Non-perturbative study of the three-body system using the Bethe-Salpeter approach

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Light Cone 2018
Jefferson Lab, US
May 17, 2018
Understanding the structure of non-perturbative few-body systems, from an fundamental point of view is important for applications in hadron physics, e.g. for studies of the nucleon.

One important aspect is to obtain a reliable solution directly in Minkowski space, so that dynamical observables such as form factor can be calculated.

In this talk, the solutions of the Bethe-Salpeter equation for a bound-state system of three bosons, bounded through a (two-body) zero range interaction, using three different approaches are discussed:

- LF projection, i.e. only retaining the valence component, in Minkowski space.
- Solution of the BS equation in Euclidean space, through Wick rotation
- Solution of the BS equation in Minkowski space by direct integration (preliminary results).
Three-body problem with zero-range interaction

- Three-body Bethe-Salpeter equation (Frederico, PLB 282 (1992) 409):
  \[ \nu(q, p) = 2iF(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p - q - k)^2 - m^2 + i\epsilon} \nu(k, p) \]

- Equal-mass case, bare propagators.
- \( \nu(q, p) \) is one of the Faddeev components of the total vertex function.
- \( F(M_{12}) \): two-body scattering amplitude characterized by scattering length \( a \) and \( M_{12}^2 = (p - q)^2 \).
  - 1) \( a < 0 \): Borromean system, no two-body bound state, 2) \( a > 0 \): two-body bound state exists.
• **LF equation:**
  - After the LF projection, i.e. introducing $k = k_0 \pm k_z$ and integrating over $k_-$, one obtains the three-body LF equation (Carbonell and Karmanov, PRC 67 (2003) 037001):

$$
\Gamma(k_\perp, x) = \frac{F(M_{12})}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1 - x - x')} \int_0^\infty \frac{d^2k_\perp'}{M_0^2 - M_3^2} \Gamma(k_\perp', x')
$$

with $M_0^2 = (k_0^2 + m^2)/x' + (k_\perp^2 + m^2)/x + ((k_\perp' + k_\perp)^2 + m^2)/(1 - x - x')$

• **Euclidean BS equation:**
  - Through a change of variables $k = k' + \frac{p}{3}$ and $q = q' + \frac{p}{3}$, and a subsequent Wick rotation (Ydrefors et al, PLB 770 (2017) 131):

$$
v_E(q'_4, q'_v) = \frac{2F(-M_{12}^2)}{(2\pi)^3} \int_{-\infty}^\infty dk'_4 \int_0^\infty dk'_v \frac{dk'_4 \Pi(q'_4, q'_v, k'_4, k'_v)}{(k'_4 - \frac{i}{3} M_3)^2 + k'_v^2 + m^2} v_E(k'_4, k'_v),
$$

with $M_{12}^2 = (\frac{2}{3} i M_3 + q'_4)^2 + q'_v^2$. The kernel $\Pi$ is here given by

$$
\Pi(q'_4, q'_v, k'_4, k'_v) = \frac{k'_v}{2q'_v} \log \frac{(k'_4 + q'_4 + \frac{i}{3} M_3)^2 + (q'_v + k'_v)^2 + m^2}{(k'_4 + q'_4 + \frac{i}{3} M_3)^2 + (q'_v - k'_v)^2 + m^2}.
$$

• Both the equations can be solved with standard methods, e.g. by using splines.
Direct method

- Direct integration of the BS equation, treating explicitly the singularities.
- The same approach was used by Carbonell and Karmanov (PRD 90 (2014) 056002) to solve the two-body problem (finite-range interaction).
- The equation for the vertex function, \( v(q_0, q_v) \) can be written in the "non-singular" form

\[
v(q_0, q_v) = \frac{\mathcal{F}(M_{12})}{(2\pi)^4} \int_0^\infty k_v^2 dk_v \left\{ i \frac{[\Pi(q_0, q_v; \varepsilon_k, k_v) v(\varepsilon_k, k_v) + \Pi(q_0, q_v; -\varepsilon_k, k_v) v(-\varepsilon_k, k_v)]}{2\varepsilon_k} 
- 2 \int_0^0 dk_0 \left[ \frac{\Pi(q_0, q_v; k_0, k_v) v(k_0, k_v) - \Pi(q_0, q_v; -\varepsilon_k, k_v) v(-\varepsilon_k, k_v)}{k_0^2 - \varepsilon_k^2} \right] \right\},
\]

using, e.g.,

\[
[k_0^2 - k_v^2 - m^2 + i\varepsilon]^{-1} = PV[k_0^2 - \varepsilon_k^2]^{-1} - i\pi/(2\varepsilon_k)\left[\delta(k_0 - \varepsilon_k) + \delta(k_0 + \varepsilon_k)\right].
\]
Above,

where \( \varepsilon_k = \sqrt{k_v^2 + m^2}, k_v = |\vec{k}| \) and the kernel \( \Pi \) only has weak, logarithmic, singularities. For \( a < 0 \) (considered here) \( F(M_{12}) \) has no pole.

- The singularities at \( k_0 = \pm \varepsilon_k \) were subtracted.
- We have solved the above equation by using a spline expansion for \( v \), i.e.

\[
v(q_0, q_v) = \sum_{ij} C_{ij} S_i(q_0) S_j(q_v).
\]
The (complete) BS equation gives a stronger bound system compared to the LF one for all $a$.

For $a < 0$ (i.e. a Borromean system) the solution with the smallest $M_3^2$, i.e. the formal ground state, is physical.

However, for $a > 0$, i.e. a two-body bound state exists, the lowest state is unphysical.

$M_3^2 > -\infty$: No Thomas collapse in the non-relativistic sense, i.e. an effective short-range repulsion.

The higher-Fock state contributions beyond the valence to the kernel can be interpreted as an effective three-body force of relativistic origin.
The LF and (Euclidean) BS vertex functions cannot be directly compared with each other.

However, we can define the transverse amplitudes

\[
A^{\text{LF}}(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = A_1^{\text{LF}} + A_2^{\text{LF}} + A_3^{\text{LF}} = -\frac{\sqrt{2\pi}}{4} \\
\times \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\Gamma(\vec{k}_{1\perp}, x_1)}{x_1 x_2 (1-x_1-x_2)} \left( \frac{\Gamma(\vec{k}_{2\perp}, x_2)}{M_0^2 - M_3^2} \right)
\]

and

\[
A^{\text{EBS}}(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = A_1^{\text{EBS}} + A_2^{\text{EBS}} + A_3^{\text{EBS}} = \\
- i \int dk_{14}^z dk_{24}^z [v_E(k_{14}, k_{1v}) + v_E(k_{24}, k_{2v}) + v_E(k_{34}, k_{3v})] \Pi_1 \Pi_2 \Pi_3
\]

where \( \Pi_j^{-1} = (k_{j4}^2 - i\frac{1}{3} M_3)^2 + k_{jz}^2 + k_{j\perp}^2 + m^2 \)
In both frameworks the first excited state has one node, and the ground state has no node. This confirms these assignments.

The extra contributions included in the "full" BS solution has a significant impact on the transverse amplitude, especially for the first excited state.
The three-body binding energy (for fixed $a$) is calculable both in Minkowski and Euclidean spaces.

In the table are shown for three cases ($a m = -1.28, -1.5, -1.705$), the obtained eigenvalue using the $B_3$ from the Euclidean calculation.

<table>
<thead>
<tr>
<th>$a m$</th>
<th>$B_3 / m$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.28$</td>
<td>0.006</td>
<td>$0.999 - 0.0544i$</td>
</tr>
<tr>
<td>$-1.5$</td>
<td>0.395</td>
<td>$1.000 + 0.0023i$</td>
</tr>
<tr>
<td>$-1.705$</td>
<td>1.001</td>
<td>$0.997 + 0.106i$</td>
</tr>
</tbody>
</table>

Results good for the case $a m = -1.5$, but the error in the imaginary is getting quite large for more strongly bound system.

One reason for the non-zero imaginary part could be the use of finite cuts, i.e. $k_{\text{max}} / m = 6.0$ and $k_{0\text{max}} / m = 13.0$ (first two cases) and $k_{0\text{max}} / m = 15.0$ (third case).

Euclidean solution obtained without cuts, i.e. using a mapping.
The figure shows the real and imaginary parts of $v(q_0, q_v)$ at fixed $q_v/m = 0.5$, for the case $B_3/m = 0.395$.

It is seen that there are four peaks (either singularities or branch cuts). It turns out that they have the positions $q_0 = M_3 \pm \sqrt{q_v^2 + 4m^2}$ and $q_0 = M_3 \pm q_v$, shown by red dashed lines. These are thus moving peaks depending on $q_v$.

The non-smooth behavior of $v$ makes the solution of this problem numerically very challenging.
Transverse amplitude in Minkowski space

\[
L_1(k_{1\bot}, k_{2\bot}) = \int_{-\infty}^{\infty} dk_{1z} \left\{ \frac{i\pi}{2k_{10}} \left[ v(k_{10}, k_{1\nu}) \chi(k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot}) + v(-k_{10}, k_{1\nu}) \chi(-k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot}) \right] \right. \\
- \int_{0}^{\infty} dk_{10} \frac{v(k_{10}, k_{1\nu}) \chi(k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot}) - v(-k_{10}, k_{1\nu}) \chi(-k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot})}{k_{10}^2 - k_{20}^2} \\
- \left. \int_{0}^{\infty} dk_{10} \frac{v(-k_{10}, k_{1\nu}) \chi(-k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot}) - v(-k_{10}, k_{1\nu}) \chi(-k_{10}, k_{1z}, k_{1\bot} ; k_{2\bot})}{k_{10}^2 - k_{20}^2} \right\},
\]

with \( k_{10} = \sqrt{k_{1z}^2 + k_{1\bot}^2 + m^2} \) and \( \chi \) is a known function having only weak, square root, singularities.
The figure compares (as an example) the modulus of the transverse amplitudes for the case $B_3/m = 0.395$.

The agreement between the two approaches is good.

Even though the Minkowski space amplitude, $v(q_0, q_v)$, has a non-smooth behavior, a smooth transverse amplitude is obtained.
One alternative in order to avoid the numerical difficulties with the direct method, could be to use the Nakanishi integral representation. This has been used successfully in the two body-case where the BS amplitude is written in the form:

\[ \Phi(k, p) = \int_{-1}^{1} dz' \int_{0}^{\infty} \frac{g(\gamma', z'; \kappa^2)}{(\gamma' + \kappa^2 - k^2 - (p \cdot k)z' - i\epsilon)^3} \] \[ \kappa^2 = m^2 - M^2 / 4 \]  

Similarly, in the three-body case, Nakanishi integral representations could be used for \( v(q, p) \) and \( F(M_{12}) \), and thus produce a non-singular integral equation. This is planned for the near future.
Conclusions

- We have in this work studied a system of three bosons interacting through a zero-range potential using three approaches. Namely, 1) using the valence LF equation in Minkowski space, 2) Solving the 4-dimensional Euclidean BS equation, 3) Solving the 4-dimensional BS equation by direct integration in Minkowski space.

- The contributions beyond the valence have large impact both on binding energies and transverse amplitudes. These contributions can be interpreted as an effective three-body force of relativistic origin.

- The direct method is of great interest since it can give a BS amplitude defined in Minkowski space, needed to compute dynamical observables.

- This is work is in progress. However, we have shown that the binding energy (at least for modest $B_3$) is in fair agreement with the Euclidean. The transverse amplitudes are also in fair agreement.

- Unfortunately, the method is numerically very challenging due the treatment of the many singularities.

- One way to solve this could be to use a Nakanishi integral representation (similarly to the two-body case) for the BS amplitude, and it will be done in the near future.