Lattice QCD
&
Parton Distributions

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- Let me try to explain some basics of gauge theory on the lattice & their physical interpretation

Then you come up with the answer of why we do Lattice QCD

- Slowly, we will move towards concepts of correlation functions (two-point, three-point [connected and disconnected insertions], four-point functions) to calculate mass, charge, form factors, parton distributions on the lattice)

- Other than going too technical, I will follow path of asking simple questions. Aim is to discuss a very small segment of LQCD and be more interactive

- Most important:
  Feel free to interrupt during the lecture

Natural Units

(Something very interesting next slide)

kg, lb, m, sec…these are very human centered measurement units

- Much better way is to tie units to nature itself
- We work in natural units $\hbar = c = 1$
- Convenient…..but is there a more fundamental reason ?

SR : time = distance / c

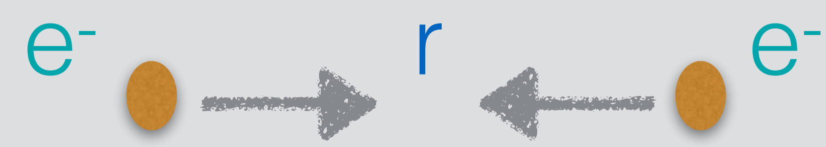
QM: energy = $\hbar$/time

$$[length]^{-1} = [time]^{-1} = [mass] = [temperature] = [energy] = \text{GeV}$$

For example: lecture duration 1 hr ~ 10^{27} GeV^{-1} is what the light takes to travel in terms of 1 proton length (mass $m_p c^2$ ~ 1 GeV)

Well….up to a certain scale……..

(Something even More interesting next slide)



F(elec) $\sim q^2 \frac{1}{r^2}$

strength ~1/137 (pure number)

F(grav) $\sim -G_N \frac{m_e^2}{r^2}$

strength (not a pure number) $\sim (10^{-33} cm)^2 \sim (L_P)^2$

$$\frac{1}{L_p} \sim 10^{19}\ GeV$$

e-e scattering through graviton: QM probability ~ (amplitude)^2

amplitude~ $G_N \times E^2$

tiny for $E << E_p$

bigger than 1(!!) for $E > E_p$

Not allowed in QM !!

So, one can't (NAIVELY) extrapolate understanding of ordinary gravity at large scale to very short scale (Planck Scale)

NOT END OF STORY
IF TIME PERMITS —LATER

To approach a plausible solution string theory comes into play (and exactly why "string", not point, straight line, triangle or some other shapes)

However, idea of "string" didn't come first to solve QG, but came to explain QCD phenomenon

Problem

Actually, similar type of argument can explain why Higgs particle required to solve problem with massive W-boson and how ~1989 people predicted

$80 < \text{Higgs mass} < 200$ GeV

(If one just doesn't think in terms of Mexican-hat potential)

Clue: massive particle has 3-spin, massless particle has 2-spin(helicity).

[possible discussion after lecture if interested]

Lattice QCD Setup

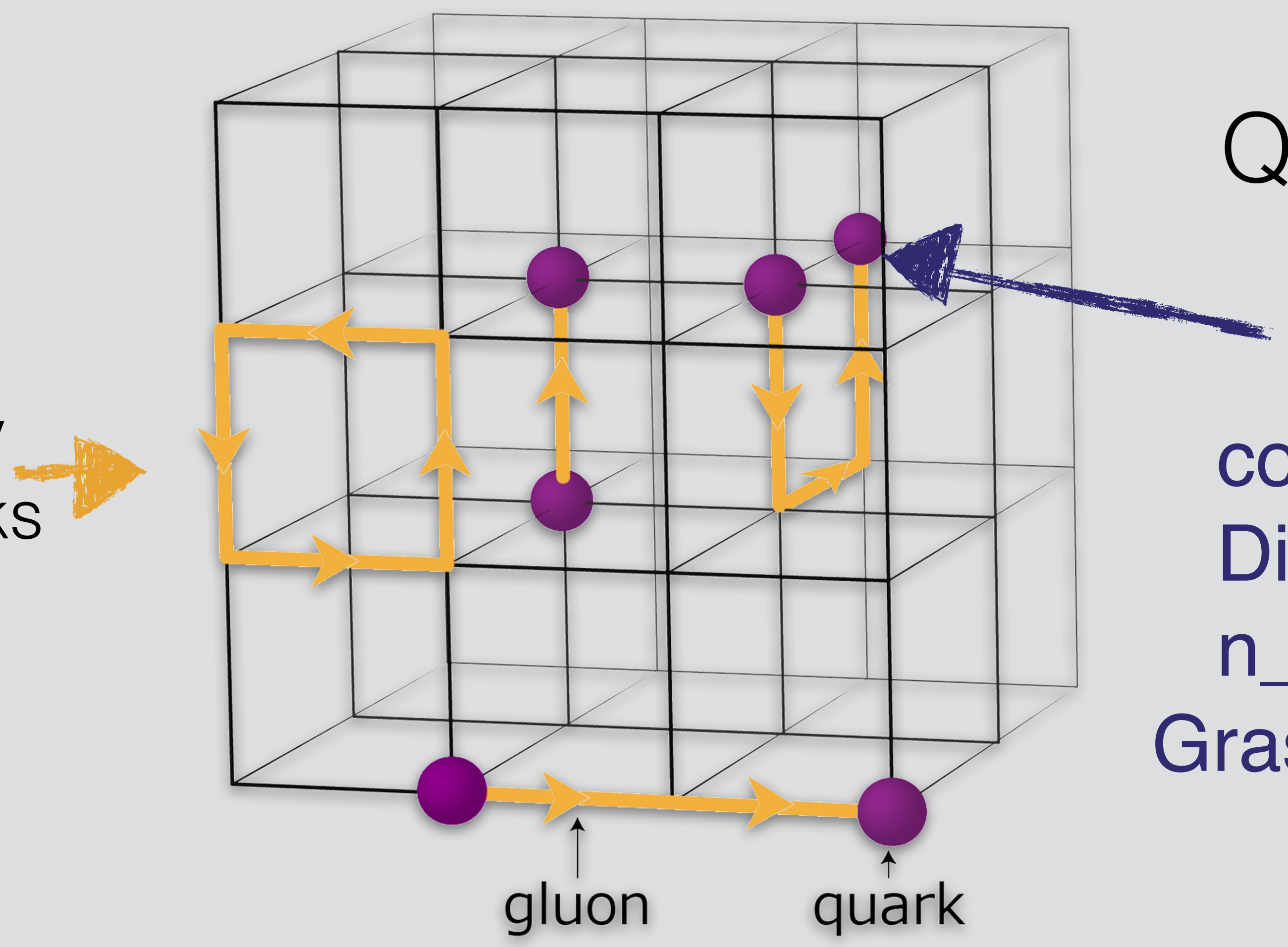
The consistent way of describing QCD on the lattice is the following:

1. discretization of spacetime (Euclidean) by a hypercubic lattice with cutoff, Λ called lattice regularization,
2. discretization of continuum QCD action,
3. quantization of QCD using path integral formalism,
4. application of Monte-Carlo simulation to calculate expectation values of different operators.

Lattice QCD Setup

★ Hyper-cubic lattice: 4D lattice for which distances between sites are same in all directions

Plaquette: elementary square closed by 4 links



Quark fields

$$\psi^{(f)}_{\alpha,c}(x)$$

color 3-vectors, Dirac 4-spinors, n_f vectors and Grassmann variables

Gauge field variables $U_\mu(x) \in SU(3)$

3x3 complex, unitary matrices on each link

★ Fields on site

What does it mean in comparison with continuum QFT and how to approximate continuum fields??

$$\phi(x)_{cont} \Longrightarrow \phi_x \; (lattice)$$

★ Gauge fields attributed to links on lattice

What is the physics?

Problem

Calculate number of sites, links and plaquettes for a symmetric hypercubic ***d*** dimensional lattice lattice of size ***L*** with periodic boundary conditions

★ Need action invariant under local gauge transformation with SU(3) matrix $\Omega(x)$

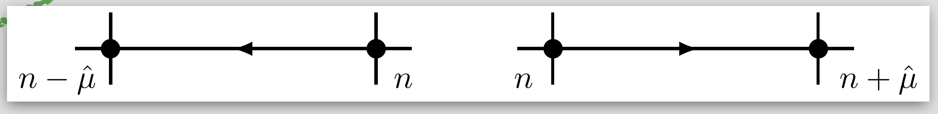
$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\Omega(x)^\dagger$$
$$A_\mu(x) \to A'_\mu(x) = \Omega(x)A_\mu(x)\Omega(x)^\dagger + i(\partial_\mu\Omega(x))\Omega(x)^\dagger$$
$$S[\psi', \bar{\psi}', A'] = S[\psi, \bar{\psi}, A]$$

★ Rotation invariance $\overline{\psi}(x)\Omega(x)^\dagger\Omega(x)\psi(x) = \overline{\psi}(x)\psi(x)$

★ Consider discretized version of lattice fermion action

Discretized version of derivative $\partial_\mu\psi(x)$

★
$$S_f[\psi, \bar{\psi}] = a^4 \sum_{n\in\Lambda} \bar{\psi}(n)\left(\sum_{\mu=1}^{4}\gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\psi(n)\right)$$



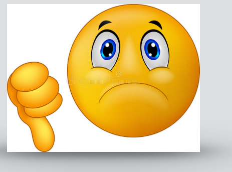
Not gauge invariant

Why? Maths next page

★ Gauge invariance of lattice action

$$\bar{\psi}(n)\psi(n) \to \bar{\psi}'(n)\psi'(n) = \bar{\psi}(n)\Omega(n)^{\dagger}\Omega(n)\psi(n) = \bar{\psi}(n)\psi(n)$$

$$\bar{\psi}(n)\psi(n+\hat{\mu}) \to \bar{\psi}'(n)\psi'(n+\hat{\mu}) = \bar{\psi}(n)\Omega(n)^{\dagger}\Omega(n+\hat{\mu})\psi(n+\hat{\mu})$$



★ We need to connect quark fields at different sites with gauge link

$$U_{\mu}(n) \to U'_{\mu}(n) = \Omega(n)U_{\mu}(n)\Omega(n+\hat{\mu})^{\dagger}$$

★ Then we recover gauge invariance

$$\bar{\psi}'(n)U'_{\mu}(n)\psi'(n+\hat{\mu}) = \bar{\psi}(n)\Omega(n)^{\dagger}U'_{\mu}(n)\Omega(n+\hat{\mu}')\psi(n+\hat{\mu})$$

$$= \bar{\psi}(n)\Omega(n)^{\dagger}\Omega(n)U_{\mu}(n)\Omega(n+\hat{\mu})^{\dagger}\Omega(n+\hat{\mu})\psi(n+\hat{\mu})$$

$$= \bar{\psi}(n)U_{\mu}(n)\psi(n+\hat{\mu})$$

★ $U_\mu(n)$ can be related to gauge transporter in continuum, a path ordered exponential integral of gauge field A_μ along $\mathcal{C}_{xy}$ connecting x and y

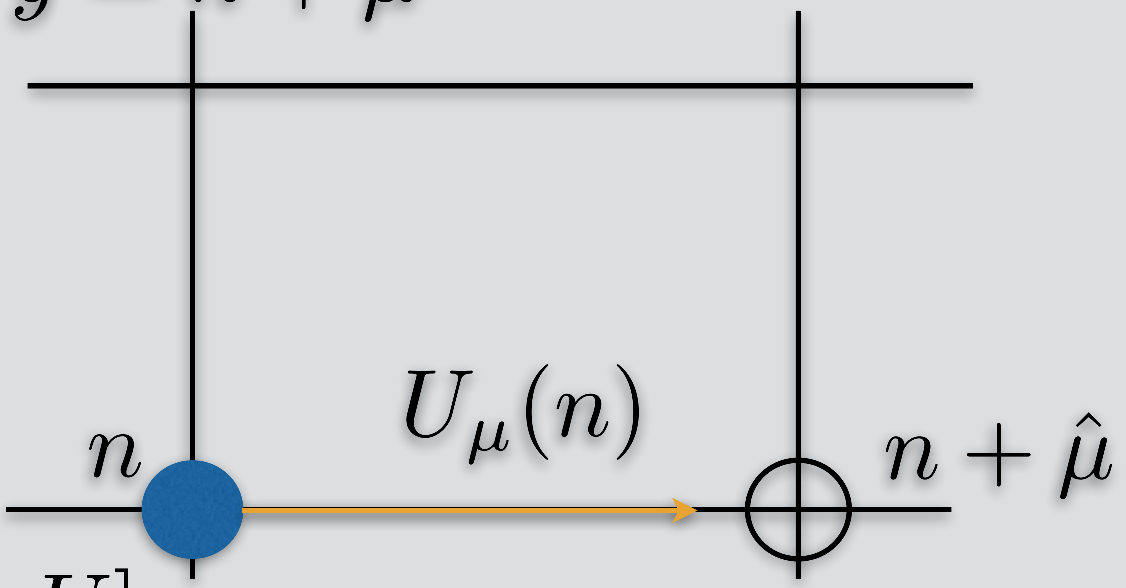
$$G_{(x,y)} = \mathcal{P} e^{i \int_{\mathcal{C}_{xy}} A \cdot ds}$$

Problem ::
What does actually this gauge transporter allow the quark to change?
Clue: Related to why U_\mu is a unitary NXN matrix !

★ *Along a link from* $x = n$ *to* $y = n + \hat{\mu}$

$$G(n, n+\hat{\mu}) = e^{iaA_\mu(n)}$$

$$= U_\mu(n) = 1 + iaA_\mu(n) + \mathcal{O}(a^2)$$



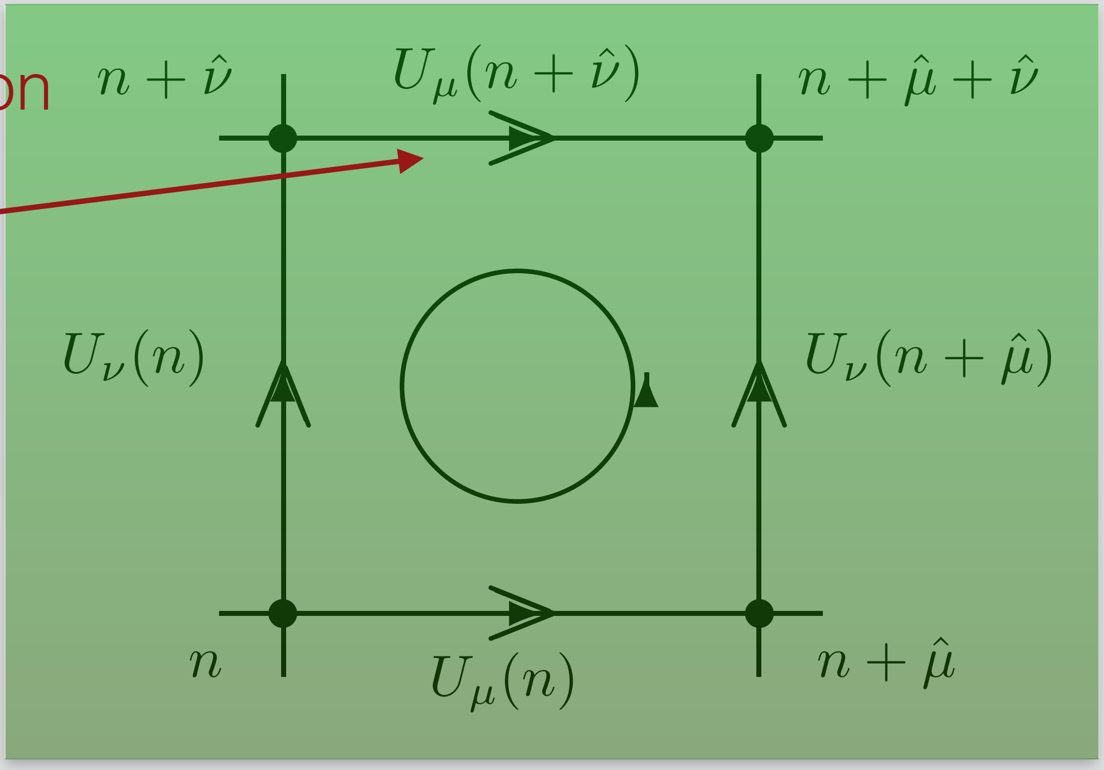
★ Then gauge invariant $S_f[\psi, \bar{\psi}, U]$

$$= a^4 \sum_{n \in \Lambda} \bar{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a} + m\psi(n) \right)$$

★ Now construct lattice gauge action with gauge invariant plaquette

opposite direction

★
$$
\begin{aligned}
U_{\mu\nu}(n) &= U_\mu(n)\, U_\nu(n+\hat{\mu}) \\
&\quad \times U_{-\mu}(n+\hat{\mu}+\hat{\nu})\, U_{-\nu}(n+\hat{\nu}) \\
&= U_\mu(n)\, U_\nu(n+\hat{\mu})\, U_\mu(n+\hat{\nu})^\dagger\, U_\nu(n)^\dagger
\end{aligned}
$$



★ Wilson gauge action in terms of sum over all plaquettes

$$
\begin{aligned}
S_G[U] &= \frac{2}{g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} Re\ Tr[1 - U_{\mu\nu}(n)] \\
&= \frac{a^4}{2g^2} \sum_{n\in\Lambda} \sum_{\mu<\nu} Tr[F_{\mu\nu}(n)^2] + \mathcal{O}(a^2)
\end{aligned}
$$

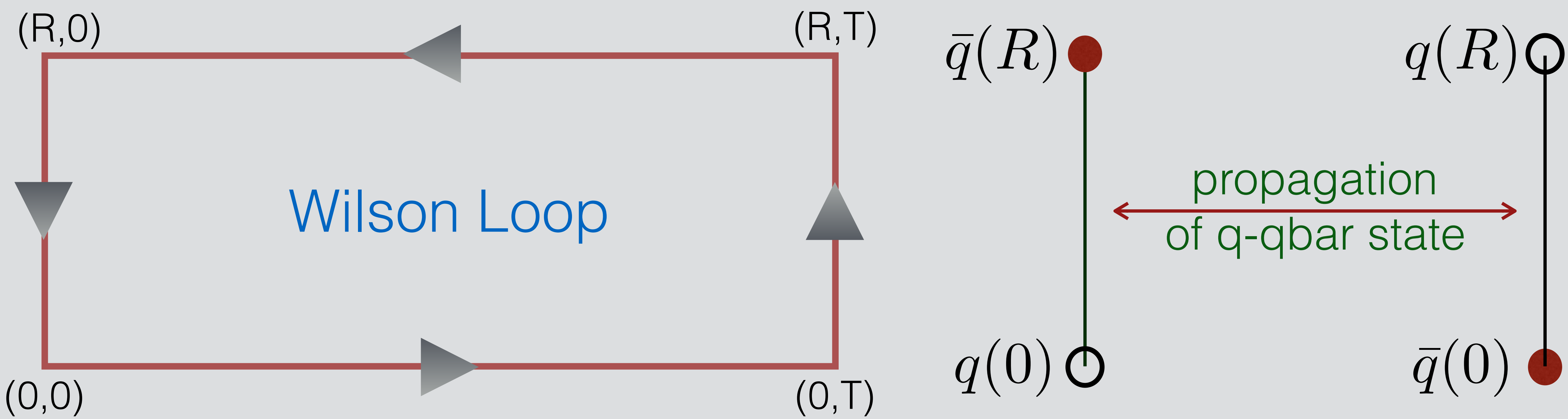
Other complicated gauge actions involve longer closed loops

Wilson Loop and Confinement

★ Wilson loop (average)

$$W(C) \equiv \langle \frac{1}{N}\, tr\, U(C) \rangle$$

$$= Z^{-1}(\beta) \int \prod_{x,\mu} dU_\mu(x) e^{-\beta S[U]} \frac{1}{N} tr\, U(C)$$



★ For T>>R, W(C) related to energy of interaction of static (WHY) quarks

$$W(R \times T) = e^{-E_0(R).T} \quad (T >> R)$$

★ Using strong-coupling expansion (expansion in $1/g^2$ or β)

plaquette average

$$W(\partial p) = \langle \frac{1}{N} trU(\partial p) \rangle$$

area of minimal surface in leading order β

$A_{min}(C)$=R X T

★ The area law:

$$W(C) \to e^{-\sigma A_{min}(C)} \quad (for\ large\ C)$$

★ Potential energy is linear function of the distance between quarks

$$E(R) = \sigma R$$

string tension (energy of string per unit length)

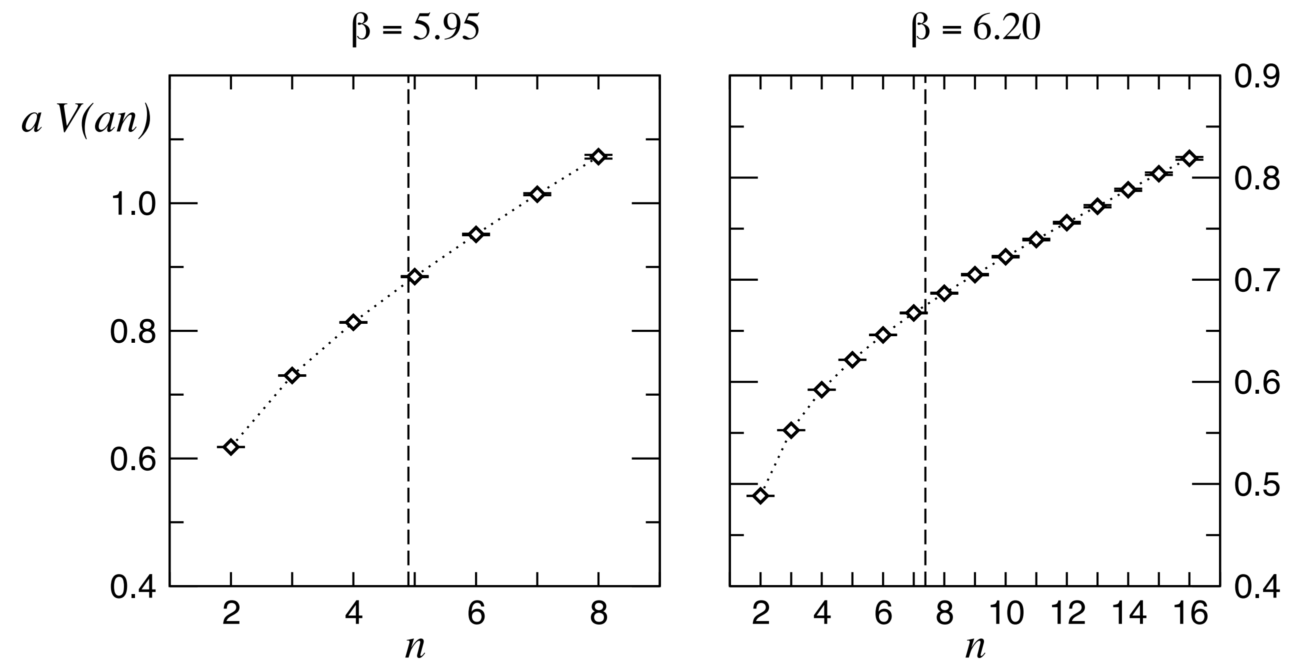
★ $$\sigma = \frac{1}{a^2} ln \frac{2N^2}{\beta} = \frac{1}{a^2} ln(2Ng^2)$$ $$W(\partial p) = \frac{\beta}{2N^2}$$

inverse of plaquette average for $N \geq 3$

Problem

★ Can you guess, what will be the average of plaquette for SU(2) case ?

LATTICE SPACING CAN BE CALCULATED FROM THIS FIG



Example for evaluation of the static potential: linearly rising!

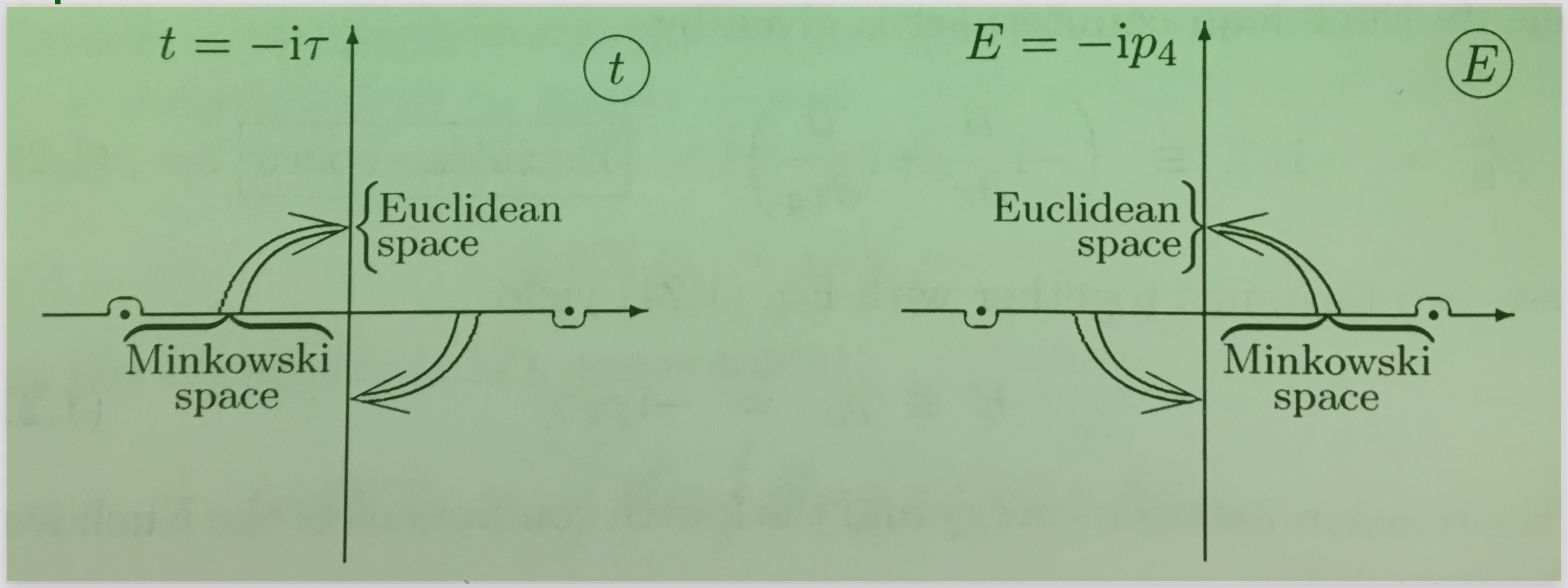
★ For Coulomb like potential one gets ***perimeter law***

$$W(C) \rightarrow e^{-const.L(C)} \quad \text{(for large C)}$$ (no confinement)

Euclidean Rotation

- Free propagator in Minkowski space

$$G(x-y) = \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}$$



- Passing into Euclidean variables

$$G_E(x-y) = \int \frac{d^d p}{(2\pi)^d} e^{ip(y-x)} \frac{i}{p^2 + m^2}$$

No $i\epsilon$ prescription required

- **Wick rotation:** Switching from a (1,*d*)-spacetime quantum theory to a (1+*d*)Euclidean quantum theory to compute observables and then switching back

Lattice Formulation

1\. Observables in lattice QCD are then expressed in terms of the path integral as

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \prod_n d\psi(n) \prod_n d\bar{\psi}(n) \mathcal{O}(U, \psi, \bar{\psi}) e^{-\left(S_G[U] + S_F[U,\psi,\bar{\psi}]\right)}$$

2\. Integrate out the Grassmann variables:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \prod_{n,\mu} dU_\mu(n) \mathcal{O}(U, G[U]) \det M[U] e^{-S_G[U]}$$

Importance Sampling

$$G(U,x,y)^{ij}_{\alpha\beta} \equiv \langle \psi^i_\alpha(x) \bar{\psi}^j_\beta(y) \rangle = M^{-1}(U)$$

Quark Propagator

3\. Generate an ensemble of gauge configurations

$$P[U] \propto \det M[U] e^{-S_G[U]}$$

DESCRIBES THE ROLES OF QUARK LOOPS IN THE VACUUM

4\. Calculate observable

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(U^n, G[U^n])$$

What We Actually Measure on Lattice

★ **Euclidean correlator** $\langle O_2(t)O_1(0)\rangle_T = \frac{1}{Z_T} tr[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1]$

$$\langle O_2(t)O_1(0)\rangle_T = \frac{1}{Z_T}\sum_{m,n}\langle m|e^{-(T-t)\hat{H}}\hat{O}_2|n\rangle\langle n|e^{-t\hat{H}}\hat{O}_1|m\rangle$$

$$= \frac{1}{Z_T}\sum_{m,n} e^{-(T-t)E_m}\langle m|\hat{O}_2|n\rangle e^{-tE_n}\langle n|\hat{O}_1|m\rangle$$

$$= \frac{\sum_{m,n} < m|\hat{O}_2|n >< n|\hat{O}_1|m > e^{-t\Delta E_n} e^{-(T-t)\Delta E_m}}{1 + e^{-T\Delta E_1 + e^{-T\Delta E_2}} + ...}$$

★ **Define** $\Delta E_n = E_n - E_0$

$$\lim_{T\to\infty}\langle O_2(t)O_1(0)\rangle_T = \sum_n < 0|\hat{O}_2|n >< n|\hat{O}_1|0 > e^{-tE_n}$$

★ $$\frac{1}{Z_T} tr[e^{-(T-t)\hat{H}}\hat{O}_2 e^{-t\hat{H}}\hat{O}_1] = \frac{1}{Z_T}\int \mathcal{D}[\phi] e^{-S_E[\phi]} O_2[\phi(.,t)] O_1[\phi(.,0)]$$

Integrand of the operators on LHS translated to functionals of the fields and then weighted with Boltzman factor containing classical Euclidean action

Correlation Functions

★ Spin 1/2 interpolation fields

$$\chi_i(\delta x) = \epsilon_{abc}[u^T(a\alpha x)\Gamma_1(\alpha\beta)d(b\beta x)]\Gamma_2(\delta\gamma)u(c\gamma x)$$

open Dirac index

$$\overline{\chi_i}(\delta x) = -\epsilon_{abc}\bar{u}(c\gamma x)\Gamma_2(\delta\gamma)[\bar{d}(b\beta x)\Gamma_1(\alpha\beta)\bar{u}^T(a\alpha x)]$$

$$\chi_1, \quad \bar{\chi}_1 \quad : \quad \Gamma_1 = C\gamma_5 \quad \text{and} \quad \Gamma_2 = 1$$

$$\chi_2, \quad \bar{\chi}_2 \quad : \quad \Gamma_1 = C \quad \text{and} \quad \Gamma_2 = \gamma_5$$

$$\chi_3, \quad \bar{\chi}_3 \quad : \quad \Gamma_1 = C\gamma_5\gamma_4 \quad \text{and} \quad \Gamma_2 = 1$$

★ The nucleon two-point correlation function is defined as:

$$G_{\alpha\beta}(t, \vec{p}, \vec{x}_0) = \sum_x e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \langle 0|T\Big(\chi_\alpha(x)\bar{\chi}_\beta(x_0)\Big)|0\rangle$$

★ $\chi(\bar{\chi})$ annihilation(creation) interpolation field

$\alpha,\ \beta$ Dirac indices

★ Insert complete sets $\sum_{n,\vec{q},s} |n,\vec{q},s\rangle \langle n,\vec{q},s| = 1$

Use Fourier transform $\sum_x e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} = N\delta_{\vec{p},\vec{q}}$

★ Then nucleon two-point correlation function reads

$$
\begin{aligned}
G_{\alpha\beta}(t,\vec{p}) &= N\sum_{n,\vec{q},s} \delta(\vec{p}-\vec{q}) e^{-i(\vec{p}-\vec{q})\cdot\vec{x}_0} e^{-E_{n,\vec{q}}(t-t_0)} \\
&\quad \langle 0|\, \chi_\alpha(x_0)\, |n,\vec{q},s\rangle \langle n,\vec{q},s|\, \bar{\chi}_\beta(x_0)\, |0\rangle \\
&= N\sum_{n,s} e^{-E_{n,\vec{p}}(t-t_0)} \langle 0|\, \chi_\alpha(x_0)\, |n,\vec{p},s\rangle \langle n,\vec{p},s|\, \bar{\chi}_\beta(x_0)\, |0\rangle
\end{aligned}
$$

Number of lattice sites

sum over n contains contribution form positive and negative parity excited-states

★ To obtain nucleon ground-state matrix element we need to suppress these contributions

Excited-States Contaminations

★Re-write two-point function as

$$G_{\alpha\beta}(t,\vec{p}) = N\sum_s \Big(e^{-E_p^{0,+}(t-t_0)} \langle 0|\, \chi_\alpha(x_0)\, |0,\vec{p},s,+\rangle \langle 0,\vec{p},s,+|\, \bar{\chi}_\beta(x_0)\, |0\rangle$$
$$+e^{-E_p^{0,-}(t-t_0)} \langle 0|\, \chi_\alpha(x_0)\, |0,\vec{p},s,-\rangle \langle 0,\vec{p},s,-|\, \bar{\chi}_\beta(x_0)\, |0\rangle \Big)$$

$|0,\vec{p},s,+\rangle$ is the positive-parity nucleon ground-state with energy $e^{-E_p^{(0,+)}}$

★ Taking trace with positive-parity projection operator $\Gamma_4 \equiv \Gamma_e \equiv \frac{1\pm\gamma_4}{2}$

$$\Gamma_{\beta\alpha}G_{\alpha\beta}(t,\vec{p}) = a^6|\phi^+|^2 e^{-E_p^{0,+}(t-t_0)} \frac{E_p^{0,+}+m^+}{E_p^{0,+}}$$
$$+a^6|\phi^-|^2 e^{-E_p^{0,-}(t-t_0)} \frac{(m^- - \sqrt{(m^-)^2+\vec{p}^2})}{E_p^{0,-}}$$

★ If one has $\frac{\vec{p}^2}{m^-} \ll 1$,

$$\mathrm{Tr}\left[\Gamma_e G(t,\vec{p})\right] = a^6|\phi^+|^2 e^{-E_p^{0,+}(t-t_0)} \frac{E_p^{0,+}+m^+}{E_p^{0,+}} - a^6|\phi^-|^2 \frac{e^{-E_p^{0,-}(t-t_0)}}{E_p^{0,-}} \frac{1}{2} \frac{\vec{p}^2}{(m^-)^2}$$

★ For a final nucleon state at rest ($\vec{p}=0$),

$$G_{NN}(t,\vec{p},\Gamma_e) \equiv \mathrm{Tr}[\Gamma_e G(t,\vec{p})] = a^6|\phi_0^+|^2 e^{-m^+(t-t_0)}$$

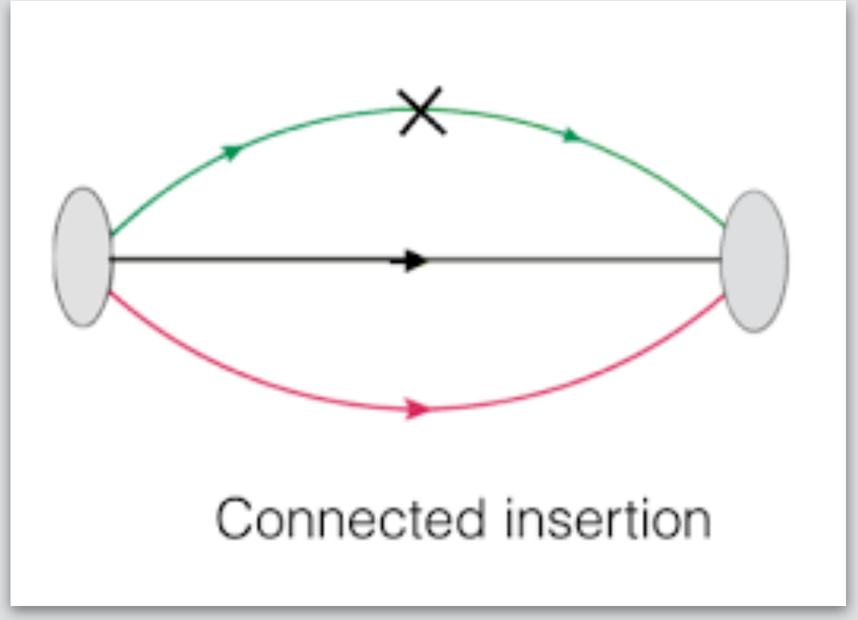
★ Negative parity states are not completely suppressed unless for zero nucleon momentum

Contamination exponentially suppressed in long time limit as negative-parity ground-state has higher mass and energy than positive-parity ground state.

Three-point Correlation Function

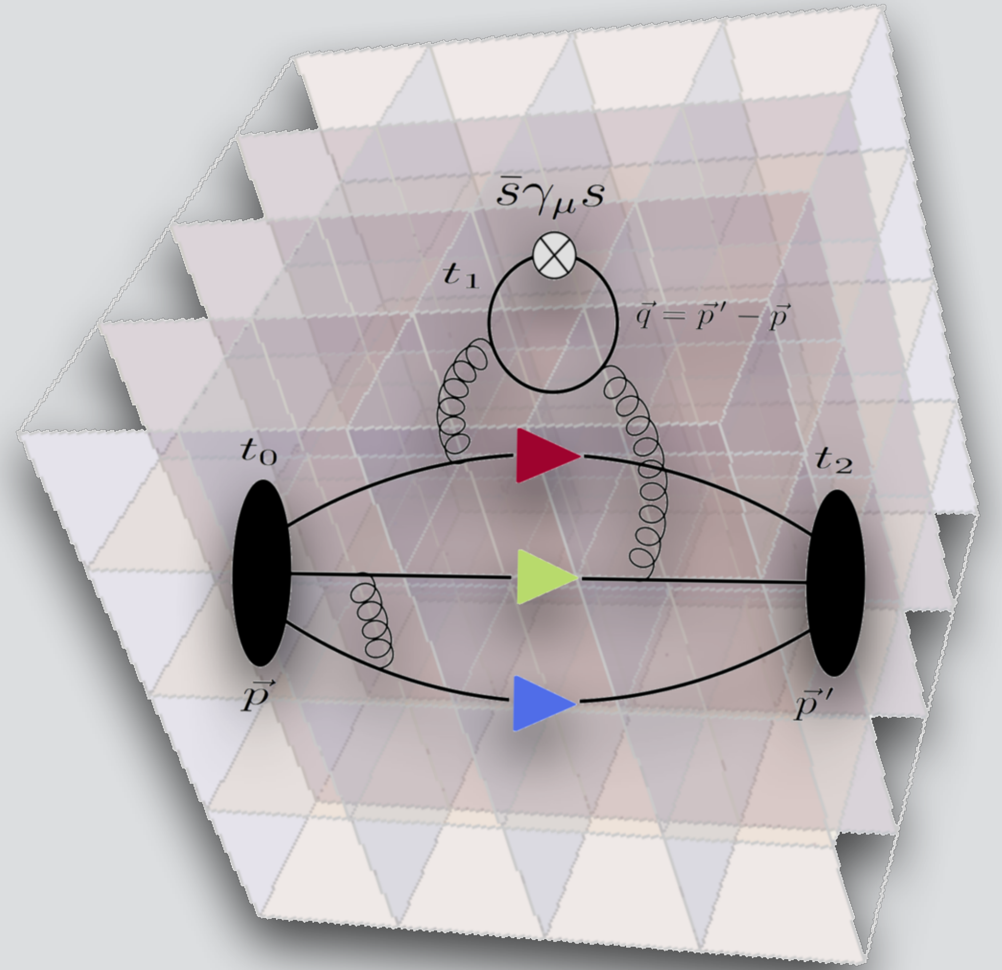
1. Connected Insertions:

Current connected to the nucleon through the quark lines



2. Disconnected Insertions:

Self contracted quark loop correlated with valence quarks in the nucleon propagator by fluctuating background gauge fields



Schematic Representation

★ Consider nucleon three-point correlator with

source momentum $\vec{p'}$

sink momentum $\vec{p}$

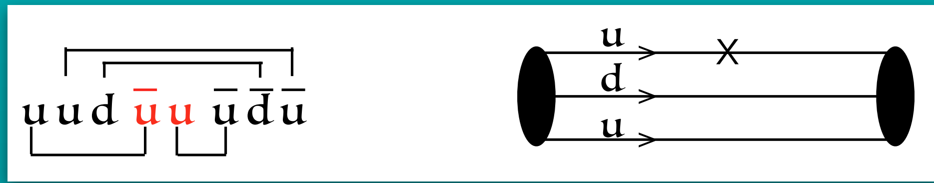
$$\mathcal{C}^{\mathcal{O}_q}_{\alpha\beta}(t,\tau;\vec{p},\vec{p}') = \langle \chi_\alpha(t,\vec{p})\mathcal{O}_q(\tau)\bar{\chi}_\beta(0,\vec{p}')\rangle$$

★ Insert operator with q-flavored LOCAL current

$$\mathcal{O}_q(\tau) = \sum_{\vec{y},v,w} \bar{q}^a_\alpha(v)\mathbb{O}^{ab}_{\alpha\beta}(v,w;\vec{y},\tau)q^b_\beta(w)$$

★ Positive-parity contracted three-point correlator

$$\begin{aligned}
\mathcal{C}^{\mathcal{O}_q}(t,\tau;\vec{p},\vec{p}') &= \Gamma_{\beta\alpha}\langle \chi_\alpha(t,\vec{p})\mathcal{O}_q(\tau)\bar{\chi}_\beta(0,\vec{p}')\rangle \\
&= \sum_{\vec{y},v,w}\sum_{\vec{x},\vec{z'}} e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{p}'\cdot\vec{z'}}\epsilon_{abc}\epsilon_{a',b',c'}(C\gamma_5)_{\gamma\delta}(\gamma_5 C^{-1})_{\rho\sigma}\Gamma_{\beta\alpha} \\
&\quad \langle u^a_\alpha(\vec{x},t)u^b_\gamma(\vec{x},t)d^c_\delta(\vec{x},t)\bar{q}^d_\lambda(v)\mathbb{O}^{de}_{\lambda\kappa}(v,w,\vec{y},\tau)q^e_\kappa(w) \\
&\quad \bar{u}^{a'}_\beta(\vec{z'},0)\bar{d}^{b'}_\rho(\vec{z'},0)\bar{u}^{c'}_\sigma(\vec{z'},0)\rangle
\end{aligned}$$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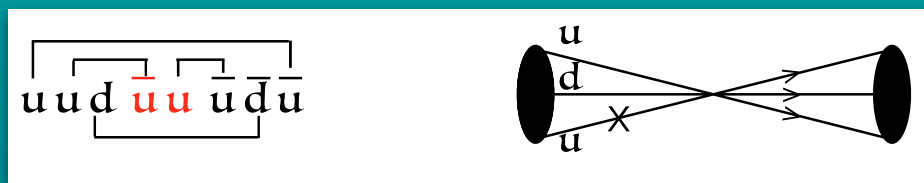
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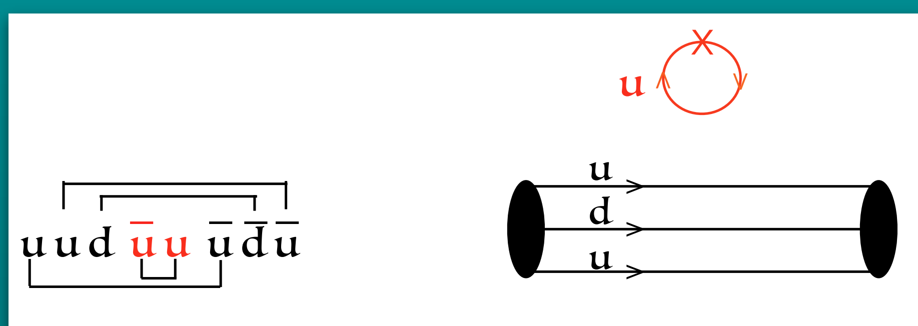
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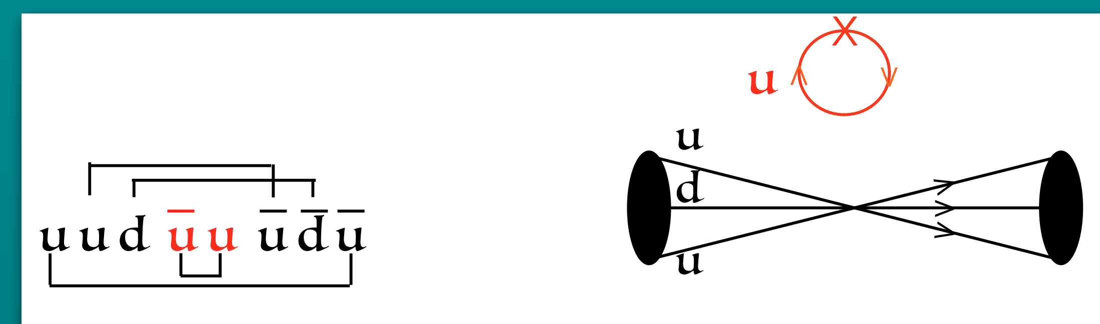
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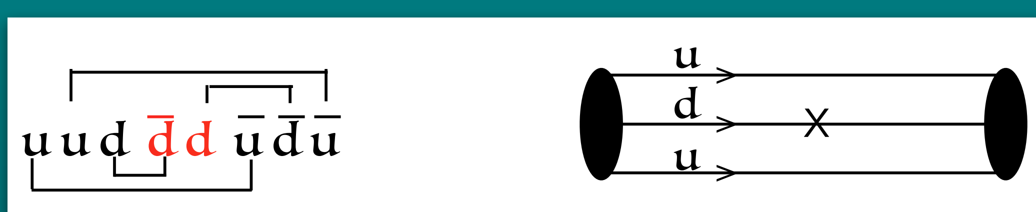
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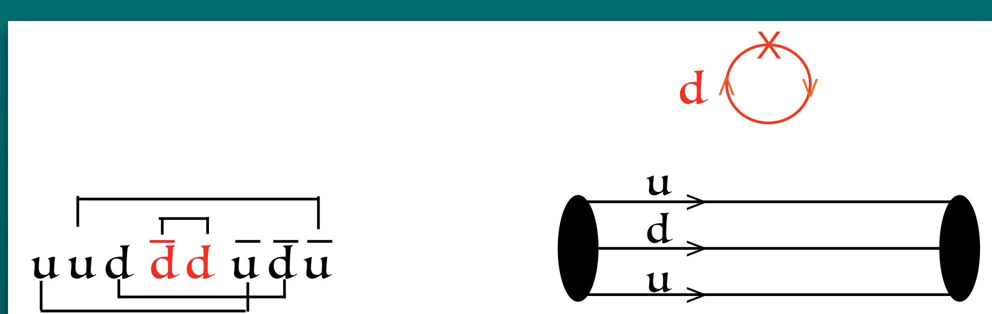
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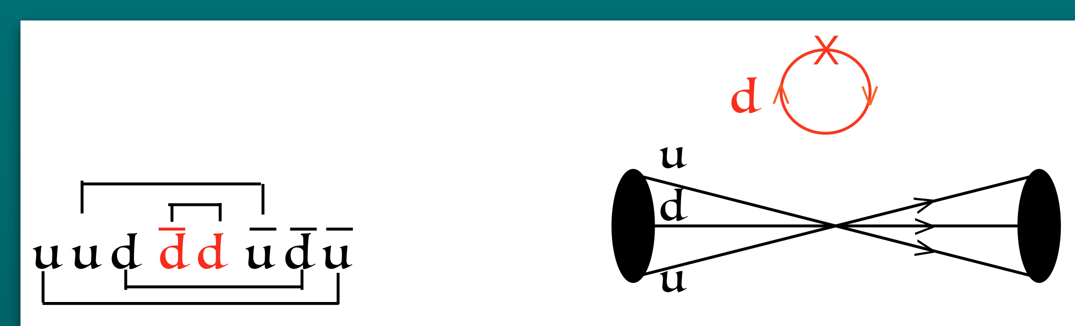
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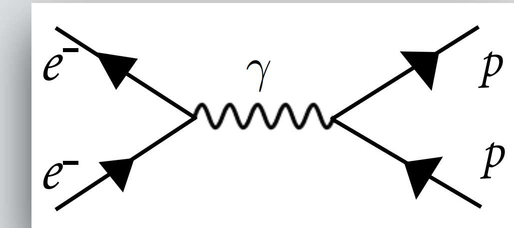
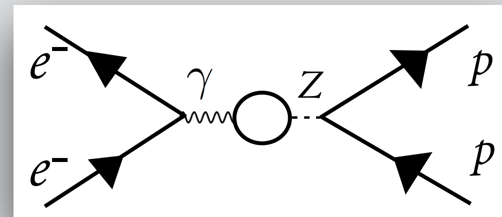
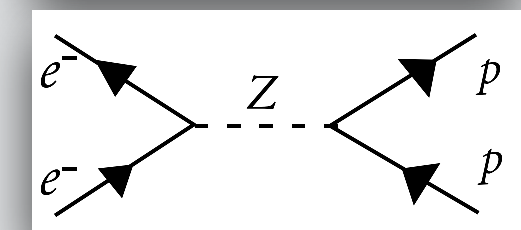
An Example LQCD Calculation
(Strange Quark Electromagnetic Properties)

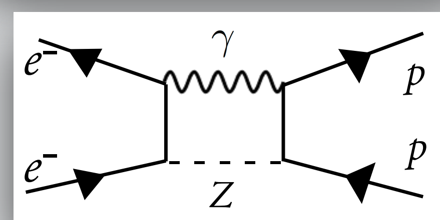
* Zel'dovich (1957): EM interaction with parity violation

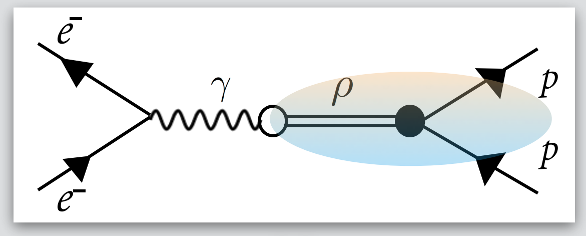


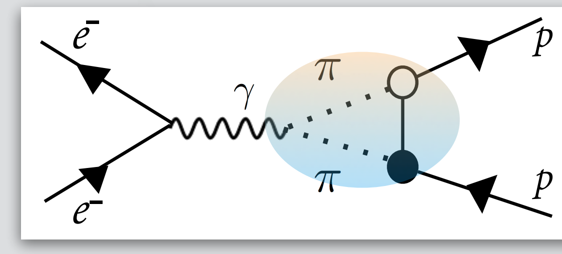


$$\mathcal{M}_Z = \frac{G_F}{2\sqrt{2}}(g_V^i l^\mu + g_A^i l^{\mu 5})(J_\mu^Z + J_{\mu 5}^Z)$$









* Kaplan, Manohar (88):

$$G_{E,M}^{Z,p(n)}(Q^2) = \frac{1}{4}\left[(1-4\sin^2\theta_W)(1+R_V^{p(n)})G_{E,M}^{\gamma,p(n)}(Q^2) - (1+R_V^{n(p)})G_{E,M}^{\gamma,n(p)}(Q^2) - (1+R_V^{(0)})G_{E,M}^{s}(Q^2)\right]$$

* McKeown and Beck (89):

$$\begin{aligned} A_{PV}^p = & -\frac{G_F Q^2}{4\sqrt{2}\pi\alpha}\frac{1}{[\epsilon(G_E^p)^2 + \tau(G_M^p)^2]} \\ & \times \{(\epsilon(G_E^p)^2 + \tau(G_M^p)^2)(1-4\sin^2\theta_W)(1+R_V^p) \\ & - (\epsilon G_E^p G_E^n + \tau G_M^p G_M^n)(1+R_V^n) \\ & - (\epsilon G_E^p G_E^s + \tau G_M^p G_M^s)(1+R_V^{(0)}) \\ & - \epsilon'(1-4\sin^2\theta_W)G_M^p G_A^e\} \end{aligned}$$

Unknown

Strange Quark Contribution

- s - quark contribution arises from vacuum: sign and magnitude related to nonperturbative structure of nucleon

- Nonzero strange electric FF $G^s{}_E$ at $Q^2 > 0$ implies different spatial distribution of s and s quarks

- Background in Q_{weak} experiment arises from magnetization of strange quark [strange magnetic FF $G^s{}_{E,M}$]

- *$G^s{}_{E,M}(Q^2)$ essential for determination of neutral weak FFs*

- Experimental results (G0, HAPPEX, A4, SAMPLE) of $G^s{}_{E,M}$ quite uncertain

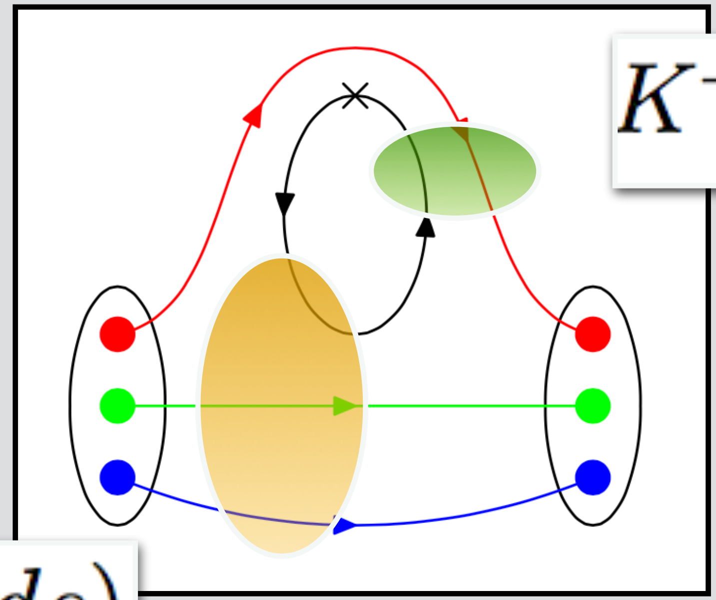
Strange Quark Contribution

* Electromagnetic current *C - odd*

*Sensitive to difference between contributions from s and $\bar{s}$

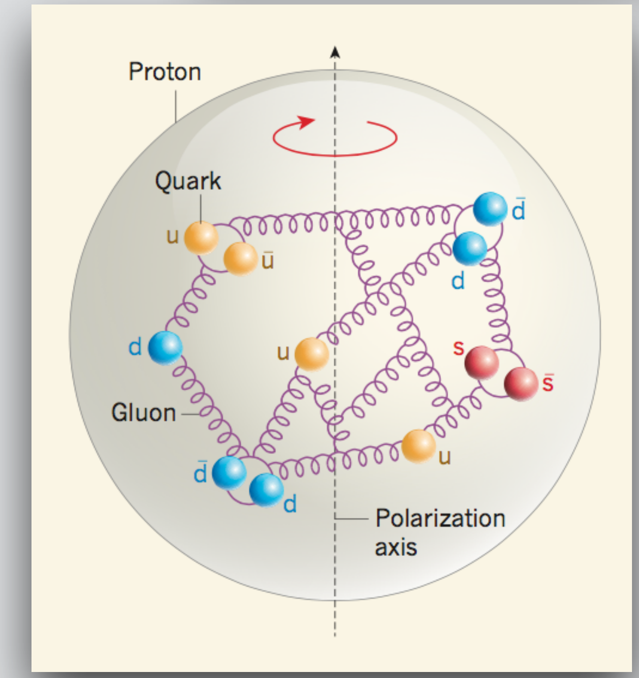
* Requires mechanisms beyond simple $g \longrightarrow s\bar{s}$ fluctuations

* Example: Meson-Baryon fluctuation: $N \rightarrow K^+\Lambda$ fluctuation



$K^+(u\bar{s})$

$\Lambda(uds)$



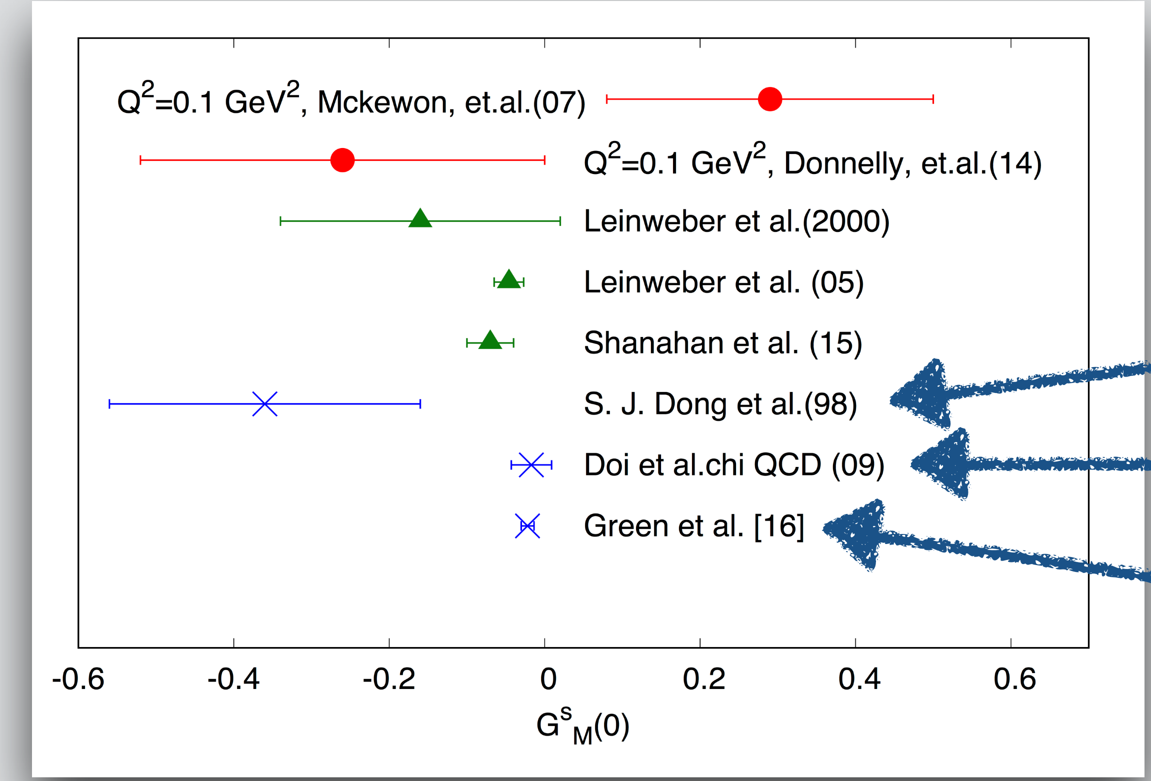
$m_{K^+} < m_\Lambda$

Theory & Experiment: $G^s_M(Q^2)$

RED: Analysis of world expt. data

GREEN: Indirect calculation

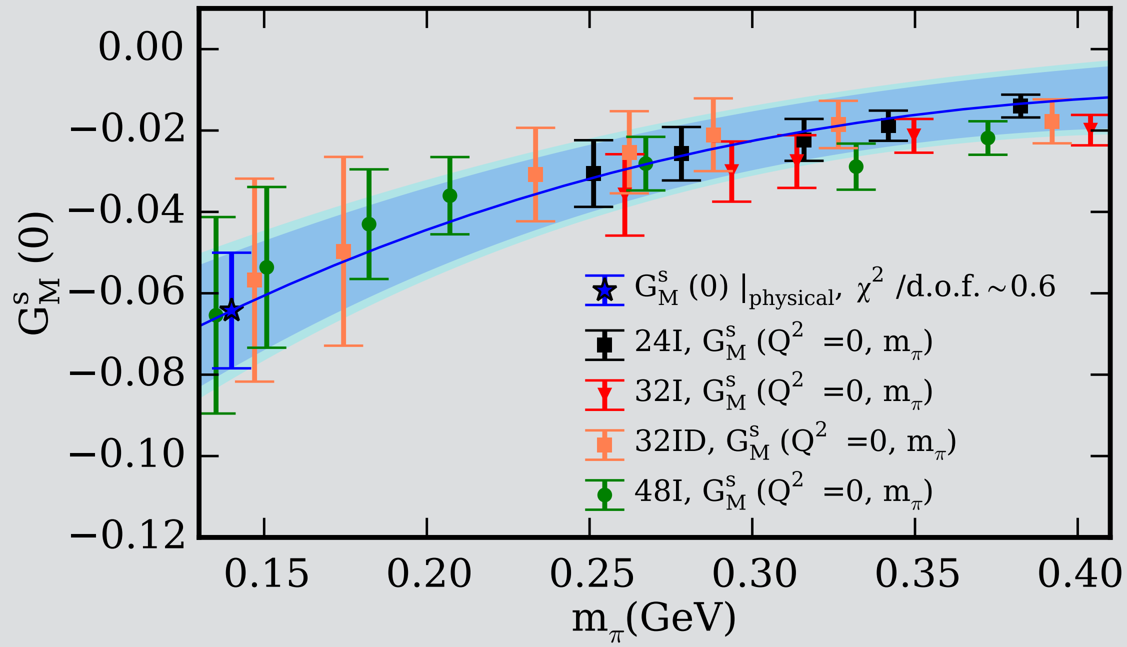
BLUE: Lattice QCD

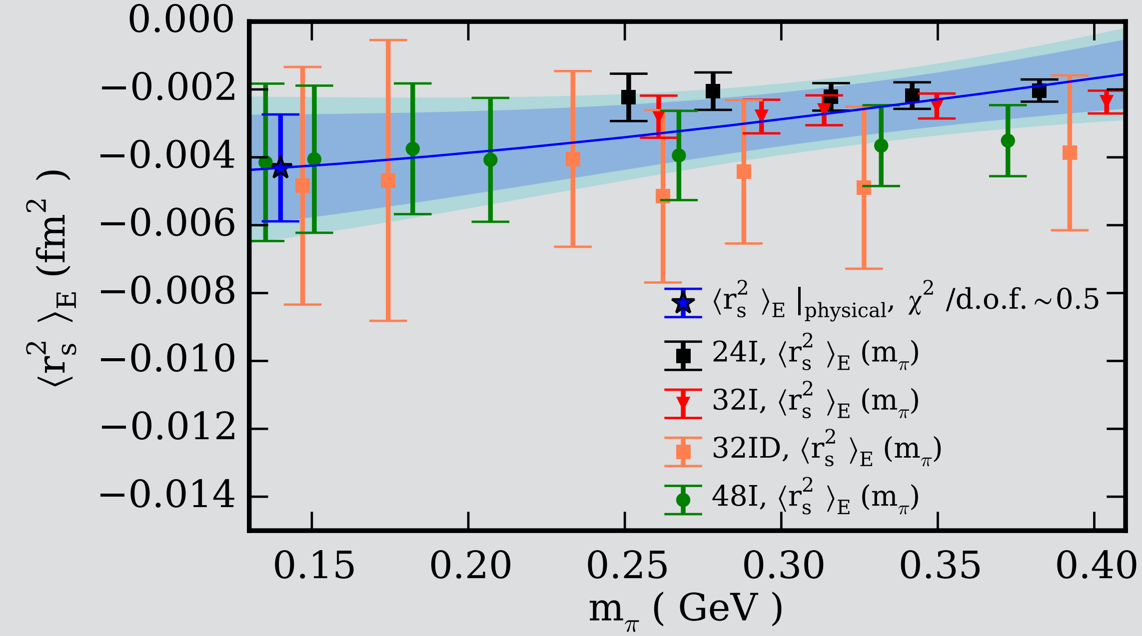


Quenched

Pion mass ~ 600 MeV

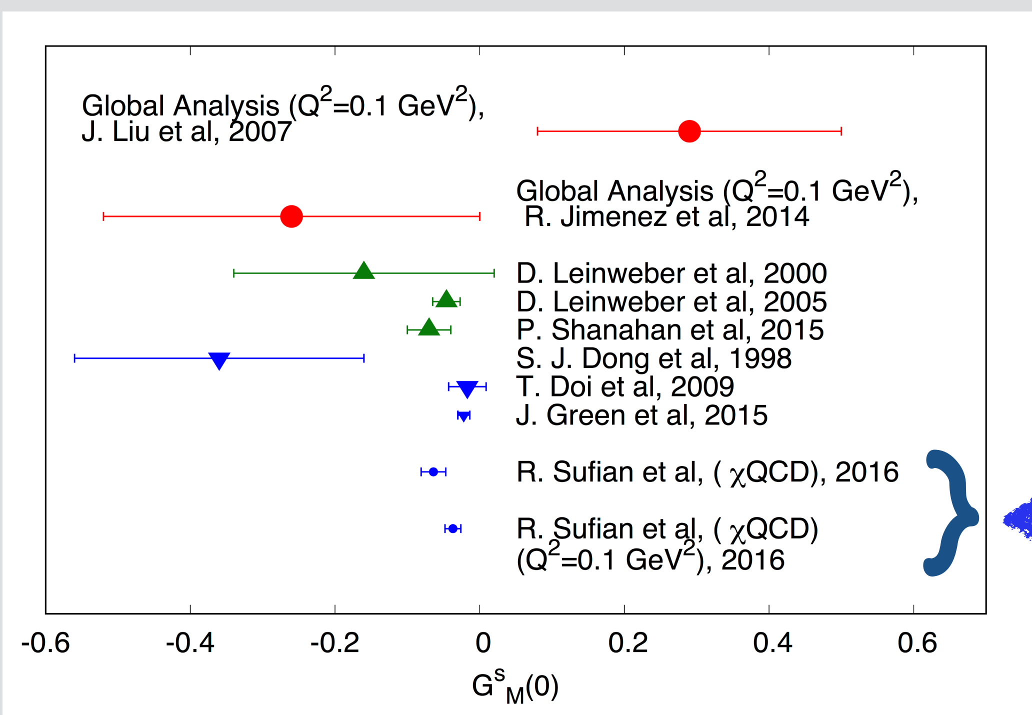
Pion mass ~ 315 MeV





$G^s_M(0)$ |physical = - 0.064(14)(09)

$r^2_{s,E}$= -0.0043(16)(14) fm^2



Most precise and accurate estimates of G^S_M

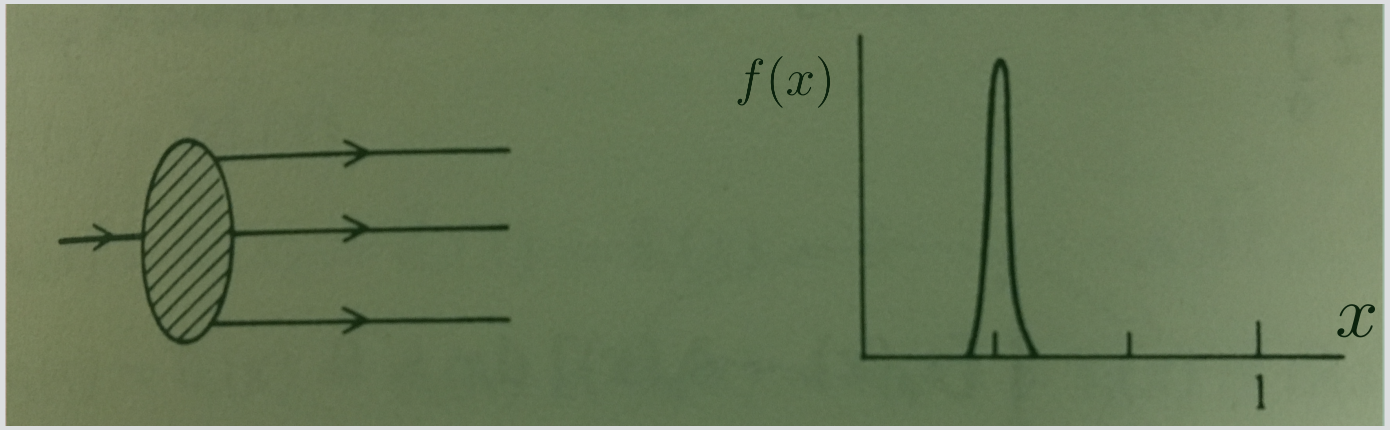
RSS,Yang et .al
PRL 2017

Parton Distribution Functions (PDFs)

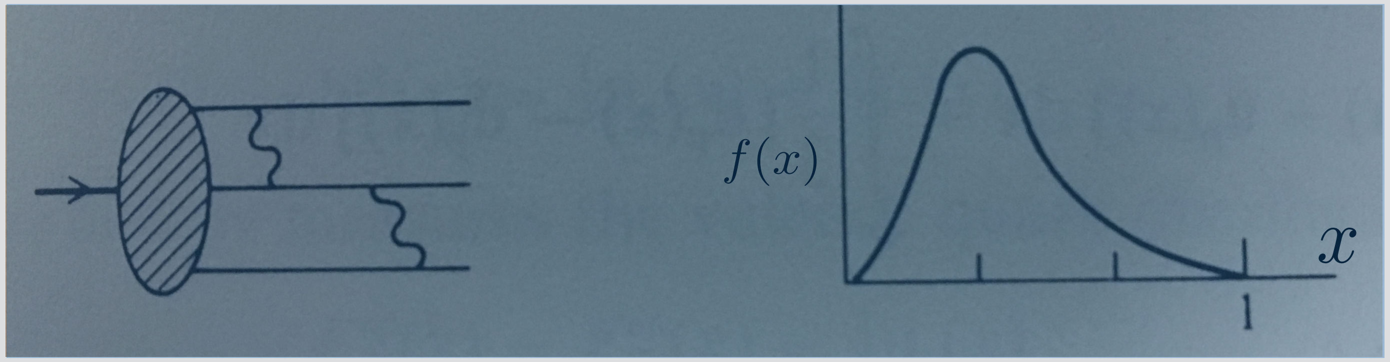
Parton density function describes the probability to find a Parton of type "a" in a hadron "A", carrying a momentum fraction ξ of the hadron

3 free quarks in nucleon
PDF δ-function

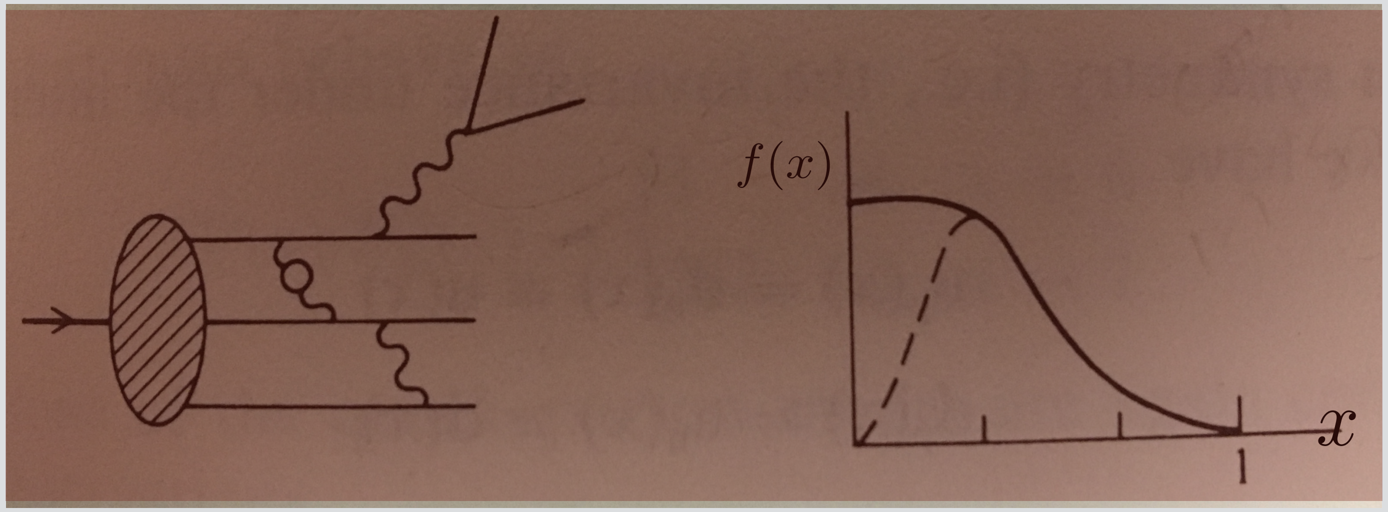
$$f(x) \sim \delta(x - 1/3)$$



Interactions between quarks with gluons exchange smears the distribution



$q\bar{q}$ sea at small x

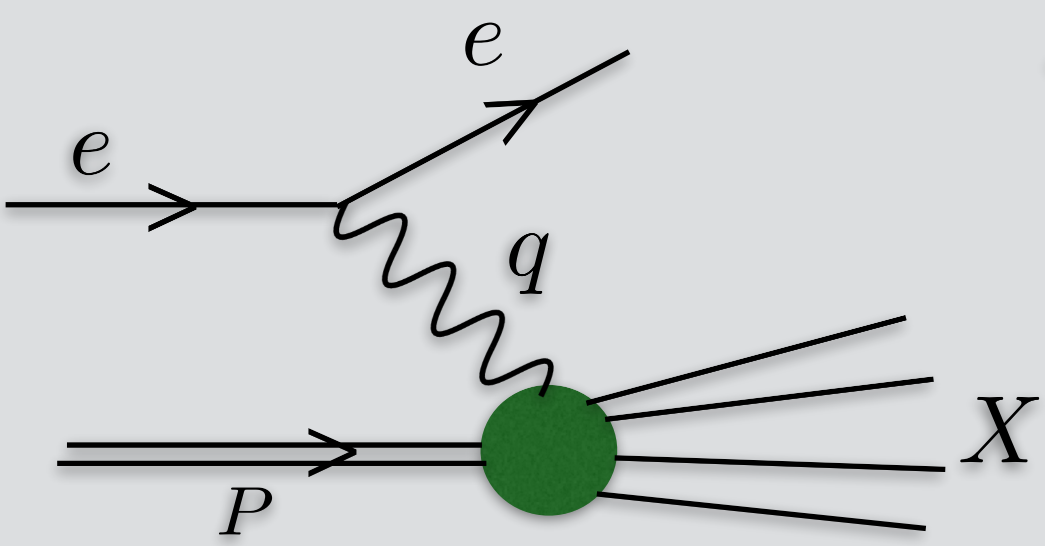


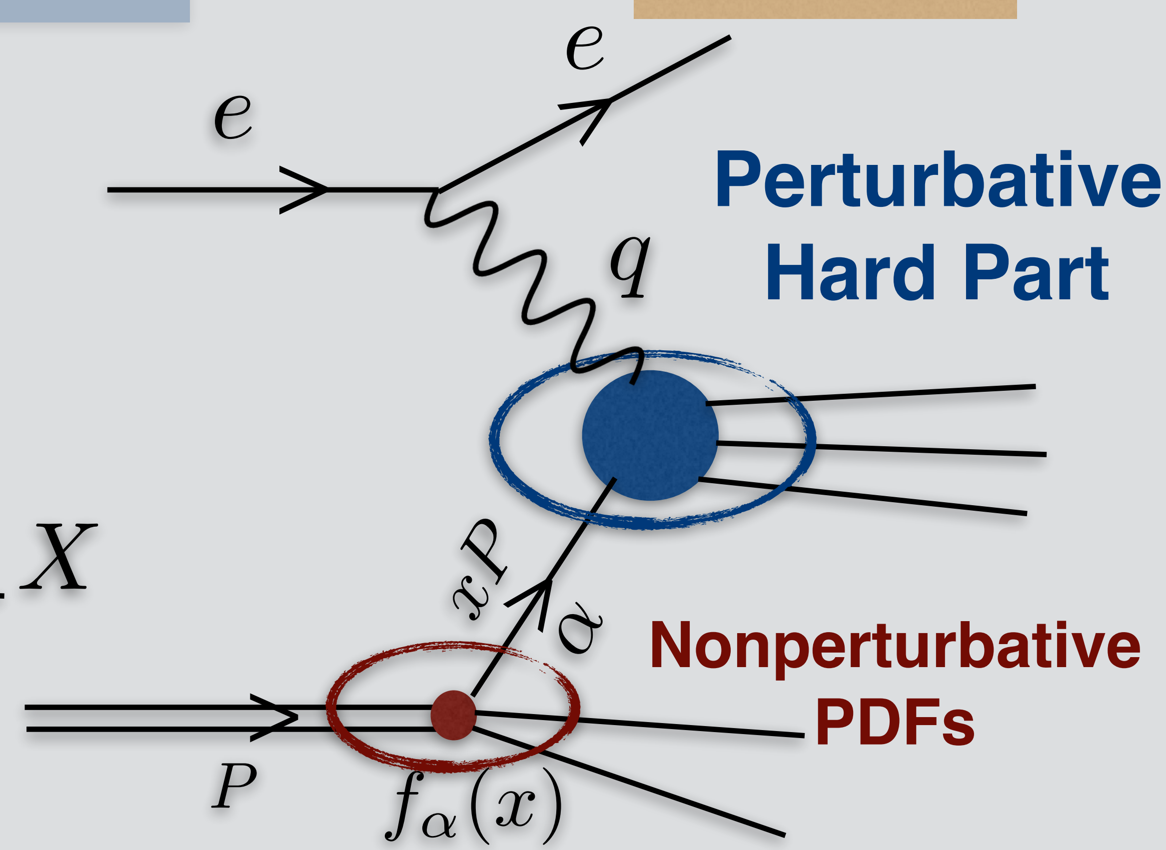
What would be the distribution for a point-like nucleon?

Parton Distribution Functions (PDFs)

$$\sigma^{DIS}(x, Q^2, \sqrt{s}) = \sum_{\alpha=q,\bar{q},g} C_\alpha\left(x, \frac{Q^2}{\mu^2}, \sqrt{s}\right) \otimes f_\alpha(x, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$$

wish the nucleon wave function were know !!





DIS

Parton Picture

x :Bjorken-x

$\sqrt{s}$:Collision Energy

Q :Momentum Transfer

μ :Factorization Scale

Factorization scale μ introduced. WHY ?

Factorization

$$\sigma_{eA\to X}(x,Q^2)=\sum_a\int_0^1 d\xi_1 d\xi_2\,\phi_{a/A}(\xi_1,\mu)\hat{\sigma}_{ea\to X}(\xi_2,Q^2,\mu)\delta(x-\xi_1\xi_2)+\text{Power corrections}$$

$$=\sum_a\int_x^1\frac{d\xi}{\xi}\phi_{a/A}(\xi,\mu)\hat{\sigma}_{ea\to X}(x/\xi,Q^2,\mu)+\text{Power corrections}$$

do not properly factorize
Higher twist contribution

Problem: How is the ξ related to x_{bj}

Does it have anything to do with LO and/or NLO…so on

1. There are quantum corrections to this factorization for QCD

2. These quantum fluctuations can have arbitrary energy

3. Factorization in trouble if the energy of the **virtual partonic** states of the same scale as the Q^2

4. Factorization scale μ describes which fluctuations should be included in the PDFs and which can be included in the hard scattering part

Wilson Lines…

Properties

1. Hermiticity:

$$\Phi_y^\dagger[a,b;A] = \Phi_{-y}[b,a;A].$$

2. Causality: Path ordering can glue paths together

$$\Phi_y[b,c;A]\Phi_y[a,b,;A] = \Phi_y[a,c;A].$$

3. Unitarity:

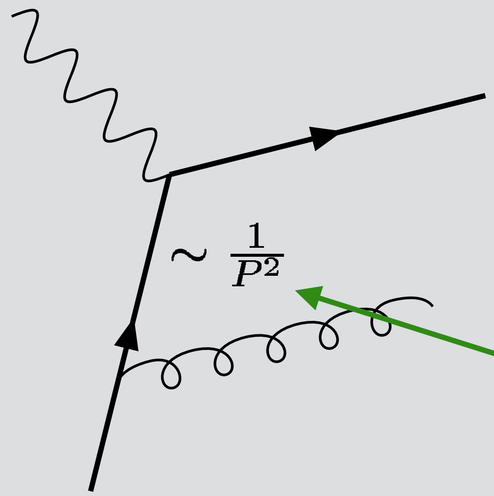
$$\Phi_y^\dagger[a,b;A]\Phi_y[a,b;A] = 1.$$

WHY WILSON LINE??

Asymptotic freedom states all higher order corrections
in perturbation theory should be small for hard particles

But the coupling strength is large if interactions are with soft particles.
In scattering experiments there are soft,i.e. low energy, gluon radiation.

Near threshold, large logarithms coming from soft radiation become the leading corrections to scattering cross section



Interaction of a quark with a photon including a gluon correction

$$p^2 = 0$$

1. If momentum of gluon small, internal quark propagator almost on–shell.

2. Can lead to large logarithmic corrections after the infrared divergences are canceled.

3. This gluon radiation consists of an infinite number of soft gluons, which would make perturbation theory an unusable method for computing physical cross sections.

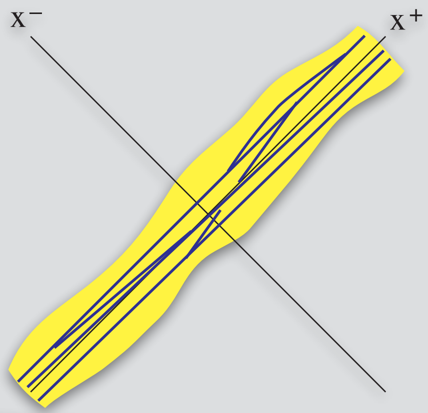
★ With some approximations all the soft radiations can be described by a vacuum expectation value of a single path ordered exponential

$$\mathcal{P} e^{ig \int dz_\mu A^a_\mu(z) t_a}$$

★ Path described by z_μ
Classical paths of the parton that emits and absorbs gluons

★ Wilson lines describe only soft gluons. So to use the Wilson lines we have to separate the soft gluons, which we describe by the Wilson line, from the hard gluons, whose contributions can be calculated using ordinary perturbation theory.

Unrenormalized quark distribution



$$f^{(0)}_{j/A}(x) = \frac{1}{4\pi}\int dy^- e^{-ixP^+y^-} \times \langle P^+, \vec{0}_T | \overline{\psi}_{0,j}(0, y^-, \vec{0}_T)\gamma^+ \psi_{0,j}(0,0,\vec{0}_T) | P^+, \vec{0}_T\rangle$$

Problem: +- why?

operator is not local but bi-local

$(0, y^-, \vec{0}_T)$ and $(0, 0, \vec{0}_T)$

The gauge-invariant definition is

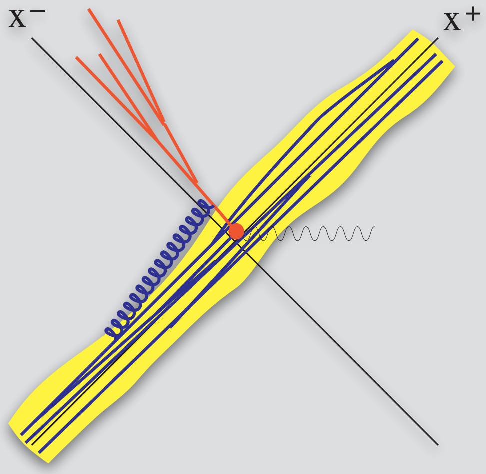
$$f^{(0)}_{j/A}(x) = \frac{1}{4\pi}\int dy^- e^{-ixP^+y^-} \langle P^+, \vec{0}_T | \overline{\psi}_{0,j}(0, y^-, \vec{0}_T)\gamma^+ \mathcal{O}_0 \; \psi_{0,j}(0,0,\vec{0}_T) | P^+, \vec{0}_T\rangle$$

$$\mathcal{O}_0 = \mathcal{P}\exp\left(ig_0 \int_0^{y^-} dz^- \, A^+_{0,a}(0, z^-, \vec{0}_T)\, t_a \right)$$

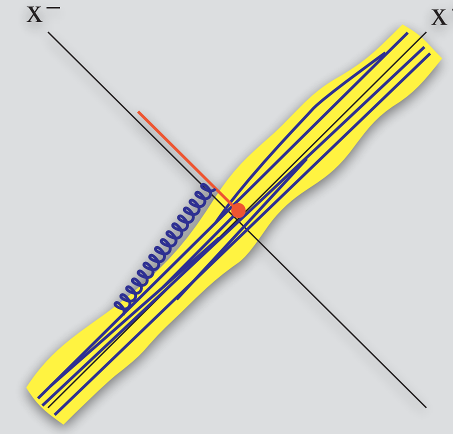
Re-write

$$\mathcal{O}_0 = \overline{\mathcal{P}}\exp\left(-ig_0 \int_{y^-}^{\infty} dz^- \, A^+_{0,a}(0, z^-, \vec{0}_T)\, t_a \right) \times \mathcal{P}\exp\left(ig_0 \int_0^{\infty} dz^- \, A^+_{0,a}(0, z^-, \vec{0}_T)\, t_a \right).$$

insert here
and also complete sets
between two exponentials



DIS



Parton distribution

A virtual photon knocks out a quark, which emerges moving in the minus direction and develops into a jet of particles.

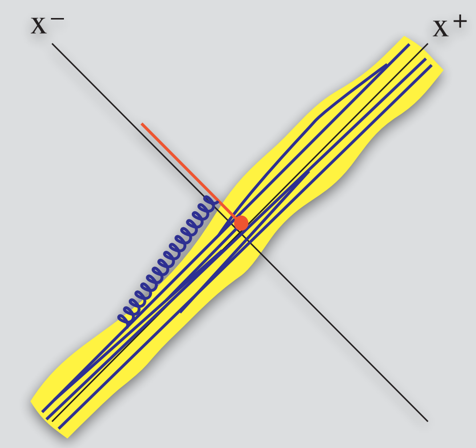
Illustrates the **amplitude** associated with the quark distribution function

quark distribution function including a sum over intermediate states $|N\rangle$

$$f_{i/h}(\xi,\mu_F) = \frac{1}{2}\int \frac{dy^-}{2\pi}\, e^{-i\xi p^+ y^-} \sum_N \langle p|\bar{\psi}_i(0,y^-,\mathbf{0})\gamma^+ F_2|N\rangle \langle N|F_1\psi_i(0)|p\rangle$$

Operator ψ_i annihilates a quark in the nucleon

$$F_1 \;=\; \mathcal{P}\exp\left(-ig\int_0^\infty dz^- A_a^+(0,z^-,\mathbf{0})\,t_a\right)$$



Parton distribution

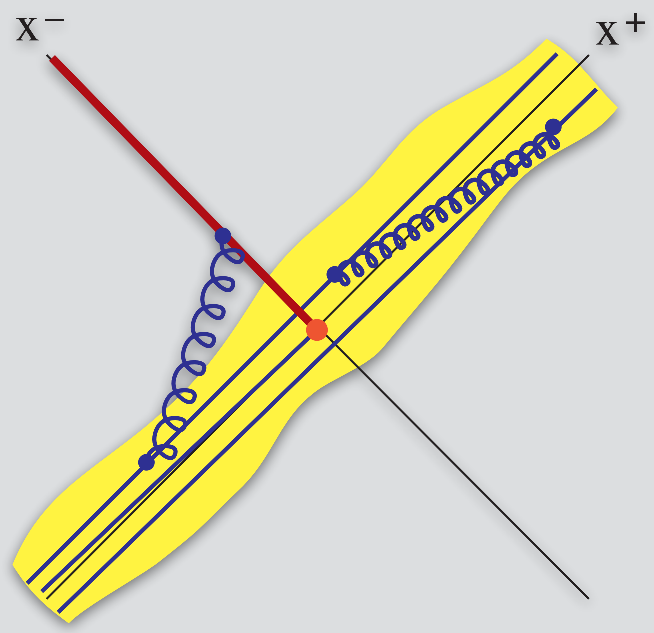
F_1 stands in for the quark moving in the x^- direction.

★ The gluon field A evaluated along a lightlike line in x^- direction absorbs longitudinally polarized gluons from the color field of the proton

★ Thus the physics of deeply inelastic scattering is built into the definition of the quark distribution function

A quark moving in the plus direction is struck
and exits to infinity with almost the speed of light in the minus direction

As it goes, the struck quark interacts with the gluon field of the hadron.



We can now see that the role of the operator $\mathcal{O}$ is to replace the struck quark with a fixed color charge that moves along a light-like line in the minus-direction, mimicking the motion of the actual struck quark in a real experiment.

What are Good Lattice “Cross Sections” (LCSs)

Single hadron matrix elements:

Ma & Qiu
PRL (2018)

1. Calculable using lattice QCD with Euclidean time

2. Well defined continuum limit ($a \to 0$), UV finite

3. Share the same perturbative collinear divergences with PDFs

4. Factorizable to PDFs with IR-safe hard coefficients
   with controllable power corrections

Lattice Calculable + Renormalizable + Factorizable

$$\sigma_n(\omega,\xi^2,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2) \times K_n^a(x\omega,\xi^2,x^2P^2,\mu^2) + \mathcal{O}(\xi^2\Lambda_{QCD}^2)$$

Nonperturbative PDFs of flavor $a = q, g$

Perturbatively Calculable Hard Coefficients

$$f_{\bar{a}}(x,\mu^2) = -f_a(-x,\mu^2)$$

P and ξ → Collision Kinematics

$\mathcal{O}_n$ → Dynamical Features of LCSs

Factorization holds for any finite ω and $P^2\xi^2$ if ξ is short distance

★ Hadron matrix elements: $\sigma_n(\omega, \xi^2, P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle$

$$\omega \equiv P \cdot \xi, \ \ \xi^2 \neq 0, \ \ \xi_0 = 0$$

★ Current-current correlators

$$\mathcal{O}_{j_1 j_2}(\xi) \equiv \xi^{d_{j_1}+d_{j_2}-2} \, Z_{j_1}^{-1} \, Z_{j_2}^{-1} j_1(\xi) \, j_2(0)$$

d_j : Dimension of the current

Z_j : Renormalization constant of the current

Z_j already known for the lattice ensembles being used

★ Different choices of currents

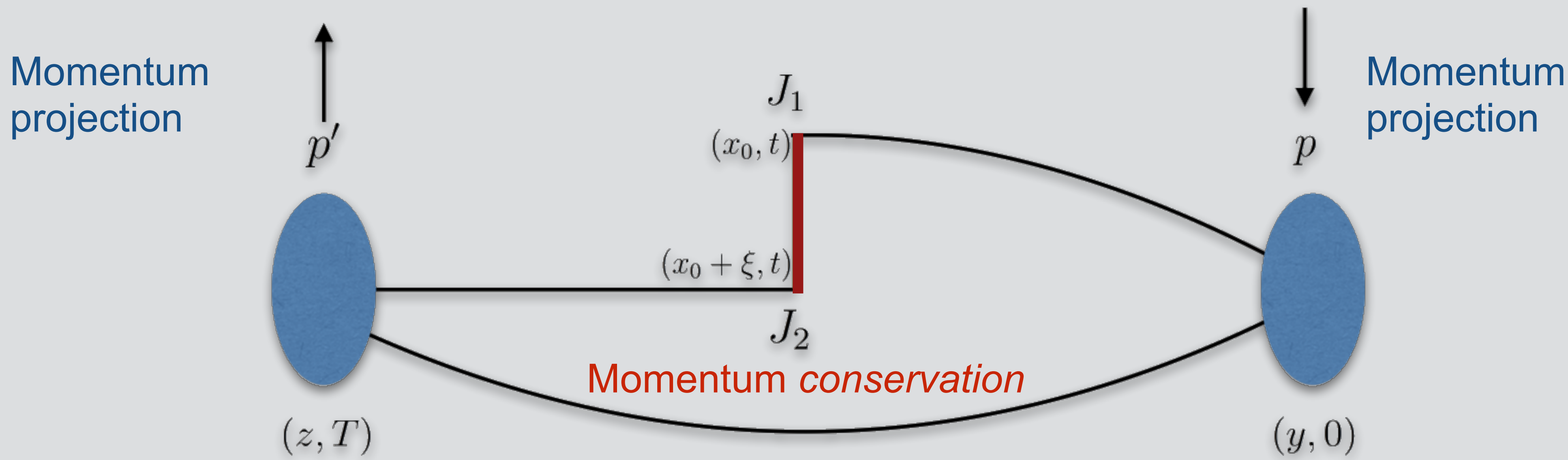
$$j_S(\xi) = \xi^2 Z_S^{-1}[\overline{\psi}_q \psi_q](\xi), \qquad j_V(\xi) = \xi Z_V^{-1}[\overline{\psi}_q \gamma \cdot \xi \psi_q](\xi),$$

$$j_{V'}(\xi) = \xi Z_{V'}^{-1}[\overline{\psi}_q \gamma \cdot \xi \psi_{q'}](\xi), \qquad j_G(\xi) = \xi^3 Z_G^{-1}[-\frac{1}{4} F^c_{\mu\nu} F^c_{\mu\nu}](\xi), \ldots$$

flavor changing current

gluon distribution

Example Lattice Setup for Pion Using LCSs
Challenges and Questions



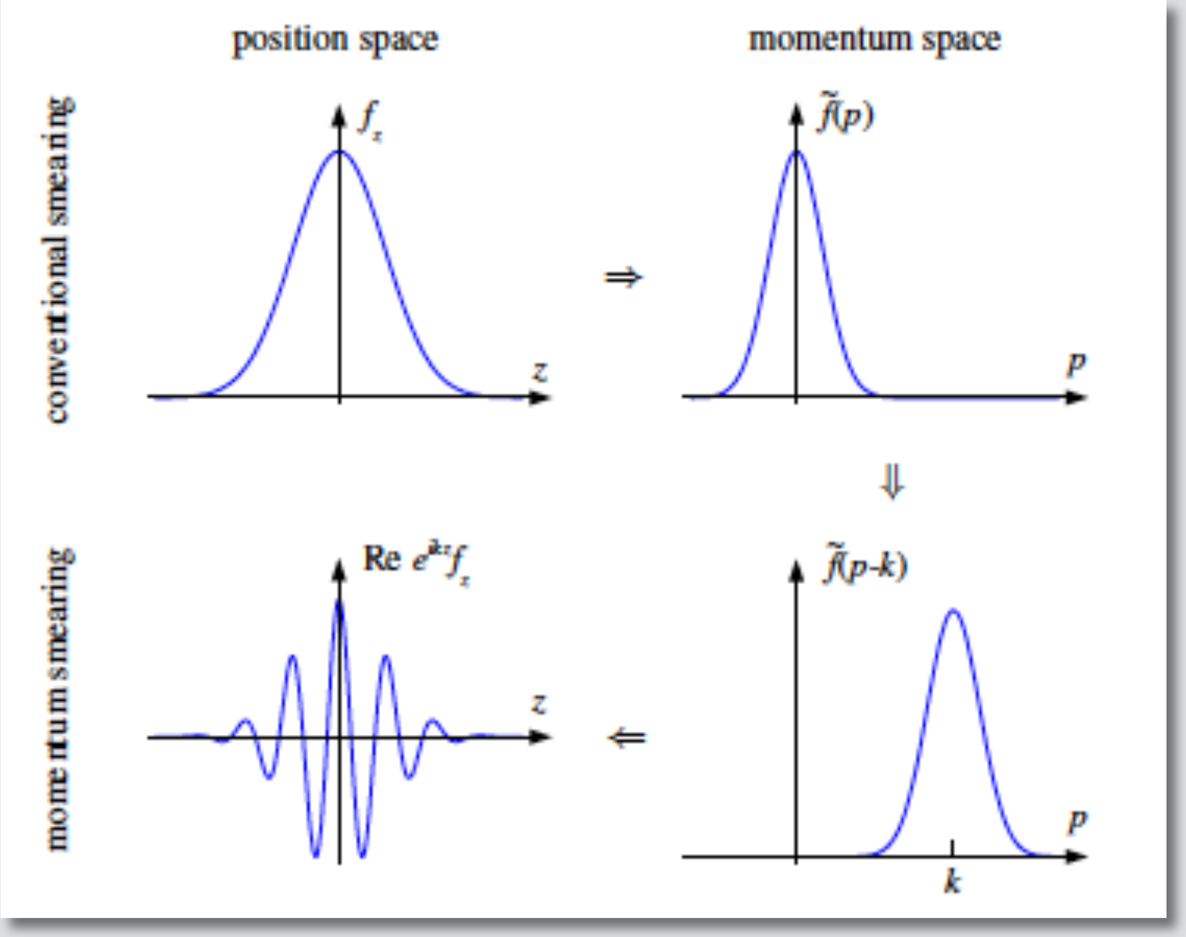
$$
\begin{aligned}
&\langle \Pi(-p')|\mathcal{O}_{J_1}(x_0)\mathcal{O}_{J_2}(\xi)|\Pi(-p')\rangle = \\
&\quad = \sum_{y,z} e^{i(p'.z-p.y)}\langle \bar{q}\,\Gamma_\Pi\, q(z,T)\,\bar{q}\,J_2\,q(x_0+\xi,t)\,\bar{q}\,J_1\,q(x_0,t)\,\bar{q}\,\Gamma_\Pi\,q(y,0)\rangle \\
&\quad = \sum_{y,z} e^{i(p'.z-p.y)}\mathrm{tr}[J_2\,D^{-1}(x_0+\xi,t;x_0,t)\,J_1\,D^{-1}(x_0,t;y,0)\,\Gamma_\Pi \\
&\qquad \times D^{-1}(y,0;z,T)\Gamma_\Pi\,D^{-1}(z,T;x_0+\xi,t)],
\end{aligned}
$$

Numerical Challenge

- Pion computationally less demanding than nucleon
- But signal-to-noise ratio is a problem

$$
C_{\sqrt{\sigma^2}}(t,\vec{p}) \rightarrow \begin{cases} e^{-m_\pi t} & \text{Mesons} \\ e^{-(3m_\pi/2)t} & \text{Baryons} \end{cases}
$$

High spatial momentum and lattice systematics



Boosted interpolating operators

Bali *et al.*, Phys. Rev. D 93, 094515 (2016)

Inverse Problem - *common to all LQCD approaches*

$$\sigma_n(\omega, \xi^2, P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x, \mu^2) K_n^a(x\omega, \xi^2, x^2 P^2, \mu^2) + \mathcal{O}(\xi^2 \Lambda_{\mathrm{QCD}}^2)$$

Calculate on Lattice

Extract PDF

Calculate in PQCD

★ Momentum space matrix elements

$$\tilde{\sigma}_n(\tilde{\omega}, q^2, P^2) \equiv \int \frac{d\xi^4}{\xi^4} e^{iq.\xi} \sigma_n(\omega, \xi^2, P^2)$$

$$\tilde{\sigma}_n = \sum_a f_a \otimes \tilde{K}_n^a + \mathcal{O}(\Lambda_{QCD}^2/q^2)$$

★ Requires many different LSCs with different currents

★ x-dependence of pion valence distribution can be obtained from $\tilde{\omega} = 1/x$

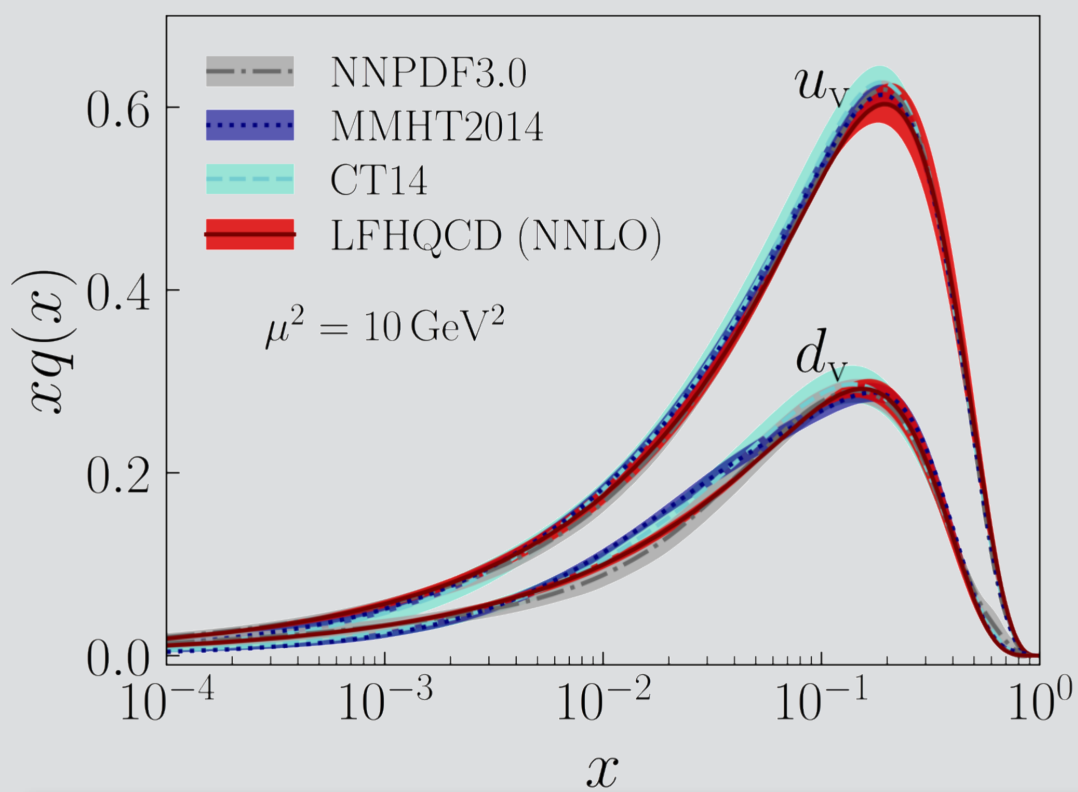
★ Low-x is not accessible unless the hadron is moving very fast (common problem to all LQCD approach)

Questions: Pion Valence Distribution

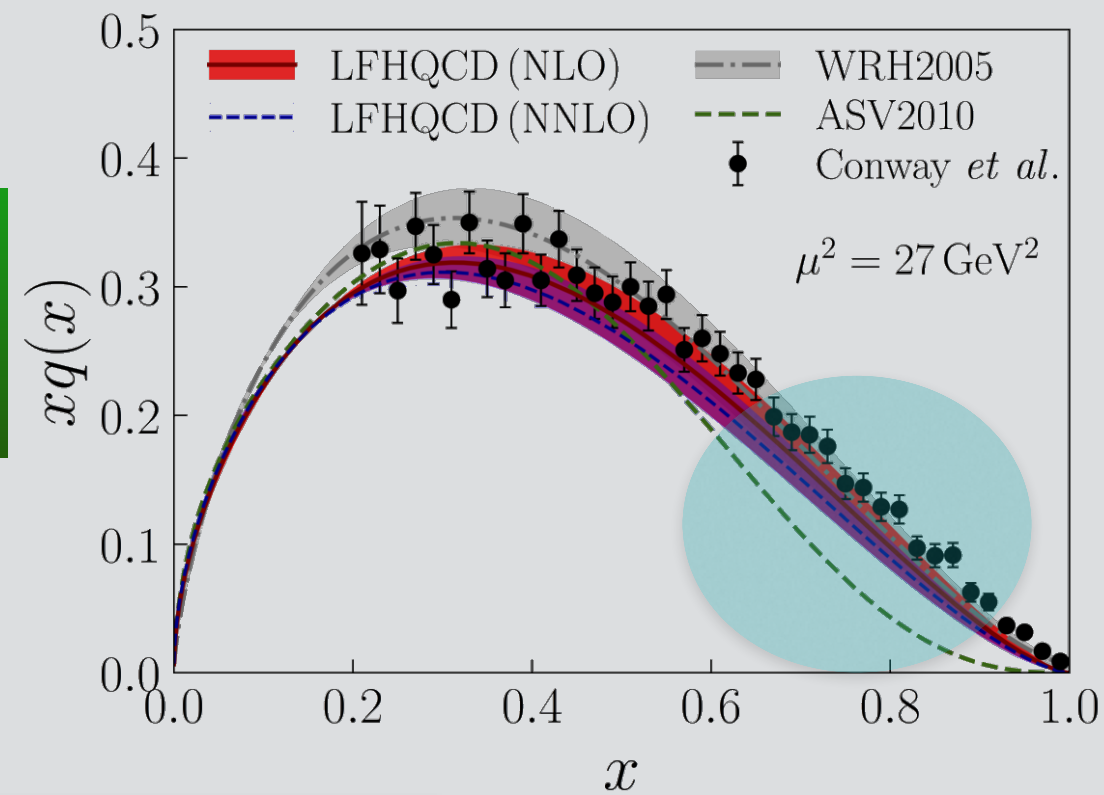
Large-x behavior of pion valence distribution an unresolved problem

★ Perturbative QCD, Dyson-Schwinger model $(1-x)^2$ fall-off

★ Nambu-Jona-Lasino (NJL) model, Duality arguments $(1-x)^1$ fall-off



Light-Front Holographic QCD



de Téramond, Liu, ***RSS***, Dosch, Brodsky, Deur
PRL (2018)

★ Lattice QCD can play vital role in understanding large x-behavior

Quasi Parton Distributions on the Lattice

- Quasi PDFs (X. Ji, PRL (2013))

$$\widetilde{q}(x,\Lambda,p_z) = \int \frac{dz}{2\pi} e^{-ixzp_z} p_z h(z,p_z),$$

Lorentz invariance broken

$$h(z,p_z) = \frac{1}{4p_\alpha} \sum_{s=1}^{2} \langle p,s|\, \bar{\psi}(z)\gamma_\alpha e^{ig\int_0^z A_z(z')dz'} \psi(0)\, |p,s\rangle$$

- Λ is an UV cut-off scale, such as the inverse lattice spacing $1/a$
- Because p is finite, x can be larger than unity.

(Convince yourself from the above expression)

- Quasi-PDF calculated at finite momentum on the lattice has proposed matching

$$\widetilde{q}(x,\Lambda,p_z) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}, \frac{\mu}{p_z}, \frac{\Lambda}{p_z}\right)_{\mu^2=Q^2} q(y,Q^2) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{p_z^2}, \frac{M^2}{p_z^2}\right)$$

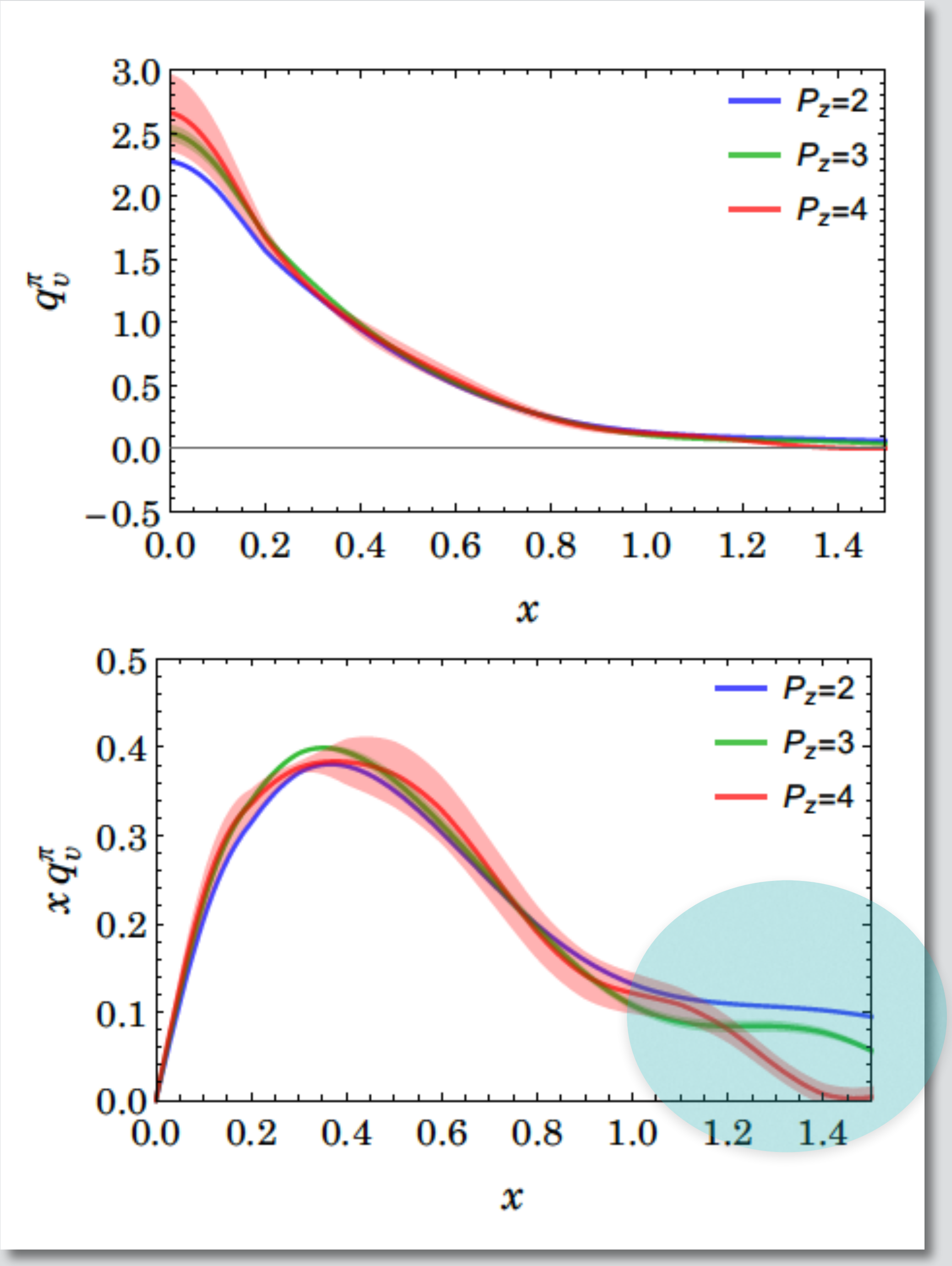
Z is a matching kernel

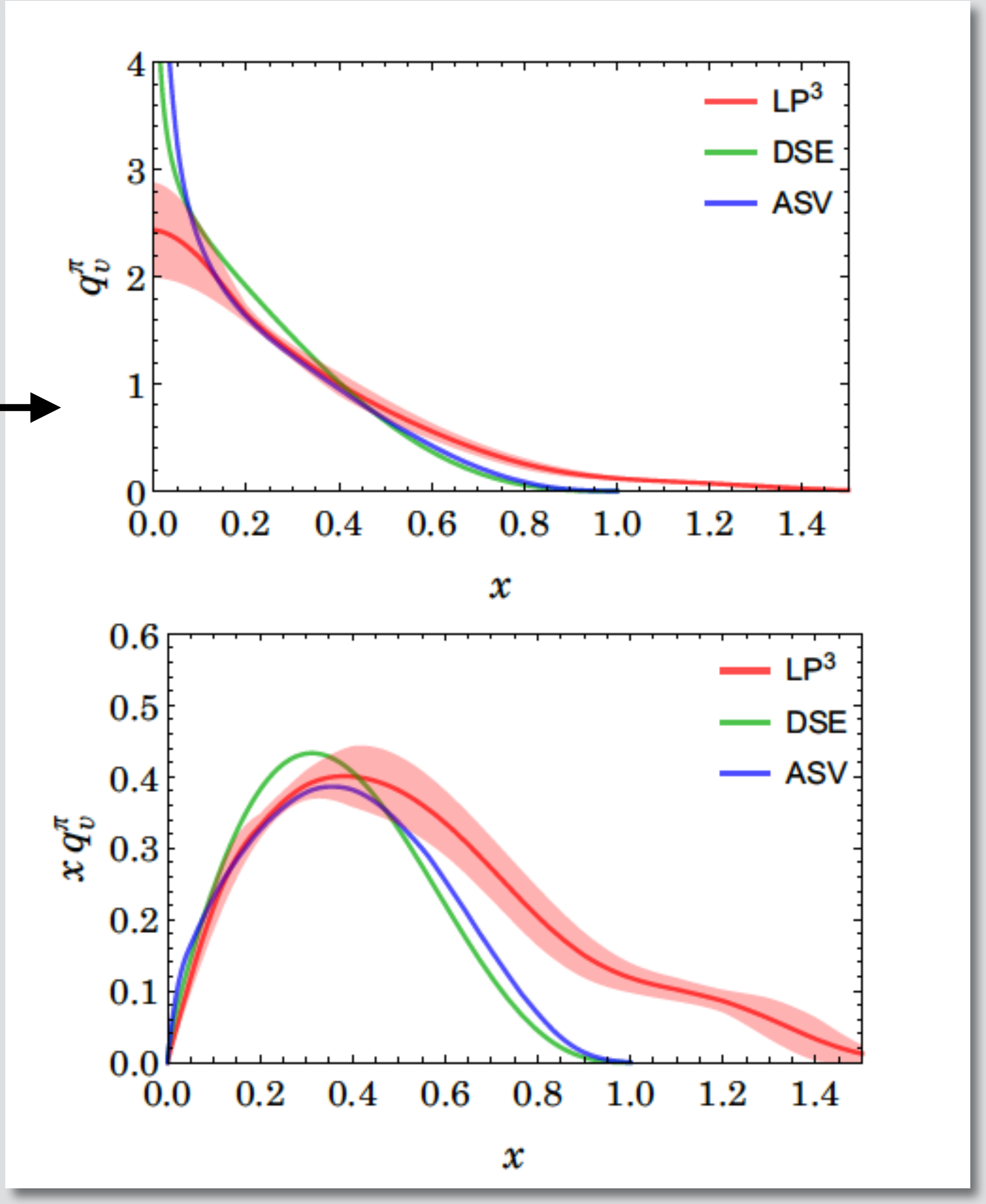
Power-law UV divergence from Wilson line in the non-local operator grows as **z/a**

Quasi-Distribution of Pion

$m_\pi \simeq 300\,\mathrm{MeV}$

LP3, arXiv:1804.01483





Quasi-Distribution of Nucleon (physical pion mass)

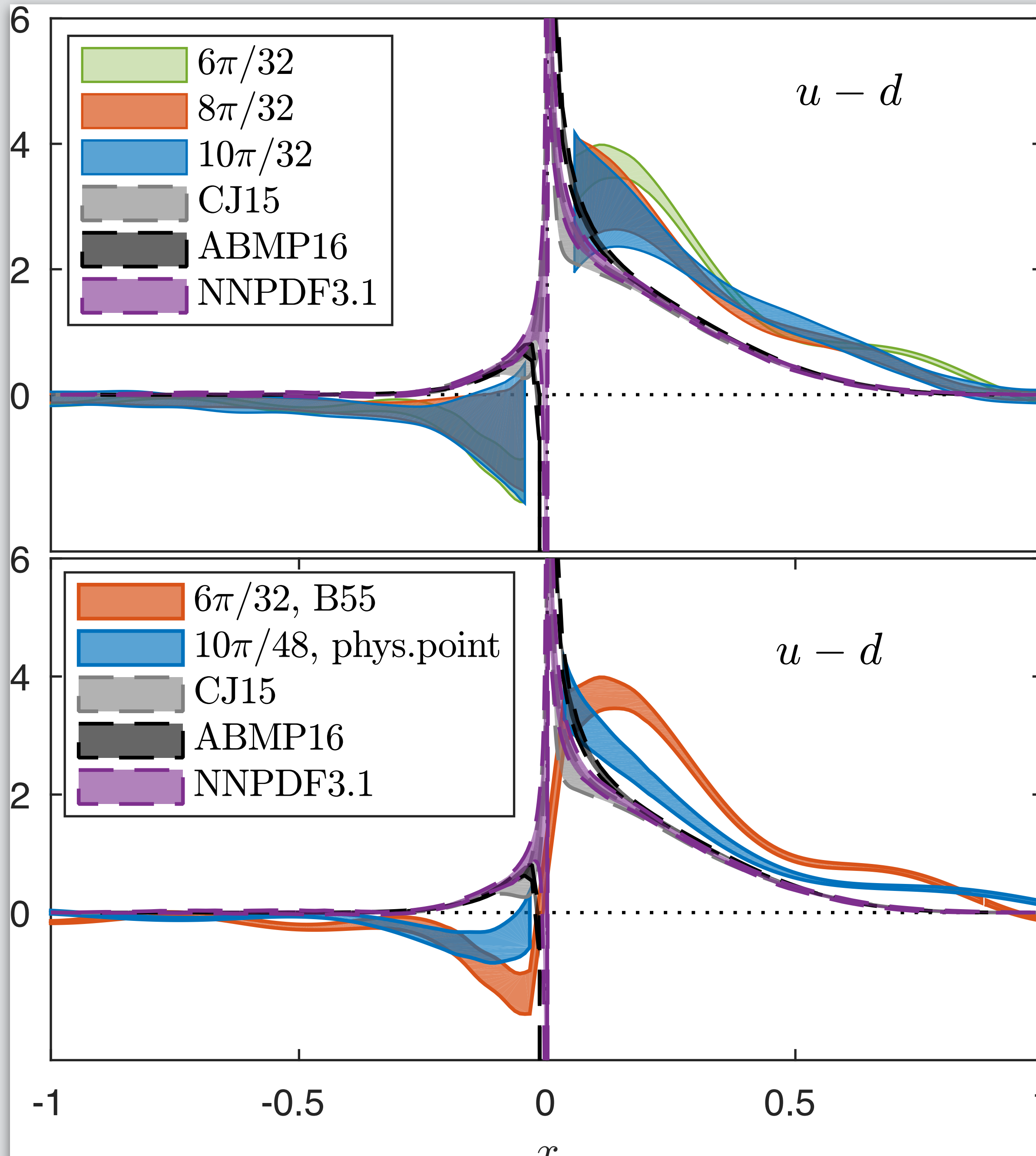


FIG. 6: Top: Comparison of unpolarized PDF from the B55 ensemble against phenomenological estimates. Notation as in Fig. 4. Bottom: Comparison of unpolarized PDF between results of this work (blue band) and of the B55 ensemble (orange band) at nucleon momentum ∼1.4 GeV.

ETM Collaboration

arXiv:1803.02685

region outside x>1 not shown

★ Pseudo-PDFs (A. Radyushkin, PLB (2017))

$$\overline{h}(\nu, z^2) \equiv h(z, p_z)$$

Lorentz invariant Ioffe time $\nu = z \cdot p$

★ The pseudo-PDF is then defined by the Fourier transform

$$\mathcal{P}(x, z^2) = \int \frac{d\nu}{2\pi} e^{-ix\nu} \overline{h}(\nu, z^2)$$

★ Featureof canceling UV divergence as

$$\mathcal{M}(\nu, z^2) = \frac{\overline{h}(\nu, z^2)}{\overline{h}(0, z^2)}$$

★ Ioffe time PDF $\mathcal{M}(\nu, z^2) = Q(\nu, \mu^2) + \mathcal{O}(z^2)$

$$q(x, \mu^2) = \int \frac{d\nu}{2\pi} e^{-ix\nu} Q(\nu, \mu^2)$$

Inverse problem again

Hadronic Tensor Method (K. F. Liu, PRL 1994, PRD 200))

The definition of hadronic tensor in the Minkowski space is

Structure function calculation

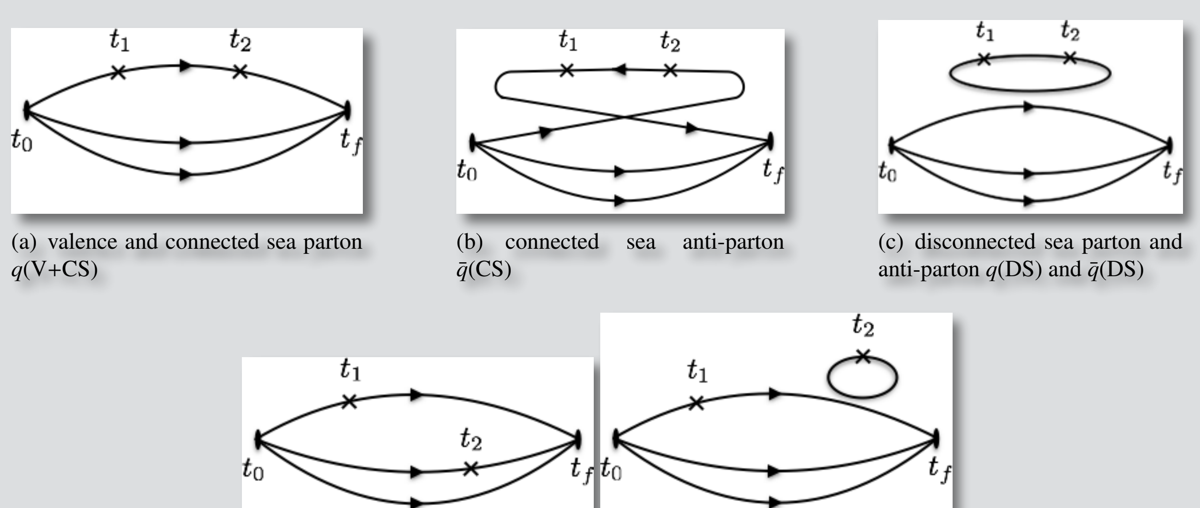
$$
\begin{aligned}
W_{\mu\nu}(q^2,\nu) &= \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\langle p|J_\mu^\dagger(z)J_\nu(0)|p\rangle_{\text{spin ave.}} \\
&= \frac{1}{2}\sum_n \int \prod_{i=1}^{n}\left[\frac{d^3p_i}{(2\pi)^3 2E_{pi}}\right]\langle N|J_\mu(0)|n\rangle\langle n|J_\nu(0)|N\rangle_{\text{spin ave.}}(2\pi)^3\delta^4(p_n-p-q).
\end{aligned}
$$

$$
C_4(\vec{p},\vec{q},\tau) = \sum_{\vec{x}_f} e^{-i\vec{p}\cdot\vec{x}_f}\sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)}\langle\chi_N(\vec{x}_f,t_f)J_\mu(\vec{x}_2,t_2)J_\nu(\vec{x}_1,t_1)\bar{\chi}_N(\vec{0},t_0)\rangle,
$$

Euclidean

$$
C_2(\vec{p},\tau) = \sum_{\vec{x}_f} e^{-i\vec{p}\cdot\vec{x}_f}\langle\chi_N(\vec{x}_f,t_f)\bar{\chi}_N(\vec{0},t_0)\rangle,
$$

$$
\tilde{W}(\vec{p},\vec{q},\tau) \overset{t_f\gg t_2,t_1\gg t_0}{=} \frac{E_N \mathrm{Tr}[\Gamma_e C_4(\vec{p},\vec{q},\tau)]}{m_N \mathrm{Tr}[\Gamma_e C_2(\vec{p},\tau)]}
$$



(a) valence and connected sea parton q(V+CS)

(b) connected sea anti-parton $\bar{q}$(CS)

(c) disconnected sea parton and anti-parton q(DS) and $\bar{q}$(DS)

(d) suppressed by $O(1/Q^2)$

(e) suppressed by $O(1/Q^2)$

Figure 1. Topologically distinct diagrams in the Euclidean-path integral formulation.

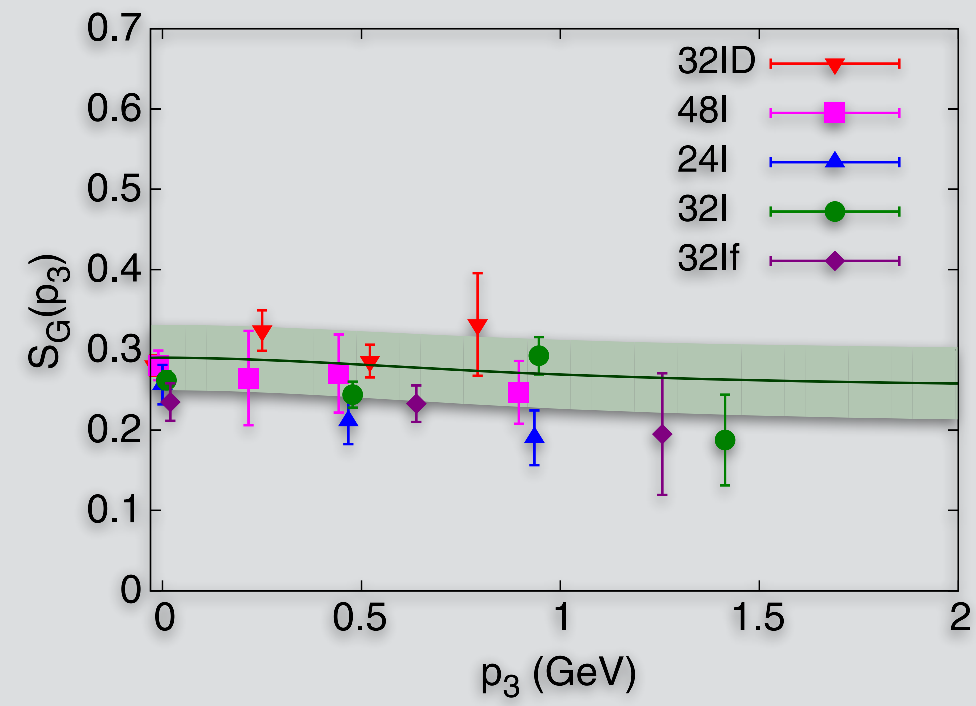
Other methods………

- ★ Inversion Method Through Compton Amplitude (A. Chambers, et al PRL (2017))
- ★ Position-space correlators V. M. Braun & D. Müller (2008))

★ One LQCD (+ LaMET) example:

First moment of gluon helicity distribution

$$\Delta G = \int dx \frac{i}{2xP^+} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \times \langle PS | F_a^{+\alpha}(\xi^-) \mathcal{L}^{ab}(\xi^-, 0) \tilde{F}^+_{\alpha,b}(0) | PS \rangle$$



Yang, RSS, et .al
PRL 2017

Glue Spin and Helicity in the Proton from Lattice QCD

★ On the lattice

$$\vec{S}_g = 2 \int d^3x \mathrm{Tr}(\vec{E}_c \times \vec{A}_c)$$

★ Matching to LaMET

$$S_G(|\vec{p}|, \mu) = \left[1 + \frac{g^2 C_A}{16\pi^2}\left(\frac{7}{3}\log\frac{(\vec{p})^2}{\mu^2} - 10.2098\right)\right]\Delta G(\mu) + \frac{g^2 C_F}{16\pi^2}\left(\frac{4}{3}\log\frac{(\vec{p})^2}{\mu^2} - 5.2627\right)\Delta\Sigma(\mu) + O(g^4) + O\left(\frac{1}{(\vec{p})^2}\right). \tag{10}$$

ΔG is 0.22, which is much smaller than unity

Convergence problem !!

But promising result as the first Calculation

PDF calculation on lattice is very challenging and that is why so interesting !

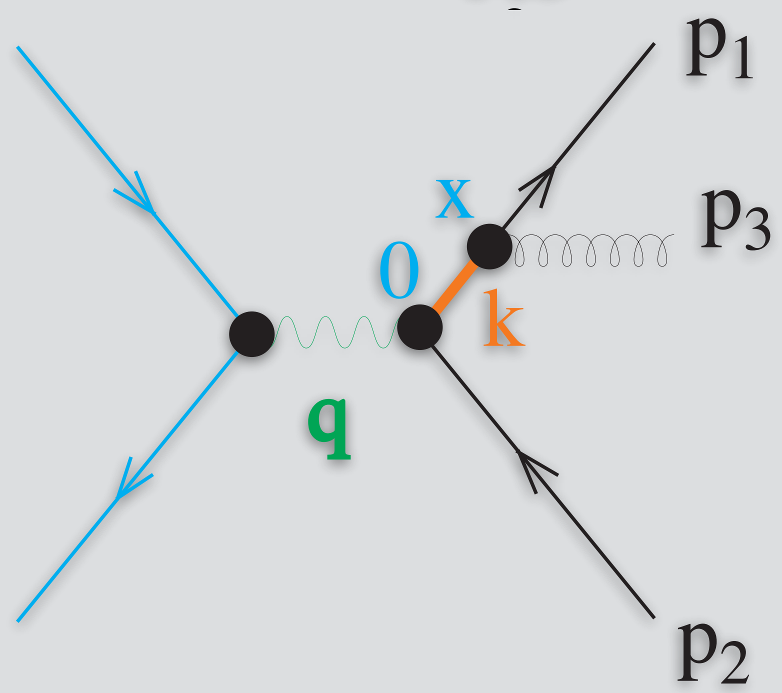
Email: sufian@jlab.org

Thank You !!

Singularities and Wilson Lines…

Example

Define $p_1^\mu + p_3^\mu = k^\mu$



Simple choice k^+ large and $\mathbf{k}_T = \mathbf{0}$.

Algebra $k^- = \dfrac{\mathbf{p}_{3,T}^2}{2p_1^+} + \dfrac{\mathbf{p}_{3,T}^2}{2p_3^+}$

Two cases

★ $\mathbf{p}_{3,T}$ becomes small with fixed p_1^+ and p_3^+

Gluon momentum is nearly collinear with the quark momentum.

★ $\mathbf{p}_{3,T}$ and p_3^+ both become small with $p_3^+ \propto |\mathbf{p}_{3,T}|$

So that the gluon momentum is soft.

current conservation at the nucleon vertex requires that $q_\mu J^\mu = 0$,

$$\begin{aligned} q_\mu J^\mu &= \bar{u}(p') \left[F_1 \not{q} + i \frac{1}{2M} F_2 q_\mu \sigma^{\mu\nu} q_\nu + q^2 F_3 \right] u(p) = 0 \\ &\Rightarrow F_3 = 0 \end{aligned}$$

The first term can be shown to vanish by applying the Dirac equation to the spinors

The second term is zero because $\sigma^{\mu\nu}$ is totally antisymmetric, while

leaving $q^2 F_3 = 0$.

- **General definition of the nucleon form factor**

$$\langle N(P')|J^{\mu}_{\mathrm{EM}}(0)|N(P)\rangle = \bar{u}(P')\left[\gamma^{\mu}F_1^N(Q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2M}F_2^N(Q^2)\right]u(P)$$

$$\begin{aligned}\Gamma^{\nu} &= K_1\gamma^{\nu} + iK_2\sigma^{\nu\alpha}(p'-p)_\alpha + iK_3\sigma^{\nu\alpha}(p'+p)_\alpha + \\ &\quad K_4(p'-p)^{\nu} + K_5(p'+p)^{\nu}\end{aligned}$$

Term proportional to $p'^{\mu} + p^{\mu}$

absorbed into a combination of the terms proportional to γ^{μ} and $\sigma^{\mu\nu}(p'_\nu - p_\nu)$.

term proportional to $\sigma^{\mu\nu}(p'_\nu + p_\nu)$ can be absorbed into the $p'^{\mu} - p^{\mu}$ term

leaving three independent form factors and the following expression for the vertex factor

$$\Gamma^{\nu} = F_1\gamma^{\nu} + i\frac{F_2}{2M}\sigma^{\nu\alpha}q_\alpha + F_3\frac{p'^{\nu} - p^{\nu}}{M}$$

QCD Beta Function

$$\mu_R^2 \frac{d\alpha_s}{d\mu_R^2} = \beta(\alpha_s) = -(b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \cdots)$$

$$b_0 = (11C_A - 4n_f T_R)/(12\pi) = (33 - 2n_f)/(12\pi)$$

$$b_1 = (17C_A^2 - n_f T_R(10C_A + 6C_F))/(24\pi^2)$$

$$\alpha_s(\mu_R^2) = (b_0 \ln(\mu_R^2/\Lambda^2))^{-1}$$

perturbatively defined coping diverges at this scale

1 Light Cone Coordinates: Definitions, Identities

A four-vector is not bold-faced (e.g. p, k), a three-vector is bold-faced with a vector symbol (e.g. $\vec{\mathbf{p}}$, $\vec{\mathbf{k}}$), and a transverse two-vector is bold-faced without a vector symbol (e.g. $\mathbf{p}$, $\mathbf{k}$). Minkowski four-vectors are written with parentheses, (); light-cone four-vectors with brackets, [].

$$p = (p^0, p^z, \mathbf{p}) = [p^+, p^-, \mathbf{p}]. \tag{1}$$

We will use non-symmetrized lightcone coordinates:

$$p^+ = p^0 + p^z \tag{2}$$

$$p^- = p^0 - p^z \tag{3}$$

$$\mathbf{p} = \mathbf{p}. \tag{4}$$

The inverse transformation is then

$$p^0 = \frac{1}{2}(p^+ + p^-) \tag{5}$$

$$p^z = \frac{1}{2}(p^+ - p^-) \tag{6}$$

$$\mathbf{p} = \mathbf{p}. \tag{7}$$

The Minkowski dot product in lightcone coordinates is:

$$p \cdot k = p^0 k^0 - p^z k^z - \mathbf{p} \cdot \mathbf{k} = \frac{1}{2}(p^+ k^- + p^- k^+) - \mathbf{p} \cdot \mathbf{k}. \tag{8}$$

The length of a vector using lightcone coordinates is then:

$$p \cdot p = p^+ p^- - \mathbf{p} \cdot \mathbf{p} \tag{9}$$

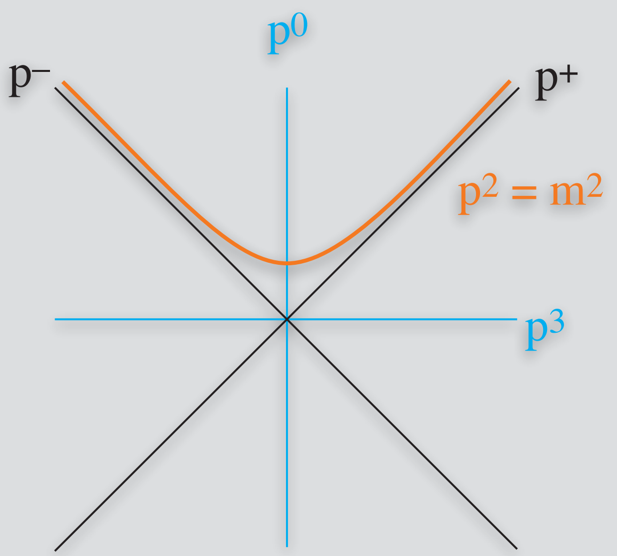
The probability for an initial state splitting is proportional to α_s.

Using null plane components, the covariant square of p^μ is

$$p^2 = 2p^+p^- - \mathbf{p}_T^2.$$

Thus, for a particle on its mass shell, p^- is

$$p^- = \frac{\mathbf{p}_T^2 + m^2}{2p^+}.$$



Note also that, for a particle on its mass shell,

$$p^+ > 0\,, \qquad p^- > 0\,.$$

Integration over the mass shell is

$$(2\pi)^{-3}\int \frac{d^3\vec{p}}{2\sqrt{\vec{p}^2+m^2}}\cdots = (2\pi)^{-3}\int d^2\mathbf{p}_T\int_0^\infty \frac{dp^+}{2p^+}\cdots. \tag{24}$$

We also use the plus/minus components to describe a space-time point x^μ: $x^\pm = (x^0 \pm x^3)/\sqrt{2}$. In describing a system of particles moving with large momentum in the plus direction, we are invited to think of x^+ as "time." Classically, the particles in our system follow paths nearly parallel to the x^+ axis, evolving slowly as it moves from one $x^+ = const.$ plane to another.

We relate momentum space to position space for a quantum system by Fourier transforming. In doing so, we have a factor $\exp(ip\cdot x)$, which has the form

$$p\cdot x = p^+x^- + p^-x^+ - \mathbf{p}_T\cdot\mathbf{x}_T. \tag{25}$$

Thus x^- is conjugate to p^+ and x^+ is conjugate to p^-. That is a little confusing, but it is simple enough.

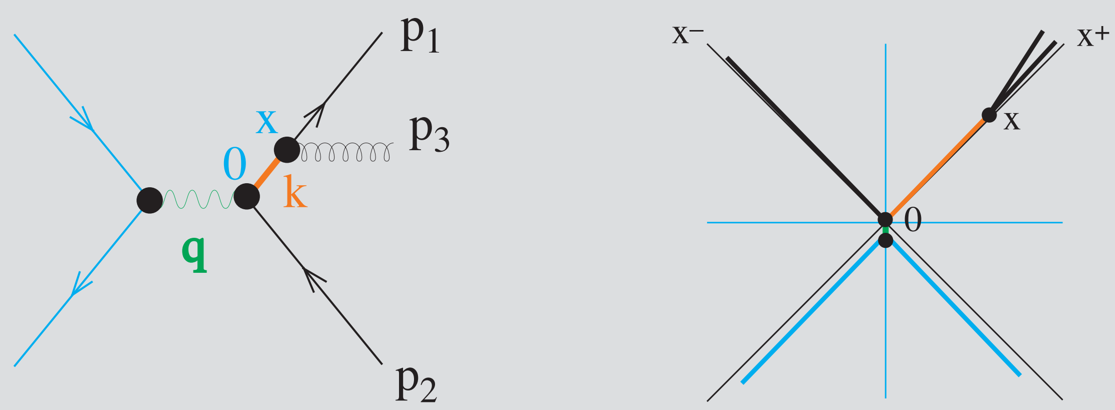


Figure 5: Correspondence between singularities in momentum space and the development of the system in space-time.

2.4 Space-time picture of the singularities

We now return to the singularity structure of $e^+e^- \to q\bar{q}g$. Define $p_1^\mu + p_3^\mu = k^\mu$. Choose null plane coordinates with k^+ large and $\mathbf{k}_T = \mathbf{0}$. Then $k^2 = 2k^+k^-$ becomes small when

$$k^- = \frac{\mathbf{p}_{3,T}^2}{2p_1^+} + \frac{\mathbf{p}_{3,T}^2}{2p_3^+} \tag{26}$$

becomes small. This happens when $\mathbf{p}_{3,T}$ becomes small with fixed p_1^+ and p_3^+, so that the gluon momentum is nearly collinear with the quark momentum. It also happens when $\mathbf{p}_{3,T}$ and p_3^+ both become small with $p_3^+ \propto |\mathbf{p}_{3,T}|$, so that the gluon momentum is soft. (It also happens when the quark becomes soft, but there is a numerator factor that cancels the soft quark singularity.) Thus the singularities for a soft or collinear gluon correspond to small k^-.

Now consider the Fourier transform to coordinate space. The quark propagator in Fig. 5 is

$$S_F(k) = \int dx^+ dx^- d\mathbf{x}\ \exp(i[k^+x^- + k^-x^+ - \mathbf{k}\cdot\mathbf{x}])\ S_F(x). \tag{27}$$

When k^+ is large and k^- is small, the contributing values of x have small x^- and large x^+. Thus the propagation of the virtual quark can be pictured in space-time as in Fig. 5. The quark propagates a long distance in the x^+ direction before decaying into a quark-gluon pair. That is, the singularities that can lead to divergent perturbative cross sections arise from interactions that happen a long time after the creation of the initial quark-antiquark pair.

There are three ways to view this result. First, we have the formal argument given above. Second, we have the intuitive understanding that after the initial quarks and gluons are created in a time Δt of order $1/\sqrt{s}$, *something* will happen with probability 1. Exactly what happens is long-time physics, but we don't care about it since we sum over all the possibilities $|N\rangle$. Third, we can calculate at some finite order of perturbation theory. Then we see infrared infinities at various stages of the calculations, but we find that the infinities cancel between real gluon emission graphs and virtual gluon graphs. An example is shown in Fig. 7.

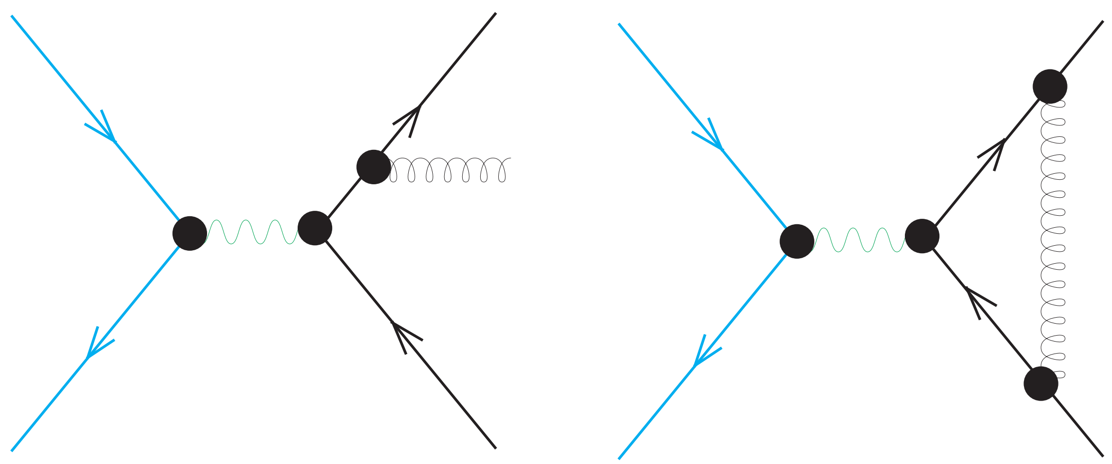


Figure 7: Cancellation between real and virtual gluon graphs. If we integrate the real gluon graph on the left times the complex conjugate of the similar graph with the gluon attached to the antiquark, we will get an infrared infinity. However the virtual gluon graph on the right times the complex conjugate of the Born graph is also divergent, as is the Born graph times the complex conjugate of the virtual gluon graph. Adding everything together, the infrared infinities cancel.

We see that the total cross section is free of sensitivity to long-time physics. If the total cross section were all you could look at, QCD physics would be a little boring. Fortunately, there are other quantities that are not sensitive to infrared effects. They are called infrared safe quantities.

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To formulate the concept of infrared safety, consider a measured quantity that is constructed from the cross sections,

$$\frac{d\sigma[n]}{d\Omega_2 dE_3 d\Omega_3 \cdots dE_n d\Omega_n}, \tag{29}$$

to make n hadrons in e^+e^- annihilation. Here E_j is the energy of the jth hadron and $\Omega_j = (\theta_j, \phi_j)$ describes its direction. We treat the hadrons as effectively massless and do not

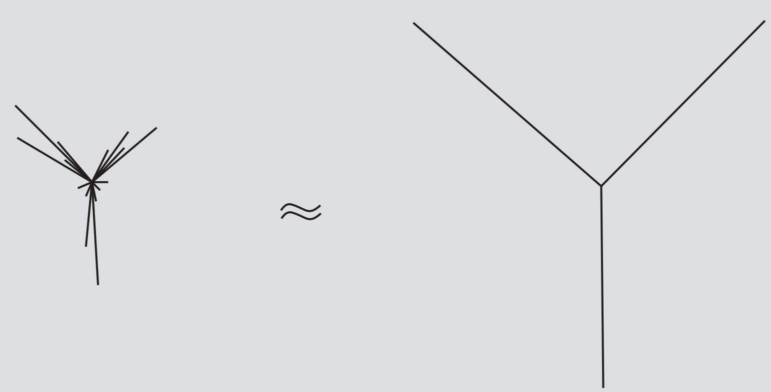

Figure 8: Infrared safety. In an infrared safe measurement, the three jet event shown on the left should be (approximately) equivalent to an ideal three jet event shown on the right.