A novel method of analysis of Experimental Data

CASE STUDY:
Multipole Extraction from Nucleon Resonances

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Jefferson Lab, May 4, 2007

I. Introduction

Typical problem: To determine experimentally certain parameters of a theory or model

- The experimental results which are characterized by a statistical and systematic uncertainty are employed and the parameters are determined by minimization of the χ^2.

- For complex physical problems the χ^2 is a hypersurface with complex topology. The experimentally surveyed surface is known with limited precision. Identifying the minima is a difficult task. Associating uncertainties to the extracted parameters is even more difficult.

Case Study: Extraction of multipole amplitudes from the electroexcitation of the $\Delta^{+}(1232)$

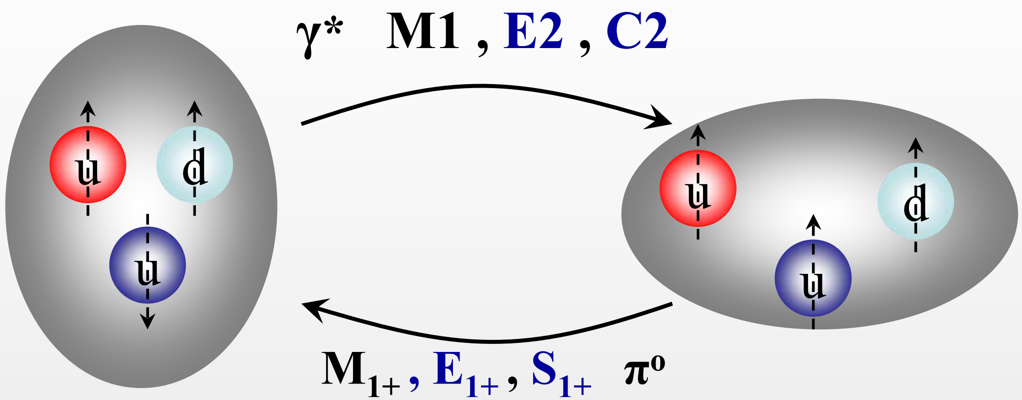
- For the discussion of the method the underlying physics is not really important
- It is a typical difficult problem:
  - We have a good understanding of the problem and a reliable parameterization of the process
  - The connection of the parameters of interest to the observables is indirect and convoluted.
  - Constraints are imposed (unitarity in our case) which interconnect the degrees of freedom
- Issue extensively studied and extensively published.

The signal for deformation in the $N \Rightarrow \Delta$ transition

p(qqq)

$I = \frac{1}{2} \quad J = \frac{1}{2}$

938 MeV



Δ(qqq)

$I = \frac{3}{2} \quad J = \frac{3}{2}$

1232 MeV

Spherical $\Rightarrow$ M1

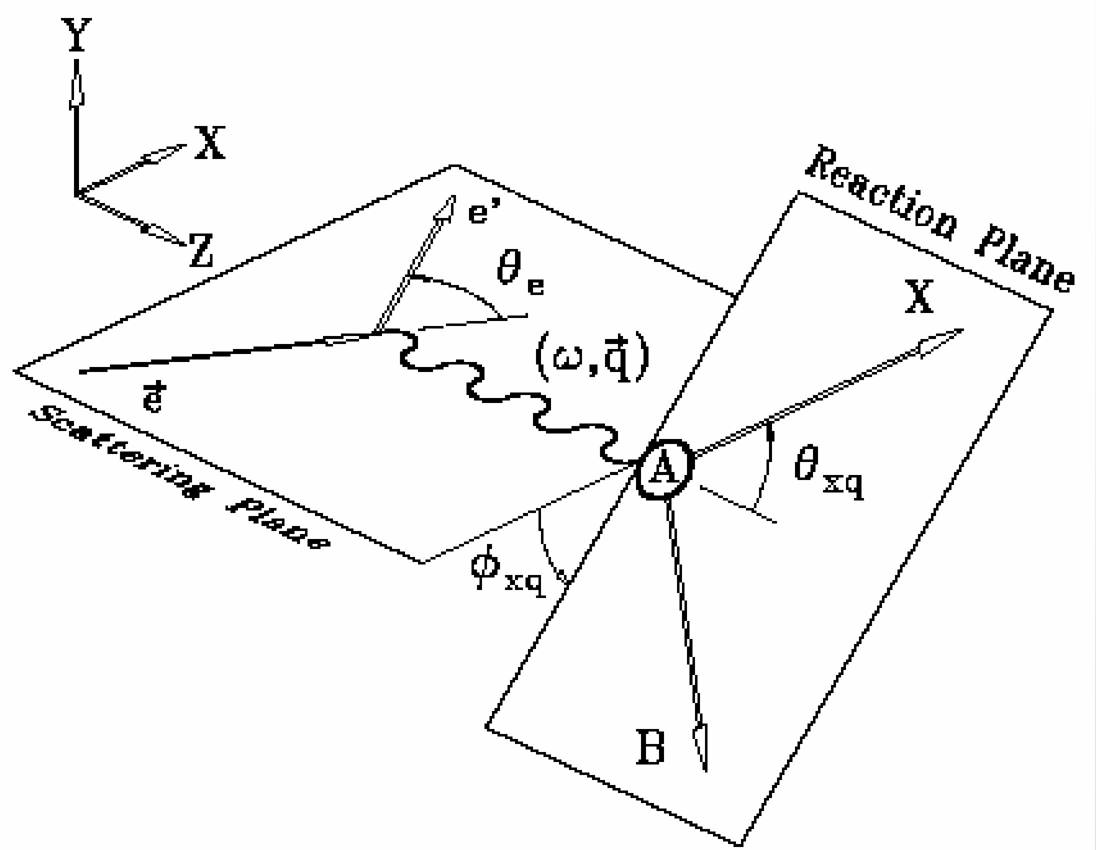
Deformed $\Rightarrow$ M1 , E2 , C2

Deformation signal

$$\mathrm{CMR} = \mathrm{Re}\left(\frac{S_{1+}^{3/2}}{M_{1+}^{3/2}}\right)$$

$$\mathrm{EMR} = \mathrm{Re}\left(\frac{E_{1+}^{3/2}}{M_{1+}^{3/2}}\right)$$

Using the precision of the electromagnetic probe



observable

$$\sigma = J_\Omega \, \Gamma_v \frac{p_{\rm cm}}{k_{\rm cm}} \left(R_T + \epsilon_L R_L + \epsilon \, R_{TT} \cos 2\phi_{X\gamma} - v_{LT} R_{LT} \cos \phi_{X\gamma} - h \, v'_{LT} R'_{LT} \sin \phi_{X\gamma} \right)$$

Extract the relevantInformation from the Interference Responses

$$R_{LT} = -\sin\theta^*_{pq}\mathrm{Re}\left\{S^*_{0+}\left[M_{1-} - M_{1+} + 3E_{1+}\right] - \left[2S^*_{1+} - S^*_{1-}\right]E_{0+} \right.$$
$$\left. -6\cos\theta^*_{pq}\left[S^*_{1+}\left(M_{1-} - M_{1+} + E_{1+}\right) + S^*_{1-}E_{1+}\right]\right\}$$

$$R'_{LT} = -\sin\theta^*_{pq}\mathrm{Im}\left\{S^*_{0+}\left[M_{1-} - M_{1+} + 3E_{1+}\right] - \left[2S^*_{1+} - S^*_{1-}\right]E_{0+} \right.$$
$$\left. -6\cos\theta^*_{pq}\left[S^*_{1+}\left(M_{1-} - M_{1+} + E_{1+}\right) + S^*_{1-}E_{1+}\right]\right\}$$

Primarily: Sensitive to C2

$$R_{TT} = 3\sin^2\theta^*_{pq}\left[\frac{3}{2}\left|E_{1+}\right|^2 - \frac{1}{2}\left|M_{1+}\right|^2 \right.$$
$$\left. -\mathrm{Re}\left\{E^*_{1+}\left[M_{1+} - M_{1-}\right] + M^*_{1+}M_{1-}\right\}\right]$$

Primarily: Sensitive to E2

Caveats and words of caution

- Multipoles are not observables; cross sections are.
- Need to obtain the information needed by fitting the multipoles (while comparing the cross section).
- Multipoles do not provide an orthogonal basis, they are correlated

E_L+ , E_L- , M_L+ , M_L- , L_L+ , L_L- $0 \leq L \leq Lcut$

$\Updownarrow$

F1, F2, F3, F4, F5, F6 (CGLN)

$\Updownarrow$

Response Functions: R_T, R_L, R_{TT}, R_{LT}, ...

$\Updownarrow$

OBSERVABLES:
Cross sections, asymmetries, etc.

Extracting Multipoles

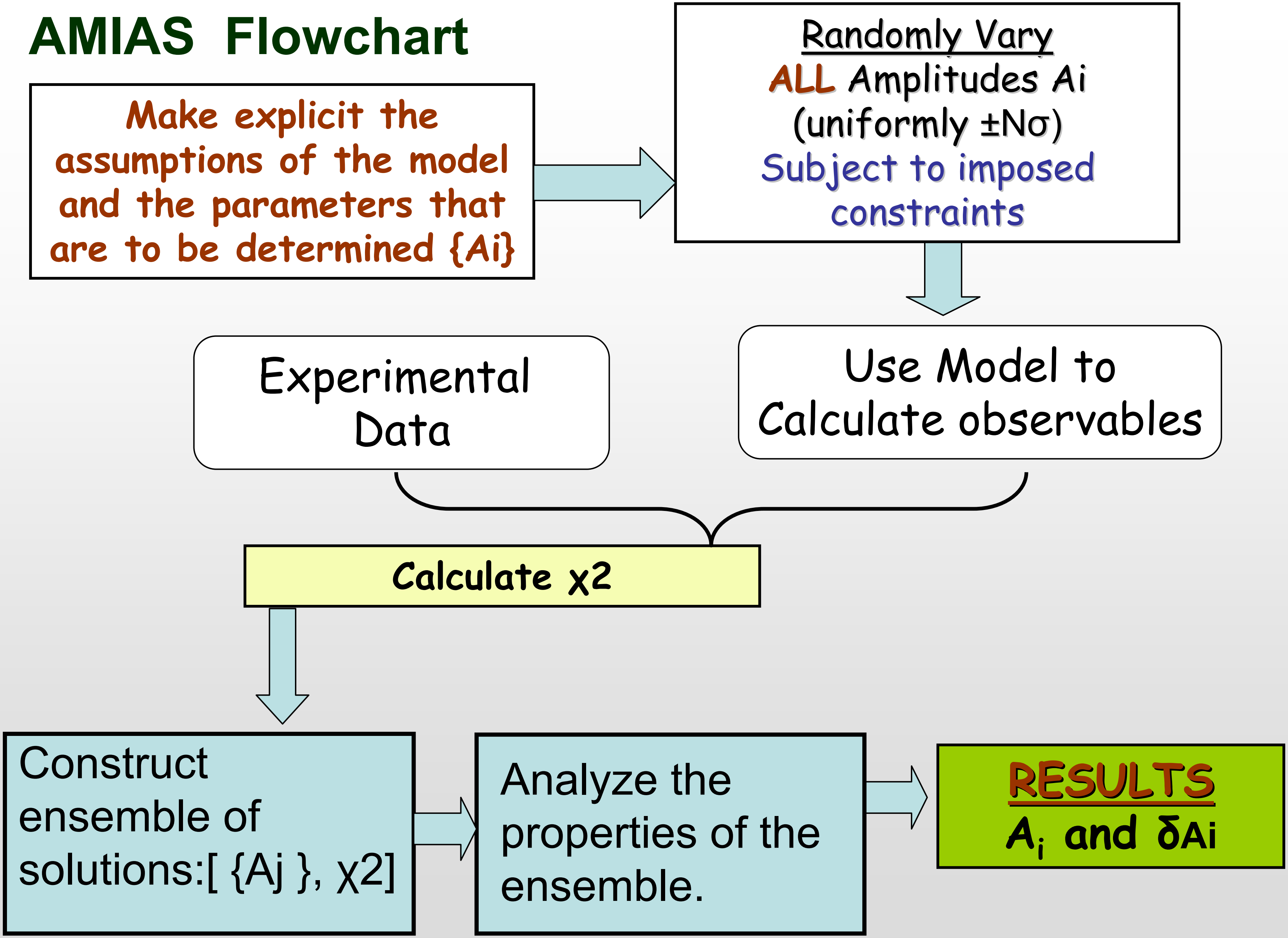
Usual procedure followed:

1. Extract the multipoles of interest (typically only 3 to 4, "dominant") by χ^2 minimization fitting. However the data base is not rich enough to fit all multipoles.
2. Assume an Lmax and keep multipoles below that multipolarity
3. "Not fitted" multipoles are fixed by a model.
4. One could, although not often practiced, estimate the model error of the extracted multipoles by using all available "reasonable" models.

- Extracted amplitudes and their ratios (EMR, CMR) are characterized by statistical, systematic and model error. The separation of the three is not clean
- Model error often dominates.
- Model error is a guestimate, and therefore it does not provide a quantitative criterion for determining the acceptance or rejection of a theory/model

II. The AMIAS Method

AMIAS Flowchart



PaStyl Flowchart

L = 0…5
Total = (36-5) complex Multipoles

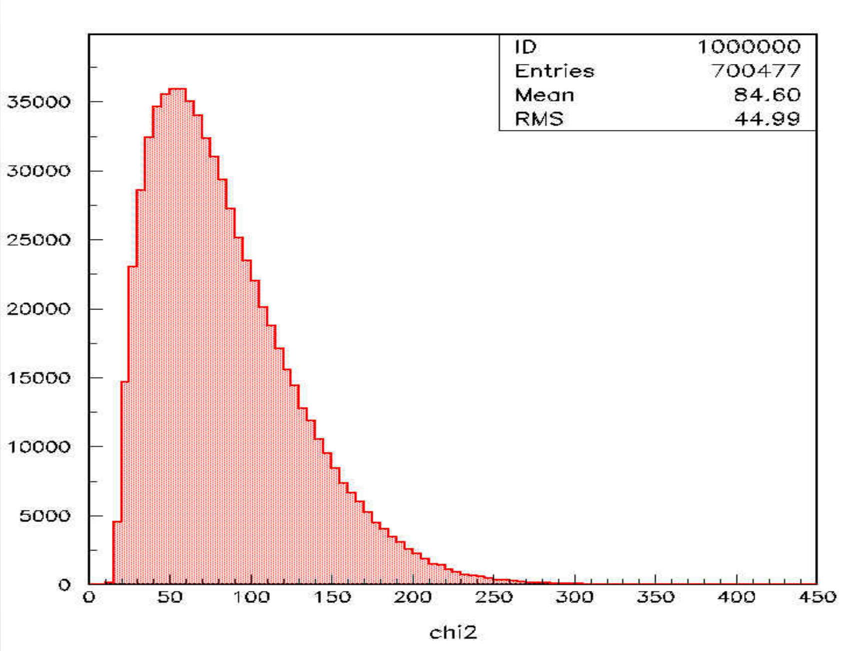
Random Variation
of **ALL** Amplitudes Ai
(uniformly $\pm 1\sigma$, $\pm 2\sigma$, …)
Unitarization

Calculation of Cross Sections

Experimental Data

Calculation of $\chi 2$

Will of course result in solutions with varying $\chi 2$

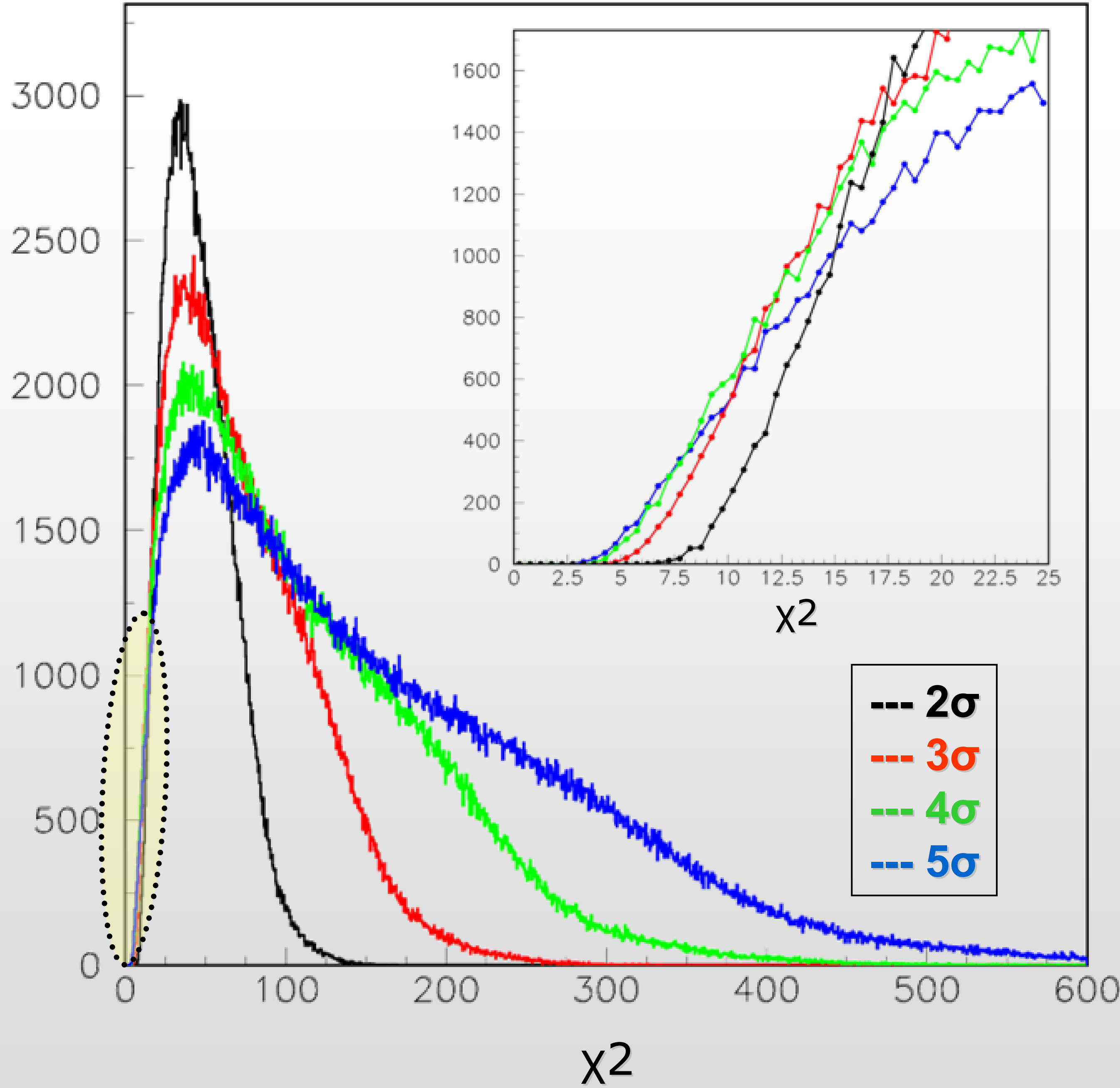


χ^2 -Distribution

Variation
of
ALL Amplitudes

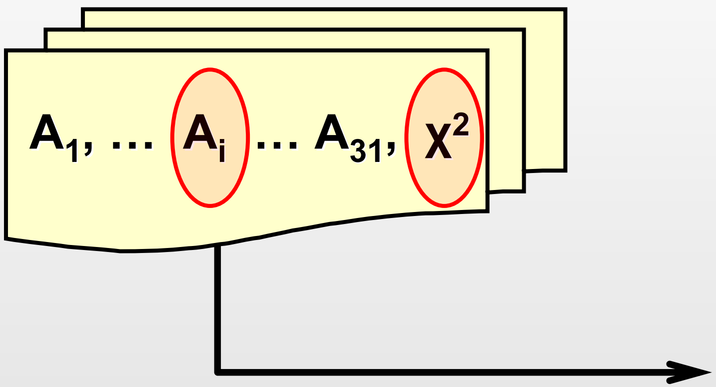
Wider range in the variation yields different distributions

After a sufficiently wide range in the variation a CONVERGENCE in χ^2 is reached.

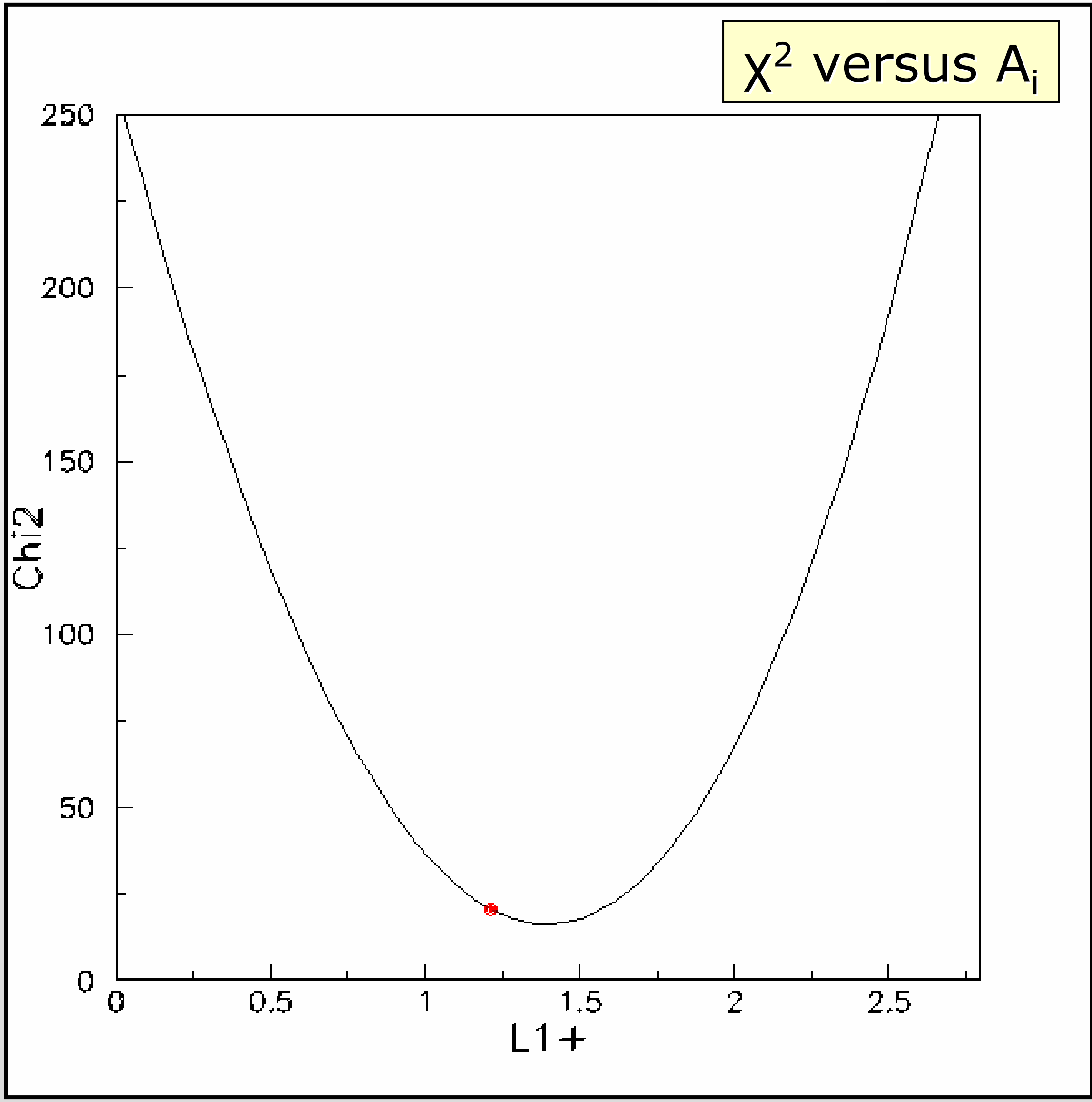


Sensitivity on Multipole A_i

For each solution we can project the dependence of a given amplitude on $\chi 2$

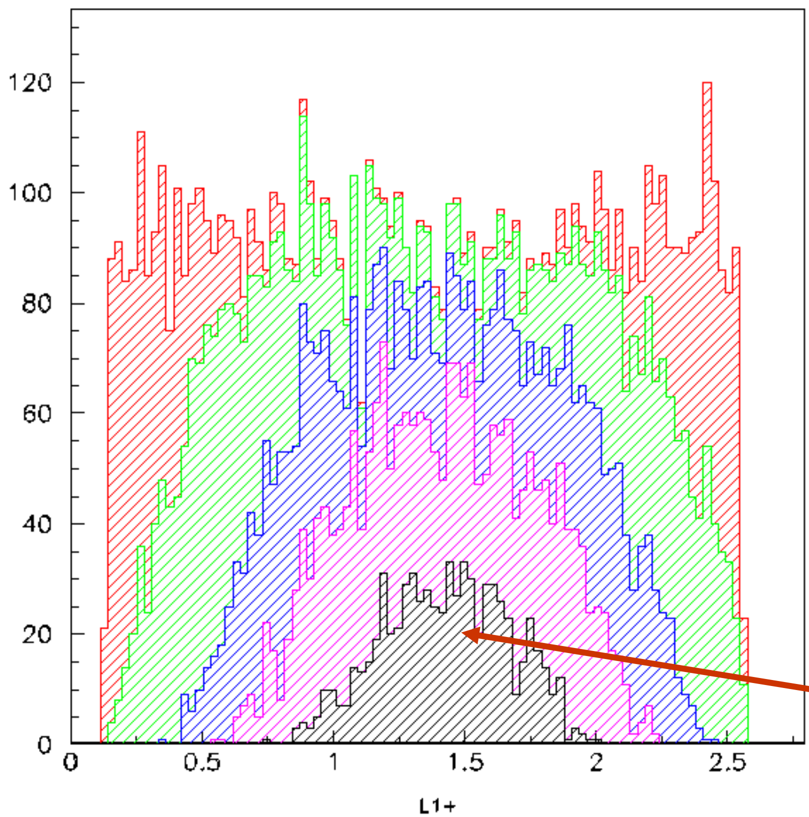


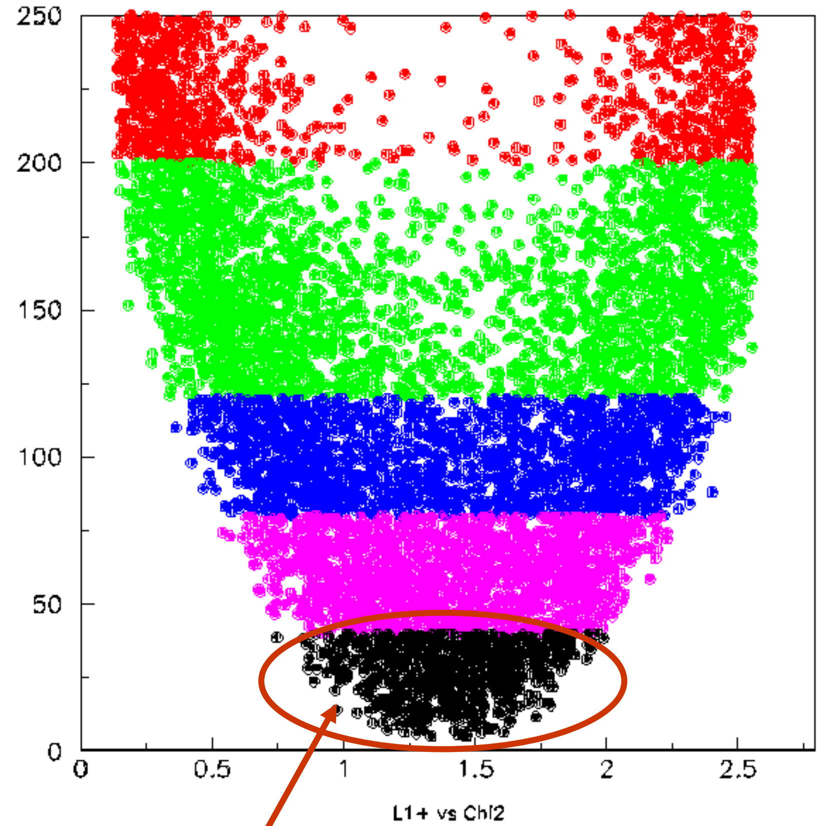
A_i is uniformly distributed (varied)



Applying χ^2 Cut on **SENSITIVE** Amplitude A_i

A_i Distribution





PROJECTION

ALL VALUES

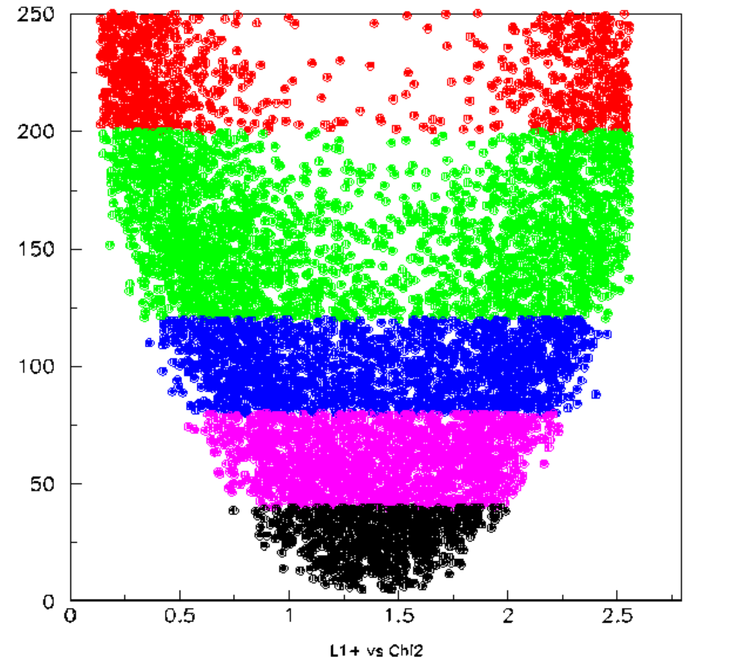
$\chi^2 < 200$

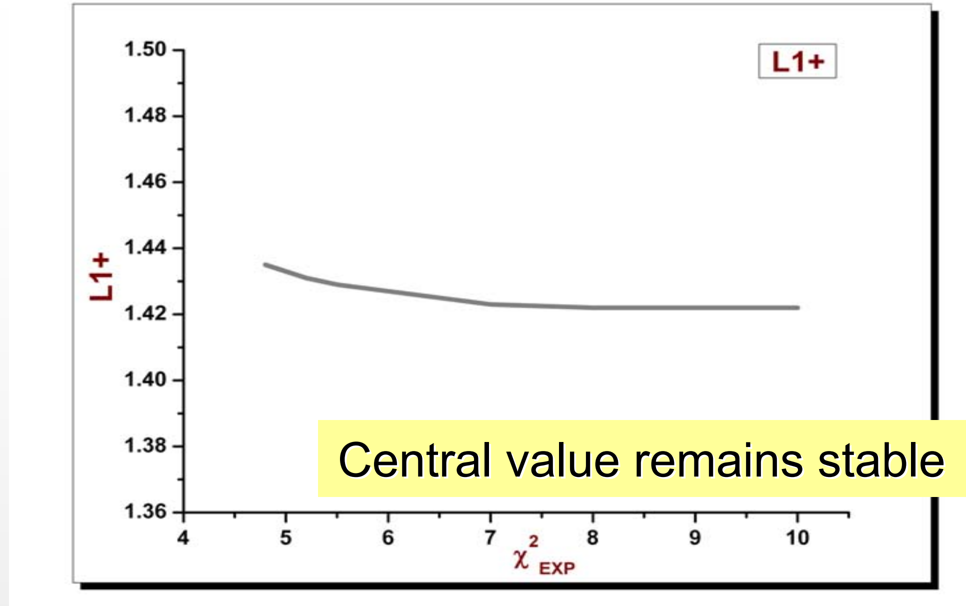
$\chi^2 < 120$

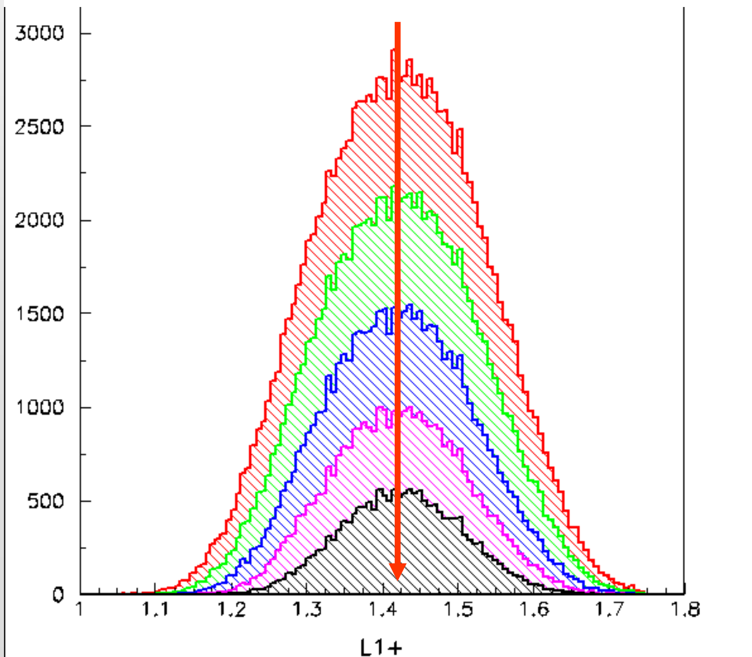
$\chi^2 < 80$

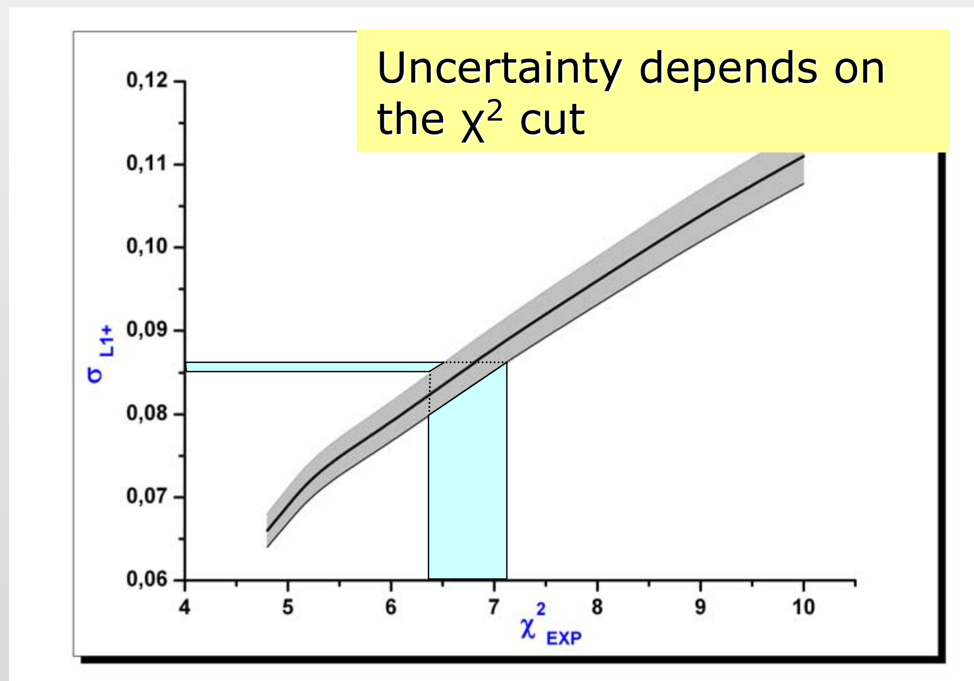
$\chi^2 < 40$

Microcanonical ensemble

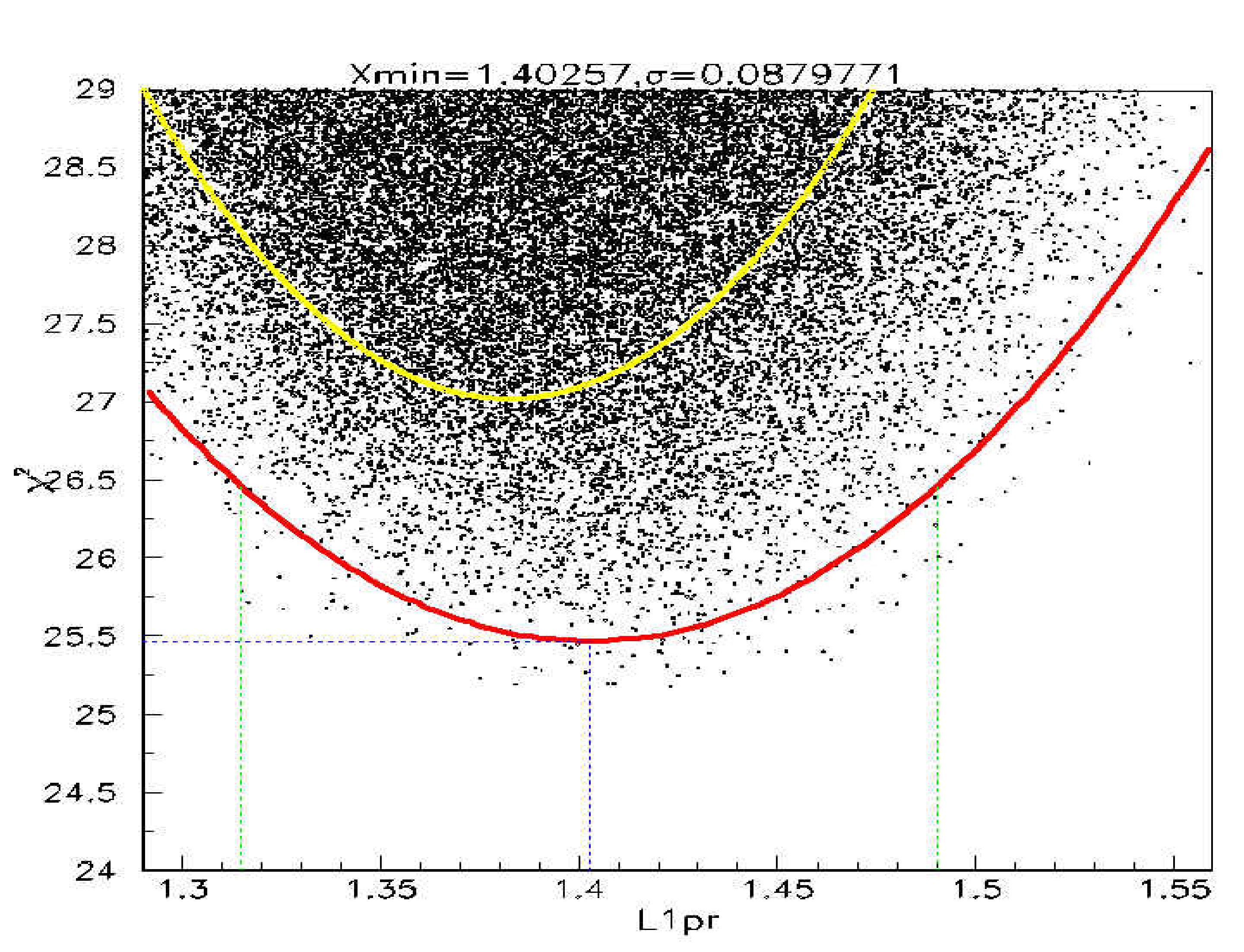

250
200
150
100
50
0
0
0.5
1
1.5
2
2.5
L1+ vs Chi2


L1+
1.50
1.48
1.46
1.44
1.42
1.40
1.38
1.36
4 5 6 7 8 9 10
χ^2_{EXP}
Central value remains stable

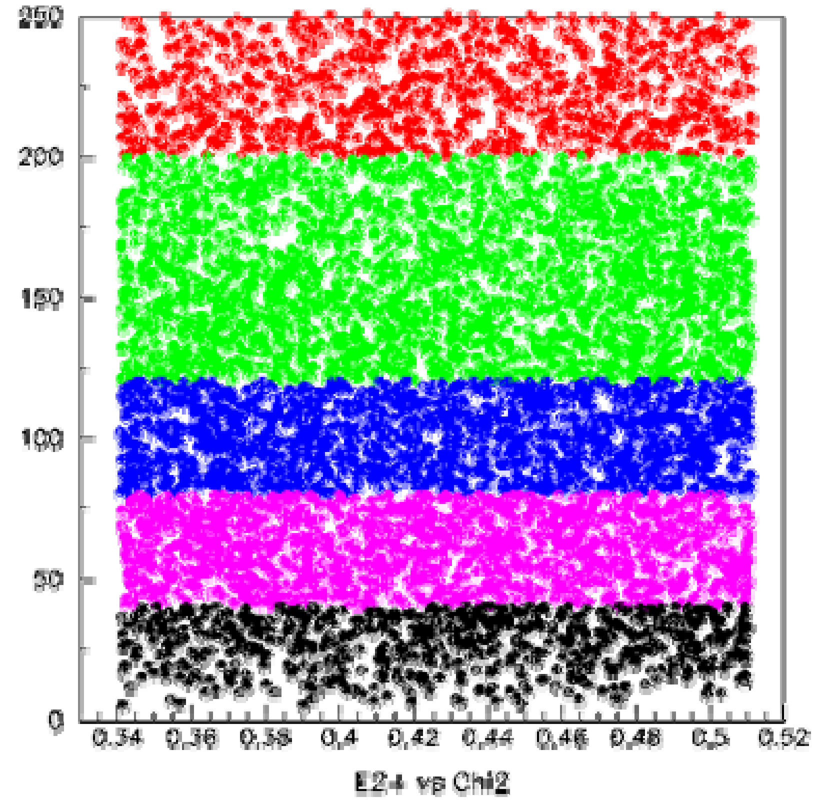

3000
2500
2000
1500
1000
500
0
1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8
L1+


Uncertainty depends on the χ^2 cut
σ_{L1+}
0,12
0,11
0,10
0,09
0,08
0,07
0,06
4 5 6 7 8 9 10
χ^2_{EXP}

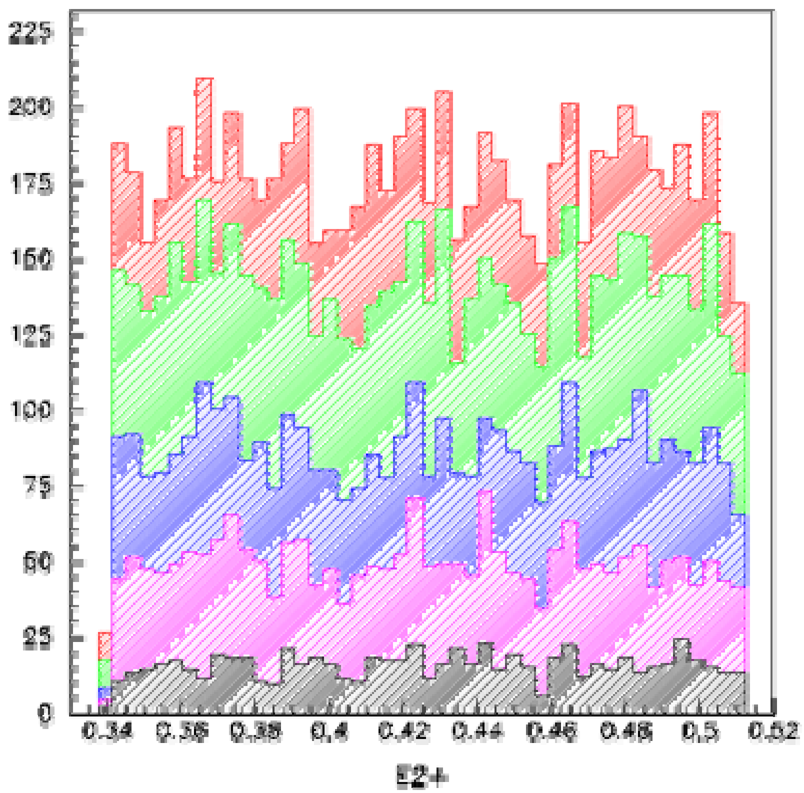
Correlations: Standard fitting



Applying χ^2 Cut on
NON SENSITIVE
Amplitude A_i



A_i Distribution

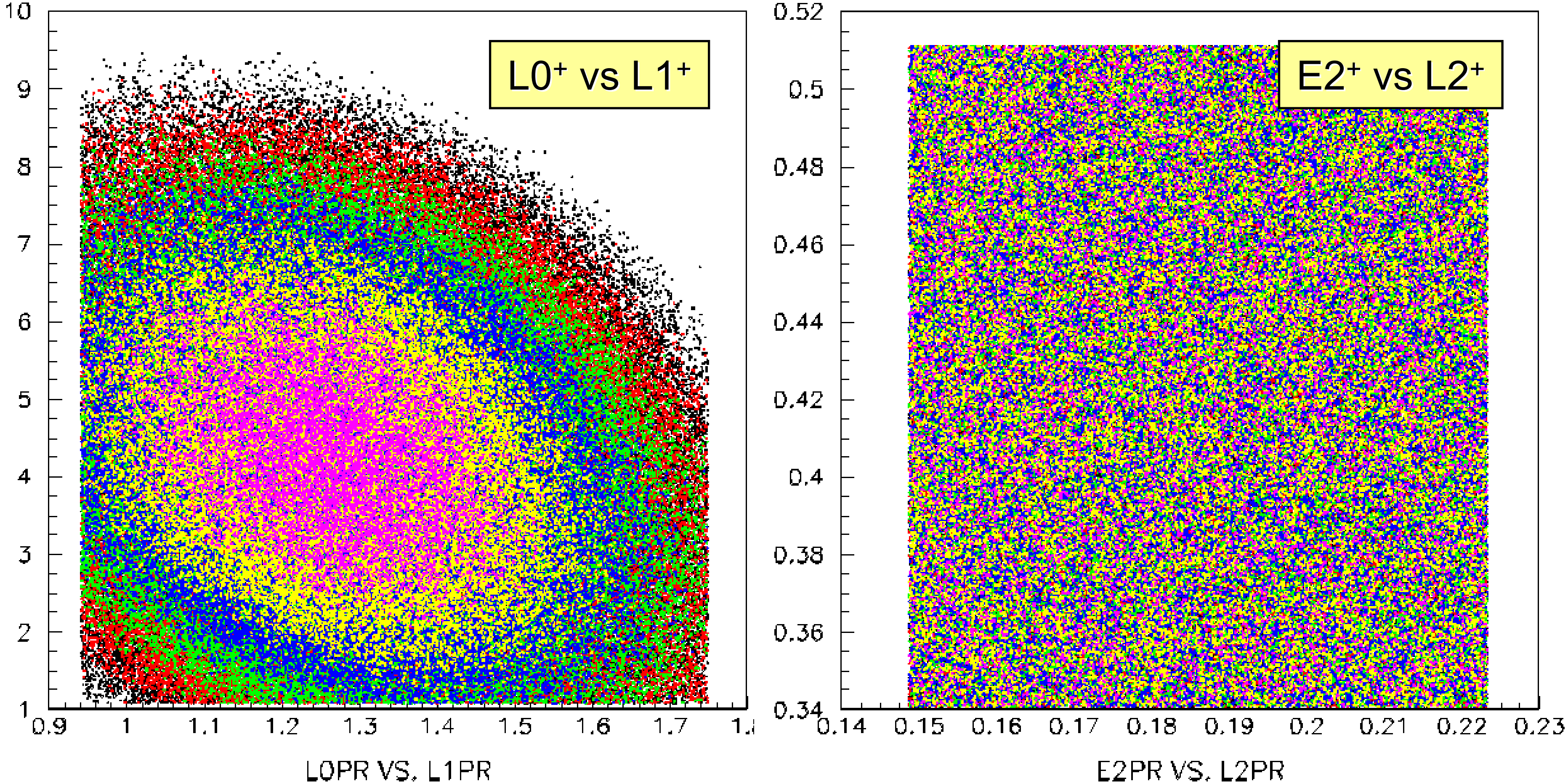


PROJECTION

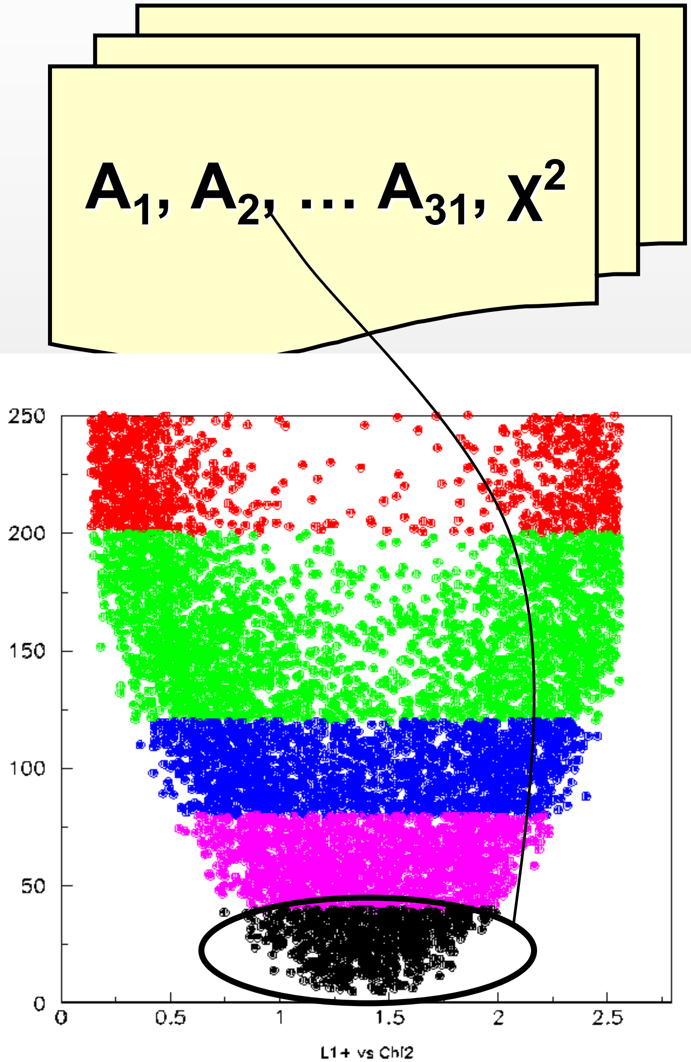
ALL VALUES
$\chi^2 < 200$
$\chi^2 < 120$
$\chi^2 < 80$
$\chi^2 < 40$

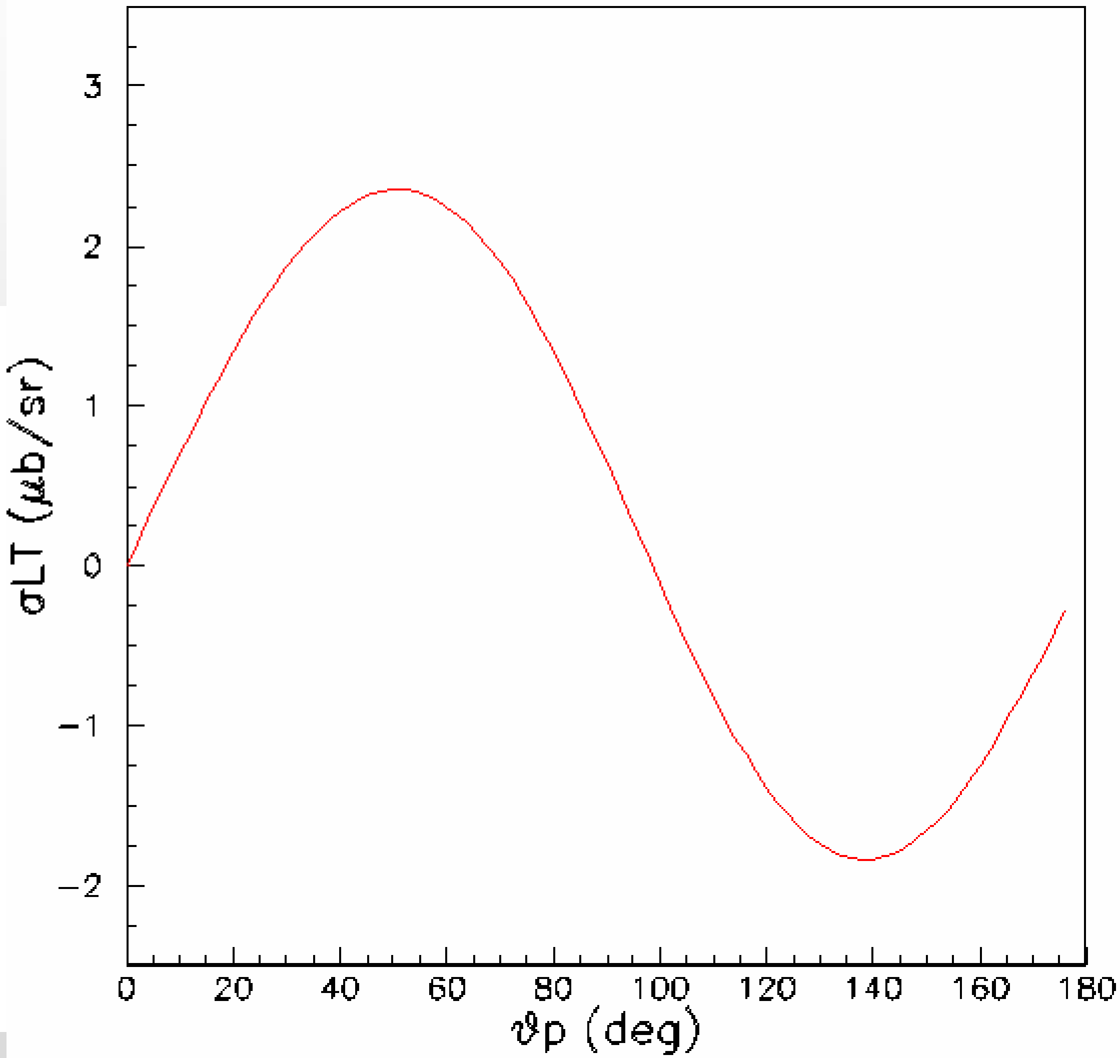
Correlations

Amplitude Correlations are automatically included through randomization in the ensemble and can be easily investigated.



Explore the allowed solutions in a given observable or derived quantity

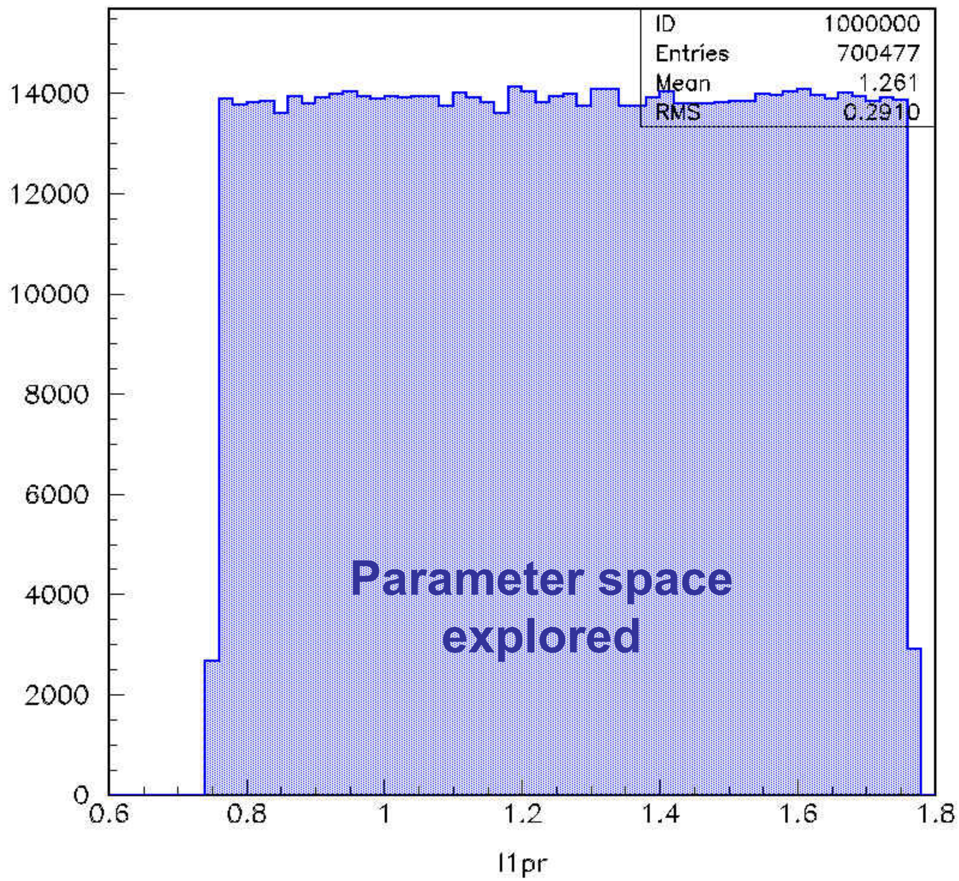


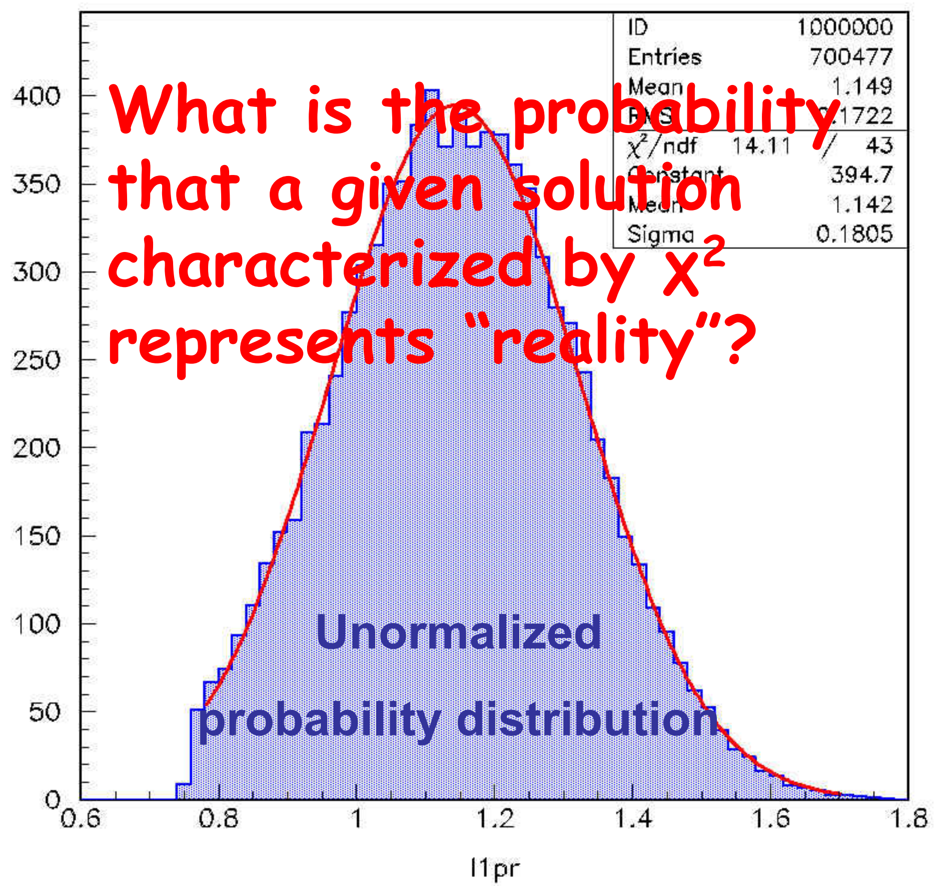


Canonical Ensemble: Use all solutions, weighted by the probability that they represent “reality”

Sensitive Amplitude
L1+

$$erfc\left(\frac{\chi^2-\chi^2_{\min}}{\chi^2_{\min}}\right)$$

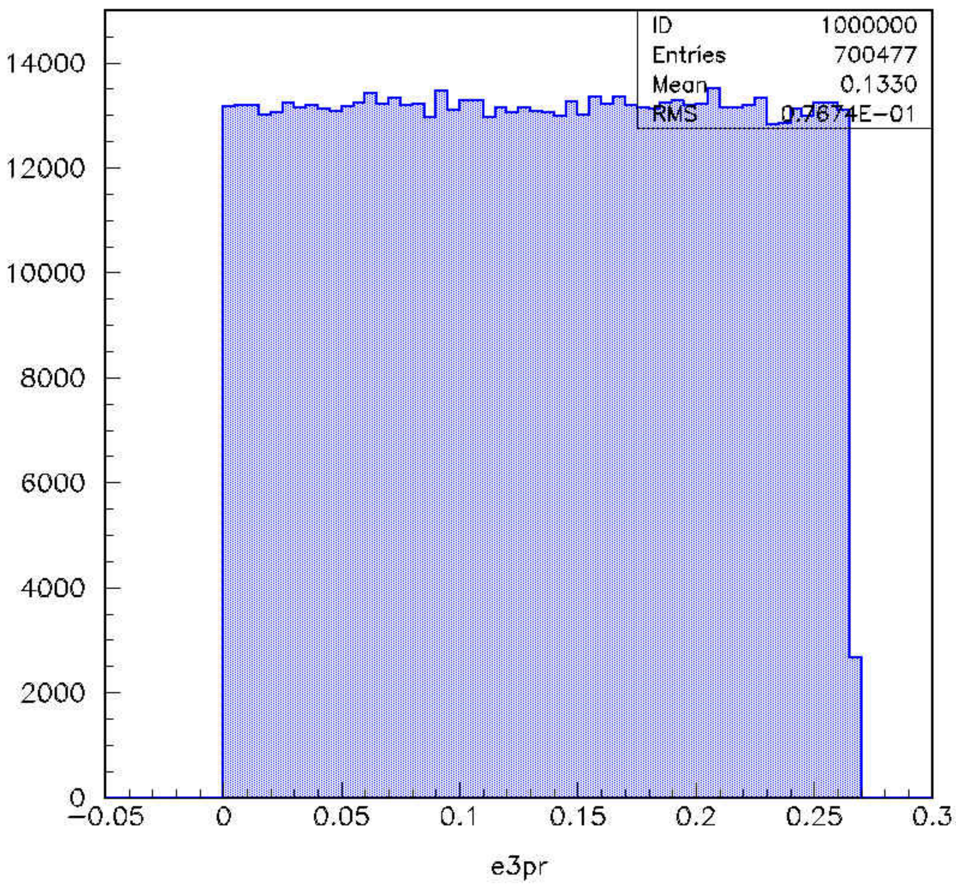


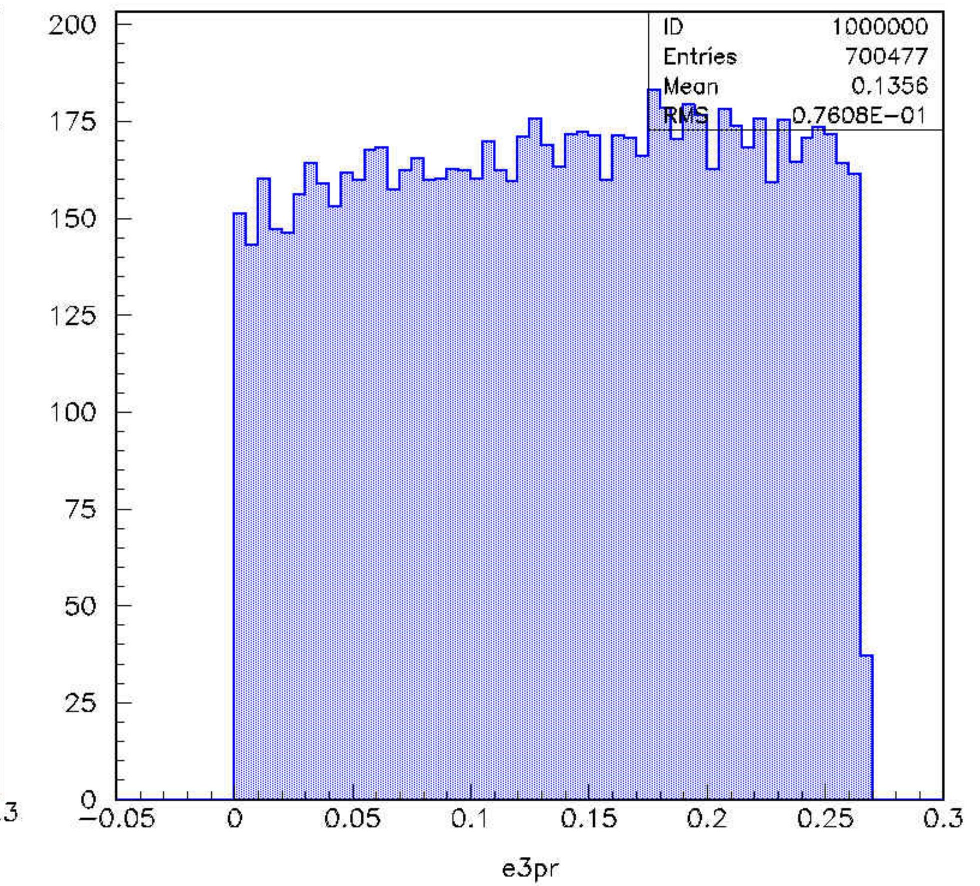


Canonical Ensemble: Use all solutions, weighted by the probability that they represent “reality”

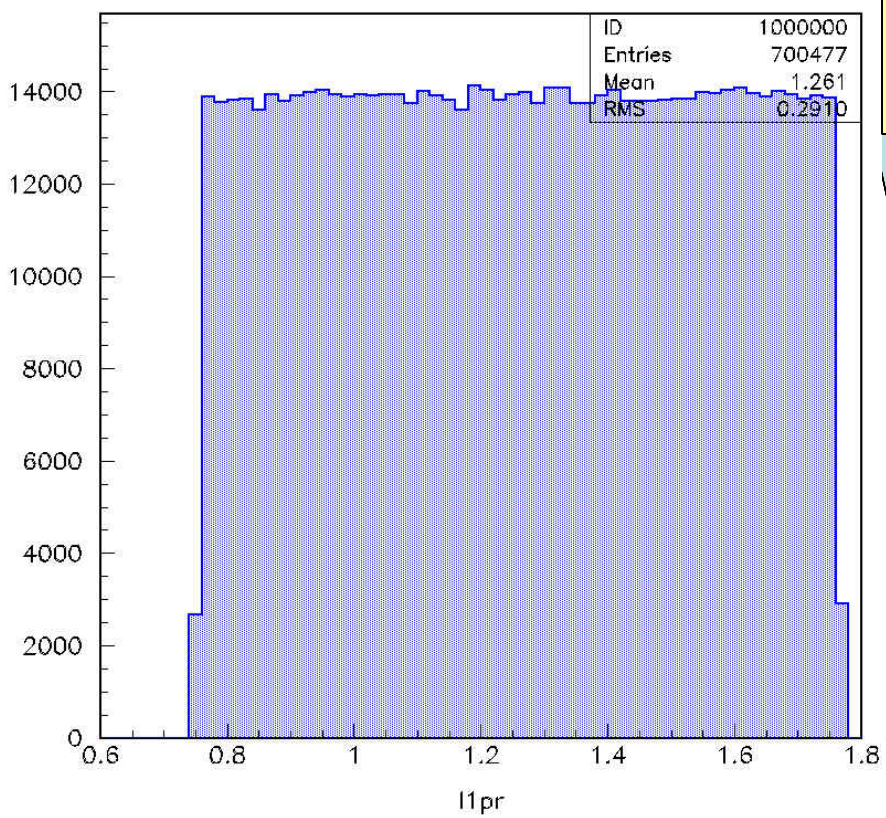
Non Sensitive Amplitude
E3+

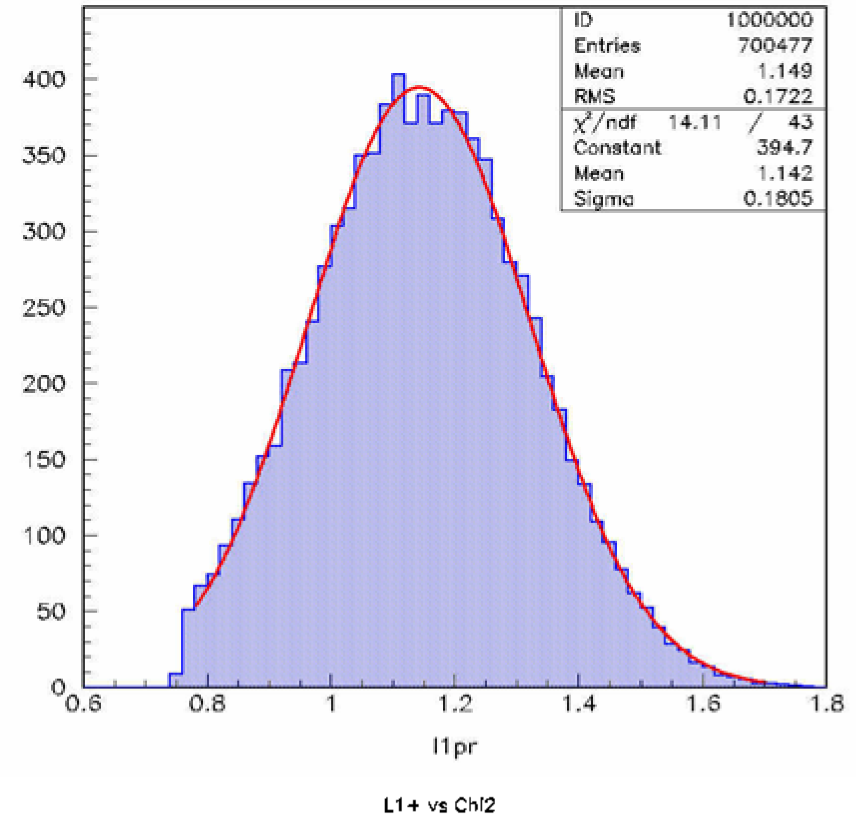
$$erfc\left(\frac{\chi^2-\chi^2_{\min}}{\chi^2_{\min}}\right)$$





INSTEAD OF PROJECTING
Out the “best solutions”
(Microcanonical ensemble)





Weigh the significance of each solution by its likelihood to be correct (canonical ensemble)

Microcanonical and canonical Ensemble of Solutions

- **Microcanonical:** easier to understand, pedagogical
- **Canonical:** rigorous, far more efficient computationally
- They yield equivalent results

→ Use Canonical ensemble of solutions

III. Validation of AMIAS

How Reliable is the extraction of Multipoles?

Test it with pseudodata.

Algorithm:

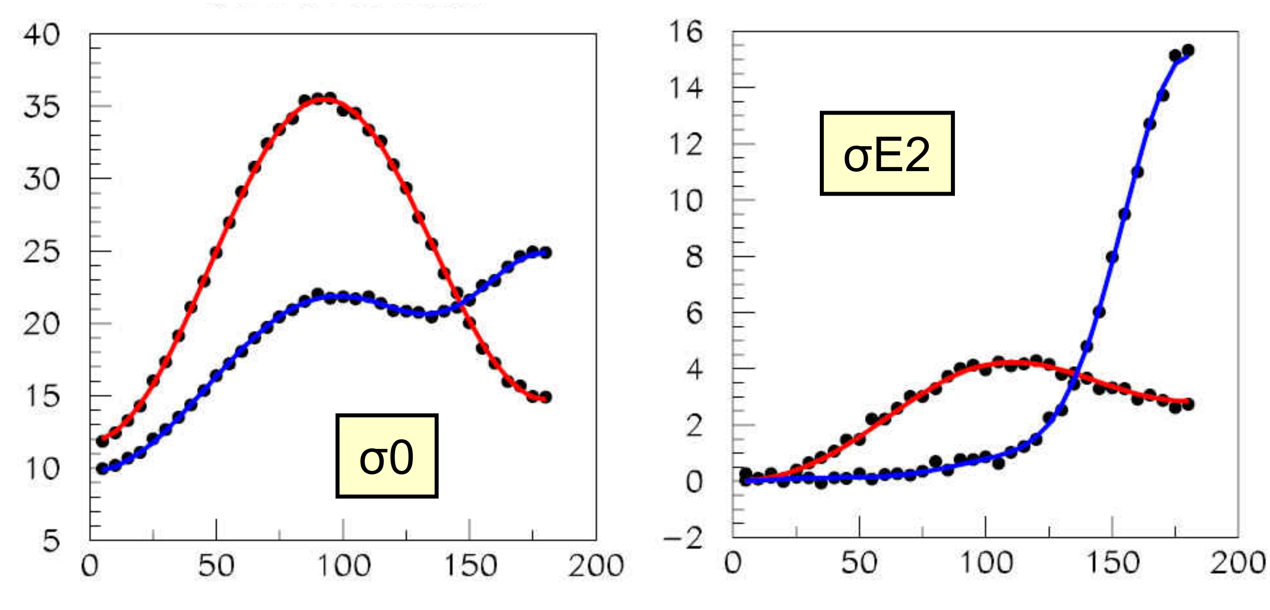
- Step1: Generate pseudodata
- Step2: Analyze them
- Step 3:Demonstrate that you get input values

The extraction of Multipole values is reliable!

Multipole	Generator	Set A	Set B	Set C
M_{1+}	27.248	27.23 ± 0.13	27.229 ± 0.013	27.249 ± 0.001
L_{0+}	3.500	3.70 ± 0.23	3.515 ± 0.022	3.502 ± 0.002
L_{1+}	1.048	1.03 ± 0.08	1.047 ± 0.008	1.048 ± 0.001
E_{1+}	1.481	1.49 ± 0.18	1.489 ± 0.017	1.482 ± 0.002
E_{0+}	4.225	3.68 ± 1.02	4.278 ± 0.135	4.239 ± 0.013
M_{1-}	4.119	4.47 ± 1.31	4.161 ± 0.126	4.124 ± 0.013
L_{1-}	1.205	1.05 ± 0.43	1.170 ± 0.080	1.203 ± 0.008
E_{2-}	1.024	1.07 ± 0.45	1.053 ± 0.061	1.027 ± 0.006
L_{2+}	0.007	0.02 ± 0.01	0.008 ± 0.001	0.008 ± 0.001
E_{2+}	0.006	0.01 ± 0.01	0.009 ± 0.001	0.007 ± 0.001

Q^2=0.127 (GeV/c)^2 , W=1232 MeV), kinematics

Norms. The three sets are identical except for the statistical accuracy

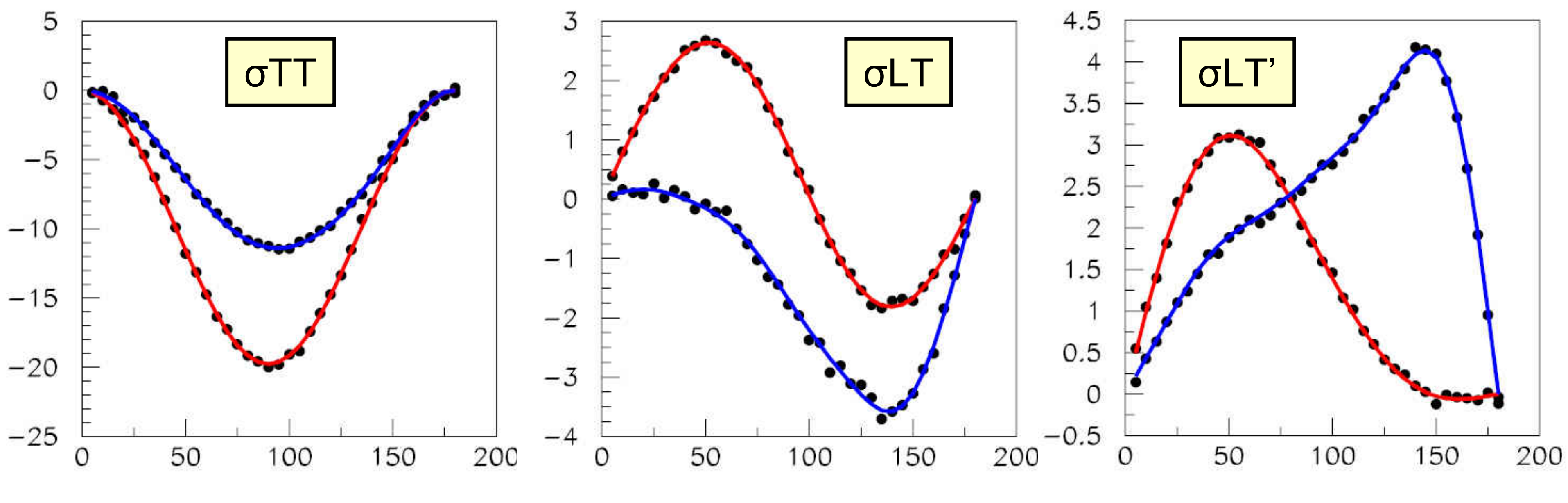


Pseudo-Data Set

π^0 p Channel (220 points)

π^+ n Channel (180 points)

Q^2=0.127 $(GeV/c)^2$, W=1232 MeV), kinemtics



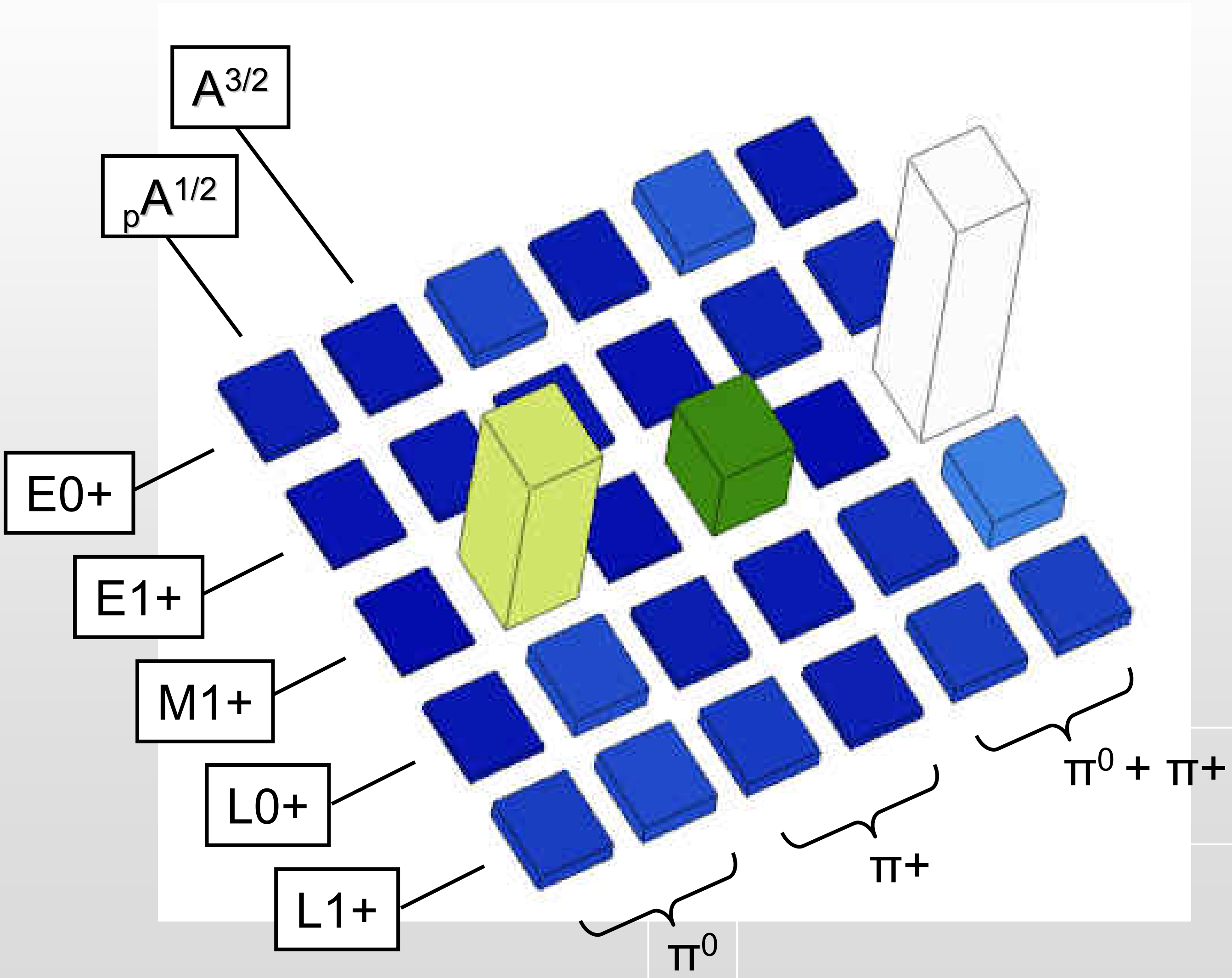
Reproduction of Values & identification of sensitivities

Analysis of the Pseudo Data (amias_ps4_400d)
Both channels π^0 p (220 points) & π^+ n (180 points)
Q^2=0.127 $(GeV/c)^2$ W=1232 MeV

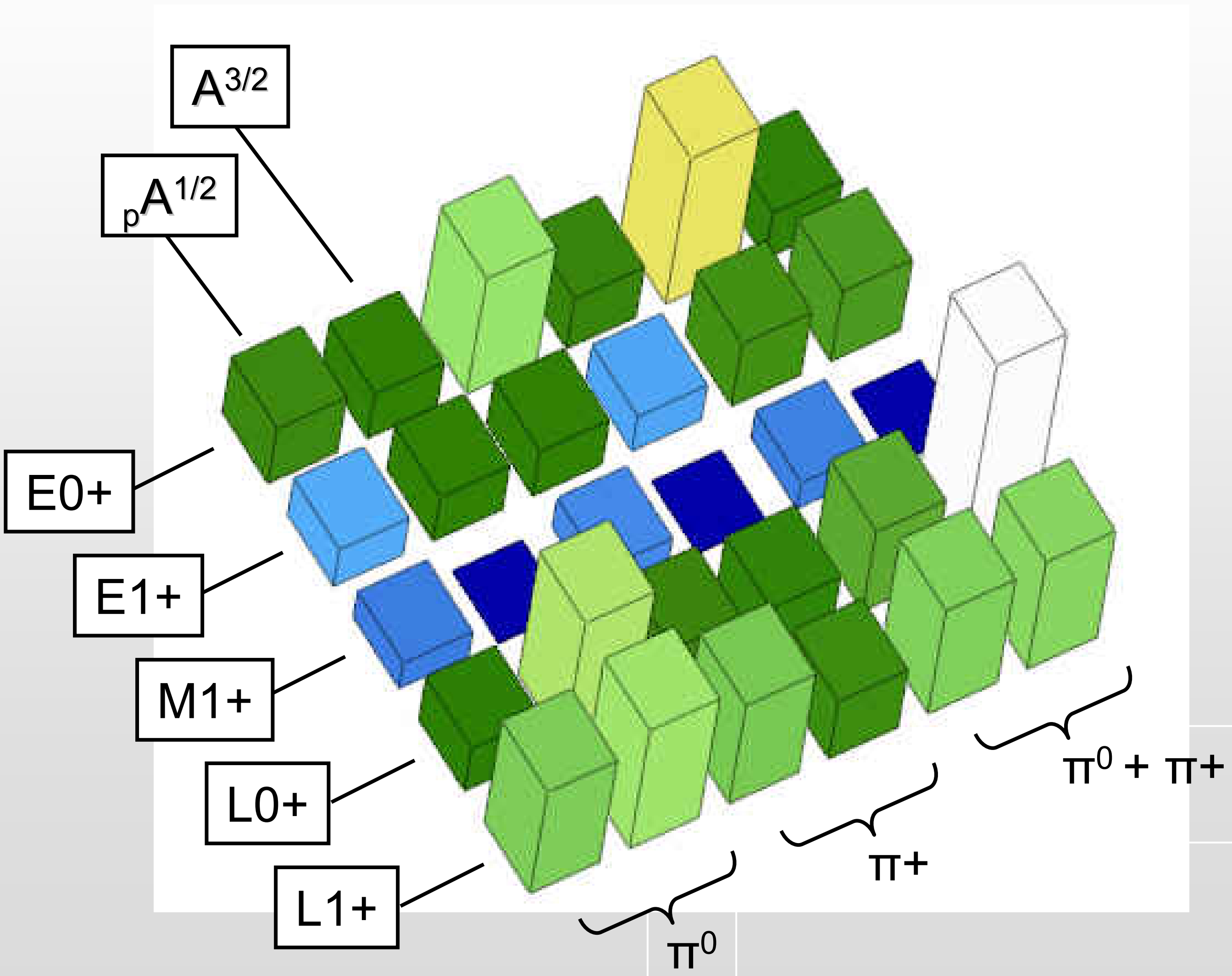
	${}_pA^{1/2}$				$A^{3/2}$			
Multipole	**MAID**	**σ_0 / A_0**	**Ratio**	**Sens**	**MAID**	**σ_0 / A_0**	**Ratio**	**Sens**
|E0+|	**6.731**	0.520 / 6.716	8 %	12.9	**8.697**	1.880 /10.39	18 %	5.5
|E1+|	**1.137**	0.190 / 1.205	16 %	6.3	**0.869**	0.150 / 0.941	16 %	6.3
|E2+|	**0.282**	0.070 / 0.299	23 %	4.3	**0.426**	---------------	------	-----
|M1+|	**0.981**	0.500 / 1.690	30 %	3.4	**41.115**	0.290 / 40.79	0.7 %	140.7
|M1-|	**1.304**	---------------	------	-----	**4.237**	1.200 / 2.480	48 %	2.1
|L0+|	**1.477**	0.180 /1.620	11 %	9.0	**5.395**	0.300 / 5.512	5.4 %	18.4
|L1+|	**0.410**	0.048 / 0.472	10 %	9.8	**1.345**	0.140 / 1.394	10 %	10.0
|L1-|	**3.253**	0.520 / 3.085	17 %	5.9	**3.143**	1.140 / 2.530	45 %	2.2

MAID: Input MAID-2003 value (multipole norm)
σ_0: Uncertainty found (gauss fit)
A_0: Central value for multipole (gauss fit)
Ratio: σ_0 / A_0
Sens: Multipole Sensitivity (= A_0 / σ_0)

Sensitivity Analysis with Pseudo-Data



Sensitivity Analysis with Pseudo-Data



How Reliable is the extraction of Uncertainties?

Testing of the reproduction of central values is straight forward.

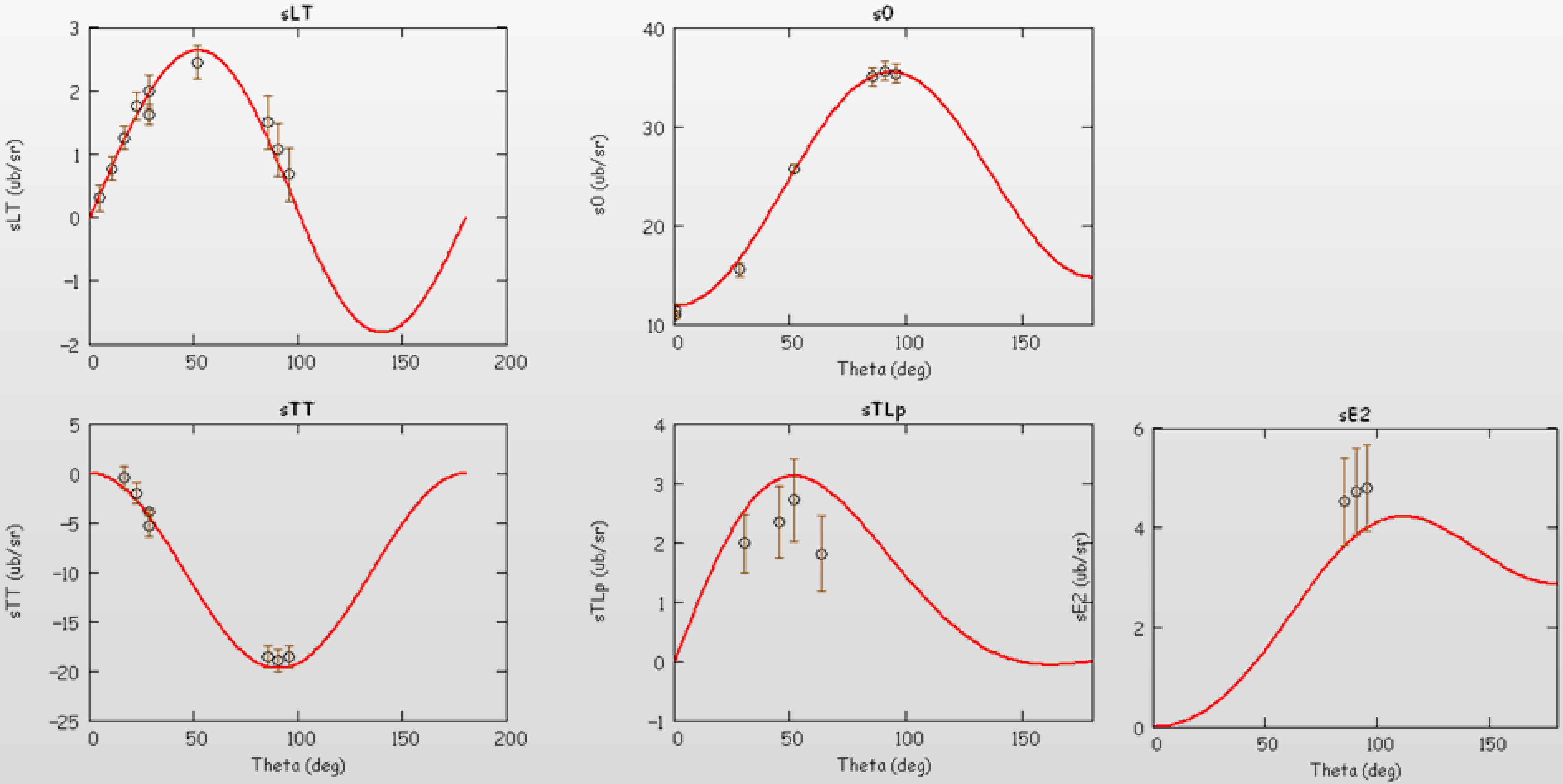
But how do we test the validation of the extracted uncertainties?

Test it with pseudodata. Algorithm:

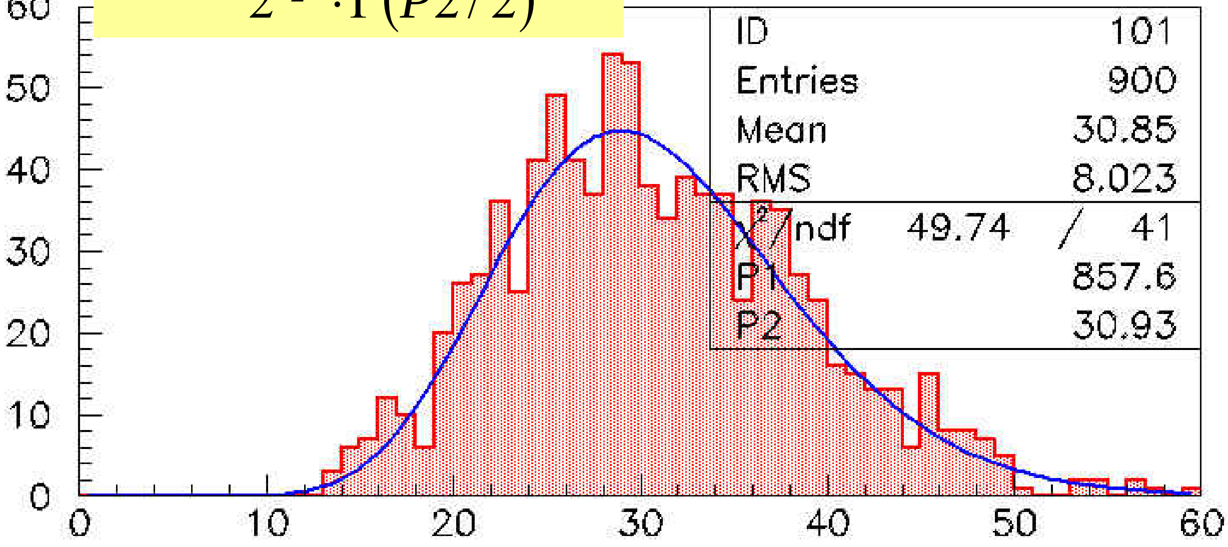
- Step1: Generate ensemble of pseudodata
- Step2: Generate "master distribution"
- Step 3: Analyze "sample data"
- Step 4: Demonstrate consistency and reliability

Bates-Mainz Data (Q^2=0.127 (GeV/c)^2 , W=1232 MeV)

Take the published solution to our data to be "the truth". Then generate pseudodata. An ensemble of pseudodata sets.



$$f(x) = \frac{x^{\frac{P2}{2}-1} \cdot e^{-x/2}}{2^{\frac{P2}{2}} \cdot \Gamma(P2/2)} \cdot P1$$



Pseudo Data

MAID2003 modified to Bates-Mainz Data Set

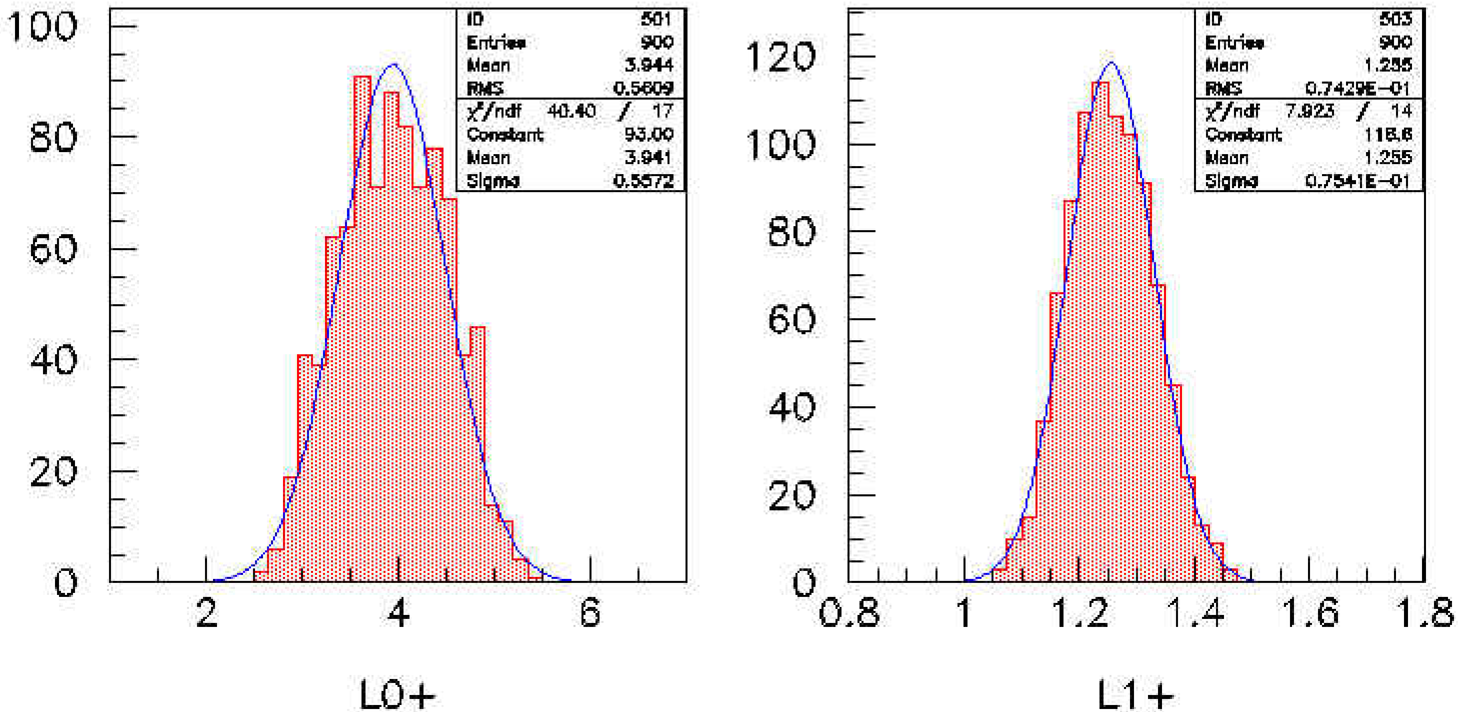
31 Experimental Points

X^2 Fit

Each point in the distribution corresponds to an "experiment".

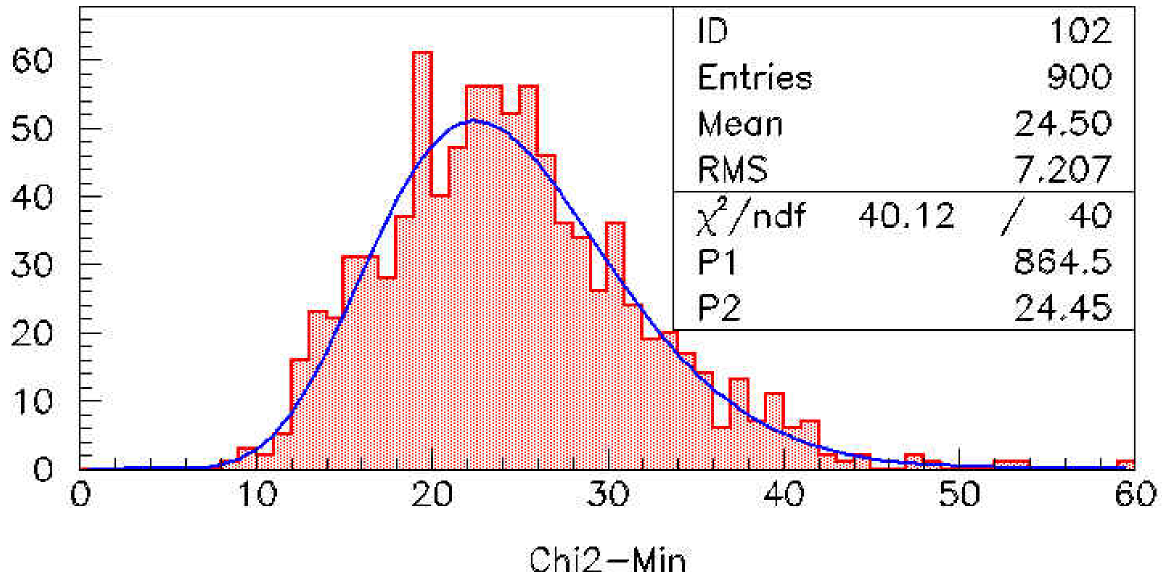
Proceed to analyze it as described earlier and drive from it a "solution". We can then plot the histogram of the X^2 of the derived solutions. This an ensemble of the solutions resulting from the sampling of the master distribution

The distribution of the derived solutions gives an absolute definition of the uncertainty

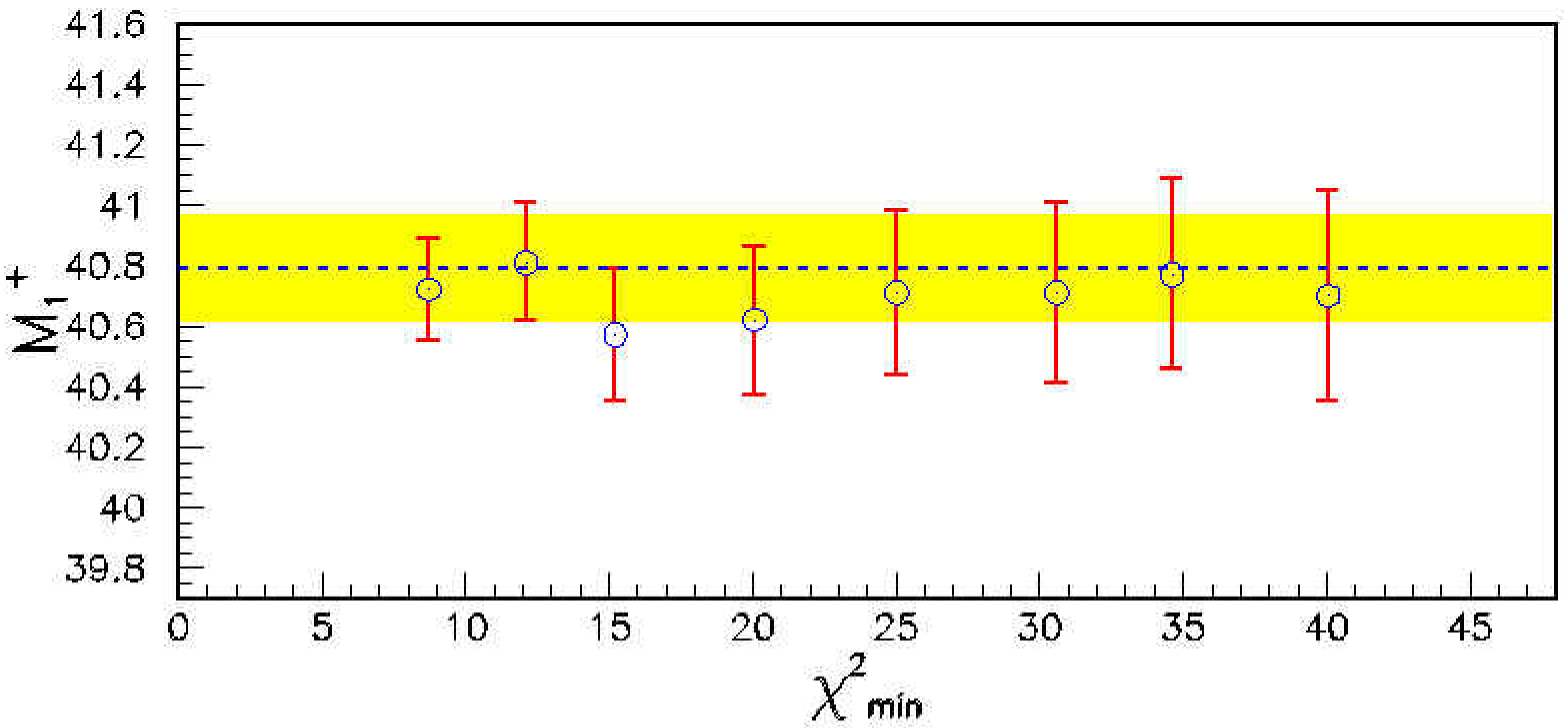


Central Value
+
Deviation

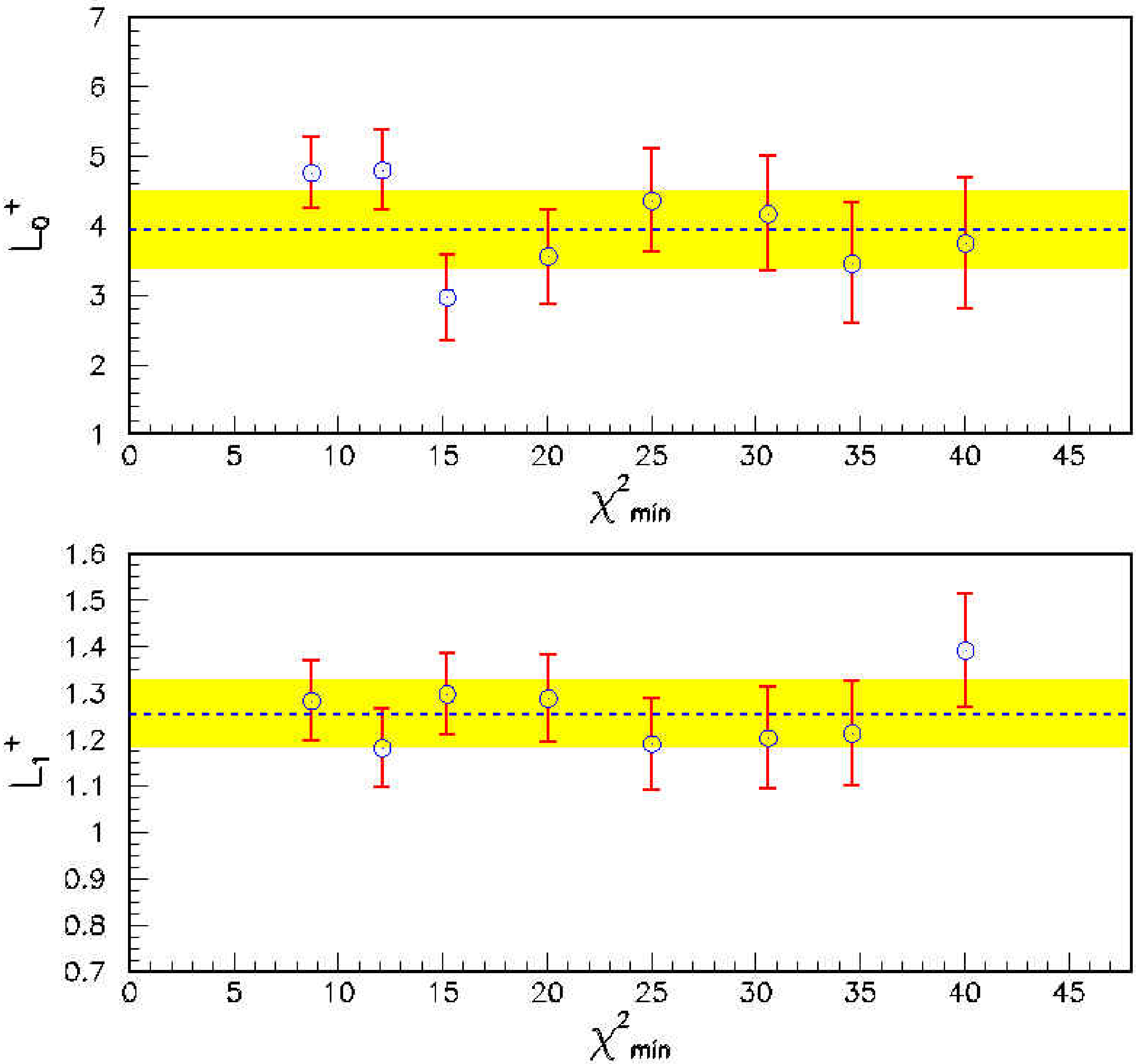
Global Set
(900 runs)



ID 102
Entries 900
Mean 24.50
RMS 7.207
χ^2/ndf 40.12 / 40
P1 864.5
P2 24.45
Chi2−Min



M_1^+
χ^2_{min}



L_0^+
χ^2_{min}
L_1^+
χ^2_{min}

Study of Pseudodata

- We reproduce both the values and the uncertainties to the appropriate statistical significance.

- The extracted uncertainties have precise statistical meaning and the resulting distributions can be interpreted precisely in terms of confidence levels.

- AMIAS allows the extraction of maximum information and identification of sensitivities

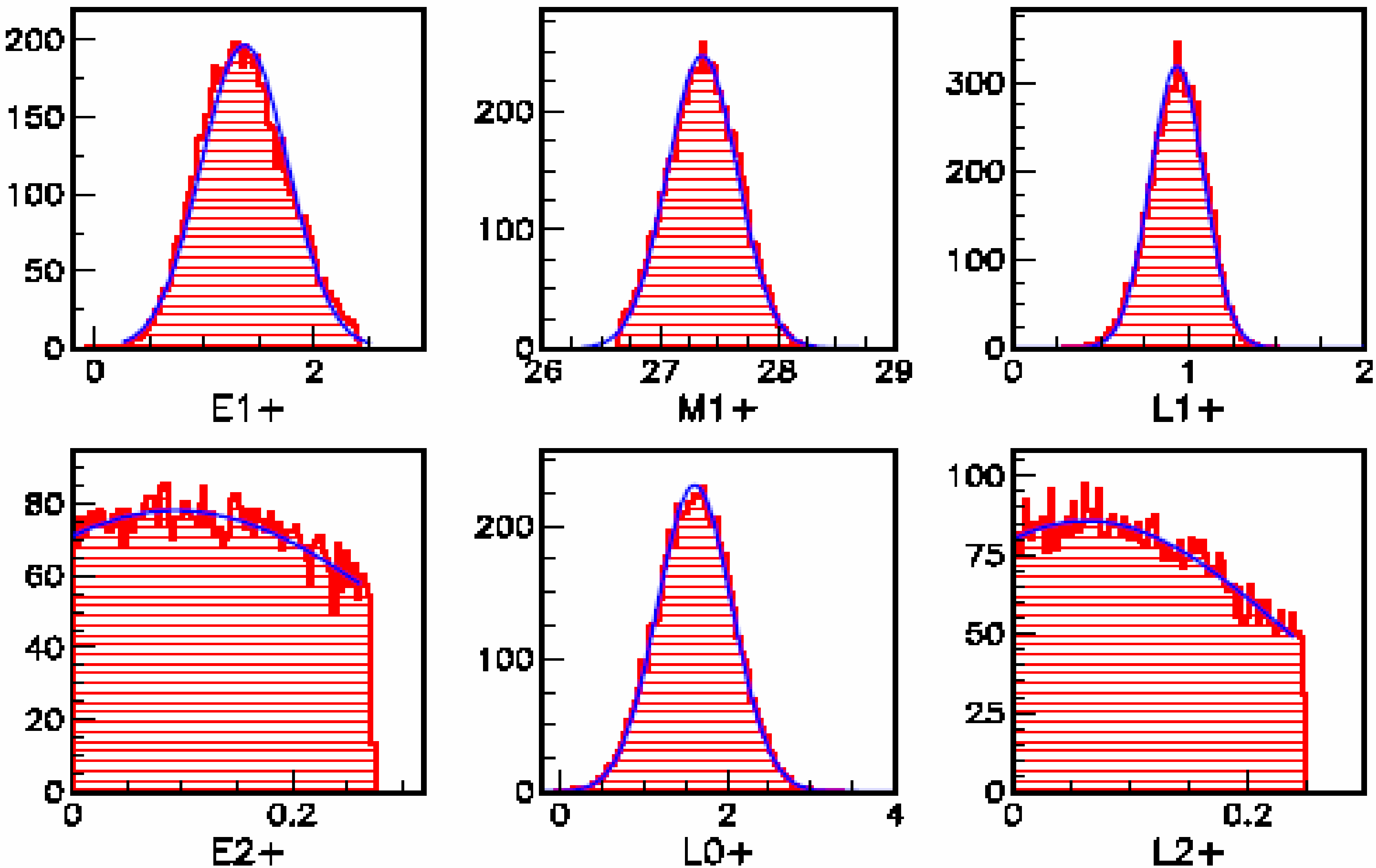
Bates-Mainz Data
(Q^2=0.127 $(GeV/c)^2$, W=1232 MeV)

1^{st} TEST CASE

Apply the Model Independent Analysis for Multipole Extraction

Lcut = 5

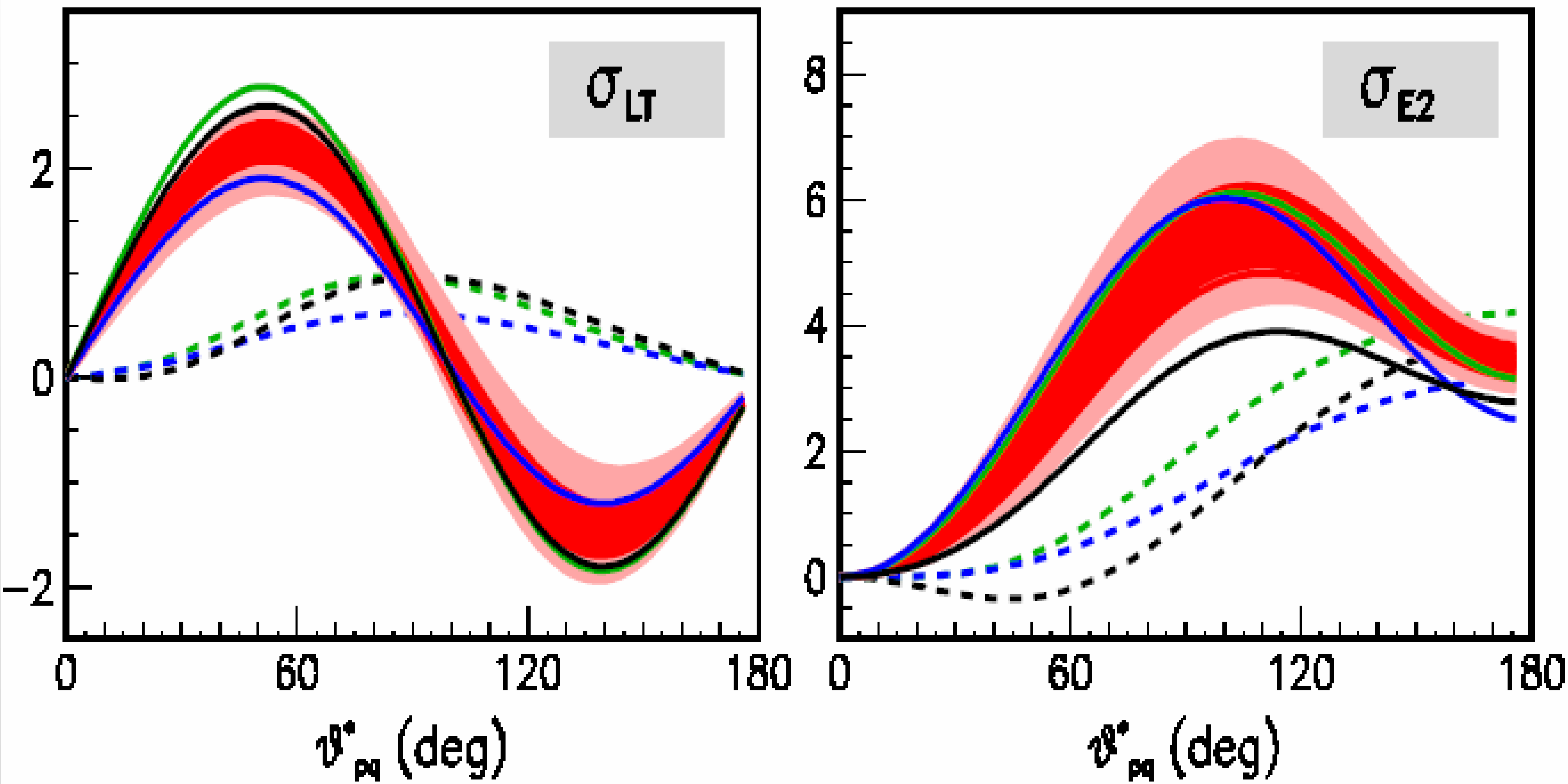
Probability Distributions

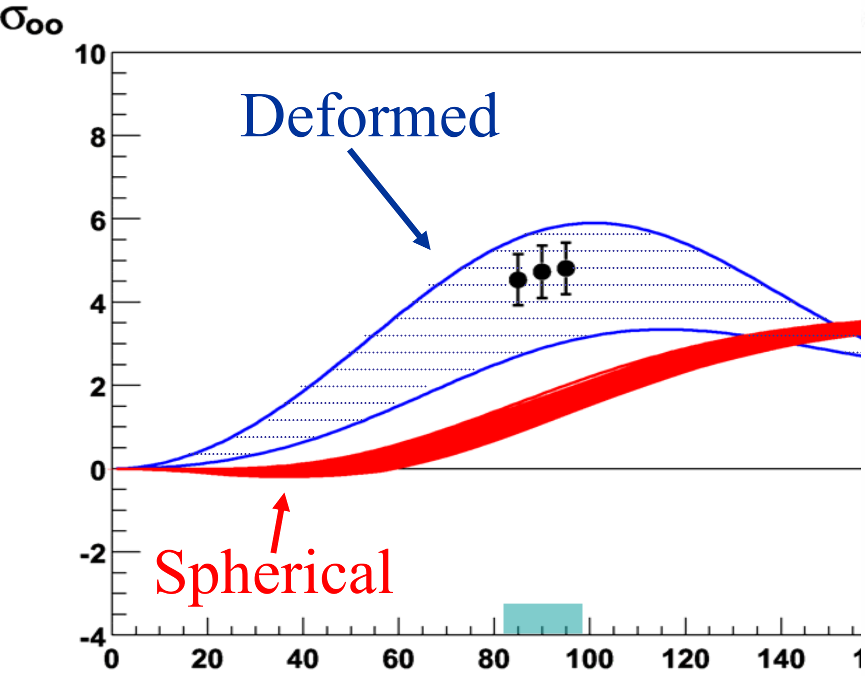


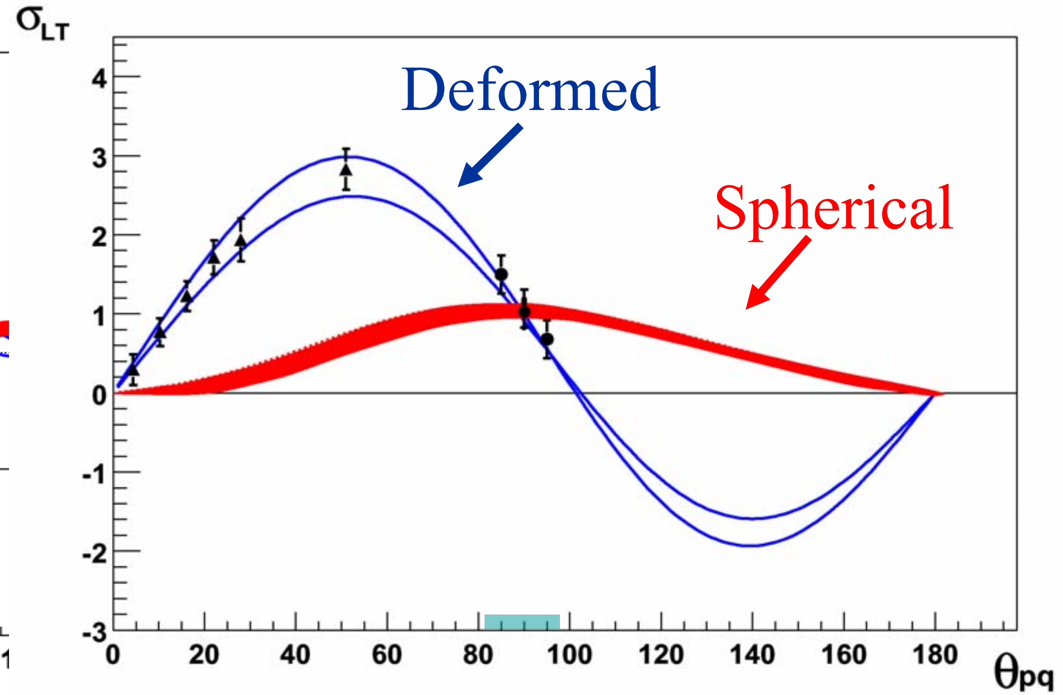
Extracted Values

	Full Data Set	Reduced Data Set	
Multipole	Extracted Value	Extracted Value	Change
M_{1+}	27.36 ± 0.15	27.21 ± 0.17	$+0.5$ %
L_{1+}	0.93 ± 0.08	0.91 ± 0.08	$+2.2$ %
L_{0+}	1.61 ± 0.22	2.67 ± 0.41	-66.0 %
E_{1+}	1.37 ± 0.20	1.36 ± 0.18	$+1.0$ %
E_{0+}	2.95 ± 1.13	–	–
L_{1-}	1.00 ± 0.52	0.47 ± 0.21	$+112.0$ %
E_{2+}	0.09 ± 0.11	–	–
L_{2+}	0.06 ± 0.09	0.04 ± 0.04	$+33.3$ %

Derived Uncertainties







CMR & EMR	$-6.27 \pm 0.32_{stat+sys} \pm 0.10_{mod}$	$-2.00 \pm 0.40_{stat+sys} \pm 0.27_{mod}$

- Ad hoc
- Not quantitative
- model dependent

CNP Eur. Phys. J. A18, 141 (2003)
N. Sparveris et al PRL **94**, 022003 (2005)

Conclusions

- The new method is a model independent analysis for identifying sensitivities and extracting Multipole values from experimental data on Nucleon Resonances.

- The method has been examined extensively with pseudodata and with limited set of experimental data. It is stable and robust.

Remaining Issues (work in progress)

- Self adapting randomization width
- Additional variation of phases with respect to unitarization
- Extend the method to handle W dependence