Renormalized 4pi effective action

M.E. Carrington
Brandon University

Collaborators: Christopher Phillips and Doug Pickering

Outline

Introduction

strong coupling $\rightarrow$ can't use perturbation theory

different approaches (for example):

• lattice calculations

$\rightarrow$ *continuum and infinite volume limits*

• continuum methods

- Schwinger-Dyson equations
- renormalization group (RG)
- n-particle irreducible ($n$pi) effective theories

1pi effective action for scalar theories:

generating functional with a source term

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i)}$$

effective action:

$$\delta W/\delta J_i = \phi^i$$

$$\Gamma[\phi] = W[J] - J_i\phi_i$$

ϕ determined self-consistently $\frac{\delta\Gamma}{\delta\phi}\Big|_{\phi=\bar{\phi}} = 0$

short-hand notation: $\int dx\, J(x)\,\varphi(x) \to J_i\varphi_i \to J\varphi$

2pi effective action:

generating functional with local and bi-local sources

$$Z[J,B] = e^{iW[J,B]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i+\frac{1}{2}\varphi_i B_{ij}\varphi_j)}$$

$\longrightarrow$ $\Gamma[\phi, G]$ is a functional of the 1- and 2-point functions

ϕ and G determined self-consistently from variational principle

$$\left.\frac{\delta\Gamma}{\delta\phi}\right|_{\substack{\phi=\bar\phi \\ G=\bar G}} = 0 \quad \text{and} \quad \underbrace{\left.\frac{\delta\Gamma}{\delta G}\right|_{\substack{\phi=\bar\phi \\ G=\bar G}} = 0}_{\text{dyson equation}}$$

npi effective action

we could add higher order source terms
$\rightarrow$ more variational vertices

3pi $\Gamma[\phi, G, U]$, 4pi $\Gamma[\phi, G, U, V] \cdots$

in general: npi Γ is a functional of n variational vertices

each determined self-consistently from its eom

$\Rightarrow$ set of coupled integral equations

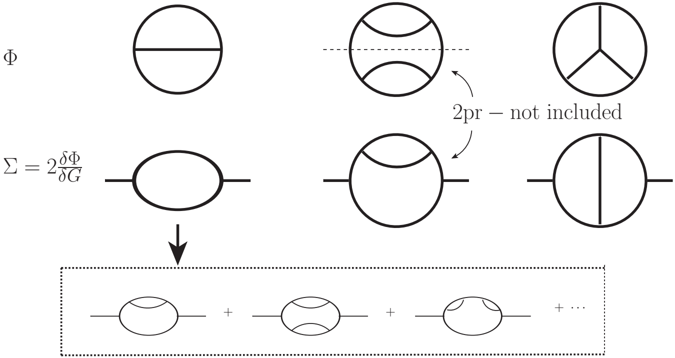
compare $\Gamma[G,\phi]$ to $\Gamma_{1\mathrm{pi}}[\phi]$

- $\Gamma[\phi, G]$ depends on the self consistent propagator

$\rightarrow$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams

$\rightarrow$ non-perturbative

- $\Gamma[\phi, G]$ is 2pi - no double counting



Key features:

- non-perturbative

  infinite resummations of selected classes of diagrams

  physics example:
  - transport coefficients in gauge theories at leading order
  require vertex corrections (LPM effect)
  - $\sigma_{\rm qed}$ *from lowest order 3pi effective action*
  not obtainable from 2pi effective action at any loop order

- action based approximation

  truncation occurs at the level of the action
  $\rightarrow$ symmetries of original theory

- renormalizability

  basic problem: self-consistent sets of integral equations plagued by overlapping ultraviolet divergences

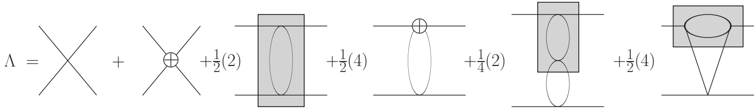
  1pi: introduce momentum independent counterterms
  same structure as bare interactions in the Lagrangian
  2pi: counterterms ... *(with some adjustments)*

  - cannot apply same method to higher order approximations

  $\rightarrow$ new method based on renormalization group (RG)

npi renormalization – an example

4-loop 2pi 4-kernel $\Lambda = 4\frac{\delta^2\Phi_{\rm int}}{\delta G^2}$



counterterms appear in positions to cancel 1-loop divergences

- but there is no one $\delta\lambda_1$ that works

this is typical of npi theories - combinatorics are different

Resolution for 2pi

need 2 vertex ct's . . .

1) they both come from the action

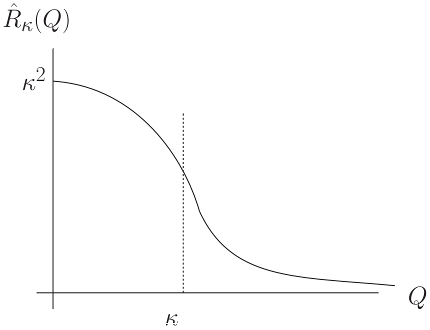
2) at $L \to \infty$ loops they are equal

H. van Hees, J. Knoll, Phys. Rev. **D65**, *025010 (2002);*
J-P Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. **A736**, *149 (2004);*
J. Berges, Sz. Borsányi, U. Reinosa, J. Serreau, Annals Phys. **320**, *344 (2005).*

BUT unknown how to use counterterms beyond the 2pi level
must develop another method to renormalize

Renormalization group method

add to the action a non-local regulator term $\Delta S_\kappa[\varphi] = -\frac{1}{2} R_\kappa \varphi^2$



$$R_\kappa = \frac{Q^2}{e^{Q^2/\kappa^2} - 1}$$

$R_\kappa(Q) \sim \kappa^2$ for $Q \ll \kappa$

fluctuations $Q \ll \kappa$ suppressed

$R_\kappa(Q) \to 0$ for $Q \geq \kappa$

fluctuations $Q \gg \kappa$ unaffected

$\Rightarrow$ family of theories indexed by the continuous parameter κ

$\kappa \to \infty$
all fluctuations suppressed
regulated action $\to$ classical action

$\kappa \to 0$
fluctuations are smoothly taken into account as κ is lowered to zero
regulated action $\to$ full quantum action

J.-P. Blaizot, A. Ipp, N. Wschebor, Nucl. Phys. A **849**, *165 (2011)*
J.-P. Blaizot, J.M. Pawlowski and U. Reinosa, Phys. Lett. B **696**, *523 (2011)*

generating functionals defined in the usual way

$$Z_\kappa[J,B] = \int [d\varphi] \exp\left\{ i\Big(S[\varphi] - \frac{1}{2}\hat{R}_\kappa \varphi^2 + J\varphi + \frac{1}{2} B\varphi^2 + \cdots \Big) \right\}$$

calculate 1pi, 2pi, $\cdots$ effective action

action depends on κ: $\Phi_\kappa = i\Gamma_\kappa$

$$\text{action flow eqn:} \quad \partial_\kappa \Phi_\kappa = \frac{1}{2}\partial_\kappa R_\kappa\, G$$

C. Wetterich, Phys. Lett., B 301, 90 (1993).

key idea

standard formalism:

- variational vertices from solving 'S-Dyson-like' integral equations
- obtained by taking functional derivatives of $\Gamma(\phi, G, V \cdots)$
- *beyond 2pi level these equations have divergences that can't be absorbed by a finite set of counterterms*

RG formalism:

variational vertices obtained from solving flow equations

- obtained by taking functional derivatives of $\partial_\kappa \Gamma_\kappa(\phi, G, V \cdots)$

4pi flow equations

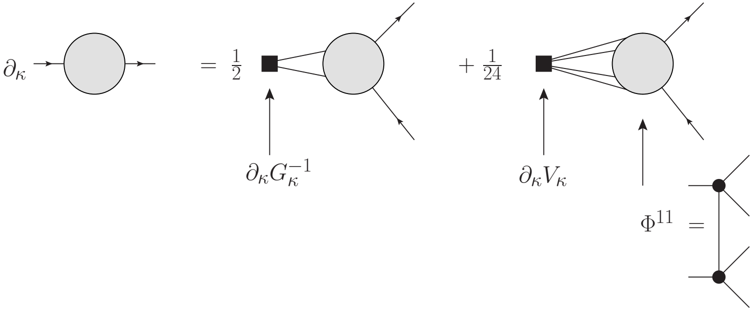
$$\partial_\kappa \Phi^{(nm)}_{\text{int}\cdot\kappa}\Big|_{\substack{G=G_\kappa\\ \phi=o}} = \frac{1}{2}\int dQ\,\partial_\kappa\left(R_\kappa+\Sigma_\kappa\right)G^2_\kappa(Q)\,\Phi^{(n,m+1)}_{\text{int}\cdot\kappa}(Q,\quad) + \frac{1}{4!}\int dQ_i\,\partial_\kappa\,V_\kappa\,G^4_\kappa(Q_i)\,\Phi^{(n+1,m)}_{\text{int}\cdot\kappa}(Q_i,\quad)$$

$\Phi^{(nm)}_{\text{int}\cdot\kappa}$ is a kernel with $2m+4n$ legs

- example $m=1$ and $n=0$ gives a 2-point function (self energy)

note we have an infinite hierarchy of coupled equations ...

our calculation = 4 loop 4pi effective action
first flow equation:



Method

solve hierarchy of differential flow eqns for κ dependent n-point fcns

role of kappa:

$\kappa \to \infty$ regulated action $\to$ classical action

$\kappa \to 0$ regulated action $\to$ full quantum action

$\to$ method to solve flow equations:

1. choose an uv scale $\kappa = \mu$ (defn of bare parameters)

theory is classical at this scale (all fluctuations suppressed)

$\to$ n-point functions are known functions of the bare parameters

2. solve differential flow equations starting from bc's at $\kappa = \mu$

$\to$ obtain the n-point fcns at $\kappa = 0$ (the quantum solutions)

Technicalities

KEY:

bc's chosen at $\kappa = \mu$ $\leftarrow$ classical scale where theory is simple

rc's are imposed at $\kappa = 0$ $\leftarrow$ this is the full quantum theory

3 Issues:

1. Tuning: definition of physical parameters ($\kappa = 0$)

$\rightsquigarrow$ constrains initial conditions on the flow equations ($\kappa = \mu$)

2. Consistency: can we satisfy both the bc's and the rc's?

(A): flow equations $\to$ vertex fcns up to κ independent constant

$\to$ can satisfy bc with an appropriate choice of this constant

(B): the rc's are satisfied if

$$\lim_{P_i \to 0} \big[\Lambda_0(P_1, P_2 \cdots P_m) - \Lambda_0(0, 0 \cdots 0)\big] = \text{constant}$$

looks obvious . . .

sub-divergences could give something ill defined like $\infty \times 0$
can satisfy condition if the truncation is performed correctly

3. Truncation:

it is obvious that hierarchy of flow eqns truncates with action

but actually: can truncate as soon as we find a kernel that satisfies

$\lim_{P_i \to 0} \left[\Lambda_0(P_1, P_2 \cdots P_m) - \Lambda_0(0, 0 \cdots 0) \right] = \text{constant}$

$\Rightarrow$ *no sub-divergence in quantum n-point function*

KEY to truncation:

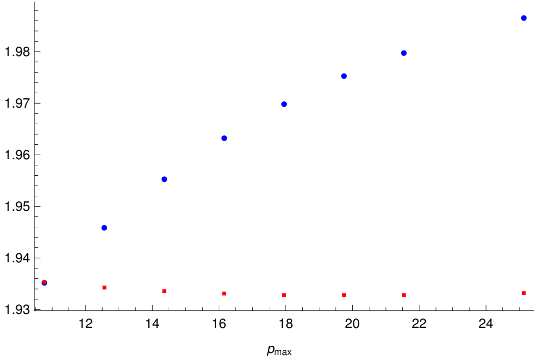
kernel with a sub-divergence must be obtained from its flow eqn

kernel without a sub-divergence doesn't have to be flowed
- substitute directly into previous flow equation

4-loop 4pi calculation

1. [(3 legs) $\times$ (4 dims)] + (4 $\times$ 1 inte vars) = 16 loops
2. memory constraints $\rightarrow$ spherical coordinates
   $\rightarrow$ 13 loops (some angles are "free")
   4 matsubara frequencies
   5 angles (very weak dependence)
   4 momentum magnitudes
3. use symmetries (for example under leg permutations)
4. must store a 9 dimensional array for the variational 4 vertex

Results



V(0) (red squares) and bare coupling (blue dots)
as a function of p_{max} with Δp held fixed - *arXiv:1901.00840*

Conclusions

- 2pi can be renormalized with counterterms

at ≥ 4 loop require two vertex and two mass counterterms
... can't be generalized to higher order theories

- functional renormalization group regulator $\Rightarrow \quad (m_b, \lambda_b)$

all divergences are absorbed into bare parameters of the lagrangian

agrees with counterterm renormalization for the 2pi calculation

method generalizes to higher order nPI

further 4pi numerical calculations are in progress