Quarkonium production in proton-nucleus collisions at collider energies

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At small value of Bjorken $x$, Small-$x$/Color Glass Condensate (CGC) framework is essential.

Semihard Saturation scale: $Q_{sA}^2 (x) \sim A^{1/3} \left( \frac{1}{x} \right)^{0.3} \gg \Lambda_{QCD}^2$

$k_\perp < Q_s$: “Saturation regime" in Gluon TMD $\phi$.

Energy/rapidity dep. of $\phi \Rightarrow$ NONLINEAR BK eq. or JIMWLK eq.
Consider **Forward Quarkonium** productions in pA collisions:

\[ p + A \rightarrow c\bar{c}, \ b\bar{b} \Rightarrow J/\psi, \ \psi(2S), \ \Upsilon \] + X

LO kinematics: \( x_{1,2} = \frac{m_{\perp}}{\sqrt{s}} e^{\pm y} \) \((p_{\perp} = 1, \ y = +2)\)

<table>
<thead>
<tr>
<th>( \sqrt{s} )</th>
<th>200 GeV</th>
<th>5.02 TeV</th>
<th>8 TeV</th>
</tr>
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<tbody>
<tr>
<td>( J/\psi : x_2 )</td>
<td>2.20 \times 10^{-3}</td>
<td>8.78 \times 10^{-5}</td>
<td>5.51 \times 10^{-5}</td>
</tr>
<tr>
<td>( \psi(2S) : x_2 )</td>
<td>2.58 \times 10^{-3}</td>
<td>1.03 \times 10^{-4}</td>
<td>6.46 \times 10^{-5}</td>
</tr>
<tr>
<td>( \Upsilon : x_2 )</td>
<td>6.44 \times 10^{-3}</td>
<td>2.56 \times 10^{-4}</td>
<td>1.61 \times 10^{-4}</td>
</tr>
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</table>

**Forward Qarkoniums production in pA collisions can probe Small-\( x \) gluons!** \( \Rightarrow P_{\perp} \) distribution should reflect the saturation effect.
Target nucleus ⇒ Multiple scattering effect.

If $Q_s A \sim m v \sim M v / 2$ in $Q \bar{Q}$ rest frame → $v$-expansion is not ensured.

In the very forward rapidity region, owing to Lorentz time dilation,

$$\frac{1}{m v} \frac{M \cosh y}{M} \gg \frac{1}{p_\perp} \sim \frac{1}{Q_s A}$$

The hadronization is effectively frozen when the $Q \bar{Q}$ passes through the nucleus. [Qiu, Sun, Xiao, Yuan (2013)]

Yes, in principle. ✓ CEM, ✓ NRQCD
The scattering contributions are coherent as a whole.

\[
M_{s_1s_2;ij}(q, p) = \frac{g^2}{(2\pi)^4} \int d^2k_\perp d^2k_{1\perp} \frac{\rho_p(k_{1\perp})}{k_{1\perp}^2} \int d^2x_\perp d^2y_\perp e^{ik_\perp \cdot x_\perp} e^{i(P_\perp - k_\perp - k_{1\perp}) \cdot y_\perp} \times \bar{u}_{s_1,i}(q) \left[ T_g(k_{1\perp}) t^b W^{ba}(x_\perp) + T_{q\bar{q}}(k_{1\perp}, k_\perp) U(x_\perp) t^a U^\dagger(y_\perp) \right] v_{s_2,j}(p)
\]

- Wilson lines ⇒ **Multiple gluon scatterings** in Eikonal approximation
- In xsection, the nuclear wave function can be expressed by

\[
S_{xg}(r_\perp) = \frac{1}{N_c} \left\langle \text{Tr} \left[ U(x_\perp) U^\dagger(y_\perp) \right] \right\rangle_{xg}
\]
Energy and nuclear dependence

- Energy/Rapidity dependence $\Rightarrow$ rc-BK eq. [Balitsky (2006)]

$$S_{Y=0}(r_\perp) = \exp\left[-\frac{(r_\perp Q_{s,0})^\gamma}{4} \ln\left(\frac{1}{|r_\perp|\Lambda} + e_c \cdot e\right)\right]$$

- Its IC is fixed by HERA-DIS global data analysis. [AAMQS][Lappi, Mantysaari (2013)]

- For Nucleus, change the saturation scale as

$$Q_{sA,0}^2(b) = A\hat{T}_A(b)S_{\perp}Q_{sp,0}^2 \equiv NQ_{sp,0}^2$$

Using Glauber Model with

1. $S_\perp = \sigma_{NN}^{inel} \Rightarrow N_{MB} \approx 4 - 8.$

2. $S_\perp = \int_p d^2b \Rightarrow N_{MB} \approx 1.6 - 1.9$ [Lappi, Mantysaari (2013)]

Note: $N_{MB}$ is model dependent.
Hadronization from $Q\bar{Q}$ to $\psi$

✓ Color Evaporation Model (CEM) [Fujii, Gelis, Venugopalan (2006)]

$$\frac{d\sigma_\psi}{d^2 P_\perp dy} = F_\psi \int_{2m_c}^{2M_D} dM \frac{d\sigma_{Q\bar{Q}}}{dM d^2 P_\perp dy}$$

- Sum over all possible quantum states of $Q\bar{Q}$.
- $Q\bar{Q}$ converts into $\psi$ with the probability $F_{Q\bar{Q}\rightarrow\psi}$.

✓ Non-Relativistic QCD (NRQCD) [Kang, Ma, Venugopalan (2013)]

$$\frac{d\sigma_\psi}{d^2 P_\perp dy} = \sum_\kappa \frac{d\sigma_\kappa}{d^2 P_\perp dy} \times \left\langle O_\kappa^\psi \right\rangle_{\text{LDMEs}}$$

- Important LDMEs: $3 S_1^{[1]}$, $1 S_0^{[8]}$, $3 S_1^{[8]}$, $3 P_0^{[8]}$
**$J/\psi$ production in CGC + CEM**

[Fujii and KW (2013)]

- The $P_\perp$ broadening at low $P_\perp$ can be described in CGC+CEM.
**J/ψ suppression in CGC + CEM**

- \( Q_{sA}^2 = (4 - 6)Q_{sp}^2 \) [Fujii, KW (2013)]

- Optical Glauber approach [Ducloue, Lappi, Mantysaari (2015)]
Factorization assumed.

The contribution of CS channel is enhanced in pA but relatively small contribution. (10% in pp, 15% – 20% in pA at small-$p_\perp$)
$J/\psi$ suppression in CGC + NRQCD

[Ma, Venugopalan, Zhang (2015)]
$\psi(2S)$ production

$\psi(2S)$ suppression in pA collisions is an interesting issue. Initial state interaction (saturation effect) is the same for $J/\psi$ and $\psi(2S)$ due to similar kinematics and physical parameters. $\Rightarrow$ Final state interaction is required.

✓ Comovers produced in pA collisions [Ferreiro (2015)]
✓ Soft color clouds induced by $c\bar{c}$ traversing the nucleus $\Leftarrow$ Improved CEM. [Ma, Vogt (2016)]

$$\frac{d\sigma_\psi}{d^2P_\perp dy} = F_\psi \int_{m_\psi}^{2m_D-\Lambda} dM \left( \frac{M}{m_\psi} \right)^2 \frac{d\sigma_{c\bar{c}}}{dMd^2P'_\perp dy} \bigg|_{P'_\perp = \frac{M}{m_\psi} P_\perp}$$
$\psi(2S)$ production

[Ma, Venugopalan, KW, Zhang in preparation]

- $Q_s^2 = 2Q_{sp}^2$ fixed for MB events.
- Local “temperature" in $Q\bar{Q}$ rest frame: $\Lambda \sim \mathcal{O}(\Delta E_{\psi(2S)})$
Quarkonium Production

**ϒ production**

Large mass scale $M \Rightarrow$ Large phase space $s \gg M^2 \gg p_\perp^2 \Rightarrow$ Parton shower (Sudakov resummation) effect is important. see [Berger, Qiu, Wang (2004)][Sun, Yuan, Yuan (2012)]

![Diagram of parton shower](image)

**Small-x vs Sudakov**

- $s \gg p_\perp^2$ : Small-$x$ resummation

$$\frac{\alpha_s N_c}{2\pi^2} \ln \frac{1}{x_g} \sim O(1) \Rightarrow \text{BFKL or BK/JIMWLK evolution}$$

- $M^2 \gg p_\perp^2$ : Sudakov resummation

$$\frac{\alpha_s N_c}{2\pi} \ln^2 \frac{M^2}{p_\perp^2} \sim O(1) \Rightarrow \text{CSS evolution}$$
Sudakov implementation in CGC + CEM

Soft gluons emission on top of the BK resummation modifies $p_\perp$-dependence.

\[
\frac{d\sigma_{Q\bar{Q}}}{d^2 q_{Q\perp} d^2 q_{\bar{Q}\perp} dy_q dy_{\bar{Q}}} = \frac{\alpha_s^2 S_\perp}{16\pi^2 C_F} \int d^2 l_\perp d^2 k_\perp \frac{\Xi_{\text{coll}}(k_{2\perp}, k_\perp - zl_\perp)}{k_{2\perp}^2}
\times F_{\text{TMD}}(l_\perp) F_{Yg}(k_\perp) F_{Yg}(k_{2\perp} - k_\perp + l_\perp)
\]

where the TMD gluon distribution is given by

\[
F_{\text{TMD}}(M, l_\perp) = \int \frac{d^2 b_\perp}{(2\pi)^2} e^{-ib_\perp \cdot l_\perp} e^{-S_{\text{Sud}}(M, b_\perp)} x_1 G \left( x_1, \mu = \frac{c_0}{b_\perp} \right).
\]
Parton shower (Sudakov) effect is dominant for low-$P_\perp$ $\Upsilon$ production in pp collisions, while it is less pronounced in pA collisions.
Summary

- $J/\psi$ production can be described by CGC + CEM/NRQCD although the model dependence is a little bit messy.
- $\psi(2S)$ production in pA should be very sensitive to the color medium surrounding $Q\bar{Q}$. This color interaction enables us to interpret $\psi(2S)$ suppression easily in CGC + ICEM.
- Parton shower effect (Sudakov double logs) is essential to interpret data of $\Upsilon$ production.
- More importantly, full NLO calculations for quarkonium production are desired to complete.
Talk Plan
Energy dependence of the dipole amplitude \((S = 1 - T) \Rightarrow\)

Balitsky-Kovchegov equation [Balitsky (1995)], [Kovchegov (1996)]

\[
\frac{dT_Y(r)}{dY} = \mathcal{K} \otimes \left[ T_Y(r_1) + T_Y(r - r_1) - T_Y(r) - T_Y(r_1)T_Y(r - r_1) \right]
\]

- BFKL
- Recombination
Initial condition of Dipole amplitude

- **GBW model** [Golec-Biernat, Wusthoff (1998)]

\[ S_Y(r_\perp) = 1 - T_Y(r_\perp) = \exp \left[ -\frac{Q_s^2 r_\perp^2}{4} \right] \]

Not realistic model: high \( k_\perp \) contribution is strongly suppressed.

- **MV model** [McLerran, Venugopalan (1994)]

\[ S_Y(r_\perp) = \exp \left[ -\frac{r_\perp^2 Q_s^2}{4} \ln \left( \frac{1}{|r_\perp|\Lambda} + e \right) \right] \]

- **MV\(^\gamma\) and MV\(^e\) models** [AAMQS(2010)][Lappi-Mäntysaari(2013)]

\[ S_Y(r_\perp) = \exp \left[ -\frac{(x_\perp Q_s^2)^\gamma}{4} \ln \left( \frac{1}{|r_\perp|\Lambda} + e_c \cdot e \right) \right] \]
Balitsky-Kovchegov equation

\[
\frac{dT_Y(r)}{dY} = \mathcal{K} \otimes \left[ T_Y(r_1) + T_Y(r - r_1) - T_Y(r) - T_Y(r_1)T_Y(r - r_1) \right] \]

The running coupling kernel in Balitsky’s prescription:

\[
\mathcal{K}(r_\perp, r_{1\perp}) = \frac{\alpha_s(r^2)}{2\pi^2} \left[ \frac{1}{r_1^2} \left( \frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} - 1 \right) + \frac{r^2}{r_1^2r_2^2} + \frac{1}{r_2^2} \left( \frac{\alpha_s(r_2^2)}{\alpha_s(r_1^2)} - 1 \right) \right]
\]

with \( \alpha_s(r^2) = \left[ \frac{9}{4\pi} \ln \left( \frac{4C^2}{r^2\Lambda^2} + a_{\text{cutoff}} \right) \right]^{-1} \)

NLO BK equation can be used. See [Lappi, Mäntysaari (2015,2016)]
Constraint from DIS at HERA

[AAMQS(2010)][Lappi-Mäntysaari(2013)]

<table>
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<th>set</th>
<th>$Q^2_{s0,p}/\text{GeV}^2$</th>
<th>$\gamma$</th>
<th>$\alpha_s(r \to \infty)$</th>
<th>$e_c$</th>
<th>$\chi^2$/d.o.f.</th>
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<td>0.5</td>
<td>1</td>
<td>–</td>
</tr>
<tr>
<td>MV$^\gamma$</td>
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<td>1.118</td>
<td>1.0</td>
<td>1</td>
<td>1.12</td>
</tr>
<tr>
<td>MV$^e$</td>
<td>0.06</td>
<td>1</td>
<td>0.7</td>
<td>18.9</td>
<td>1.15</td>
</tr>
</tbody>
</table>
IC for the rcBK for nuclei

The initial condition of the rcBK equation for heavy nuclei is poorly understood due to lack of numerous data of $eA$ scattering. A fit to the available NMC (New Muon Collab.) data (fixed target $eA$) on the nuclear structure functions $F_{2,A}(x, Q^2)$ gives valuable information in this respect.

- Set the initial saturation at $x = 0.01$ as $Q_{s,A}^2(x_2) = c A^{1/3} Q_{s0}^2$
- For MV$^\gamma$ model, $c = 0.5$ is favored from the fit of data. [Dusling, Gelis, Lappi, Venugopalan (2009)]
Extrapolation at $x \geq 0.01$

- From QCD scaling law, [Fujii, Gelis, Venugopalan][Fujii, KW]

$$\phi_{p,Y}(k_\perp) = \phi_{p,Y_0}(k_\perp) \left( \frac{1 - x}{1 - x_0} \right)^4 \left( \frac{x_0}{x} \right)^{0.15}$$

- Fitting from collinear PDF [Ma, Venugopalan (2014)]

$$F_Y(k_\perp)_{x \geq x_0} = a(x) F_{Y_0}(k_\perp)$$

where (e.g.)

$$a(x) = \frac{x G_{CTEQ}(x, Q_0^2)}{x G_{Dipole}(x_0, Q_0^2)}$$

with the identity

$$x G_{Dipole}(x, Q_0^2) = \frac{1}{4\pi^3} \int_0^{Q_0^2} dk_\perp^2 \phi_{Y,p}^{g,g}(k_\perp)$$

$R_p$ is determined by fitting of collinear gluon PDF.

These extrapolations cause systematic uncertainty.
**Q\bar{Q} production in the CGC framework**

The $q\bar{q}$ production amplitude from the background gauge fields of $O(\rho_p^1 \rho_A^\infty)$ can be described as

$$M_{s_1s_2;ij}(q, p) = \frac{g^2}{(2\pi)^4} \int d^2k_\perp d^2k_1 \frac{\rho_p(k_1)}{k_1^2} \int d^2x_\perp d^2y_\perp e^{ik_\perp \cdot x_\perp} e^{i(P_\perp - k_\perp - k_1) \cdot y_\perp}$$

$$\times \bar{u}_{s_1, i}(q) \left[ T_g(k_1) t^b W^{ba}(x_\perp) + T_{q\bar{q}}(k_1, k) U(x_\perp) t^a U^\dagger(y_\perp) \right] v_{s_2, j}(p)$$

At high scattering energies, Eikonal approximation is valid. The interaction between incident quark (gluon) and the dense gluons in the target nucleus can be expressed as the Wilson line in the fundamental (adjoint) representation:

$$U(x_\perp) = \mathcal{P}_+ \exp \left[ ig \int_{-\infty}^{+\infty} dz^+ A^-_A(z^+, x_\perp) \cdot t \right],$$

$$W(x_\perp) = \mathcal{P}_+ \exp \left[ ig \int_{-\infty}^{+\infty} dz^+ A^-_A(z^+, x_\perp) \cdot T \right]$$
At classical level, it is required to average the squared amplitude over the distributions of the classical color sources $\rho_p$ and $\rho_A$

\[ \frac{d\hat{\sigma}_{q\bar{q}}}{d^2q_\perp d^2p_\perp dy_q dy_p} = \int \frac{d^2b_\perp}{[2(2\pi)^3]^2} \int \mathcal{D}\rho_p \mathcal{D}\rho_A W_p[\rho_p] W_A[\rho_A] \times |M_{s_1s_2;ij}(q, p)|^2 \]

with $W_p$ and $W_A$ are the weight functionals of $\rho_p$ and $\rho_A$.

- Rapidity dependence of operator $\langle O \rangle = \int \mathcal{D}\rho_p OW_p[\rho_p]$ is embodied in the weight functionals which obey the JIMWLK equation
Quarkonium production in pA collisions

\[
\frac{d\sigma_{Q\bar{Q}}}{d^2q_\perp d^2p_\perp dy_q dy_p} = \frac{\alpha_s}{(2\pi)^6 C_F} \int \frac{d^2k_1}{(2\pi)^2} \frac{\varphi_{p,Y_p}(k_{1\perp})}{k_{1\perp}^2 k_{2\perp}^2} \times \left[ \Xi^{g,g} \phi_A^{g,g}_{A,Y_A} + \int \frac{d^2k}{(2\pi)^2} \Xi^{q\bar{q},g} \phi_A^{q\bar{q},g}_{A,Y_A} + \int \frac{d^2k_\perp d^2k'}{(2\pi)^4} \Xi^{q\bar{q},q\bar{q}} \phi_A^{q\bar{q},q\bar{q}}_{A,Y_A} \right]
\]

\[
\Rightarrow \int \frac{d^2k_{1\perp}}{(2\pi)^2} \Xi(k_{1\perp},k_{2\perp},k_{\perp}) \phi_A^{q\bar{q},g}_{A,Y_A}(k_{2\perp},k_{\perp})
\]
Multipoint function

The unintegrated gluon distribution function

$$\varphi_{p,Y_p}(k_\perp) = \frac{(2\pi)^2\alpha_s S_\perp}{k_\perp^2} \int d^2x_\perp e^{ik_\perp \cdot x_\perp} \langle \rho_p(0) \rho_p(x_\perp) \rangle_{Y_p}.$$ 

and the higher multi-point Wilson line correlator in large-$N_c$ limit

$$\phi_{q\bar{q},g}^{A,Y_A}(k_{2\perp}, k_\perp) \approx S_\perp \frac{N_c k_{2\perp}^2}{4} F_{Y_A}(k_{2\perp} - k_\perp) F_{Y_A}(k_\perp)$$

Fourier transform of the fundamental dipole amplitude is defined as

$$F_Y(k_\perp) = \int d^2x_\perp e^{-ik_\perp \cdot x_\perp} \frac{1}{N_c} \langle \text{Tr} \left[ U(x_\perp) U^\dagger(0_\perp) \right] \rangle_Y.$$

Dipole amplitude
Forward production : Hybrid formalism

At forward rapidity where $x_1 \sim 1$, the phase space of the gluon distribution in projectile proton shrinks. By taking the limit $k_{1\perp} \to 0$ in the hard scattering parts and replacing the unintegrated gluon distribution function of proton with the collinear gluon distribution function (PDF), Hybrid-formula ($x_1 \sim 1$ and $x_2 \ll 1$) can be written as

$$\frac{d\sigma_{q\bar{q}}}{d^2p_{q\perp}d^2p_{\bar{q}\perp}dy_q dy_{\bar{q}}} = \frac{N_c \alpha_s S_A}{64\pi^2 C_F} \int_{k_{\perp}} \Xi_{\text{coll}}(k_{2\perp}, k_{\perp}) x_1 G(x_1, \mu) \times F_{x_2}(k_{2\perp} - k_{\perp}) F_{x_2}(k_{\perp})$$

with $p_{q\perp} + p_{\bar{q}\perp} = k_{2\perp}$ and $x_1 G(x_1, \mu) \equiv C \int_{\mu^2}^{} dk_{\perp}^2 \varphi_{p,Y_p}(k_{\perp})$
Quarkonium Production Models

- Color Evaporation Model  Fritzsch (1977) . . .
  - Constant transition rate : $F_{J/\psi}$
- Color Singlet Model  Chang (1980) . . .
  - $\alpha_s$ expansion
  - High $P_\perp$ production
  - $\alpha_s$ and $v \sim \alpha_s (mv)$ expansion : $mv^2 \ll mv \ll m$
- Power expansion  Nayak, Qiu, and Sterman (2005) . . .
  - $M/P_\perp$ and $\alpha_s$ expansion : High $P_\perp$ production

... and More!

Quarkonium production at low $P_\perp < M$

Interesting but challenging! $\Rightarrow$ Small-$x$/CGC framework provides important information of quarkonium production mechanism.
CGC framework + NRQCD

[Kang, Ma, Venugopalan (2013)]

Color singlet channel

\[
\frac{d\sigma_{q\bar{q}}^{CS}}{d^2 P_\perp dy} \propto \int \frac{d^2 k_{1\perp} d^2 k_\perp d^2 k'_\perp}{(2\pi)^6} \frac{\varphi_{p,Y_p}(k_{1\perp})}{k_{1\perp}^2} \frac{1}{2J + 1} \sum_{J_z} \mathcal{F}_{q\bar{q}}^J(P, k_{1\perp}, k_\perp) \mathcal{F}_{q\bar{q}}^{J_+}(P, k_{1\perp}, k'_\perp)
\]

\times \int d^2 x_\perp d^2 x'_\perp d^2 y_\perp d^2 y'_\perp e^{i(k_\perp \cdot x_\perp - k'_\perp \cdot x'_\perp)} e^{i(k_{2\perp} - k_\perp) \cdot y_\perp} e^{-i(k_{2\perp} - k'_\perp) \cdot y'_\perp}

\times \frac{1}{N_c} \langle \text{tr}[U(x_\perp) t_a U^\dagger (y_\perp)] \text{tr}[U(y'_\perp) t_a U^\dagger (x'_\perp)] \rangle_{Y_A}.

\approx \frac{1}{2} \left[ Q_{YA}(x_\perp, y_\perp; y'_\perp, x'_\perp) - S_{YA}(x_\perp, y_\perp) S_{YA}(y'_\perp, x'_\perp) \right]

The quadrupole amplitude

\[
Q_{YA}(x_\perp, y_\perp; y'_\perp, x'_\perp) \equiv \frac{1}{N_c} \text{tr}\langle U(x_\perp) U^\dagger(x'_\perp) U(y'_\perp) U^\dagger(y_\perp) \rangle_{Y_A}.
\]
Enhancement mechanism

[D. Kharzeev, E. Levin, M. Nardi and K. Tuchin]

1

- $J/\psi$ has $J^{PC} = 1^{--}$: odd number of gluons are needed.
- pp collisions: $g + g \rightarrow J/\psi + \text{soft-}g \rightarrow O(\alpha_s^5)$
- pA collisions: $g + g + g \rightarrow J/\psi \rightarrow O(\alpha_s^6 A^{2/3}) \sim O(\alpha_s^2)$ when $\alpha_s^2 A^{1/3} \sim 1$.

The color singlet production in pA collisions is enhanced compared to pp collisions.

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1Phys. Rev. Lett. 102, 152301 (2009)
Color Octet state

\[
\frac{d\sigma_{q\bar{q}}^{\text{CO}}}{d^2P_\perp dy} = \frac{\alpha_s S_\perp}{(2\pi)^3(N_c^2 - 1)} \int \frac{d^2k_{1\perp} d^2k_\perp}{(2\pi)^4} \frac{\varphi_{p,Y_p}(k_{1\perp})}{k_{1\perp}^2} \times F_{Y_A}(k_{2\perp} - k_\perp) F_{Y_A}(k_\perp) \Xi^{\text{CO}}
\]

- Of particular importance is that the nuclear effect is the same both in CEM and the color octet state in NRQCD with the large-$N_c$ approximation.
Conventional TMD framework

One loop calculations in NRQCD factorization framework [Sun, Yuan, Yuan (2012)]

\[
\left. \frac{d\sigma}{d^2 P_\perp dy} \right|_{|P_\perp| \ll M} = \int \frac{d^2 b_\perp}{(2\pi)^2} e^{iP_\perp \cdot b_\perp} W(M, b_\perp, x_1, x_2) + (Y - \text{term})
\]

with

\[
W(M, b_\perp, x_1, x_2) = e^{-S_{sud}(M, b_\perp)} W(M, b_\perp, C_1, C_2)
\]

\[
W(M, b_\perp, C_1, C_2) = \sigma_0 \frac{M^2}{s} \int \frac{dx}{x} \frac{dx'}{x'} C_{gg} \left( \frac{x_1}{x} \right) C_{gg} \left( \frac{x_2}{x'} \right) G(x_1, \mu) G(x_2, \mu)
\]

- TMD factorization [Collins-Soper-Sterman (1985)] [Collins (2011)]
- Y-term is power suppressed by $P_\perp/M$ at low $P_\perp$
- $\mu = \frac{c_0}{b_\perp}$ with $c_0 = 2e^{-\gamma E} \approx 1$
Collins-Soper-Sterman (CSS) formalism

Split the Sudakov factor into two parts

\[ S_{\text{Sud}}(M, b) = S_{\text{perp}}(M, b_\star) + S_{\text{NP}}(M, b) \]

with \( b_\star = \frac{b}{\sqrt{1 + (b/b_{\text{max}})^2}} \)

- Perturbative: \( b_\star \sim b \ll b_{\text{max}} \)

\[ S_{\text{perp}}(M, b) = \int_{c_0/b^2}^{M^2} \frac{d\mu^2}{\mu^2} \left[ A \ln \left( \frac{M^2}{\mu^2} \right) + B \right] \]

\[ A = \sum_{i=1} A^{(i)} \left( \frac{\alpha_s}{\pi} \right)^i, \quad B = \sum_{i=1} B^{(i)} \left( \frac{\alpha_s}{\pi} \right)^i. \quad A^{(1)} = C_A \text{ and} \]

\[ B^{(1)} = -(b_0 + \delta_8 c/2)N_c \]

with \( b_0 = \left( \frac{11}{6} N_c - \frac{n_f}{3} \right) \frac{1}{N_c} \).

- Non-perturbative: \( b > b_{\text{max}} \) [Sun, Yuan, Yuan (2012)]

\[ S_{\text{NP}}(M, b) = \exp \left[ b^2 \left( -g_1 - g_2 \ln \left( \frac{M}{2Q_0} \right) - g_1 g_3 \ln(100x_1 x_2) \right) \right] \]

\( g_1 = 0.03, \quad g_2 = 0.87, \quad g_1 g_3 = -0.17 \) with \( Q_0 = 1.6 \text{ GeV}, \quad b_{\text{max}} = 0.5 \text{ GeV} \)
Lesson from conventional TMD calculation

[Sun, Yuan, Yuan (2012)]

- Left fig: $J/\psi$ at LHC forward, RHIC mid and forward. Right fig: $\Upsilon$ at LHC forward.
- The agreement between theoretical results and experimental data for $J/\psi$ production is not as good as that for $\Upsilon$ production.