Motivations

♦ what is the light-front vacuum state?

♦ how the information about mass appears at the light-front hypersurface

♦ how to deal with the light-front infra-red singularities for $k^+ \to 0$?

♦ discussion for free field massive scalar field
quantum fields are operator-valued distributions

smeared quantum fields - local field operators - act in the Hilbert space

the vacuum sector of Hilbert space - $\mathcal{H}_0$ - consists of all states that can be created from the vacuum by local field operators

the purpose of smearing for states in $\mathcal{H}_0$ is to make sure that these states have finite norm and thus really are Hilbert space states

for local field operators no canonical quantization procedure

canonical smearing for equal-time quantization procedure

canonical smearing for light-front quantization procedure
Notation for equal-time quantization

- canonically smeared scalar fields

\[ \hat{\phi}[t,f] = \int_{\mathbb{R}^3} d^3 x f(x) \hat{\phi}(t,x), \quad x = (x, y, z) \]

- real-valued test function \( f(x) \in \mathcal{S}(\mathbb{R}^3) \) - Schwartz class

- Fourier transform of test function

\[ \tilde{f}(k) = \int_{\mathbb{R}^3} d^3 x f(x) e^{-i k \cdot x}, \quad \tilde{f} \in \mathcal{S}(\mathbb{R}^3) \]

- \( \omega^a f : \mathbb{R}^3 \to \mathbb{R} \), with \( a \in \mathbb{Z} \), \( \omega(k) = \sqrt{m^2 + k^2} \)

\[ (\omega^a f)(x) = \int_{\mathbb{R}^3} d^3 k e^{-i k \cdot x} (m^2 + k^2)^{a/2} \tilde{f}(k), \quad \omega^a f \in \mathcal{S}'(\mathbb{R}^3) \]

- inner product for test functions \((f, g) \in \mathcal{S}(\mathbb{R}^3)\)

\[ (f, g) = \int_{\mathbb{R}^3} d^3 x f(x) g(x) = \int_{\mathbb{R}^3} d^3 k \tilde{f}^*(k) \tilde{g}(k) < \infty \]
Equal-time canonical quantization

- canonical commutators at $t = 0$ we denote
  \[ \hat{\phi}[0,f] = \hat{\phi}[f] \]

\[ [\hat{\phi}[f], \hat{\pi}[g]] = i \langle f, g \rangle \mathbf{1}_{op}, \quad [\hat{\phi}[f], \hat{\phi}[g]] = 0, \quad [\hat{\pi}[f], \hat{\pi}[g]] = 0. \]

- Heisenberg time evolution equations
  \[
  \frac{d}{dt} \hat{\phi}[f] = \frac{1}{i} [\hat{\phi}[f], \hat{H}] = \hat{\pi}[f], \quad \frac{d}{dt} \hat{\pi}[f] = \frac{1}{i} [\hat{\pi}[f], \hat{H}] = -\hat{\phi}[\omega^2 f].
  \]

- canonically smeared annihilation and creation operators:
  \[
  \hat{a}[f] := \hat{\phi}[\omega f] + i \hat{\pi}[f], \quad \hat{a}^\dagger[f] := \hat{\phi}[\omega f] - i \hat{\pi}[f], \quad (\hat{a}[f])^\dagger = \hat{a}^\dagger[f],
  \]

- satisfy Heisenberg equations
  \[
  [\hat{H}, \hat{a}[f]] = -\hat{a}[\omega f], \quad [\hat{H}, \hat{a}^\dagger[f]] = \hat{a}^\dagger[\omega f],
  \]

- have canonical commutators
  \[
  [\hat{a}[f], \hat{a}^\dagger[g]] = 2 \langle f | \omega g \rangle \mathbf{1}_{op}, \quad [\hat{a}[f], \hat{a}[g]] = [\hat{a}^\dagger[f], \hat{a}^\dagger[g]] = 0,
  \]
equal equivalence for canonically smeared operators

\[ \hat\phi[f] = \frac{1}{2} \left( \hat{a}[^{-1}f] + \hat{a}[^{-1}f] \right), \quad \hat{\pi}[f] = \frac{i}{2} \left( \hat{a}[f] - \hat{a}[f] \right). \]

smeared operators lead to operator-valued distributions

\[ \hat{a}[f] = \int_{\mathbb{R}^3} d^3x \, f(x) \hat{a}(x) = \int_{\mathbb{R}^3} d^3k \, \tilde{f}^*(k) \hat{a}(k), \]

\[ \hat{a}^+[f] = \int_{\mathbb{R}^3} d^3x \, f(x) \hat{a}^+(x) = \int_{\mathbb{R}^3} d^3k \, \tilde{f}(k) \hat{a}^+(k). \]

commutators for operator-valued distributions

\[ \left[ \hat{a}(k), \hat{a}^+(p) \right] = 2\omega(k)(2\pi)^3 \delta^3(k - p) \, 1_{\text{op}}, \]

\[ \left[ \hat{a}(k), \hat{a}(p) \right] = \left[ \hat{a}^+(k), \hat{a}^+(p) \right] = 0. \]
Equal-time vacuum and diagonalization of $\hat{H}$

- vacuum state $|0\rangle$: unique, normalized, Poincaré invariant
  \[
  \langle 0|0 \rangle = 1, \quad \hat{H}|0\rangle = 0, \quad \hat{P}|0\rangle = 0, \quad \hat{M}_{\mu\nu}|0\rangle = 0,
  \]

- Heisenberg equations lead to
  \[
  \hat{H} \hat{a}[f]|0\rangle = -\hat{a}[\omega f]|0\rangle, \quad \hat{H} \hat{a}^\dagger[f]|0\rangle = \hat{a}^\dagger[\omega f]|0\rangle.
  \]

- there are eigenvectors of $\hat{H}$ with positive and negative eigenvalues
  \[
  \hat{H} \hat{a}(k)|0\rangle = -\omega(k) \hat{a}(k)|0\rangle, \quad \hat{H} \hat{a}^\dagger(k)|0\rangle = \omega(k) \hat{a}^\dagger(k)|0\rangle.
  \]

- vacuum state removes the negative eigenvalue states from $\mathcal{H}_0$
  \[
  \hat{a}(k)|0\rangle = 0 \implies \hat{a}[f]|0\rangle = 0 \implies \hat{\pi}[f]|0\rangle = i\phi[\omega f]|0\rangle,
  \]
Equal-time 1-particle states

- canonically smeared 1-particle state:

\[ |f\rangle := \hat{a}^\dagger[f]|0\rangle = 2\hat{\phi}[\omega f]|0\rangle, \]

- decomposition of the canonical commutators

\[ \langle 0|[\hat{\phi}[f], \hat{\pi}[g]]|0\rangle = i(f, g) \quad \implies \quad \langle 0|\hat{\phi}[f] \hat{\phi}[g]|0\rangle = \frac{1}{2} (f, \omega^{-1} g) < \infty \]

- inner product for canonically smeared 1-particle states

\[ \langle f|g \rangle = 4\langle 0|\hat{\phi}[\omega f] \hat{\phi}[\omega g]|0\rangle = 2(f, \omega g) < \infty, \]

- canonically smeared 1-particle states are in \( \mathcal{H}_0 \)
Distribution-valued states - momentum dependent kets

- canonically smeared 1-particle states

\[ |f\rangle = \int_{\mathbb{R}^3} d^3k \tilde{f}(k) |k\rangle \quad \implies \quad |k\rangle = \hat{a}^\dagger(k) |0\rangle \]

- momentum dependent kets \( |k\rangle \) have inner product

\[ \langle p|k\rangle = 2\omega(k) \delta^3(k - p), \]

- they are eigenvectors for translation generators operators

\[ \hat{H} |k\rangle = \omega(k) |k\rangle, \quad \hat{P}_j |k\rangle = -k^j |k\rangle = k_j |k\rangle \]

- distribution-valued states are not states in \( \mathcal{H}_0 \)
2-particle states

- canonically smeared 2-particle states:

\[ |f, g\rangle := \hat{a}^\dagger [f] \hat{a}^\dagger [g]|0\rangle, \]

- are orthogonal to vacuum and 1-particle states

\[ \langle 0|f, g\rangle = 0 \quad \text{and} \quad \langle h|f, g\rangle = 0 \]

- have finite norm

\[ \langle f, g|f, g\rangle = 4(f, \omega_f)(g, \omega_g) + 4(f, \omega_g)(g, \omega_f) \]

- accordingly

\[ |f, g\rangle \in \mathcal{H}_0 \]

- canonically smeared field operator changes number of particles by ±1

\[ \hat{\phi}[f] |g\rangle = \hat{\phi}[f] \hat{a}^\dagger [g]|0\rangle = \frac{1}{2} [g, \omega^{-1} f] + (f, g) |0\rangle. \]
Notation for light-front quantization

\[ \hbar = c = 2\pi = 1 \]

- canonically smeared scalar fields
  \[ \hat{\phi}[x^+, f] = \int_{\mathbb{R}^3} d^3\bar{x} \ f(\bar{x}) \ \hat{\phi}(x^+, \bar{x}), \quad x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad \bar{x} = (x^-, x_\perp) \]

- real-valued test function \( f(\bar{x}) \in \mathcal{S}(\mathbb{R}^3) \)

- Fourier transform of test function \( \tilde{f} \in \mathcal{S}(\mathbb{R}^3) \)
  \[ \tilde{f}(\bar{k}) = \int_{\mathbb{R}^3} d^3\bar{x} \ f(\bar{x}) \ e^{i\bar{k}\cdot\bar{x}}, \quad \bar{k} = (k^+, k_\perp), \quad \bar{k} \cdot \bar{x} = -x^-k^+ + k_\perp \cdot x_\perp \]

- new test function \( \omega^a_{LF}f : \mathbb{R}^3 \to \mathbb{R} \), with \( a \in \mathbb{Z} \), \( \omega_{LF}(k_\perp) = \sqrt{m^2 + k^2_\perp} \)
  \[ (\omega^a_{LF} f)(\bar{x}) = \int_{\mathbb{R}^3} d^3\bar{k} \ e^{i\bar{k}\cdot\bar{x}} (m^2 + k^2_\perp)^{a/2} \tilde{f}(\bar{k}), \quad \omega^a_{LF}f \in \mathcal{S}(\mathbb{R}^3) \]

- inner product for test functions \( (f, g) \in \mathcal{S}(\mathbb{R}^3) \)
  \[ (f, g) = \int_{\mathbb{R}^3} d^3\bar{x} \ f(\bar{x}) \ g(\bar{x}) = \int_{\mathbb{R}^3} d^3\bar{k} \ \tilde{f}^*(\bar{k}) \ \tilde{g}(\bar{k}) < \infty \]
Light-Front canonical quantization

- LF canonical commutators at $x^+ = 0$ we denote $\hat{\phi}[0, f] = \hat{\phi}[f]$

\[
\left[ \hat{\phi}[f], \hat{\phi}[\partial^- g] \right] = -\frac{i}{2} (f, g) \mathbf{1}_{op},
\]

- Heisenberg equation for temporal evolution

\[
d_+ \hat{\phi}[\partial^- f] = -i \left[ \hat{\phi}[\partial^- f], \hat{P}^- \right] = \frac{1}{2} \hat{\phi}[\omega_{LF}^2 f],
\]

- Kinematical infinitesimal translations

\[
\left[ \hat{\phi}[f], \hat{P}^+ \right] = -i \hat{\phi}[\partial^- f], \quad \left[ \hat{\phi}[f], \hat{P}_i \right] = -i \hat{\phi}[\partial_i f].
\]

- Preliminary 1-particle states

\[
|f\rangle := \hat{\phi}[f]|0\rangle = \int_{\mathbb{R}^3} \frac{d^3 \bar{k}}{2k^+} \tilde{f}(\bar{k}) |\bar{k}\rangle, \quad \Rightarrow \quad |\bar{k}\rangle = 2k^+ \int_{\mathbb{R}^3} d^3 \bar{x} e^{i \bar{k} \cdot \bar{x}} \hat{\phi}(\bar{x}) |0\rangle
\]
Light-front vacuum and diagonalization of $\hat{P}^+$

♦ LF vacuum state $|0\rangle$: normalized, unique and Poincaré invariant

$$\langle 0|0 \rangle = 1, \quad \hat{M}|0\rangle = 0, \quad \hat{M} = \{\hat{P}^\pm, \hat{P}_\perp, \hat{M}_{\mu\nu}\}$$

♦ diagonalization of $\hat{P}^+$

$$\hat{P}^+ \hat{\phi}[f]|0\rangle = i\hat{\phi}[\partial_- f]|0\rangle, \quad \hat{P}^+ |\vec{k}\rangle = k^+ |\vec{k}\rangle,$$

♦ nonnegative spectrum of $\hat{P}^+$ leads to the condition for $k^+ < 0$

$$|\vec{k}\rangle = 0 \quad \Longrightarrow \quad 2k^+ \int_{\mathbb{R}^3} d^3\vec{x} \, e^{i\vec{k} \cdot \vec{x}} \hat{\phi}(\vec{x}) |0\rangle = 0$$

♦ canonically smeared 1-particle states

$$|f\rangle = \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{2k^+} \Theta(k^+) \tilde{f}(\vec{k}) |\vec{k}\rangle = \phi[f]|0\rangle.$$
Momentum dependent 1-particle states

♦ decomposition of LF canonical commutator

\[ \langle 0 | \hat{\phi} [f] \hat{\phi} [\partial_- g] | 0 \rangle - \langle 0 | \hat{\phi} [\partial_- g] \hat{\phi} [f] | 0 \rangle = - \frac{i}{2} (f, g), \]

♦ inner product for momentum dependent kets

\[ \langle \bar{k} | \bar{p} \rangle = 2k^+ \delta^3 (\bar{p} - \bar{k}) \]

♦ inner product for canonically smeared 1-particle states is ill defined in $\mathcal{S}(\mathbb{R}^3)$

\[ \langle f | g \rangle = \int_{\mathbb{R}^3} \frac{d^3k}{2k^+} \Theta(k^+) \tilde{f}^* (\bar{k}) \tilde{g}(k), \]

♦ for Schlieder-Seiler test functions

\[ \mathcal{J}(\mathbb{R}^3) := \{ g \in \mathcal{S}(\mathbb{R}^3); \tilde{g}(k^+, \vec{k}_\perp) = 0, \text{ for } k^+ = 0 \} \]

inner product is finite \[ \langle f | g \rangle < \infty \]
Light-front canonically smeared 1-particle states

♦ the Heisenberg equation for the canonically smeared operators

\[
\left[ \hat{\phi}[\partial_f], \hat{H} \right] = \frac{i}{2} \hat{\phi}[\omega_{LF}^2 f], \quad \Rightarrow \quad \hat{H} \partial_f \rangle = -\frac{i}{2} \omega_{LF}^2 f \rangle,
\]

\hat{H} = \hat{P}^-

♦ LF Schrödinger equations for momentum dependent 1-particle kets (for \( k^+ > 0 \))

\[
\hat{H} |\vec{k}\rangle = \frac{m^2 + k_\perp^2}{2k^+} \ |\vec{k}\rangle.
\]

♦ the expectation value of the Hamiltonian operator is ill defined for \( f \in \mathcal{S}(\mathbb{R}^3) \)
quad (Schlieder-Seiler test functions)

\[
\langle f | \hat{H} | f \rangle = \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{2k^+} \Theta(k^+) \ |\vec{f}(\vec{k})| \frac{m^2 + k_\perp^2}{2k^+}
\]

♦ problem for the quantum mechanical interpretation of 1-particle states

♦ manifestation of light-front infra-red problem
Smearing in light-front time $x^+$

- **LF time dependent momentum 1-particle kets**, valid for $k^+ > 0$

  $|x^+, \kbar\rangle := e^{ix^+\hat{H}}|\kbar\rangle = e^{ix^+k^-}|\kbar\rangle$,

  $k^- = \frac{m^2 + k_\perp^2}{2k^+}$

- **LF time smeared 1-particle momentum dependent kets**

  $|f^+, \kbar\rangle := \int_{\mathbb{R}} dx^+ f^+(x^+) |x^+, \kbar\rangle = \tilde{f}^*_+(k^-)|\kbar\rangle$, $f^+(x) \in \mathcal{S}(\mathbb{R})$

- **totally smeared 1-particle states**

  $|f^+, \_\rangle := \int_{\mathbb{R}^3} \frac{d^3\kbar}{2k^+} \Theta(k^+) \tilde{f}(\kbar) |f^+, \kbar\rangle$

- **the LF temporal evolution of the quantum field operator**

  $\hat{\phi}[x^+, \_] = e^{i\hat{H}x^+} \hat{\phi}[\_] e^{-i\hat{H}x^+}$

- **totally smeared field operator - local field operator**

  $\hat{\phi}[f^+, \_] := \int_{\mathbb{R}} dx^+ f^+(x^+) \hat{\phi}[x^+, \_]$
Totally smeared fields - local field operators

- totally smeared 1-particle states

\[ \hat{\phi}[f_+, f_-]|0\rangle = |f_+, f_-\rangle, \]

- have finite norm

\[ \langle f_+, f_-|f_+, f_-\rangle = \int_{\mathbb{R}^3} \frac{d^3\bar{k}}{2k^+} \Theta(k^+) \left| \tilde{f}(\bar{k}) \right|^2 \left| f_+(k^-) \right|^2 < \infty, \]

- finite expectation value for the Hamiltonian operator

\[ \langle f_+, f_-|\hat{H}|f_+, f_-\rangle = \int_{\mathbb{R}^3} \frac{d^3\bar{k}}{(2\pi)^3} \frac{\Theta(k^+)}{2k^+} \left| \tilde{f}(\bar{k}) \right|^2 \left| f_+(k^-) \right|^2 \frac{m^2 + k_\perp^2}{2k^+} < \infty, \]

thus \( |f_+, f_-\rangle \in \mathcal{H}_0 \)
### Conclusions

#### vacuum state

- **Equal-time**
  - diagonalization of $\hat{H}$
  - depends on interaction (mass)

- **Light-front**
  - diagonalization of $\hat{P}^+$
  - kinematical (mass indep’t)

#### 1-particle state

- momentum dependent kets $\not\in \mathcal{H}_0$
- canonically smeared kets

- **Equal-time**
  - $\ket{f} \in \mathcal{H}_0$
  - depend on interaction (mass)

- **Light-front**
  - $\ket{f} \not\in \mathcal{H}_0$
  - kinematical (mass indep’t)

  - totally smeared LF kets
    - $\ket{f_+, f} \in \mathcal{H}_0$
    - depend on interaction (mass)
    - no LF infra-red problem
Prospects

smearing with test function is a promising tool for LF QFT

♦ LF perturbation theory

- propagators
- closed loop diagrams

■ LF non-perturbative theory

- ?? ??
smearing with test function is a promising tool for LF QFT

♦ LF perturbation theory
  • propagators
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  • ?? ??

Thank you for your attention