Lattice QCD determination of quark masses and $\alpha_s$

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Quark masses and strong coupling are fundamental parameters of the SM but cannot be directly determined from experiment because we do not have direct access to quarks.

Well-defined $m_q$ and $\alpha_s$ are scheme and scale-dependent. Convention is to use $\overline{MS}$. Compare results from multiple approaches for strong test of QCD. Lattice QCD methods are particularly accurate.

Masses/$\alpha_s$ are input to theoretical expressions for SM cross-sections e.g. $H \rightarrow c\bar{c}$.
Lattice QCD: fields defined on 4-d discrete space-(Euclidean) time. Lagrangian parameters: $\alpha_s, m_q a$

1) Generate sets of gluon fields for Monte Carlo integrn of Path Integral (inc effect of u, d, s, (c) sea quarks)

2) Calculate valence quark propagators and combine for “hadron correlators” . Fit for hadron masses and amplitudes

- Determine $a$ to convert results in physical units. Fix $m_q$ from hadron mass

*numerically extremely challenging*

- cost increases as $a \to 0, m_{u/d} \to \text{phys}$ and with statistics, volume.
Can tune bare lattice QCD mass parameters very accurately using experimentally very well-determined ground-state meson masses.
Mass parameters in Lattice QCD Lagrangian can be tuned very accurately against experimental hadron masses.

**Issue is:**

**Conversion of lattice quark masses to \( \overline{MS} \) scheme**

\[
m_{\overline{MS}}(\mu) = Z_m(\mu a)m_{latt}
\]

**Options to calculate \( Z \):**

1) **lattice QCD pert. th.** - hard to do beyond NLO
2) **Nonperturbative calculation of a quantity that can be matched to \( \overline{MS} \) using continuum QCD pert. theory**

* Error dominated by that of \( Z \)

Note: \( Z \) cancels in mass ratios, which are *completely nonperturbative* in lattice QCD in a given quark formalism. Provides critical test of procedure above.
Lattice QCD: determining quark masses and providing non-perturbative tests of the determination

$m_c, m_b$

current-current correlators

$m_c$

$m_s$
determined directly from lattice QCD

$m_s, m_u/m_d$

Green’s function in RI-SMOM scheme
Current-current correlator method for $m_c$ (and $m_b$)

Time-moments of lattice QCD correlators extrapolated to the continuum limit can be related to $s^{-1}$-moments of $R_{e^+e^-}$ and to continuum QCD perturbation theory known through $\alpha_s^3$ (NNNLO) e.g. Kuhn et al, hep-ph/0702103

\[
G_n = \sum_t (t/a)^n G(t)
\]

\[ n = 4, 6, 8, 10 \ldots \]

\[ n = 2k + 2 \]

\[
R_{e^+e^-}(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha^2/(3s)}
\]

\[
\mathcal{M}_k \equiv \int \frac{ds}{s^{k+1}} R_c(s)
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{\(R(s)\) for different energy intervals around the charm threshold region. The solid line corresponds to the theoretical prediction, the uncertainties obtained from the variation of the input parameters and of $\mu$ are indicated by the dashed curves. The inner and outer error bars give the statistical and systematical uncertainty, respectively.}
\end{figure}
Continuum QCD perturbation theory for the moments is a function of quark mass and known through $\alpha_s^3$

In lattice QCD can calculate moments not available to expt. e.g. for pseudoscalar density correlator for c quarks:

$$R_{n,latt} = \frac{G_4}{G_4^{(0)}} \quad n = 4$$

ratio to results with no gluon field improves disc. errors

$$= \frac{am_{\eta_c}}{2am_c} \left( \frac{G_n}{G_n^{(0)}} \right)^{1/(n-4)} \quad n = 6, 8, 10 \ldots$$

$$R_{n,cont} = \frac{m_{\eta_c}}{2m_c(\mu)} \frac{C_P^k}{C_{P,0}^k}$$

$$\frac{C_P^k}{C_{P,0}^k} = 1 + \sum c_i \alpha_s^i(\mu)$$

simultaneous fit to multiple moments - gives $\alpha_s, m_c$

HPQCD + Chetyrkin et al, 0805.2999, C. Mcneile et al, HPQCD,1004.4285
• Repeat calcn for \( m_q \geq m_c \) inc. ultrafine lattices

Agrees well with contnm results using \( R_{e+e^-} \)

Can determine \( m_h / m_{\eta_h} \) for heavy quarks - extrapolate

(slightly) to b.

\[
\overline{m}_b^{n_f=5}(\overline{m}_b) = 4.164(23)\text{GeV}
\]

key error is now extrapln in a
Example error budget for HISQ current-current method

TABLE IV. Error budget [31] for the $c$ mass, QCD coupling, and the ratios of quark masses $m_c/m_s$ and $m_b/m_c$ from the $n_f = 4$ simulations described in this paper. Each uncertainty is given as a percentage of the final value. The different uncertainties are added in quadrature to give the total uncertainty. Only sources of uncertainty larger than 0.05% have been listed.

<table>
<thead>
<tr>
<th>Source of Uncertainty</th>
<th>$m_c(3)$</th>
<th>$\alpha_{\text{MS}}(M_Z)$</th>
<th>$m_c/m_s$</th>
<th>$m_b/m_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perturbation theory</td>
<td>0.3</td>
<td>0.5</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Statistical errors</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$a^2 \rightarrow 0$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\delta m_{\text{uds}}$ $\rightarrow 0$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\delta m_{c}^{\text{sea}}$ $\rightarrow 0$</td>
<td>0.3</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$m_h \neq m_c$ (Eq. (15))</td>
<td>0.1</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Uncertainty in $w_0$, $w_0/a$</td>
<td>0.2</td>
<td>0.0</td>
<td>0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>$\alpha_0$ prior</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Uncertainty in $m_{\eta_s}$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.4</td>
<td>0.0</td>
</tr>
<tr>
<td>$m_h/m_c \rightarrow m_b/m_c$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.4</td>
</tr>
<tr>
<td>$\delta m_{\eta_c}$: electromag., annihil.</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$\delta m_{\eta_b}$: electromag., annihil.</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Total:</td>
<td><strong>0.64%</strong></td>
<td><strong>0.63%</strong></td>
<td><strong>0.55%</strong></td>
<td><strong>1.20%</strong></td>
</tr>
</tbody>
</table>

HPQCD, 1408.4169
**mc summary**

Good consistency between lattice actions using JJ method

- **lattice n_f=4**
- **PDG evaluation**

flavours of sea quarks inc. on lattice, adjust result to 4 using pert. th.

- HPQCD HISQ n_f = 4, JJ [1408.4169]
- ETMC n_f = 4, RI-mom [1403.4504]
- HotQCD HISQ n_f = 3, JJ [1606.08798]
- JLQCD domain-wall n_f = 3, JJ [1511.09163]
- $\chi$QCD overlap n_f = 3, RI-mom [1410.3343]
- HPQCD HISQ n_f = 3, JJ [1004.4285]

$\overline{m_c}(\overline{m_c}, n_f = 4) = 1.2715(95) \text{ GeV}$
**m_b summary**

Several different methods here. Good consistency between different methods and b-quark formalisms.

**Lattice average:**

$$\overline{m}_b(\overline{m}_b, n_f=5) \text{ (GeV)}$$

**PDG evaluation**

- **Lattice average:** 4.178(14) GeV

- **flavours of sea quarks** inc. on lattice. Use pert. th. to inc. to 5

- **HPQCD NRQCD JJ** $n_f=4$ [1408.5768]

- **HPQCD HISQ JJ** $n_f=3$ [1004.4285]

- **HPQCD HISQ ratio** $n_f=4$ [1408.4169]

- **HPQCD NRQCD $E_0$** $n_f=3$ [1302.3739]

- **ETMC ratio method** $n_f=4$ [1603.04306]
Agrees with that from current-current correlator method - test of pert. th.

\[
\left( \frac{m_{q_1, \text{latt}}}{m_{q_2, \text{latt}}} \right)_{a=0} = \frac{m_{q_1, \overline{MS}(\mu)}}{m_{q_2, \overline{MS}(\mu)}}
\]

completely nonperturbative determination of ratio gives:

\[
\frac{m_b}{m_c} = 4.541(26)
\]

see also: HPQCD, 1004.4285; ETM, 1603.04306; HotQCD, 1606.08798
Mass ratio can be obtained directly from lattice QCD if the same quark formalism is used for both quarks. Not possible with any other method ...

\[ \frac{m_c}{m_s} = 11.652(65) \]

**summary from hotQCD, 1606.08798**

- **N_f = 2 + 1 + 1**
  - HPQCD'15
- **N_f = 2 + 1**
  - MILC'14
  - ETMC'14
  - this paper
  - \( \chi \text{QCD}'15 \)
- **N_f = 2**
  - HPQCD'10
  - ETMC'10
  - Durr'12
Combining $m_c$ and $m_c/m_s$ leads to 1% accuracy in $m_s$ - also compare to RI-SMOM scheme determination

\[
\overline{m}_s(3\text{GeV}, n_f = 3) = 84.1(5)\text{MeV}
\]

- HPQCD 1408.4169  mc/ms+mc
- ETMC 1403.4504  RI-MOM
- RBC/UKQCD 1411.7017  RI-SMOM
- Durr et al 1011.2403  RI-MOM
- HPQCD 0910.3102  mc/ms+mc
- HPQCD (pert) 0511160  lattice pert
- Also hotQCD:1606.08798  83.6(1.5) MeV
Alternative method to determine (light) quark masses: RI-SMOM scheme

Impose a ‘MOM’ renormalisation scheme directly on the lattice, i.e. fix an amputated vertex function to its tree-level value (in Landau gauge).

Match to $\overline{MS}$ perturbatively - $\alpha_s^2 = \text{NNLO}$

Important improvement:

‘Non-exceptional’ kinematics:

$$p_1^2 = p_2^2 = (p_1 - p_2)^2 = \mu^2$$

much smaller systematic errors from non-pert effects AND pert. matching

RBC/UKQCD, 0712.1061

Sturm et al, 0901.2599;
Gorbahn, Jäger, 1004.3997
Calculate lattice $Z_m$ - multiply by tuned lattice bare mass and pert. matching to $\overline{MS}$

Agrees well with result expected from JJ method for $m_c$ and $m_c/m_s$

HPQCD, Lytle et al, preliminary, 1511.06547
Lattice QCD determination of $m_s/m_{ud}$ requires consideration of em effects via charged/neutral $\pi/K$

$$m_{ud} = \frac{m_u + m_d}{2}$$

$$\frac{m_s}{m_{ud}} = \text{LO chi-PT}$$

$$\frac{M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2}{M_{\pi^+}^2} = 25.9$$

Summary from 1607.00299

PDG average dominated by lattice = 27.3(7)
Lattice QCD determination of $\alpha_s$

Lattice QCD Lagrangian has parameter $g^2 = \text{lattice scheme coupling at scale } \pi/a$ - determination of $a$ fixes the coupling.

However, again it is conversion to $\overline{MS}$ which is issue.

\[ Q = a_0 + a_1 \alpha_s(\mu) + a_2 \alpha_s(\mu)^2 + \ldots \]

Calculate in lattice QCD. Could be e.g. continuum limit of 4th moment of JJ correlator. Minimal experl uncty

Choice of $\mu$ depends on Q e.g. $\sim 3m_h$ in JJ. Fixing it requires determination of $a$ using e.g. a hadron mass.

Perturbative expansion - using continuum QCD pert. th. in $\overline{MS}$ if Q is cont. quantity. Needs to be high-order.
So far, only one analysis is available which involves the determination of the fine structure constant \( \alpha_s(M_Z, n_f = 5, M_Z) \), PDG

\[
\alpha_s(M_S, n_f = 5, M_Z), \text{ PDG} = 0.1181(11)
\]

Lattice results are most precise; several different methods contribute.

**Figure 9.2:** Quantum chromodynamics

small Wilson loops

JJ

small Wilson loops

Schro. functional
ghost-gluon vertex

static quark potential

Effect of inc. pert. theory order in JJ method
Conclusions

Lattice QCD results available from multiple quark formalisms and methods now. - good consistency

\( m_c(m_c) \) is determined to 1% and 
\( m_b(m_b) \) to 0.5% from continuum and lattice methods.

\( \alpha_s(M_{Z}) \) to 1% from lattice - multiple methods

1% accurate \( m_c/m_s \) ratio allows 1% in \( m_s \) also, along with RI-SMOM methods

Tests of perturbation theory from completely non-perturbative mass ratios and JJ/RI-SMOM comparison

Future improvements from higher order pert th. (?possible) and finer lattices to push up \( \mu \) values.
Backup slides
Current-current correlator method for lattice $m_c$
HPQCD + Chetyrkin et al, 0805.2999, C. Mcneile et al, HPQCD,1004.4285

- Substitute time-moment of lattice charmonium correlator for experiment. In principle can use any current J now.
- For HISQ quarks pseudoscalar $\eta_c$ correlator is most accurate. J is absolutely normalised.

**step 1:** calculate $\eta_c$ correlators by combining lattice charm quark propagators

**step 2:** large time - fit to exponential, gives $\eta_c$ mass

**step 3:** tune lattice quark mass so $\eta_c$ mass correct.

**step 4:** calculate time moments to compare to QCD pert. theory. Emphasises short-time contribns.
Further check of JJ method:
compare vector moments (after normalising current) to those extracted from 
\[ R_{e^+e^-} \]
Agreement is a 1% test of (lattice) QCD.
Also gives charm quark contribution to anomalous magnetic moment of the muon
\[ \alpha^c_\mu = 14.4(4) \times 10^{-10} \]

HPQCD, 1208.2855, 1403.1778

see also ETMC, 1111.5252
Alternative determinations of $m_b$  

Current-current correlator method using vector bottomonium correlators calculated with improved NRQCD $b$ quarks

$n_f = 2+1+1$ HISQ sea quarks

$n_f = 5$ HISQ sea quarks

Larger moment numbers more nonrelativistic - use 18
Update and improved method

Use improved $n_f = 2+1+1$ gluon field configs, more accurate lattice spacing determination etc etc.

Determine $m_c$ at higher scales by using multiple $m_h$

$$
\tilde{R}_n = \frac{1}{m_c} \left( \frac{G_n}{G_n^{(0)}} \right)^{1/(n-4)}
$$

$tuned \ m_c$  $G_n$ at $m_h$  $\rightarrow \frac{1}{m_c(\mu)} \frac{C_k^P}{C_k^{P.0}}$

$\mu = 3m_h$

$$
\frac{m_{0h}}{m_{0c}} = \frac{m_h(\mu)}{m_c(\mu)}
$$

$$
m_c(m_c) = 1.2715(95)\text{GeV}
$$