Fermion-antifermion phenomenology in Minkowski space

Jorge H. A. Nogueira

Università di Roma ‘La Sapienza’ and INFN, Sezione di Roma (Italy)
Instituto Tecnológico de Aeronáutica, (Brazil)

Supervisors: Profs. T. Frederico (ITA) and G. Salmè (INFN)

Collaborators: Dr. E. Ydrefors, Prof. W. de Paula and Dr. C. Mezrag

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   - Bethe-Salpeter equation
   - Nakanishi integral representation
   - Light-front projection

2. Two-body bound state within the BSE
   - Bosonic BSE in Minkowski space
   - The interaction kernel
   - Fermion-antifermion BSE in Minkowski space
   - The mock pion

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General goals

- Bethe-Salpeter equation to study non-perturbative systems;
- Fully covariant relativistic description in Minkowski space;
- Understand step-by-step the degrees of freedom;
- How bad is to ignore the crosses in the BSE kernel?
- Introducing color factors and the large $N_c$ limit;
- Make the numerics feasible;
- No Fock space truncation;
- Phenomenological studies within the approach;
The BSE for the bound state with total four momentum $p^2 = M^2$, composed of two scalar particles of mass $m$ reads

$$
\Phi(k,p) = S(p/2 + k)S(p/2 - k) \int \frac{d^4k'}{(2\pi)^4} iK(k,k',p)\Phi(k',p),
$$

$$
S(k) = \frac{i}{k^2 - m^2 + i\epsilon} : \text{Feynman propagator}
$$

The kernel $K$ is given as a sum of irreducible Feynman diagrams (ladder, cross-ladder, etc).

E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951)

N. Nakanishi, Graph Theory and Feynman Integrals (Gordon and Breach, New York, 1971)
Nakanishi integral representation

- General representation for N-leg transition amplitudes;
- 2-point correlation function: Kallen-Lehmann spectral representation;
- For the vertex function (Bound state) - 3-leg amplitude:

\[
\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g(\gamma', z'; \kappa^2)}{(\gamma' + \kappa^2 - k^2 - (p \cdot k)z' - i\epsilon)^3}, \quad \kappa^2 = m^2 - M^2 / 4
\]

where \( \gamma \equiv |k_\perp|^2 \in [0, \infty) \) and \( z \equiv 2\xi - 1 \in [-1, 1] \) with \( \xi \in [0, 1] \)

- All dependence upon external momenta in the denominator;
- Allows to recognize the singular structure and deal with it analytically;
- Weight function \( g(\gamma', z') \) is the unknown quantity to be determined numerically;

Light-front projection

- Much easier to treat Minkowski space poles properly;
- Simpler dynamics of the propagators/amplitudes within LF (See talk by Prof. Ji);
- Easy connection with LFWF:
  - Introduce the LF variables $k_\pm = k_0 \pm k_z$;
  - Valence LFWV from the BS amplitude:

\[
\psi_{n=2/p}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}} \xi (1 - \xi) \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \Phi(k, p),
\]

- Corresponding to eliminate the relative LF time $t + z = 0$;
- Essential in this approach to solve BSE directly in Minkowski space;
The Nakanishi integral representation (NIR) gives the Bethe-Salpeter amplitude $\chi$ (BSA) through the weight function $g$;  

The Light-Front projection of the BSA gives the valence light-front wave function (LFWF) $\Psi_2$;  

The inverse Stieltjes transform gives $g$ from the valence LFWF;  

Applying the NIR on both sides of the BSE and integrating over $k_-$ leads to the integral equation:

$$
\int_0^\infty d\gamma' \frac{g(\gamma', z; \kappa^2)}{[\gamma + \gamma' + z^2m^2 + (1 - z^2)\kappa^2]^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz' V(\alpha; \gamma, z, \gamma', z')g(\gamma', z'; \kappa^2)
$$

where $V$ is expressed in terms of the BS interaction kernel.

- Ladder approx. - agreement among different groups [1];
- Cross-ladder impact; suppression with color dof [2];
- Scattering length; Spectroscopy and LF momentum distributions of the excited states [3];
- Agreement with BSE in Euclidean space [4];

One example to support the hypothesis

**Figure:** Coupling constant for various values of the binding energy $B$ obtained by using the Bethe-Salpeter ladder (L) and ladder plus cross-ladder (CL) kernels, for an exchanged mass of $\mu = 0.5m$. In the upper panels are shown the results computed with no color factors. Similarly, in the lower panels are compared the results for $N = 2, 3$ and $4$ colors.

- Suppression is already pretty good for $N_c = 3$ - might support the truncation at the ladder...at least within this system.

Fermion-antifermion BSE in Minkowski space

- Introducing spin

\[ \Phi(k,p) = S(p/2 + k) \int d^4k' F^2(k - k') iK(k,k') \Gamma_1 \Phi(k',p) \hat{\Gamma}_2 S(k - p/2) \]

where \( \Gamma_1 = \Gamma_2 = 1 \) (scalar), \( \gamma_5 \) (pseudo), \( \gamma^\mu \) (vector)

\[ iK^\mu_\nu(k,k') = -i g^2 \frac{g^\mu_\nu}{(k - k')^2 - \mu^2 + i\epsilon}, \quad F(k - k') = \frac{(\mu^2 - \Lambda^2)}{[(k - k')^2 - \Lambda^2 + i\epsilon]} \]

- Taking benefit from orthogonality properties for the decomposition

\[ \Phi(k,p) = \sum_{i=1}^4 S_i(k,p) \phi_i(k,p) \]

where the spin dependent structures are \( S_1 = \gamma_5 \), \( S_2 = \frac{p}{M} \gamma_5 \), \( S_3 = \frac{k \cdot p}{M^3} p \gamma_5 - \frac{1}{M} k \gamma_5 \) and \( S_4 = \frac{i}{M^2} \sigma^{\mu\nu} p_\mu k_\nu \gamma_5 \)

- The scalar amplitudes \( \phi_i \) are represented by the NIR;
  - In the equal mass case, symmetry under the exchange of the particles simplifies the problem;
  - \( g_j(\gamma', z'; \kappa^2) \) expanded as Laguerre(\( \gamma \)) \( \times \) Gegenbauer(\( z \)).
Extra singular contribution of the fermionic system

The coupled integral equation system is given by:

\[ \psi_i(\gamma, z) = g^2 \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma' \ g_j(\gamma', z'; \kappa^2) \ L_{ij}(\gamma, z, \gamma', z'; p) \]

- \( S_i \) operators + fermionic propagators: \((k^-)^n\) extra singularities;
- Singularities have generic form:

\[ C_n = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^n \ S(k^-, \nu, z, z', \gamma, \gamma') \quad n = 0, 1, 2, 3 \]

- End-point singularities can be analytically treated by

\[ I(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{[\beta x - y \mp i\epsilon]^2} = \pm \frac{2\pi i \delta(\beta)}{[-y \mp i\epsilon]} \]

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Yan et al PRD 7 (1973) 1780

J. Nogueira (ITA, Brazil / ’La Sapienza’, Italy)  
Few-body with BSE  

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Coupling Constants

- Vector coupling as a function of the binding energy for $\mu/m = 0$;

- Dots: Kernel regularized by a cutoff;
  - No analytical treatment of the singularities;

- Agreement also with results in Euclidean space (for the scalar exchange) - see [2];

High-momentum tails

- LF amplitudes $\psi_i$ times $\gamma / m^2$ at fixed $z = 0$ ($\xi = 1/2$);
  - Thin lines $B/m = 0.1$ and thick 1.0;
  - Solid: $i = 1$, Dashed: $i = 2$, dash-dot: $i = 4$, $\psi_3 = 0$ for $z = 0$;
  - As expected for the pion valence amplitude;

Valence probabilities

- By properly normalizing the BSE we can study the valence probabilities of the bound states;
- Taking, for instance, $\mu/m = 0.15$ and a cutoff $\Lambda/m = 2$ for the vertex form factor (fermionic case):

  \[
  \begin{array}{c|cc}
  B/m & P^F_{val} & P^B_{val} \\
  \hline
  0.01 & 0.96 & 0.94 \\
  0.1 & 0.78 & 0.80 \\
  1.0 & 0.68 & 0.67 \\
  \end{array}
  \]

- Results are similar for massless vector exchange;
- Very low $P^F_{val}$: higher Fock components are extremely important;
- Lack of color confining kernel might be playing a role;
Mock pion

- Guideline for the mock pion input parameters:
  - Gluon effective mass $\approx 500$ MeV (Landau Gauge LQCD) \[1\];
  - $m_q \approx 250$ MeV \[2\]; $M_\pi = 140$ MeV fixed;
  - $\Lambda/m = 1, 2, 3$
  - $\alpha_s = g^2/(4\pi)(1 - \mu^2/\Lambda^2)^2$; Reasonable rescaled coupling constant in the infrared region \[3\];
- Transverse and longitudinal momentum valence distributions for different sets of parameters:

![Graph 1]

![Graph 2]

\[1\] Oliveira, Bicudo, JPG 38 (2011) 045003; Duarte, Oliveira, Silva, Phys. Rev. D 94 (2016) 014502
\[4\] de Paula, TF, Pimentel, Salmè, Viviani, EPJC 77 (2017) 764; de Paula et al, in preparation
3D LF amplitudes

- Dynamical observables: the LFWF components;
  - \( \frac{B}{m} = 1.35, \frac{\mu}{m} = 2.0, \frac{\Lambda}{m} = 1.0, m_q = 215 \text{ MeV} \): \( f_\pi = 96 \text{ MeV}, \) \( P_{val} = 0.68 \)
- Other observables are straightforward to compute once you have BS amplitude solution;
Conclusions

- NIR + LF projection → Showing to be essential tools to deal with the BSE in Minkowski space;
  - Many systems already treated and several subtle points under control;
  - Approach is even more robust - now able to deal properly with all singularities within fermionic systems;
  - Valence is far from enough once spin dof is included;
  - Approach gives you all the information beyond the valence;
  - Ladder approx. supported by the color suppression of the non-planar diagrams;

- Simple and direct connection with physical observables;
- More sensitive numerics - e.g. derivatives of the basis;
  - Exploration of new numerical methods is important;
- Still the most stable method within Minkowski space (see talk by E. Ydrefors);
- Essential features need to be included;
  - Confining kernel;
  - Self-energies and vertex corrections;
Outlook

- Address color confinement to the kernel;
  - 1st step: ansatz that connects to the superconformal LFH confining potential - simply (and unique) harmonic oscillator;
  - $q\bar{q}$ bound state in 1+1 dimensions and how it matches with DLCQ;
    - Can bring some understanding of some features within the approach;
  - More formal possibility: summation of H graphs in the kernel;
- Self-energies, vertex corrections;
  - Other approaches (LFH, LQCD, DSE) can inspire a first step;
  - Start by simpler phenomenological ways of implementing it, such as by models inspired by Lattice QCD (see talk by Prof. Frederico);
  - DSE fully in Minkowski space by means of spectral representation;
- Unequal mass case with possible applications for other mesons;
- Form factors, PDFs, TMDs, Fragmentation functions...
- Fermion-boson BSE was solved (in preparation);
  - Possible phenomenological applications for baryons;

Thank you!