mu^2) \quad \text{for } i \in \mathbb{I}_{nL}, \ i \neq g \]

\[ D^{(n)}_h(z, \mu^2) = D^{(n)}_h(z, \mu^2) = D_g(z, \mu^2) \otimes \frac{\alpha_s}{2\pi} C_F \frac{1 + (1 - z)^2}{z} \left[ \ln \frac{\mu^2}{m_h^2} - 1 - 2 \ln z \right] \]

Similarly one can obtain the matching condition for the gluon

\[ D^{(n)}_g(z, \mu^2) = D^{(nL)}_g(z, \mu^2) \left( 1 - \frac{T_F \alpha_s}{3\pi} \ln \frac{\mu^2}{m_h^2} \right) \]

Matching conditions

\[ \rightarrow \text{for FFs are non-zero at NLO (at variance with PDFs)} \]

\[ \rightarrow \text{can be used to generate radiatively charm and bottom contributions to } F^h_2 \]

\[ \rightarrow \text{are required when evolving the perturbative charm FF through the bottom threshold} \]

In practice

\[ \rightarrow \text{the FF heavy-quark intrinsic component (non-vanishing if not active) is non-negligible} \]

\[ \rightarrow \text{one usually treats (discontinuously) heavy-quark FFs as unknown functions} \]

\[ \rightarrow \text{inconsistency: heavy-quark FF effects should not depend on the flavour scheme} \]

\[ \rightarrow \text{alleviated when the scale of the process is above the heavy quark threshold} \]
General-mass flavour number scheme

1. Combine a FFN scheme (at $Q \lesssim m_h$) with a ZM-VFN scheme (at $Q \gg m_h$)
   a GM-VFN scheme operates as a tower of FFN schemes increasing $n_L$ as the scale increases over each quark mass threshold

2. Demand that physical observables are continuous across the thresholds

$$\mathcal{F}_{2}^{GM}(z, m_{h}^{2}, \mu^{2}) = \sum_{j}^{n_{L}} C_{2,j}^{GM}(n_{L})(z, m_{h}^{2}, \mu^{2}) \otimes D_{j}^{(n_{L})}(z, \mu^{2})$$

$$= \sum_{i}^{n_{L}+1} C_{2,i}^{GM}(n_{L}+1)(z, m_{h}^{2}, \mu^{2}) \otimes D_{j}^{(n_{L}+1)}(z, \mu^{2})$$

3. Demand the matching conditions for the FFs

$$D_{i}^{(n_{L}+1)}(z, \mu^{2}) = \sum_{j}^{n_{L}} A_{ij}^{(n_{L})}(\mu^{2}/m_{h}^{2}) \otimes D_{j}^{(n_{L})}(z, \mu^{2})$$

4. Combine the two to obtain the condition

$$C_{2,j}^{GM}(n_{L})(z, m_{h}^{2}, \mu^{2}) = \sum_{i}^{n_{L}+1} C_{i,2}^{GM}(n_{L}+1)(z, m_{h}^{2}, \mu^{2}) \otimes A_{i,j}^{(n_{L})}(\mu^{2}/m_{h}^{2})$$

5. Ensure that the condition above is satisfied at all orders
Comparing various flavour schemes

\[ \frac{d\sigma^{ZM}}{dz} = \sum_{i=q,g,h} \hat{\sigma}_i^{ZM}(Q) \otimes D_i^{ZM}(Q) \]

\[ \frac{d\sigma^{GM}}{dz} = \sum_{i=q,g,h} \hat{\sigma}_i^{GM}(Q, m_h) \otimes D_i^{GM}(Q) \]

\[ \frac{d\sigma^{M}}{dz} = \sum_{i=q,g} \hat{\sigma}_i^{M}(Q, m_h) \otimes D_i^{M}(Q) + \hat{\sigma}_h^{M}(Q, m_h) \otimes D_h^{M} \]
Impact of higher-order QCD corrections

\[ e^+e^- \rightarrow \pi X \text{ K-factor} \]
computed with NLO FFs

\begin{align*}
Q = 10.5 \text{ GeV} & \quad & Q = 10.5 \text{ GeV} \\
\text{d}\sigma^\pi (\text{NNLO}) / \text{d}\sigma^\pi (\text{NLO}) & \quad & \text{d}\sigma^\pi (\mu=Q) / \text{d}\sigma^\pi (\mu=Q) \\
Q = 29.0 \text{ GeV} & \quad & Q = 29.0 \text{ GeV} \\
\text{d}\sigma^\pi (\text{NLO}) / \text{d}\sigma^\pi (\text{LO}) & \quad & \\
Q = 91.2 \text{ GeV} & \quad & Q = 91.2 \text{ GeV} \\
\end{align*}

\textit{K-factor smaller at NNLO/NLO than at NLO/LO}

indication of large unresummed logarithms at small and large values of \( z \)

renormalization scale variations:
estimate of missing higher-order corrections at most \( \sim 5\% \) at NNLO at the lowest \( Q \)
thetical uncertainties under control
1.4 Other processes
SIDIS cross section

\[ \ell(m) + p(P) \xrightarrow{\gamma} \ell(m') + h(p_h) + X \]

\[ \frac{d\sigma^h}{dx dy dz} = 2\pi \alpha_{em}^2 \left[ \frac{1 + (1 - y)^2}{y} 2F_1^h + \frac{2(1 - y)}{y} F_L^h \right] \]

\[ 2F_1^h = \sum_{q, \bar{q}} e_q^2 \left\{ q \otimes D_q^h + \frac{\alpha_s}{2\pi} \left[ q \otimes C_{qq}^1 \otimes D_q^h + q \otimes C_{gq}^1 \otimes D_g^h + g \otimes C_{qq}^1 \otimes D_q^h \right] \right\} \]

\[ F_L^h = \frac{\alpha_s}{2\pi} \sum_{q, \bar{q}} e_q^2 \left[ q \otimes C_{qq}^L \otimes D_q^h + q \otimes C_{gq}^L \otimes D_g^h + g \otimes C_{qq}^L \otimes D_q^h \right] \]

separate information on \( D_q \) and \( D_{\bar{q}} \), no direct information on \( D_g \)
additional convolution with a PDF
The quantity usually measured by experiments is the SIDIS multiplicity

\[ M^h(x, z, Q) = \frac{d\sigma^h/dx dy dz}{d\sigma^h_{DIS}/dx dy} \]

Define two regions in the Breit frame (the frame in which the virtual photon energy vanishes and its spatial momentum is antiparallel with the initial proton’s momentum)

- **current fragmentation region**: \( \theta < \pi/2 \)
- **target fragmentation region**: \( \theta > \pi/2 \)

\( \theta \) (angle between the spatial momentum of the detected hadron and the virtual photon)

**fracture functions** describe the partonic structure of the initial hadron after it has produced the detected hadron \( (p \to h + i + X) \)

Fracture functions do not contribute to the cross section when the direction of the detected hadron’s momentum is within the current fragmentation region.
COMPASS multiplicities: pions [PLB 764 (2017) 1]

3-D kinematic binning in $x$, $y$, and $z$ (317 kinematic bins)
very weak $y$ dependence, strong $z$ dependence
COMPASS multiplicities: pions

3-D kinematic binning in $x$, $y$, and $z$ (317 kinematic bins)
very weak $y$ dependence, strong $z$ dependence
COMPASS multiplicities: kaons

3-D kinematic binning in $x$, $y$, and $z$ (317 kinematic bins)
very weak $y$ dependence, strong $z$ dependence
COMPASS multiplicities: kaons

3-D kinematic binning in $x$, $y$, and $z$ (317 kinematic bins)
very weak $y$ dependence, strong $z$ dependence
Hadron production in $pp$ collisions

$$p(p_a) + p(p_b) \rightarrow h(p_h) + X$$

$$E_h \frac{d^3 \sigma}{dp_h^3} = \sum_{a,b,c} f_a \otimes f_b \otimes \hat{\sigma}_{ab}^c \otimes D_h^c$$

$$= \sum_{i,j,k,l} \int \frac{dx_a}{x_a} \int \frac{dx_b}{x_b} \int \frac{dz}{z^2} f^{i/p_a}(x - a) f^{j/p_b} D_{h/k}^k(z) \hat{\sigma}_{ij}^{kl} \delta(\hat{s} + \hat{t} + \hat{u})$$

can verify and improve constraints on the charge-sign and flavour separation of quark FFs provided by SIA and SIDIS

constrain gluon FFs significantly better than data from SIA and SIDIS, owing to the occurrence of the gluon FF at LO in the calculation

the detected hadron $h$ will sometimes be a soft remnant from one of the initial state hadrons (as in SIDIS)

a current fragmentation region analogous to SIDIS is free of initial hadron remnants depending on the kinematical region, uncertainties from the PDFs can be sizeable

available data sets are in a kinematical regime where higher order perturbative corrections (possibly requiring resummation) and/or higher twist effects are large
Hadron production in $pp$ collisions

$E d^3\sigma/dq^3$ [pb/GeV$^2$]

- WA70 ($\sqrt{s} = 23$ GeV)
- E706 ($\sqrt{s} = 31.5$ GeV)
- AFS ($\sqrt{s} = 63$ GeV)
- R806 ($\sqrt{s} = 62.8$ GeV)
- PHENIX ($\sqrt{s} = 62$ GeV)
- PHENIX ($\sqrt{s} = 200$ GeV)
- UA2 ($\sqrt{s} = 540$ GeV)
- ALICE ($\sqrt{s} = 7$ TeV) $\times 0.1$

$(h^+ + h^-)/2$

- UA1 ($\sqrt{s} = 200$ GeV)
- UA1 ($\sqrt{s} = 500$ GeV)
- UA1 ($\sqrt{s} = 900$ GeV)
- CDF ($\sqrt{s} = 630$ GeV)
- CDF ($\sqrt{s} = 1800$ GeV)
- CMS ($\sqrt{s} = 7$ TeV)
1.4 Summary of Lecture 1
Summary

1. Parton Distribution Functions are required in factorisation formulas
   → if hadrons (typically nucleons) are prepared in the initial state
   → unpolarised/polarised PDFs required depending on their polarisation
   → unpolarised/polarised PDFs need to be determined from the data

2. Fragmentation Functions are required in factorisation formulas
   → if hadrons are identified in the final state
   → FFs are the time-like counterparts of PDFs
   → inherit from PDFs evolution equations
   → FFs need to be determined from the data

3. Variety of processes used to access FFs
   → single hadron production in SIA, SIDIS and $pp$ collisions
   → SIA remains the theoretically cleanest process among the three
   → SIA is also the less challenging, as it does not involve PDFs

4. Need to put together all this information in a global QCD analysis
   → devise a suitable fitting methodology
   → parametrisation, representation of uncertainties, fit validation
   → fitting methodology valid for both PDFs and FFs