Light-Front Simulations of True Muonium

Hank Lamm
with Rich Lebed
17 May 2018
LIGHTCONE 2018
Within the Standard Model, *lepton universality* is broken only by the Higgs interaction.
Within the Standard Model, *lepton universality* is broken only by the Higgs interaction. But $m_\nu$ implies this isn’t the end of the story.
What’s the deal with muon physics?¹²³

**mu·on prob·lem:** (n) the curious observation that discrepancies exist in muon sector

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The ultimate discriminator: true muonium

I don't know about μ?
Well, I'm feeling μ+μ

- Taylor Swift
on Muon Problem
Wait? I can bind muons?

- Proposed in 1961, still undetected

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6 Y. Ji and H. Lamm. “Discovering True Muonium in \( \eta, \eta' \to (\mu^+\mu^-)\gamma \)”. In: in prep. (2018).
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- *Nearly* purely leptonic, *nearly* purely QED bound state

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...but many channels suggested

---

4 S. J. Brodsky and R. F. Lebed. “Production of the Smallest QED Atom: True Muonium ($\mu^+\mu^-$)”.


6 Y. Ji and H. Lamm. “Discovering True Muonium in $\eta, \eta' \rightarrow (\mu^+\mu^-)\gamma$”. In: in prep. (2018).
Difficult, but it’s what we pay experimentalists for

DIRAC, HPS are existing fixed targets
LEMAM is proposed
µ⁺µ⁻ collider
Proposed e⁺e⁻ collider at BINP
REDTOP is a proposed η/η' factory

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How can we construct a LF model for true muonium?

- Fock state expansion with partons

\[ |\Psi\rangle \equiv \sum_n \int [d\mu_n] |\mu_n\rangle \langle \mu_n| \Psi; M, P^+, P_\perp, S^2, S_z; h \rangle \]

\[ \equiv \sum_n \int [d\mu_n] |\mu_n\rangle \Psi_n|_h(\mu) , \] (1)
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- \( H_{LC} = P^- P^+ = M^2 \)

- Acting on a state with this gives

\[ \sum_{\lambda_j'} \int_D dx'_j d^2 k_{\perp,j}' \langle x_i, k_{\perp,i}; \lambda_i | H_{LC} | x_j, k_{\perp,j}; \lambda_j', \rangle \psi(x'_j, k_{\perp,j}'; \lambda_j') = M^2 \psi(x_i, k_{\perp,i}; \lambda_i) \tag{2} \]
Hamiltonian operator have sparse structure

Gauge fixing required for defining $H_{LC}$. Standard choice is $A^+ = 0$

$$H_{LC} = T + V + S + C + F$$ \hfill (3)

<table>
<thead>
<tr>
<th>Sector</th>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\gamma\rangle$</td>
<td>0</td>
<td>•</td>
<td>V</td>
<td>V</td>
<td>F</td>
</tr>
<tr>
<td>$</td>
<td>e\bar{e}\rangle$</td>
<td>1</td>
<td>V</td>
<td>•</td>
<td>S</td>
<td>V</td>
</tr>
<tr>
<td>$</td>
<td>\mu\bar{\mu}\rangle$</td>
<td>2</td>
<td>V</td>
<td>S</td>
<td>•</td>
<td>•</td>
</tr>
<tr>
<td>$</td>
<td>e\bar{e}\gamma\rangle$</td>
<td>3</td>
<td>F</td>
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<td>•</td>
<td>•</td>
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<td>$</td>
<td>\mu\bar{\mu}\gamma\rangle$</td>
<td>4</td>
<td>F</td>
<td>•</td>
<td>V</td>
<td>S</td>
</tr>
</tbody>
</table>

The Hamiltonian matrix for two-flavor QED, where $n$ labels Fock states. The vertex, seagull and fork interactions are denoted by $V$, $S$, $F$ respectively.

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Use effective interactions, or you’ll have a bad time

**IF** we truncate Fock space, we have **model wavefunctions** *(Yay)* but breaks **gauge invariance** *(Boo)*

---

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- Consider a two sector $H_{LC}$ operator:

$$H_{LC} = \begin{pmatrix} H_{PP} & H_{PQ} \\ H_{QP} & H_{QQ} \end{pmatrix}$$

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- A generic $Q$ state can be rewritten as

$$\hat{Q}\left|\Psi\right\rangle = \hat{Q}(\omega - H_{LC})^{-1}\hat{Q}H_{LC}\hat{P}\left|\Psi\right\rangle$$  \hspace{1cm} (5)

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  \]  
  (5)
- With this, we write an effective Hamiltonian **without** $Q$ states
  \[
  H^{eff}(\omega)_{LC} = \hat{P}H_{LC}\hat{P} + \hat{P}H_{LC}\hat{Q}(\omega - H_{LC})^{-1}\hat{Q}H_{LC}\hat{P}
  \]  
  (6)

---

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  \] (6)
- Replace \(\omega\) by \(f(x', k'_\perp, x, k_\perp)\) to remove divergences

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TMSWIFT: Two-particle Hamiltonian Solver

---TMSWIFT Parameters---

Output File: OUT/truem_7p^4Ei_PWz=3, CwFtn0 o.o.A50
Number of Flavors: 3
J, Z: 0
Alpha: 0.3
N_tot: 27
N_mu Discretization Flag: clenshaw_curtis
N_theta Discretization Flag: clenshaw_curtis
Annihilation Flag: 0
Flavor Mixing Flag: 0
Asymptotic G2 Flag: 0
NUMPS Flag: 1

Mass B: 1
P Bohr 0: 1.15
N_mu 0: 3
N_theta 0: 3
Lambda 0: 12

Mass 1: 0.5
P Bohr 1: 0.075
N_mu 1: 3
N_theta 1: 3
Lambda 1: 10.1

Mass 2: 2
P Bohr 2: 0.3
N_mu 2: 3
N_theta 2: 3
Lambda 2: 90

---
Regularization improves singlet state $\Lambda$ dependence

- Remove remaining uncancelled divergence in $\langle \uparrow \downarrow | H_{\text{eff}} | \downarrow \uparrow \rangle$

---

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$$\lim_{k'_{\perp} \to \infty} \delta G_2(x, k_{\perp}; x', k'_{\perp}) = \frac{1}{(k'_{\perp})^2} \to \delta (r)$$  \hspace{1cm} (7)

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$$\lim_{k'_\perp \to \infty} \delta G_2(x, k'_\perp ; x', k'_\perp) = \frac{1}{(k'_\perp)^2} \to \delta(r)$$

\[ M^2 \leq \begin{array}{c}
M^2 \text{ for (top) } 1^3S^0_1, \ (bottom) \ 1^1S^0_0 \text{ as function of } \Lambda_\mu. \ (\circ) \text{ are from}^9, \ (\blacksquare) \text{ remove } \delta G_2. \\
\end{array} \]

All the bound states!

Units of $m_\mu$ for $m_e = 1/2m_\mu$, $\alpha = 0.3$, $\Lambda_e = 1/2\Lambda_\mu$, $\Lambda_\mu = m_\mu \alpha/2$

- Spectroscopic notation $^{2S+1}L^J_{J_z}$
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- Spectroscopic notation $^{2S+1}L^J_{J_z}$
- Rotational invariance is made **worse** for regularization
- **Renormalization effects** seen in positronium
- Discrepancy comes from excluding high momentum electron states
Eigenvalue shifts \( \Delta M^2 \equiv M_{\mu\mu}^2 - M_0^2 \) (in units of \( m_\mu^2 \)) for \( 1^3S_1^0 \). The dashed line IF is the instant-form prediction, using the non-relativistic wave function, while the light-front (LF) points are obtained by taking \( \alpha = 0.3 \).
$N, \Lambda$ limits appear regulated

- Extract $M^2$ and $f_i$ from finite-size and finite-cutoff through Padé expansion

$$M^2(N, \Lambda) = \frac{M^2_\Lambda + \frac{b}{N} + \frac{c}{N^2}}{1 + \frac{d}{N} + \frac{e}{N^2}}$$

$M^2(N,\Lambda_{\mu})$ of Triplet State for $\alpha=0.2$
Results for the $M^2(\alpha)$ and $f_{V,P}(\alpha)$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$M^2(1^1S_0)$</th>
<th>$f_V(1^1S_0)$</th>
<th>$M^2(1^3S_1)$</th>
<th>$f_P(1^3S_1)$</th>
<th>$C_{\text{HFS,LF}}$</th>
<th>$C_{\text{HFS,ET}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3.99989993(3)</td>
<td>4.18(10) x 10^{-5}</td>
<td>3.9989996(3)</td>
<td>3.893(6) x 10^{-5}</td>
<td>0.76(77)</td>
<td>0.5834</td>
</tr>
<tr>
<td>0.02</td>
<td>3.9995997(2)</td>
<td>1.1(4) x 10^{-4}</td>
<td>3.996002(2)</td>
<td>1.088(7) x 10^{-4}</td>
<td>0.79(42)</td>
<td>0.5837</td>
</tr>
<tr>
<td>0.03</td>
<td>3.9990987(4)</td>
<td>2.05(9) x 10^{-4}</td>
<td>3.999101(2)</td>
<td>1.93(6) x 10^{-4}</td>
<td>0.74(34)</td>
<td>0.5841</td>
</tr>
<tr>
<td>0.04</td>
<td>3.998397(4)</td>
<td>3.15(5) x 10^{-4}</td>
<td>3.998404(5)</td>
<td>3.07(7) x 10^{-4}</td>
<td>0.76(56)</td>
<td>0.5847</td>
</tr>
<tr>
<td>0.05</td>
<td>3.9974914(4)</td>
<td>4.466(2) x 10^{-4}</td>
<td>3.9975098(3)</td>
<td>3.95(2) x 10^{-4}</td>
<td>0.74(2)</td>
<td>0.5855</td>
</tr>
<tr>
<td>0.07</td>
<td>3.995068(3)</td>
<td>7.404(7) x 10^{-4}</td>
<td>3.9951351(8)</td>
<td>5.908(5) x 10^{-4}</td>
<td>0.7(4)</td>
<td>0.5877</td>
</tr>
<tr>
<td>0.1</td>
<td>3.98987(6)</td>
<td>1.273(2) x 10^{-3}</td>
<td>3.990137(3)</td>
<td>9.16(3) x 10^{-4}</td>
<td>0.67(2)</td>
<td>0.5922</td>
</tr>
<tr>
<td>0.2</td>
<td>3.9576(6)</td>
<td>3.9(2) x 10^{-3}</td>
<td>3.9614(5)</td>
<td>1.9(2) x 10^{-3}</td>
<td>0.6(2)</td>
<td>0.6204</td>
</tr>
<tr>
<td>0.3</td>
<td>3.8996(6)</td>
<td>1.02(3) x 10^{-2}</td>
<td>3.91538(4)</td>
<td>2.39(2) x 10^{-3}</td>
<td>0.49(2)</td>
<td>0.6735</td>
</tr>
</tbody>
</table>

\[
f_V(P) = \int \frac{dx}{\sqrt{x(1-x)}} \frac{d^2k_\perp}{(2\pi)^3} \left[ \psi_{J=0}^J(k_\perp, x, \uparrow \downarrow) \mp \psi_{J=0}^J(k_\perp, x, \downarrow \uparrow) \right]
\]

\[\text{Dirac-Coulomb, \ } O(\alpha^7)\text{\ TMSWIFT}\]

---

Only minor issues with $\alpha$-dependence

Fit parameters of $M^2(\alpha) = \left( \sum_{\beta} N_{\beta} \alpha^\beta \right)^2$. The perturbative predictions are $N_2 = -\frac{1}{4}, N_{4s} \approx -0.328, N_{4t} \approx 0.255$

<table>
<thead>
<tr>
<th>$E_n$</th>
<th>$\alpha$</th>
<th>$N_0$</th>
<th>$N_2$</th>
<th>$N_4$</th>
<th>$N_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^1S_0$</td>
<td>[0.01,0.3]</td>
<td>1.999999998(2)</td>
<td>-0.2500(2)</td>
<td>-0.37(5)</td>
<td>-0.04(21)</td>
</tr>
<tr>
<td></td>
<td>[0.01,0.1]</td>
<td>1.9999999990(2)</td>
<td>-0.25004(2)</td>
<td>-0.35(2)</td>
<td>0.08(10)</td>
</tr>
<tr>
<td>$1^3S_1$</td>
<td>[0.01,0.3]</td>
<td>1.999999998(2)</td>
<td>-0.24990(8)</td>
<td>0.39(3)</td>
<td>-0.78(8)</td>
</tr>
<tr>
<td></td>
<td>[0.01,0.1]</td>
<td>1.9999999979(6)</td>
<td>-0.24993(5)</td>
<td>0.38(3)</td>
<td>-0.60(26)</td>
</tr>
</tbody>
</table>

Fit parameters of $f_i(\alpha) = N \alpha^\beta$. The perturbative prediction is $\beta = 3/2$.

<table>
<thead>
<tr>
<th>$f_i$</th>
<th>$\alpha$</th>
<th>$N$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_V$</td>
<td>[0.01,0.3]</td>
<td>0.0412(9)</td>
<td>1.510(7)</td>
</tr>
<tr>
<td></td>
<td>[0.01,0.1]</td>
<td>0.0411(3)</td>
<td>1.509(3)</td>
</tr>
<tr>
<td>$f_P$</td>
<td>[0.01,0.3]</td>
<td>0.022(3)</td>
<td>1.37(4)</td>
</tr>
<tr>
<td></td>
<td>[0.01,0.1]</td>
<td>0.0240(8)</td>
<td>1.394(10)</td>
</tr>
</tbody>
</table>
Studying $\Psi(\mu) = a\mu^{-\kappa}$ can guide phenomenology

Use to build models of mesons

Strongly-coupled QED seems to indicate that $\kappa \to 3.5$ for dominant term, and $\kappa \to 2.5$ for subdominant term.
True muonium has pushed LF an inch further up the hill

- **True muonium** presents novel LF testing ground
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- Leptonic loops, masses, decay rates, and wavefunctions computed
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- **True muonium** presents novel LF testing ground
- Leptonic loops, masses, decay rates, and wavefunctions computed
- TMSWIFT allows for renormalization implementations and new Fock states