Non-perturbative constraints on the matrix elements of the energy-momentum tensor

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Outline

1. Form factors in local QFT
2. The form factors of $T^{\mu\nu}$
3. A distributional matching approach
4. Summary and outlook
1. Form factors in local QFT

• Form factors $F(q^2)$ parametrise the non-perturbative characteristics of matrix elements

  → e.g. spin structure, charge distribution, ...

• In order to fully understand the properties of form factors one therefore requires a non-perturbative approach

• “Local QFT” – define a QFT using a series of physical motivated axioms

  → axioms hold independently of the coupling regime, hence non-perturbative properties can be derived

• One of the key features of this framework is that quantised fields $\varphi(x)$ are distributions [Streater, Wightman; Haag] – this subtlety is important for consistently defining matrix elements and charges

[R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and all that (1964)]
[R. Haag, Local Quantum Physics, Springer-Verlag (1996)]
2. The form factors of $T^{\mu\nu}$

- Due to the properties of $T^{\mu\nu}$ the matrix elements for spin-$\frac{1}{2}$ momentum eigenstates can be written in the following Lorentz covariant manner:

$$\langle p'; m'; M\mid T^{\mu\nu}(0)\mid p; m; M\rangle = \bar{u}_{m'}(p') \left[ \frac{1}{4} \gamma^{\mu}(p + p')^{\nu} A(q^2) + \frac{1}{8M} (p + p')^{\mu\nu}\rho q_{\rho} B(q^2) \right. \\
+ \left. \frac{1}{M} (q^\mu q^\nu - q^2 g^{\mu\nu}) C(q^2) \right] u_m(p) \delta^{(+)}_{M}(p) \delta^{(+)}_{M}(p')$$

where: $|p; m; M\rangle := \delta^{(+)}_{M}(p)|p; m\rangle$, $\delta^{(+)}_{M}(p) := 2\pi \theta(p^0) \delta(p^2 - M^2)$

$$\langle p'; m'; M\mid p; m; M\rangle = (2\pi)^4 \delta^{(+)}(p' - p) \delta^{(+)}_{M}(p') \delta_{m'm}$$

- Physical states have the form:

$$|\Psi^{g}_{M,m}\rangle = \int \frac{d^4p}{(2\pi)^4} \delta^{(+)}_{M}(p) g(p) |p; m\rangle = \int \frac{d^3p}{(2\pi)^3 2p^0} g(p) |\Gamma_{M}^{+}\rangle |p; m\rangle$$

- For simplicity we consider massive canonical spin states, where $m$ is the rest frame spin projection. Could equally well use other spin states (e.g. helicity spin)
3. A distributional matching approach

- **Approach**: Use the distributional properties of the matrix elements to impose constraints on the form factors [PL, Chiu, Brodsky, 1707.06313]

(i) Compute the matrix elements of $P^\mu$ and $J^i$ using the Poincaré transformation properties of the states

→ **Spacetime translations**: $e^{iP\cdot a}|p; k; M\rangle = e^{ip\cdot a}|p; k; M\rangle$

\[
\langle p'; m'; M|P^\mu|p; m; M\rangle = p^\mu (2\pi)^4 \delta^4 (p' - p)\delta_{M}^{(+)}(p')\delta_{m' m}
\]

→ **Lorentz transformations**: $e^{-\beta\cdot J}|p; k; M\rangle = \sum_{l} \mathcal{D}^{s}_{lk}(\beta)|\Lambda(\beta)p; l; M\rangle$

\[
\langle p'; m'; M|J^i|p; m; M\rangle = (2\pi)^4 \delta^{(+)}_{M}(p') \left[ S^i_{m' m} - i\delta_{m' m}\epsilon^{ijk} p^j \frac{\partial}{\partial p^k} \right] \delta^4 (p' - p)
\]

(ii) Compare these expressions with those obtained from the form factor decomposition of the matrix elements
3. A distributional matching approach

- In order to define the Poincaré charges in a consistent manner one must smear the currents with appropriate test functions [Kastler et al.]

\[
P^\mu = \lim_{\substack{d \to 0 \\ R \to \infty}} \int d^4x \ f_{d,R}(x) T^{0\mu}(x)
\]

\[
f_{d,R}(x) := \alpha_d(x_0) F_R(x)
\]

\[
\int dx_0 \alpha_d(x_0) = 1 \quad \alpha_d(x_0) \xrightarrow{d \to 0} \delta(x_0)
\]

\[
F_R(0) = 1 \quad F_R(x) \xrightarrow{R \to \infty} 1.
\]

- One can then relate the matrix elements of the charges to those of the energy-momentum tensor:

\[
\langle p'; m'; M | P^\mu | p; m; M \rangle = \lim_{\substack{d \to 0 \\ R \to \infty}} \tilde{f}_{d,R}(q) \langle p'; m'; M | T^{0\mu}(0) | p; m; M \rangle
\]

\[
\langle p'; m'; M | J^i | p; m; M \rangle = -i\epsilon^{ijk} \lim_{\substack{d \to 0 \\ R \to \infty}} \frac{\partial \tilde{f}_{d,R}(q)}{\partial q_j} \langle p'; m'; M | T^{0k}(0) | p; m; M \rangle
\]

- Identical definitions can also be used to define light-like charges [Jegerlehner]


3. A distributional matching approach

- **$P^\mu$ matrix element**: matching the coefficients of the distributions with the same momentum dependence implies

  \[
  A(q^2)\delta^4(q) = \delta^4(q) \\
  q^j B(q^2)\delta^4(q) = 0 \\
  q^j q^l C(q^2)\delta^4(q) = 0
  \]

  \[
  \lim_{n \to \infty} \int d^4 q \, \delta^{\{0\}}_n(q) A(q^2) = 1 \\
  A(0) = 1
  \]

- **$J^i$ matrix element**: performing the same procedure one obtains

  \[
  A(q^2)\partial^k \delta^4(q) = \partial^k \delta^4(q) \\
  A(q^2)\delta^4(q) = \delta^4(q) \\
  B(q^2)\delta^4(q) = 0 \\
  q^l \left[ A(q^2) + B(q^2) \right] \partial^k \delta^4(q) = 0 \\
  q^j q^l C(q^2)\partial^k \delta^4(q) = 0, \quad (l \neq k) \\
  q^j C(q^2)\delta^4(q) = 0
  \]

  \[
  A(0) = 1 \\
  B(0) = 0
  \]

The first set of constraints are a subset of these.
3. A distributional matching approach

• This matching approach recovers the standard results for the form factors in the $q \to 0$ limit, but no choice of frame, wavepacket, operator component, or spin component $m$ is required.

• **What about the boost matrix elements?** Interestingly, by using the matrix elements for the boost operator [Bakker, Leader, Trueman]:

$$\langle p'; m'; M | K^i | p; m; M \rangle = (2\pi)^4 \delta^{(+)}_M (p') \left[ \frac{(p \times \sigma_{m'm})^i}{2(p^0 + M)} + i \delta_{m'm} p^0 \frac{\partial}{\partial p_i} \right] \delta^4 (p' - p)$$

one obtains *precisely the same* constraints as those derived using the angular momentum operator $J^i$.

• These findings demonstrate that the $q \to 0$ constraints imposed on the form factors $A(q^2)$, $B(q^2)$ and $C(q^2)$ are actually a consequence of the physical on-shell requirement of the states, and the manner in which they transform under Poincaré transformations.

[Bakker, Leader, Trueman; hep-ph/0406139]
4. Summary and outlook

• Using a distributional matching approach one can derive low-energy ($q \to 0$) constraints on the form factors associated with the matrix elements of $T^{\mu\nu}$ [PL, Chiu, Brodsky, 1707.06313]

• This approach involves expanding the matrix elements of Poincaré charges in two different ways, and then comparing the coefficients

• The same set of constraints: $A(0)=1$, $B(0)=0$ are derived from the angular momentum and boost matrix elements
  → these constraints are a consequence of the on-shellness and Poincaré transformation properties of the states

• This procedure has potentially interesting generalisations
  → form factors associated with different currents
  → matrix elements involving states with higher spin
Different spin states:

**Canonical:**  
\[ |p, s\rangle = B(v)|0, s \rangle = B(v)D_{m1/2}^{1/2}(R(s))|0, m\rangle \]

**Jacob-Wick helicity:**  
\[ |p, \lambda\rangle_{jW} = R_z(\phi)R_y(\theta)R_z(-\phi)B_z(v)|0, m = \lambda\rangle \]

**Wick helicity:**  
\[ |p, \lambda\rangle = R_z(\phi)R_y(\theta)B_z(v)|0, m = \lambda\rangle \]

- Wick helicity states have more complicated matrix elements:

\[
W\langle p'; m'; M | J^i | p; m; M \rangle_W = (2\pi)^4 \delta^{(+)}_M(p') \left[ m \delta_{m'm} \frac{(\delta^i_1 p^1 + \delta^i_2 p^2)|p|}{(p^1)^2 + (p^2)^2} \right. - \left. i\delta_{m'm} \epsilon^{ijk} p^j \frac{\partial}{\partial p_k} \right] \delta^4(p' - p)
\]

- In the spin-\(1/2\) case: \(S^i_{m'm} = \frac{1}{2}\sigma^i_{m'm}\) but for higher spin states this matrix is more complicated [Bakker, Leader, Trueman]

[Bakker, Leader, Trueman; hep-ph/0406139]
Definitions used:

\[ h(q) \partial^k \delta^4(q) = h(0) \partial^k \delta^4(q) - (\partial^k h)(0) \delta^4(q) \]

\[ q = p' - p, \quad \bar{p} = \frac{1}{2}(p' + p) \]

\[
\frac{\partial}{\partial q_k} \left\{ [\bar{u}_{m'} (\bar{p} + \frac{1}{2} q) u_m (\bar{p} - \frac{1}{2} q)] q^0 = \frac{\bar{p} \cdot q}{\bar{p}^0} \right\} \bigg|_{q = 0} = \frac{i}{(\bar{p}^0 + M)} \epsilon^{klm} \bar{p}^l \sigma^m_{m'} \sigma^n_{m''},
\]

\[
\bar{u}_{m'} (\bar{p}) \sigma^{jk} u_m (\bar{p}) = 2 \epsilon^{jkl} \left[ \bar{p}^0 \sigma^l_{m'm} - \frac{\bar{p}^l (\bar{p} \cdot \sigma_{m'm})}{\bar{p}^0 + M} \right],
\]

Form factor calculation – angular momentum case:

\[
(2\pi)^4 \left[ \frac{1}{2} \sigma^i_{m'm} + i \delta_{m'm} \epsilon^{ijk} \bar{p}^j \frac{\partial}{\partial q_k} \right] \delta^4(q) =
\]

\[
(2\pi) \delta \left( q^0 - \frac{\bar{p} \cdot q}{\bar{p}^0} \right) \lim_{R \to \infty} \frac{1}{2} \sigma^i_{m'm} \hat{f}_{d,R}(q) A(q^2) + i \delta_{m'm} \epsilon^{ijk} \bar{p}^j \frac{\partial}{\partial q_k} \delta^4(q) -
\]

\[
\left[ \frac{\bar{p}^0}{2M} \sigma^i_{m'm} - \frac{\bar{p}^0 (\bar{p} \cdot \sigma_{m'm})}{2M(\bar{p}^0 + M)} \right] \hat{f}_{d,R}(q) B(q^2) - \epsilon^{ijk} \frac{\bar{p}^l \bar{u}_{m'} (\bar{p}) \sigma^j_{m'm} u_m (\bar{p}) q^0}{8M\bar{p}^0} \frac{\partial}{\partial q_k} \left[ A(q^2) + B(q^2) \right]
\]

\[
+ i \delta_{m'm} \epsilon^{ijk} \frac{q^0 q^j}{\bar{p}^0} \frac{\partial}{\partial q_k} \hat{f}_{d,R} C(q^2) + \epsilon^{ijk} \left[ \frac{q^j \bar{p}^k}{(\bar{p}^0)^2} \delta_{m'm} - \frac{\bar{p} q^j q^j (\bar{p} \times \sigma_{m'm})^k}{2M(\bar{p}^0)^2(\bar{p}^0 + M)} \hat{f}_{d,R}(q) C'(q^2) \right]
\]
Local QFT approaches are defined by a core set of axioms:

**Axiom 1 (Hilbert space structure).** The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\mathcal{P}_+^\uparrow$.

**Axiom 2 (Spectral condition).** The spectrum of the energy-momentum operator $P^\mu$ is confined to the closed forward light cone $\mathcal{V}^+ = \{ p^\mu \mid p^2 \geq 0, p^0 \geq 0 \}$, where $U(a,1) = e^{ip^\mu a^\mu}$.

**Axiom 3 (Uniqueness of the vacuum).** There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.

**Axiom 4 (Field operators).** The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type $(\kappa)$) which have components $\varphi^{(\kappa)}_i(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.

**Axiom 5 (Relativistic covariance).** The fields $\varphi^{(\kappa)}_i(x)$ transform covariantly under the action of $\mathcal{P}_+^\uparrow$:

$$U(a, \alpha) \varphi^{(\kappa)}_i(x) U(a, \alpha)^{-1} = S^{(\kappa)}_{ij}(\alpha^{-1}) \varphi^{(\kappa)}_j(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\mathcal{L}_+^\uparrow$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \mathcal{L}_+^\uparrow$.

**Axiom 6 (Local (anti-)commutativity).** If the support of the test functions $f, g$ of the fields $\varphi^{(\kappa)}_i, \varphi^{(\kappa)}_m$ are space-like separated, then:

$$[\varphi^{(\kappa)}_i(f), \varphi^{(\kappa)}_m(g)]_\pm = \varphi^{(\kappa)}_i(f)\varphi^{(\kappa)}_m(g) \pm \varphi^{(\kappa)}_m(g)\varphi^{(\kappa)}_i(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi^{(\kappa)}_i, \varphi^{(\kappa)}_m$. 

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**A. Wightman**


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**R. Haag**

Quantum fields $\varphi(x)$ are distributions – *what difference does this make?*

→ This means that they cannot be evaluated at a single point (e.g. think of the Dirac delta $\delta(x)$ at $x=0$)

→ Need to 'average them out' over some spacetime region $A$

$$\mathcal{M}_\varphi := \int_A d^4x \varphi(x)f(x)$$

Can think of this as the performance of a measurement $M_\varphi$ in the region $A$ where $f(x)$ is non-zero

• But why? – Heisenberg's uncertainty principle! $\Delta x \Delta p \sim \frac{\hbar}{2}$
The distributional nature of form factors implies that these objects are not in general continuous. Nevertheless, form factors $F(q^2)$ are seemingly measured at specific values of $q^2$. In order to reconcile these points of view one must recognise that one cannot ever physically measure a form factor at a specific value of $q^2$, since this would require an experiment with infinite precision.

In practice, a measurement of $F(q^2)$ at $q^2 = Q^2$ is really a measurement of an averaged-out quantity $F(Q^2; \Delta)$ in some small but non-vanishing region $[Q^2 - \Delta, Q^2 + \Delta]$.

$F(Q^2; \Delta)$ is the convolution of $F(q^2)$ with a test function $f_\Delta(q^2)$, which characterises the resolution $\Delta$ of the experiment

$$\bar{F}(Q^2; \Delta) := (F * f_\Delta)(Q^2)$$