Nucleon PDFs in small boxes

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Based on:
Novel idea: PDFs on the lattice

PDFs from QCD: the only non-perturbative way to study QCD is lattice QCD.

Lattice QCD is defined by…
- Discretization
- Euclidean vs Minkowski
- Quark masses
- Finite volume

\[ t_M \rightarrow -it_E \]
Novel idea: PDFs on the lattice

PDFs from QCD: the only non-perturbative way to study QCD is lattice QCD.

Lattice QCD is defined by...
- Discretization
- Euclidean vs Minkowski
- Quark masses
- Finite volume

Focus of this talk…
Scheme to extract PDFs from the lattice

PDFs on the lattice

There are different techniques:

- **Wilson lines**: \( \langle N | \bar{q} \, W \, q | N \rangle \) \( \propto \) Ji (2013), Radyushkin (2017)

- **two current operators**: \( \langle N | J(0, \xi) \, J(0) | N \rangle \) \( \propto \) Ma & Qiu (2018), Braun et al. (2008, 2018)

Lattice QCD

\[
\langle N | \bar{q} \, W \, q | N \rangle_V \\
\langle N | J(0, \xi) \, J(0) | N \rangle_V
\]
There are different techniques:

- **Wilson lines:** \( \langle N|\bar{q}\,W\,q|N\rangle \) \( \propto \) Ji (2013), Radyushkin (2017)

- **Two current operators:** \( \langle N|\mathcal{J}(0,\xi)\mathcal{J}(0)|N\rangle \) \( \propto \) Ma & Qiu (2018), Braun et al. (2008, 2018)
Finite volume: Infrared limit of the theory

- Finite-volume artifacts arise from the interactions with mirror images
- Assuming $L >>$ size of the hadrons $\sim 1/m_\pi$
  - This is a purely infrared artifact
  - We can determine these artifact using hadrons as d.o.f.
Finite volume: Infrared limit of the theory

- Finite-volume artifacts arise from the interactions with mirror images.
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  - This is a purely infrared artifact.
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$$m_N(L) - m_N(\infty) \sim \langle N|\hat{V}|N\rangle_L \sim e^{-m_{\pi}L}$$

Lüscher (1985)
Finite volume effects: Matrix elements

In general, the masses and matrix elements of stable particles have been observed to have these exponentially suppressed corrections.

But matrix elements of non-local currents suffer from larger FV effects:

$$\langle N | J(0, \xi) J(0) | N \rangle_\infty : \text{generally decays as a function of } \xi$$

$$\langle N | J(0, \xi) J(0) | N \rangle_V : \text{periodic, since } \mathcal{J}(t, x) = \mathcal{J}(t, x + L e_i)$$

Expect enhanced finite volume effects to keep periodicity!
Finite volume effects: Matrix elements

Expect enhanced finite volume effects to keep periodicity!
A simple example: mass of a pion

Consider a toy model for mesons

\[ \mathcal{L}_M = \frac{\lambda}{4!} \phi^4 \]

Bare propagator is volume-independent:

\[ \Delta_0(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon} \]

so we have to go to loops… self-energy…

In a finite volume, integrals over momenta become sums:

\[ \int \frac{dk_i}{2\pi} \to \sum_{k_i} \frac{\Delta k_i}{2\pi} = \sum_{k_i} \frac{2\pi \Delta n}{2\pi L} = \frac{1}{L} \sum_{k_i} \]

\[ \int \frac{d^3k}{(2\pi)^3} \to \frac{1}{L^3} \sum_{k_i} \]
A simple example: self-energy of a pion

in infinite volume:

\[ I_\infty = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2_\pi} \]

in finite volume:

\[ I_{\text{FV}} = \frac{1}{L^3} \sum_k \int \frac{dk_4}{2\pi} \frac{1}{k^2 + m^2_\pi} = \sum \int \frac{d^4 k}{(2\pi)^4} \frac{e^{i k \cdot n L}}{k^2 + m^2_\pi} \]

finite/infinite volume difference:

\[ \delta m^2(L) \sim \delta I_{\text{FV}} = I_{\text{FV}} - I_\infty \]

\[ = \sum_{n \neq 0} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{i k \cdot n L}}{k^2 + m^2_\pi} \]

\[ \sim K_1(Lm) \sim \frac{e^{-Lm}}{(Lm)^{3/2}} \]
A simple example: self-energy of a pion

\[ m_\pi(L) = m_\pi + c \frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}} \]

\[ \chi^2/N_{\text{dof}} = 0.9 \]

\[ a_t m_\pi(\infty) = 0.06906(13) \]

\[ m_\pi \sim 390 \text{ MeV}, \quad a_s \sim 0.12 \text{ fm} \rightarrow m_\pi L \sim 3.8, 4.7, 5.6 \]
Our toy model

Consider a theory with two scalar particles

- a light one, $\varphi$, analogous to the pion
- a heavy one, $\chi$, analogous to the nucleon
- momentum independent coupling

Coupling to an external current:

$m_\varphi \ll m_\chi$

$= g$

$= g_\chi$

$= g_\varphi$

$= g_{\chi\varphi}$
Light external states

\[ \mathcal{M}_{\infty}^{(LO)}(\xi, p) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q} \cdot \xi}}{(p_E + q_E)^2 + m_\varphi^2} \]

Even at LO has an integral

Even at LO has an integral

Finite volume correction:

\[ \delta \mathcal{M}_L^{(LO)}(\xi, p) = g_\varphi^2 \sum_{n \neq 0} \int_{q_E} \frac{e^{i\mathbf{q} \cdot (\xi + iLn)}}{(p_E + q_E)^2 + m_\varphi^2} \]

\[ \delta \mathcal{M}_L^{(LO)}(\xi, p) = \frac{m_\varphi g_\varphi^2}{4\pi^2} e^{-i\mathbf{p} \cdot \xi} \sum_{n \neq 0} \frac{K_1(m_\varphi |\xi + Ln|)}{|\xi + Ln|} \sim \frac{m_\varphi g_\varphi^2}{4\pi^2} \frac{K_1(m_\varphi |L - \xi|)}{|L - \xi|} \]

\[ \delta \mathcal{M}_L^{(LO)}(\xi, p) \propto e^{-m_\varphi (L - \xi)} \frac{1}{(L - \xi)^{3/2}} \]
Light external states

\[ \mathcal{M}_\infty^{(LO)} \]

\[ \mathcal{M}_L^{(LO)} \]

Expected behavior!
Light external states

\[
\frac{\delta M_L}{M_\infty} = \begin{cases} 
0.5 & m_\pi L = 4 \\
0.0 & m_\pi L = 6
\end{cases}
\]

\[
m_\pi \xi
\]
Light external states

\[ \frac{\delta M_L}{M_\infty} \]

- \( m_\pi L = 4 \)
- \( m_\pi L = 6 \)

\[ \sim 10\% \] when \( \frac{\xi}{L} = \frac{1}{4} \)
Light external states

100% systematic uncertainty!
inaccurate...despite it being arbitrarily precise!

\[ m_\pi \xi = 2 \]
Heavy external states

Leading order

\[ \delta M^{(LO)}_L(\xi, p) \propto \frac{e^{-m_\chi (L - \xi)}}{(L - \xi)^{3/2}} \ll e^{-m_\phi (L - \xi)} \]

Next to Leading Order

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d) 

(e) \hspace{1cm} (f) \hspace{1cm} (g) \hspace{1cm} (h)
In general...

We find that in general the matrix elements...

\[ \langle M | J(0, \xi) J(0) | M \rangle_L - \langle M | J(0, \xi) J(0) | M \rangle_\infty = P_a(\xi, L)e^{-M(L-\xi)} + P_b(\xi, L)e^{-m_\pi L} + \cdots , \]

Polynomial prefactors \( \propto \frac{L^m}{|L - \xi|^n} \)

This result might be universal and have a better convergence than the EFT used, but we don’t have a proof yet...
Summary

- We presented first steps towards understanding finite-volume artifacts that arise in matrix elements of spatially non-local operators.
  - Matrix elements of spatially-separated currents, one of the approaches to determine hadron structure from lattice QCD.

- We considered a toy model involving two scalar particles to estimate the size of finite-volume corrections.
  - Lightest particle: LO contribution dominant, effects scale like: $P(\xi, L)e^{-m_\pi(L-\xi)}$
  - Heaviest particle: NLO contribution dominant, effects scale like: $P(\xi, L)e^{-m_\pi L}$
Thank you!
Backup slides
Finite volume effects: Matrix elements

Wilson line is not periodic:

\[ W[x + \xi e_i, x] \equiv U_i(x + (\xi - a)e_i) U_i(x + (\xi - 2a)e_i) \times \cdots \times U_i(x + a e_i) \]

Quark bilinears connected to Wilson Lines:

\[ \bar{q}(x + (\xi + nL)e_i) W[x + (\xi + nL)e_i, x] q(x) = \bar{q}(x + \xi e_i) W[x + \xi e_i, x] \left( W[x + Le_i, x]^n \right) q(x) \]

are no periodic. However,

\[ q(x) \text{ and } U(x) \text{ feel boundary conditions} \]

\[ \text{expect enhanced finite volume effects for large } \xi \]
Asymptotic behaviors

\[ \delta M_{L}^{(b)}(\xi, 0) = g^2 g_\varphi g_\chi \sum_{\{n,m\} \neq 0} \left[ \int_{0}^{1} dx \mathcal{I}_2 [|Ln - \xi|; M(x)] \right] \left[ \int_{0}^{1} dy \mathcal{I}_2 [|Lm - \xi|; M(y)] \right], \]

\[ \delta M_{L}^{(c)}(\xi, 0) = 2g^2 g_\varphi^2 \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\chi] \left[ \int_{0}^{1} dx (1 - x) \mathcal{I}_3 [|Lm - \xi|; M(x)] \right], \]

\[ \delta M_{L}^{(d)}(\xi, 0) = g^2 g_\chi \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\chi] \mathcal{I}_1 [|Lm - \xi|; m_\varphi], \]

\[ \delta M_{L}^{(e)}(\xi, 0) = gg_\varphi g_\chi \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\varphi] \left[ \int_{0}^{1} dx \mathcal{I}_2 [|Lm - \xi|; M(x)] \right], \]

\[ \delta M_{L}^{(f)}(\xi, 0) = gg_\chi g_\chi \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\chi] \left[ \int_{0}^{1} dx \mathcal{I}_2 [|Lm - \xi|; M(x)] \right], \]

\[ \delta M_{L}^{(g)}(\xi, 0) = gg_\chi g_\varphi \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\chi] \left[ \int_{0}^{1} dx \mathcal{I}_2 [|Lm|; M(x)] \right], \]

\[ \delta M_{L}^{(h)}(\xi, 0) = \frac{1}{2} g_\chi g_\chi g_\varphi \sum_{\{n,m\} \neq 0} \mathcal{I}_1 [|Ln - \xi|; m_\chi] \mathcal{I}_1 [|Lm|; m_\varphi]. \]
Asymptotic behaviors

\[
\begin{align*}
\delta M_{L}^{(a)}(\xi, 0) & \sim \frac{g^{2}g_{\varphi}^{2}}{128\pi^{3}m_{\varphi}} \left[ \frac{\xi^{1/2}}{(L - \xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L - \xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L - \xi) \right] e^{-m_{\varphi}L}, \\
\delta M_{L}^{(b)}(\xi, 0) & \sim \frac{g^{2}g_{\varphi}g_{\chi}}{64\pi^{3}m_{\varphi}} \left[ \frac{1}{\xi^{1/2}(L - \xi)^{1/2}} H_{1,1/2}(\xi) H_{1,1/2}(L - \xi) \right] e^{-m_{\varphi}L}, \\
\delta M_{L}^{(c)}(\xi, 0) & = \frac{g^{2}g_{\chi}^{2}m_{\chi}^{1/2}m_{\varphi}^{1/2}}{128\pi^{3}m_{\varphi}^{3/2}} \left[ \frac{(L - \xi)^{1/2}}{\xi^{3/2}} H_{1-x,3/2}(L - \xi) \right] e^{-\xi(m_{\chi}-m_{\varphi})} e^{-m_{\varphi}L}, \\
\delta M_{L}^{(d)}(\xi, 0) & = \frac{g^{2}g_{\chi}m_{\chi}^{1/2}m_{\varphi}^{1/2}}{32\pi^{3}} \left[ \frac{1}{\xi^{3/2}(L - \xi)^{3/2}} \right] e^{-\xi(m_{\chi}-m_{\varphi})} e^{-m_{\varphi}L}, \\
\delta M_{L}^{(e)}(\xi, 0) & = \frac{g_{\varphi}g_{\chi}^{2}}{64\pi^{3}} \left[ \frac{1}{\xi^{1/2}(L - \xi)^{3/2}} H_{1,1/2}(\xi) \right. \\
& \quad \left. + \frac{1}{\xi^{3/2}(L - \xi)^{1/2}} H_{1,1/2}(L - \xi) \right] e^{-m_{\varphi}L}, \\
\delta M_{L}^{(f)}(\xi, 0) & = \frac{g_{\varphi}g_{\chi}^{2}m_{\chi}^{1/2}}{64\pi^{3}m_{\varphi}^{1/2}} \left[ \frac{1}{\xi^{3/2}(L - \xi)^{1/2}} H_{1,1/2}(L - \xi) \right] e^{-\xi(m_{\chi}-m_{\varphi})} e^{-m_{\varphi}L}, \\
\delta M_{L}^{(g)}(\xi, 0) & = \frac{g_{\varphi}g_{\chi}m_{\chi}^{1/2}}{64\pi^{3}m_{\varphi}^{1/2}} \left[ \frac{1}{\xi^{3/2}(L - \xi)^{1/2}} H_{1,1/2}(L) \right] e^{-\xi m_{\chi}} e^{-m_{\varphi}L}, \\
\delta M_{L}^{(h)}(\xi, 0) & = \frac{g_{\chi}g_{\varphi}m_{\varphi}^{1/2}m_{\chi}^{1/2}}{64\pi^{3}} \left[ \frac{1}{\xi^{3/2}L^{3/2}} \right] e^{-m_{\chi}x} e^{-m_{\varphi}L},
\end{align*}
\]
Heavy external states: Next to Leading Order

\[ e^{-m \chi \xi} \]

\[ e^{-m \phi \xi} \]

\[ \frac{1}{(m \phi \xi)^{3/2}} \]

\[ m_{\pi} \xi \]

Legend:
- Red line: \( 1, \alpha = 1/2 \)
- Green line: \( x, \alpha = 3/2 \)
- Orange line: \( 1 - x, \alpha = 3/2 \)