Resonance and QCD

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Resonances are the 99%

- Most hadrons are resonances
- Play a crucial role in a wide range of phenomenology
Why lattice?

$$f_0(500)/\sigma$$
QCD-stable states are generated exactly
QCD-stable states are generated exactly
N-body forces are incorporated exactly
QCD-stable states are generated exactly
N-body forces are incorporated exactly
Resonances are generated and decay
QCD-stable states are generated exactly
N-body forces are incorporated exactly
Resonances are generated and decay
QED/weak sector can be treated perturbatively or non-perturbatively
Spectroscopy in LQCD

- *Vanilla* spectroscopy - QCD stable states [non-composite states]
- Physical or lighter quark masses [down to $m_\pi \approx 120$ MeV]
- Non-degenerate light-quark masses: $N_f=1+1+1+1$
- Dynamical QED

![Graph showing $\Delta_{QED} M_K^2 = [M_{K^+}^2 - M_{K^0}^2]_{QED}$](chart.png)

Fodor et al. [BMWc] (2016)
Vanilla spectroscopy - QCD stable states [non-composite states]

the frontier of spectroscopy - hadronic resonances [composite states]

Spectroscopy in LQCD
A pseudo-quantitative definition
(bump in cross sections/amplitude - e.g., $\pi\pi$ scattering in $\rho$-channel)

$$M_1 = \frac{8\pi E_{cm}}{p} \frac{1}{\cot \delta_1 - i}$$

Protopopescu et al. (1972)
A pseudo-quantitative definition
(bump in cross sections/amplitude - e.g., $\pi\pi$ scattering in $\rho$-channel)

- $E \rightarrow \lvert \mathcal{M}_1 \rvert$
- $0.6 < \rho(770) < 1.0$
- $E_\rho \sim 763$ MeV
- $\Gamma_\sigma \sim 156.4$ MeV

Protopopescu et al. (1972)
A counter example
(Isoscalar, scalar ππ scattering)
A counter example
(Isoscalar, scalar $\pi\pi$ scattering)

\[ E = 449(22)_{16} \text{ MeV} \]

\[ f_0(500)/\sigma \]

\[ E_\sigma = 550(24) \text{ MeV} \]

\[ \Gamma_\sigma \]

\[ E_{\text{cm}} / \text{GeV} \]
Spectroscopy recap

\[ s = E_{\text{cm}}^2 \]

**Infinite volume**
- first Riemann sheet

**Bound state**

**Re\[s\]**

**Threshold**

**Branch cut - where scattering takes place**
Spectroscopy recap

\[ s = E_{\text{cm}}^2 \]

\[ s_R = \left(E_R - \frac{i}{2} \Gamma_R\right)^2 \]
Lattice QCD

Correlation functions using:

- Wick rotation [Euclidean spacetime]: $t_M \rightarrow -it_E$
- Monte Carlo sampling
- Lattice spacing: $a \sim 0.03 - 0.15$ fm
- Finite volume
- Quark masses: $m_q \rightarrow m_q^{\text{phys}}$

Have we mangled QCD too much?
Finite volume spectrum

“only a discrete number of modes can exist in a finite volume”
Finite vs. infinite volume spectrum

both pictures are QCD:

“Two analytic manifestations of QCD”
Experiment

amplitude analysis → partial wave amplitudes → analytic continuation → poles
Lüscher formalism

spectrum satisfy: \[ \det[F^{-1}(E_L, L) + \mathcal{M}(E_L)] = 0 \]

\( E_L = \) finite volume spectrum
\( L = \) finite volume
\( F = \) known function
\( \mathcal{M} = \) scattering amplitude
Lüscher formalism

spectrum satisfy: \( \det[F^{-1}(E_L, L) + \mathcal{M}(E_L)] = 0 \)

- Lüscher (1986, 1991) [elastic scalar bosons]
- Rummukainen & Gottlieb (1995) [moving elastic scalar bosons]
- Kim, Sachrajda, & Sharpe/Christ, Kim & Yamazaki (2005) [QFT derivation]
- Bernard, Lage, Meißner & Rusetsky (2008) [N\(\pi\) systems]
- Gockeler, Horsley, et al. (2012) [N\(\pi\) systems]
- RB, Davoudi, Luu & Savage (2013) [generic spinning systems]
- Feng, Li, & Liu (2004) [inelastic scalar bosons]
- Bernarda, Lage, Meißner, and Rusetsky [inelastic scalar bosons in TBC]
- Hansen & Sharpe / RB & Davoudi (2012) [moving inelastic scalar bosons]
Extracting the spectrum

Two-point correlation functions:

\[ C_{ab}^{2pt.}(t, P) \equiv \langle 0 | \mathcal{O}_b(t, P) \mathcal{O}_a^\dagger(0, P) | 0 \rangle = \sum_n Z_{b,n} Z_{a,n}^\dagger e^{-E_n t} \]

- Evaluate all Wick contraction - [distillation - Peardon, et al. (Hadron Spectrum, 2009)]
- Use a large basis of operators with the same quantum numbers
- 'Diagonalize' correlation function variationally
Isoscalar $\pi\pi$ scattering

$\rho \cot \delta_0 / \text{GeV}$

$m_\pi = 236 \text{ MeV}$

$m_\pi = 391 \text{ MeV}$

expt.

HadSpec Collaboration

Isoscalar $\pi\pi\pi$ scattering

\[ m_\pi = 236 \text{ MeV} \]

\[ m_\pi = 391 \text{ MeV} \]

\[ \mathcal{M} \sim \frac{1}{p \cot \delta_0 - ip} \rightarrow \frac{1}{p \cot \delta_0 + |p|} \]

HadSpec Collaboration
Isoscalar $\pi\pi\pi$ scattering

$\delta_0$ vs. $p^2 / \text{GeV}^2$

$m_\pi = 391 \text{ MeV}$

$m_\pi = 236 \text{ MeV}$

HadSpec Collaboration

Isoscalar $\pi\pi\pi$ scattering

HadSpec Collaboration

Isovector $\pi\pi$ scattering

$\delta_1 / ^\circ$

$E_{cm} / \text{MeV}$

$m_\pi = 391 \text{MeV}$

$m_\pi = 236 \text{MeV}$

HadSpec Collaboration

Dudek, Edwards & Thomas (2012)
Wilson, RB, Dudek, Edwards & Thomas (2015)
Comparison with experiment

$m_\pi = 140$ MeV

Bolton, RB, & Wilson (2016)
The $\rho$ vs $m_\pi$

$m_\rho = \text{Re}(E_\rho)/\text{MeV}$

$m_\pi = 536 \text{ MeV}$
$m_\pi = 700 \text{ MeV}$
$m_\pi = 391 \text{ MeV}$
$m_\pi = 236 \text{ MeV}$
$m_\pi = 140 \text{ MeV}$, Lattice QCD + $U_N$PT
$m_\pi = 140 \text{ MeV}$, Roy Equation

Lin et al. (2009)
Dudek, Edwards, Guo & Thomas (2013)
Dudek, Edwards & Thomas (2012)
Wilson, RB, Dudek, Edwards & Thomas (2015)
Advantage over experiment:
- heavy quarks make broad resonances bound
- unambiguously track poles in complex plane
The $\sigma / f_0(500)$ vs $m_\pi$

$B_\sigma = 24(4) \text{ MeV}$

$\text{RB, Dudek, Edwards & Wilson (2016)}$
The $\sigma$/$f_0(500)$ vs $m_\pi$

$\text{Im}\sqrt{s_0} = \frac{1}{2} \frac{\Gamma_\pi}{\text{MeV}}$

$E_\sigma$ / MeV

$m_\pi = 391$ MeV

$m_\pi = 236$ MeV

The $\sigma/f_0(500)$ vs $m_\pi$

\[ -\text{Im}\sqrt{s_0} = \frac{1}{2} \Gamma_\sigma/\text{MeV} \]

Going higher in energy

Coupled channels: \[ \det[F^{-1}(E_L, L) + \mathcal{M}(E_L)] = 0 \]

\[ m_\pi = 391 \text{ MeV} \]

Hansen & Sharpe / RB & Davoudi (2012)

Going higher in energy

Coupled channels: \( \det[F^{-1}(E_L, L) + \mathcal{M}(E_L)] = 0 \)

\( m_\pi = 391 \text{ MeV} \)

Hansen & Sharpe / RB & Davoudi (2012)

Going higher in energy

Coupled channels

Beyond two particles:

\[
\det \left[ 1 + \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix} \begin{pmatrix} \mathcal{K}_2 & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{df,3} \end{pmatrix} \right] = 0
\]

Hansen & Sharpe (2014)
RB, Hansen & Sharpe (2017)
scattering data

electro/photo-production data

amplitude analysis

partial wave amplitudes

analytic continuation

poles

amplitude analysis

transition amplitudes

analytic continuation

form factors

Lüscher formalism

partial wave amplitudes

analytic continuation

poles

Experiment

FV spectrum

Lattice QCD
poles
analytic continuation
partial wave amplitudes

analytic continuation
poles
form factors

electroweak amplitudes

form factors

Lattice QCD

Experiment

scattering data
amplitude analysis
partial wave amplitudes
analytic continuation
form factors

electro/photo-production data
amplitude analysis
transition amplitudes
analytic continuation
form factors

one-to-two FV matrix elements
Lellouch-Lüscher formalism
partial wave amplitudes
analytic continuation
poles

Lüscher formalism
FV spectrum
analytic continuation
poles

amplitude analysis
electroweak amplitudes
analytic continuation
form factors

one-to-two FV matrix elements
Lellouch-Lüscher formalism
partial wave amplitudes
analytic continuation
poles

amplitude analysis
electroweak amplitudes
analytic continuation
form factors
Electroweak properties

\[ |\langle 2 | \mathcal{J} | 1 \rangle_L| = \sqrt{\mathcal{H}} \mathcal{R} \mathcal{H} \]

finite volume matrix element

\[ \langle 2 | \mathcal{J} | 1 \rangle_L = \text{finite matrix element} \]

\[ \mathcal{R} = \text{known function} \]

electroweak amplitude

\[ \mathcal{H} = \text{electroweak amplitude} \]

References:
RB, Hansen (2016)
RB, Hansen (2015)
RB, Hansen, Walker-Loud (2014)
$\pi\gamma^*-\text{to-}\pi\pi$

Exploratory $\pi\gamma^*-\text{to-}\pi\pi$ / $\pi\gamma^*-\text{to-}\varphi$ calculation:

$m_\pi = 391$ MeV

Matrix element determined in 42 kinematic point: $(E_{\pi\pi}, Q^2)$

Lorentz decomposition:

$$\mathcal{H}_\pi^{\mu} = \epsilon^{\mu\nu\alpha\beta} P_{\pi,\nu} P_{\pi\pi,\alpha} \epsilon_\beta (\lambda_{\pi\pi}, P_{\pi\pi}) \frac{2}{m_\pi} A_{\pi\pi,\pi\gamma^*}$$

ππ/ρ polarization

ππ/ρ helicity

Lorentz scalar
$\pi\gamma^*$-to-$\pi\pi$ amplitude

$|M_{\pi\pi}| = A_{\pi\pi, \gamma^*}$

$E_{\pi\pi}^*/m_\pi$

$Q^2 = 0$

$Q^2 = 0.803$ GeV$^2$

elastic $\pi\pi$ amplitude
Form factor at $q$ pole

evaluated at the $q$-meson pole, $(853(2)-i 12.4(6)/2)$ MeV

\[ E_{cm} = E_\rho \]

Shultz, Dudek, & Edwards (2014)
Summary / outlook

Coupled channels

formalism understood:
Hansen & Sharpe / RB & Davoudi (2012)

few implementations to date by HadSpec

**Summary / outlook**

- **Coupled channels**
- **Electroweak form factors / structure - tetraquarks, molecules, etc.**

formalism understood:

RB, Hansen (2016)
RB, Hansen (2015)

first implementation: $\pi \gamma^*-\pi \pi / \pi \gamma^*-\phi$

Summary / outlook

- Coupled channels
- Electroweak form factors / structure - tetraquarks, molecules, etc.

Formalism understood:

RB, Hansen (2016)
RB, Hansen (2015)

Elastic form factor:
**Summary / outlook**

- **Coupled channels**
- **Electroweak form factors / structure - tetraquarks, molecules, etc.**
- **Three-particle systems**

Formalism under construction:

\[
\det \left[ 1 + \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix} \begin{pmatrix} \mathcal{K}_2 & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{df,3} \end{pmatrix} \right] = 0
\]

Hansen & Sharpe (2014)
RB, Hansen & Sharpe (2016)
Summary / outlook

- Coupled channels
- Electroweak form factors / structure - tetraquarks, molecules, etc.
- Three-particle systems
- Physical point, chiral extrapolation?

\[ m_\pi = 140 \text{ MeV} \]

Bolton, RB, & Wilson (2016)
Summary / outlook

- Coupled channels
- Electroweak form factors / structure - tetraquarks, molecules, etc.
- Three-particle systems
- Physical point, chiral extrapolation?
- Pole tracking

![Graph showing data points with error bars and labeled axes: \( \frac{1}{2} \Gamma_\sigma / \text{MeV} \) on the y-axis and \( E_\sigma / \text{MeV} \) on the x-axis. The graph includes data points for \( m_\pi = 236 \text{ MeV} \) and \( m_\pi = 391 \text{ MeV} \).]
The big picture!
Collaborators & references

formalism

Hansen
Walker-Loud
Sharpe

numerical

Wilson
Shultz
Thomas

Bolton


Back-up slides
The $\sigma / f_0(500)$ vs $m_{\pi}$

Figure 4: The $\sigma / f_0(500)$ resonance poles listed in the RPP 1996 edition (Black squares) together with those also cited in the 2010 edition [70] (Red circles). Note the much better consistency of the latter and the general absence of uncertainties in the former. The huge light gray area corresponds to the uncertainty band assigned to the PDG estimate 1996-2010 before, a very significant part of the apparent disagreement between different poles in Fig.2 is not coming from experimental uncertainties when extracting the data, but from the use of models in the interpretation of those data and unreliable extrapolations to the complex plane. Actually, different analyses of the same experiment could provide dramatically different poles, depending on the parameterization or model used to describe the data and its later interpretation in terms of poles and resonances. Maybe the most radical example are the three poles from the Crystal Barrel collaboration, lying at $(1100 \pm 300) \text{ MeV}$ [68], $(400 \pm 500) \text{ MeV}$ and $(1100 \pm 137) \text{ MeV}$ [69], corresponding to the highest masses and widths in that plot. These poles were compiled together in the RPP although they even lie in different Riemann sheets. Moreover we will see in Sect.2 that all three lie outside the region of analyticity of the partial wave expansion (Lehmann-Martin ellipse [71]).

Therefore it should be now clear that in order to extract the parameters of the pole, which lies so deep in the complex plane and has no evident fast phase-shift motion, it is not enough to have a good description of the data. As a matter of fact, many functional forms could fit very well the data in a given region, but then widely with each other when extrapolated outside the fitting region. For instance, if all data were consistent (which they are not) one can always find a good data description using polynomials, or splines, which have no poles at all. Hence, to look for the pole, the correct analytic extension to the complex plane, or at least a controlled approximation to it, is needed. Unfortunately that has not always been the case in many analyses, and thus the poles obtained from poor analytic extensions of an otherwise nice experimental analysis are at risk of being artifacts or just plain wrong determinations. This, together with the huge uncertainty attached to the $s_0$ in the RPP, is what made many people outside the community to think that no progress was made in the light scalar sector for many decades.

However, progress was being made and the other remarkable feature of Fig.2 is that by 2010 most determinations agreed on a light sigma with a mass between 400 and 550 MeV and a half...
Going higher in energy

$N^*$

$\Delta^*$

$m/m_\Omega$

$m_\pi = 391$ MeV

Edwards, Dudek, Richards, Wallace [Hadspec Collab.] (2011)
Qualitative understanding

\[ \text{Im}[s] \]

\[ \text{Re}[s] \]

\[ m_\pi \sim 350 \text{ MeV} \]

- green = first Riemann sheet
- red = second Riemann sheet

Hanhart, Peláez, and Ríos (2008)
Qualitative understanding

Im[s] = first Riemann sheet
Re[s] = second Riemann sheet

$m_\pi \sim 350$ MeV

Hanhart, Peláez, and Ríos (2008)
Qualitative understanding

Hanhart, Peláez, and Ríos (2008)

- First Riemann sheet
- Second Riemann sheet

$m_\pi \sim 350$ MeV
Determining spectrum

\[ C(t) v_n(t) = \lambda_n(t) C(t_0) v_n(t), \]
\[ \lambda_n(t) \sim e^{-E_n(t-t_0)} \]
Parametrization

\[ t(s) = \frac{1}{\rho(s)} \frac{\sqrt{s} \Gamma(s)}{m_R^2 - s - i\sqrt{s} \Gamma(s)}, \]

\[ \Gamma(s) = \frac{g_R^2 k^3}{6\pi} \frac{s}{s} \]

\[ t_{ij}^{-1}(s) = \frac{1}{(2k_i)\ell} K_{ij}^{-1}(s) \frac{1}{(2k_j)\ell} + I_{ij}(s), \]

\[ \text{Im } I_{ij}(s) = -\delta_{ij} \rho_i(s) \]

\[ K_{ij}(s) = \frac{g_i g_j}{m^2 - s} + \sum_{n=0}^{N} \gamma_{ij}^{(n)} \left( \frac{s}{s_0} \right)^n, \]

\[ K_{ij}^{-1} = \sum_{m=0}^{M} c_{ij}^{(m)} s^m, \]
$\pi \gamma^* \to \pi \pi$ amplitude
Correlation functions

Contractions:

Operators and matrix elements:

\[ C^{(3)}_{\pi\pi\pi, \mu}(P_{\pi}, P_{\pi\pi}; \Delta t, t) = \langle 0 | \Omega_{\pi}(\Delta t, P_{\pi}) \tilde{J}_\mu(t, P_{\pi} - P_{\pi\pi}) \Omega_{\pi\pi}^\dagger(0, P_{\pi\pi}) | 0 \rangle \]

\[ = e^{-(E_{\pi\pi} - E_{\pi}) t} e^{-E_{\pi}\Delta t} \langle \pi; L | \tilde{J}_\mu | \pi\pi; L \rangle + \ldots \]

\[ \Omega_{\pi} = \text{optimized} \ '\pi' \ \text{operator,} \]
\[ \text{linear combo. of} \ \sim 10 \ \text{ops.} \]

\[ \Omega_{\pi\pi} = \text{optimized} \ '\pi\pi' \ \text{operator,} \]
\[ \text{linear combo. of} \ \sim 20-30 \ \text{ops.} \]

\[ \tilde{J}_\mu = \text{electromagnetic current} \]
\( \pi \gamma^* \)-to-\( \pi \pi \pi \) amplitude

\[ P_{\pi \pi} = [111] \ A_1, \ n = 0 \]

\[ E_{\pi \pi}^*/m_\pi = 2.162(17) \]
$\pi\gamma^*\text{-to-}\pi\pi\pi$ amplitude

$P_{\pi\pi} = [111]$  $E_2$, $n = 0$

$E_{\pi\pi}^*/m_\pi = 2.205(20)$
$\pi\gamma^*$-to-$\pi\pi$ amplitude

$P_{\pi\pi} = [001]$  $E_2, n = 0$
$E_{\pi\pi}^*/m_\pi = 2.231(10)$
$\pi\gamma^*$-to-$\pi\pi$ amplitude

$P_{\pi\pi} = [011] \ B_1, \ n = 1$

$E_{\pi\pi}^*/m_{\pi} = 2.788(13)$
πγ*-to-ππ amplitude
Comparison with phenomenology
$\pi\gamma$-to-$\pi\pi$ cross section

$m_\pi \approx 400$ MeV

$\sim 7$ times larger than experiment/phenomenology

non trivial quark-mass dependence!

$\sigma_{\pi\gamma \to \pi\pi}^{l=1} / \mu$b

$E_{\pi\pi}^*/m_\pi$

Hoferichter, Kubis, & Sakkas (2012)
\( \pi \gamma \rightarrow \pi \pi \pi \) cross section

\[ m_\pi \approx 400 \text{ MeV} \quad m_\pi \approx 400 \text{ MeV} \]

\[ \sigma_{\pi^+ \gamma \rightarrow \pi^+ \pi^0} \propto \frac{q^{*}_{\pi \gamma} F_{\pi \rho}^2(m_\rho, 0)}{m_\pi^2} \times \frac{1}{\Gamma_1(m_\rho)} \]

\[ \lim_{E^*_{\pi \pi} \rightarrow m_\rho} \sigma_{\pi^+ \gamma \rightarrow \pi^+ \pi^0} \]

\[ 0.60 \times \text{ (physical)} \]

\[ 12 \times \text{ (physical)} \]

\[ m_H = 854.1 \pm 1.1 \text{ MeV} \]

\[ \gamma = 5.80 \pm 0.11 \]

\[ \Gamma_H = \frac{E_H}{m_H} = 32.4 \pm 0.6 \text{ MeV} \]

\[ E_{cm} / \text{MeV} \]

\[ m_\pi = 391 \text{ MeV} \]
On determining correlation function using small basis of operators
Extracting the spectrum

Two-point correlation functions:

\[ C_{ab}^{2pt.}(t, P) \equiv \langle 0 | \mathcal{O}_b(t, P) \mathcal{O}_a^\dagger(0, -P) | 0 \rangle \]

\[ = \sum_n \langle 0 | \mathcal{O}_b(t, P) | n, L \rangle \langle n, L | \mathcal{O}_a^\dagger(0, -P) | 0 \rangle \]

\[ = \sum_n \langle 0 | e^{t \hat{H}_{QCD}} \mathcal{O}_b(0, P) e^{-t \hat{H}_{QCD}} | n, L \rangle \langle n, L | \mathcal{O}_a^\dagger(0, -P) | 0 \rangle \]

\[ = \sum_n Z_{b,n} Z_{a,n}^\dagger e^{-E_n t} \]

\[ \text{spectrum} \]

\[ \text{time}=0 \quad \text{time}=t \]
Extracting the spectrum

Two-point correlation functions:

\[ C_{ab}^{2pt.}(t, P) \equiv \langle 0| \mathcal{O}_b(t, P) \mathcal{O}_a^\dagger(0, -P)|0 \rangle = \sum_n Z_{b,n} Z_{a,n}^\dagger e^{-E_n t} \]

- Use a large basis of operators with the same quantum numbers
- Diagonalize correlation function

30 operators

Wilson, RB, Dudek, Edwards & Thomas (2015)
Isovector \( \pi\pi \) scattering

Dudek, Edwards & Thomas (2012)
Wilson, RB, Dudek, Edwards & Thomas (2015)
The incorrect answer
$a_0(980)$ poles

$\pi\eta$-$KK$-$\pi\eta'$ in $I=1$, $m_\pi=391$MeV

Unitarized $\chi$PT

\[ \mathcal{M}_{U\chi PT} = \mathcal{M}_{LO} \frac{1}{\mathcal{M}_{LO} - \mathcal{M}_{NLO}} \mathcal{M}_{LO} \]

\[ S = 1 + 2i\sigma \mathcal{M} \]
\[ \mathcal{M} = (\text{Re}(\mathcal{M}^{-1}) - i\sigma)^{-1} \]
\[ \mathcal{M}^{-1} = \mathcal{M}_{LO}^{-1} \frac{1}{1 + \mathcal{M}_{LO}^{-1} \mathcal{M}_{NLO} + \ldots} = \mathcal{M}_{LO}^{-1} (1 - \mathcal{M}_{LO}^{-1} \mathcal{M}_{NLO} + \ldots) \]
\[ \text{Re}(\mathcal{M}^{-1}) = \mathcal{M}_{LO}^{-1} (1 - \mathcal{M}_{LO}^{-1} \text{Re}(\mathcal{M}_{NLO}) + \ldots) \]

Dobado and Pelaez (1997)
Oller, Oset, and Pelaez (1998)
Oller, Oset, and Pelaez (1999)
**LL-factor**

Relationship between amplitude and “form factor”:

\[
\mathcal{A}_{\pi\pi,\pi\gamma^*}(E_{\pi\pi}^*, Q^2) = \left( \frac{F(E_{\pi\pi}^*, Q^2)}{\cot \delta_1(E_{\pi\pi}^*) - i} \right) \sqrt{\frac{16\pi}{q_{\pi\pi}^* \Gamma(E_{\pi\pi}^*)}}
\]

\[
F(E_{\pi\pi}^*, Q^2) = \tilde{\mathcal{A}}(E_{\pi\pi}^*, Q^2; L) \sqrt{\frac{K_{\pi\pi}}{\mathcal{R}}},
\]

\[
\frac{1}{\sqrt{2E_{\pi\pi}^* K_{\pi\pi}(E_{\pi\pi}^*)}} = \sin \delta_1(E_{\pi\pi}^*) \sqrt{\frac{16\pi}{q_{\pi\pi}^* \Gamma(E_{\pi\pi}^*)}}
\]

**LL factor:**

\[
\frac{2E_{\pi}}{\mathcal{R}} = 32\pi \frac{E_{\pi} E_{\pi\pi}}{q_{\pi\pi}^*} \cos^2 \delta_1 \frac{\partial}{\partial P_{0,\pi\pi}^*} \left( \tan \delta_1 + \tan \phi P_{\pi\pi,\Lambda\pi\pi} \right) \bigg|_{P_{0,\pi\pi} = E_{\pi\pi}^*}
\]

\[
= 32\pi \frac{E_{\pi} E_{\pi\pi}}{q_{\pi\pi}^*} (\delta_1' + r\phi'),
\]
“Form factor”

Fit parametrization:

\[
h^{l\{\alpha,\beta\}}(E^*_{\pi\pi}, Q^2) = \frac{\alpha_1}{1 + \alpha_2 Q^2 + \beta_1 (E^*_{\pi\pi} - m_0^2)} + \alpha_3 Q^2 + \alpha_4 Q^4 + \alpha_5 \exp \left[ -\alpha_6 Q^2 - \beta_2 (E^*_{\pi\pi} - m_0^2) \right] + \beta_3 (E^*_{\pi\pi} - m_0^2) + \beta_4 (E^*_{\pi\pi} - m_0^4),
\]
The $\sigma/f_0(500)$ vs $m_\pi$

$$s_0 = (E_\sigma - \frac{i}{2} \Gamma_\sigma)^2, \quad g^2_{\sigma\pi\pi} = \lim_{s \to s_0} (s_0 - s) t(s)$$

UχPT - Nebreda & Peláez (2015)
The $\sigma/f_0(500)$ vs $m_\pi$

$$s_0 = (E_\sigma - \frac{i}{2} \Gamma_\sigma)^2, \quad g_{\sigma\pi\pi}^2 = \lim_{s \to s_0} (s_0 - s) t(s)$$
Increasingly complex systems

\[ D\pi - D\eta - D_s \bar{K} \text{ scattering} \]

(I=1/2 channel)

\[ \rho_i \rho_j |t_{ij}|^2 \]

\[ D\pi \rightarrow D\pi \]

\[ D\eta \rightarrow D\eta \]

\[ D\pi \rightarrow D\eta \]

\[ D\pi \rightarrow D_s \bar{K} \]

\[ D_s \bar{K} \rightarrow D_s \bar{K} \]

\[ D\eta \rightarrow D_s \bar{K} \]

\[ E_{\text{cm}}/\text{MeV} \]

Moir, Peardon, Ryan, Thomas, Wilson
$D\pi - D\eta - D_s \overline{K}$ scattering

(I=1/2 channel)
$U\chi$PT expectation
for $\sigma/f_0(500)$
$\frac{\sigma}{f_0(500)}$ vs $m_\pi$
Sketch of Lüscher
Two particles in a box

Onto two particles:

\[ L^3 \int \frac{dP_0}{2\pi} e^{iP_0 t} \left\{ \begin{array}{c} A \quad V \quad B^\dagger + A \quad V \quad V \quad B^\dagger \quad + \ldots \end{array} \right\} \]

After some massaging...

\[ = L^3 \int \frac{dP_0}{2\pi} e^{iP_0 t} \left\{ C_\infty(P) + A \quad V \quad B^\dagger + A \quad V \quad V \quad B^\dagger \quad + \ldots \right\} \]

\[ = L^3 \int \frac{dP_0}{2\pi} e^{iP_0 t} \left\{ C_\infty(P) - A(P) \frac{1}{F^{-1}(P, L) + M(P)} \right\} \]

poles satisfy: \[ \det[F^{-1}(P, L) + M(P)] = 0 \]
Chiral fits
Chiral fit

\[ \alpha_1 \equiv -2\ell_1^r + \ell_2^r, \quad \alpha_2 \equiv \ell_4^r \]

\[ \alpha_1(770 \text{ MeV}) = 14.7(4)(2)(1) \times 10^{-3} \]

\[ \alpha_2(770 \text{ MeV}) = -28(6)(3) \binom{01}{11} \times 10^{-3} \]

previos results:

\[ \alpha_1(770 \text{ MeV}) \in [9, 13] \times 10^{-3} \]

\[ \alpha_2(770 \text{ MeV}) \in [1, 12] \times 10^{-3} \]
$m_\pi$ dependence

\begin{align*}
\delta_1 / ^\circ & \\
E_{\pi\pi}^* / \text{MeV} & \\
\end{align*}

- $m_\pi = 236 \text{ MeV}, \text{ fit}$
- $m_\pi = 140 \text{ MeV}, \text{ postdiction}$
- $m_\pi = 391 \text{ MeV}, \text{ postdiction}$