Nucleon matrix elements from Moments of Correlation Functions and the Proton Charge Radius

Chia Cheng Chan (LBNL)
Chris Bouchard (Glasgow)
Kostas Orginos (JLab/WM)
David Richards (JLab)*

*Speaker
Proton EM form factors

- Nucleon Pauli and Dirac Form Factors described in terms of matrix element of vector current

\[ \langle N \mid V_\mu \mid N \rangle(q) = \bar{u}(\vec{p}_f) \left[ F_q(q^2)\gamma_\mu + \sigma_{\mu\nu} q_\nu \frac{F_2(q^2)}{2m_N} \right] u(\vec{p}_i) \]

- Alternatively, Sach’s form factors determined in experiment

\[
G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2) \\
G_M(Q^2) = F_1(Q^2) + F_2(Q^2)
\]

Charge radius is slope at \( Q^2 = 0 \)

\[
\left. \frac{\partial G_E(Q^2)}{\partial Q^2} \right|_{Q^2=0} = -\frac{1}{6} \langle r^2 \rangle = \left. \frac{\partial F_1(Q^2)}{\partial Q^2} \right|_{Q^2=0} - \frac{F_2(0)}{4M^2}
\]
2.2 Proton Form-Factor Ratio Measurements up to $Q^2 = 12$ GeV$^2$ using Recoil Polarization

Introduction

The experiment GEp 1E.70-7.0)2 was approved by PAC, in August of 200- and was the experiment that provided the original motivation for the Super Bigbite Spectrometer. It will measure the Sachs Form Factors ratio $G_p^E/G_p^M$ of the proton using the polarization transfer method in the reaction $p(e_1 e_2 p)$. The polarization of the recoil proton will be measured using a large acceptance spectrometer based on the Super Bigbite magnet that will incorporate a double polarimeter instrumented with GEM trackers and a highly segmented hadron calorimeter.

The electron will be detected in coincidence by a electromagnetic calorimeter that is sometimes referred to as "BigCal". PAC, allocated "" days of beam time for the proposed measurement and recommended a maximum value of $Q^2 = 12$ GeV$^2$.

These parameters were used to readjust the original plan of measurements which will be made at three values of $Q^2$: 5 (5 and 2 GeV$^2$, while achieving an error in the ratio $G_p^E/G_p^M$ of 0.80. The projected results are shown in Fig. 5 in which we show results from earlier $G_p^E$ measurements and the anticipated errors for the GEp experiment. The excellent precision that GEp will obtain even at 5 GeV$^2$ is clearly evident.

Additional measurements at even higher values of $Q^2$ will be evaluated after SBS commissioning.

Nucleon Charge Radius

Direct calculation of charge radius through coordinate-space moments

UKQCD, Lellouch, Richards et al., NPB444 (1995) 401

Bouchard, Chang, Orginos, Richards, Lattice 2016

Boosted interpolating operators

LHPC, Syritsyn, Gambhir, Orginos et al, lattice 2016

Distillation + Operators for hadrons in flight

Form Factor in LQCD

\[ C_{3pt}(t_{sep}, t; \vec{p}, \vec{q}) = \sum_{\vec{x}, \vec{y}} \langle 0 | N(\vec{x}, t_{sep})V_\mu(\vec{y}, t)\bar{N}(0, 0) | 0 \rangle e^{-i\vec{p} \cdot \vec{x}} e^{-i\vec{q} \cdot \vec{y}} \]

\[ \rightarrow \langle 0 | N | N, \vec{p} + \vec{q} \rangle \langle N, \vec{p} + \vec{q} | V_\mu | N \bar{p} \rangle \langle N, \bar{p} | \bar{N} | 0 \rangle e^{-E(\vec{p}+\vec{q})(t_{sep}-t)} e^{-E(\vec{p})t} \]

\[ + \text{Excited states - suppressed at large } T \]
Electromagnetic Form Factors

Wilson-clover lattices from BMW


Hadron structure at nearly-physical quark masses

Why can’t we get rid of those excited states!

Smallest non-zero $Q^2$ determined by spatial volume

$\Rightarrow$ Calculate slope of form factor directly.
Isgur-Wise Function and CKM matrix

Extract $V_{cb}$ if know intercept at zero recoil

Lattice

Calculate slope at zero recoil...

Moment Methods

- Introduce three-momentum projected three-point function
  \[ C^{3\text{pt}}(t, t') = \sum_{\vec{x}, \vec{x}'} \left\langle N_{t, \vec{x}}^{a} \Gamma_{t', \vec{x}'} \overline{N}_{0, \vec{0}}^{b} \right\rangle e^{-ikx'_z} \]

- Now take derivative w.r.t. \( k^2 \)
  \[ C'_{3\text{pt}}(t, t') = \sum_{\vec{x}, \vec{x}'} \frac{-x'_z}{2k} \sin (kx'_z) \left\langle N_{t, \vec{x}}^{a} \Gamma_{t', \vec{x}'} \overline{N}_{0, \vec{0}}^{b} \right\rangle \]

  whence
  \[ \lim_{k^2 \to 0} C'_{3\text{pt}}(t, t') = \sum_{\vec{x}, \vec{x}'} \frac{-x'_z^2}{2} \left\langle N_{t, \vec{x}}^{a} \Gamma_{t', \vec{x}'} \overline{N}_{0, \vec{0}}^{b} \right\rangle. \]

  Odd moments vanish by symmetry
Analogous expressions for two-point functions:

\[
C_{2pt}(t) = \sum_{\vec{x}} \left\langle N^{b}_{t,\vec{x}} \overline{N}^{b}_{0,\vec{0}} \right\rangle e^{-ikx_z}
\]

\[
\Rightarrow
C'_{2pt}(t) = \sum_{\vec{x}} \frac{-x_z}{2k} \sin(kx_z) \left\langle N^{b}_{t,\vec{x}} \overline{N}^{b}_{0,\vec{0}} \right\rangle
\]

\[
\Rightarrow
\lim_{k^2 \to 0} C'_{2pt}(t) = \sum_{\vec{x}} \frac{-x_z^2}{2} \left\langle N^{b}_{t,\vec{x}} \overline{N}^{b}_{0,\vec{0}} \right\rangle.
\]

Lowest coordinate-space moment ↔ slope at zero momentum
Lattice Details

• Two degenerate light-quark flavors, and strange quark set to its physical value

\[ a \approx 0.12 \text{ fm} \]
\[ m_\pi \approx 400 \text{ MeV} \]

Lattice Size : \( 24^3 \times 64 \)

• To gain control over finite-volume effects, replicate in z direction: \( 24 \times 24 \times 48 \times 64 \)
Two-point correlator

\[ \ln [C_{2pt}(t, x_z)] \]

Any polynomial moment in \( x_z \) converges

“Effective mass”

\[ \ln \frac{C_{2pt}(t, x_z)}{C_{2pt}(t, x_z + 1)} \]
Three-point correlator

\[ \ln \left[ C_{3pt}(t', x'_{z}) \right] \]

- Spatial moments push the peak of the correlator away from origin
- Larger finite volume corrections compared to regular correlators
Fitting the data...

\[ C_{3pt}^{3pt}(t, t') = \sum_{n,m} \frac{Z_n^{\dagger a}(0) \Gamma_{nm}(k^2) Z_m^b(k^2)}{4M_n(0)E_m(k^2)} e^{-M_n(0)(t-t')} e^{-E_m(k^2)t'} \]

\[ C_{2pt}(t) = \sum_m \frac{Z_m^{b\dagger}(k^2) Z_m^b(k^2)}{2E_m(k^2)} e^{-E_m(k^2)t} \]

where

\[ Z_n^{\dagger a}(0) \equiv \langle \Omega | N^a | n, p_i = (0, 0, 0) \rangle \]

\[ Z_m^b(k^2) \equiv \langle m, p_i = (0, 0, k) | \overline{N}^b | \Omega \rangle \]

\[ \Gamma_{nm}(k^2) \equiv \langle n, p_i = (0, 0, 0) | \Gamma | m, p_i = (0, 0, k) \rangle \]

**Allow for multi-state contributions in the fit**
• Now look at the functional form of derivatives:

\[
C'_2(t) = \sum_m C'^{2pt}_m(t) \left( \frac{2Z'_m(k^2)}{Z^b_m(k^2)} - \frac{1}{2[E_m(k^2)]^2} - \frac{t}{2E_m(k^2)} \right)
\]

\[
C'_3(t, t') = \sum_{n,m} C'^{3pt}_{nm}(t, t') \left\{ \frac{\Gamma'_n(k^2)}{\Gamma_{nm}(k^2)} - \frac{Z^b_n(k^2)}{Z^b_m(k^2)} - \frac{1}{2[E_m(k^2)]^2} - \frac{t'}{2E_m(k^2)} \right\}
\]

**spatially extended sources**  **Second distance scale**
In practice we use multi-exponential, Bayesian fits.
$F_1$ Form Factor

$F_1' = \frac{e}{2} \left( 1 - \frac{1}{n^2} \right)$

$n^2 (ak = 2\pi n/L)$

4 state fit

4 state slopes

2 state fit

3 state fit

5 state fit

4 state 2+3pt

$Z_0_s(0)$

$Z_0_s(4)$

$Z_0_s(16)$
Conclusions

- Moment methods allow direct calculations of slopes of form factors at momenta allowed on lattice
- Lowest (even) moment gives the slope at $Q^2 = 0$.
- Larger finite-volume effects than regular correlators (perhaps expected - no free lunch).
- Illustrated here for u-quark contribution to EM form factor; d-quark and sea-quark contributions in progress....