Never make a calculation until you already know the answer

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Jefferson Labs, Newport News, VA, 24 February 2016
Knowing an answer is possible without calculating

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem using paper and pencil. (2nd NAEP, 1977–78)

1
2
19
21
I don’t know
Knowing an answer is possible without calculating

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem using paper and pencil. (2nd NAEP, 1977–78)

1
2
19
21
I don’t know
Knowing an answer is possible without calculating

Estimate the answer to $12/13 + 7/8$. You will not have time to solve the problem using paper and pencil. (2nd NAEP, 1977–78)

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<th>Age 13</th>
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Knowing an answer is possible without calculating

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem using paper and pencil. (2nd NAEP, 1977–78)

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**Exact-calculation problem:**

$\frac{7}{15} + \frac{4}{9}$

39 54
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
Rote learning makes problem solving into a random walk
Rote learning makes problem solving into a random walk
Rote learning combines the worst of human and computer thinking

<table>
<thead>
<tr>
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<th>human chess</th>
<th>computer chess</th>
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<tr>
<td>calculation</td>
<td>1 position/second</td>
<td>$10^8$ positions/second</td>
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<tr>
<td>judgment</td>
<td>fantastic</td>
<td>minimal</td>
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Rote learning combines the worst of human and computer thinking

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<tr>
<td>Judgment</td>
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I’ve written out two antidotes to rote learning

Freely and legally available from MIT Press—with freedom to modify and redistribute.
The antidote is to avoid rigor
The antidote is to avoid rigor mortis.
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
Pictures explain most of Stirling’s formula for $n!$

\[ \ln n! \approx \int_1^n \ln k \, dk = n \ln n - n + 1; \]

\[ n! \approx e \times n^n / e^n. \]
The protrusions are the underestimate
Each protrusion is almost a triangle
Doubling each triangle makes them easier to add
The doubled triangles stack nicely

Sum of doubled triangles = \ln n
The integral along with the triangles explain most pieces of Stirling’s formula for \( n! \)

\[
\ln n! = \sum_{1}^{n} \ln k
\]

\[
\approx \quad n \ln n - n + 1
\]

\[
+ \quad \frac{1}{2} \ln n
\]

\[
n! \approx e \times \frac{n^n}{e^n} \times \sqrt{n}
\]

should be \( \sqrt{2\pi} \)
Our perceptual hardware can do one-shot learning

Dalmatian at the beach
Reasoning without our perceptual hardware leaves us less clever

Dalmatian at the beach
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. *coincidences*
   c. drag
   d. musical notes
We can understand coincidences

Imagine that every packet on a network gets a random 80-bit number (the “hash”) based on its contents and the timestamp, and that new packets are being generated at 1 megahertz ($2^{20}$ per second). How long, on average, before two packets have been generated with a common hash?

a. 1 second ($2^0$ seconds)

b. 10 days ($2^{20}$ seconds)

c. $3 \times 10^4$ years ($2^{40}$ seconds)

d. $3 \times 10^{10}$ years ($2^{60}$ seconds)
... but not if we use the exact formula

After $n$ packets,

$$p_{	ext{no coincidence}} = \left(1 - \frac{1}{2^{80}}\right) \left(1 - \frac{2}{2^{80}}\right) \left(1 - \frac{3}{2^{80}}\right) \cdots \left(1 - \frac{n-1}{2^{80}}\right).$$

For what $n$ does $p$ get small enough, say $1/e$?
An approximate scaling analysis provides more insight.

$n$ packets make $n^2/2$ “handshakes.”
An approximate scaling analysis provides more insight

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Each handshake has a $2^{-80}$ chance of being a collision.
An approximate scaling analysis provides more insight

\( n \) packets make \( n^2/2 \) “handshakes.”

Each handshake has a \( 2^{-80} \) chance of being a collision.

\[
p_{\text{no coincidence}} \approx (1 - 2^{-80})^{n^2/2} \\
\approx \exp \left( -\frac{n^2}{2} \times 2^{-80} \right).\]
An approximate scaling analysis provides more insight

$n$ packets make $n^2/2$ “handshakes.”

Each handshake has a $2^{-80}$ chance of being a collision.

$$p_{\text{no coincidence}} \approx (1 - 2^{-80})^{n^2/2} \approx \exp\left(-\frac{n^2}{2} \times 2^{-80}\right).$$

Therefore, $p_{\text{no coincidence}}$ drops to $1/e$ when

$$n \approx 2^{40}.$$
We can understand coincidences by using proportional reasoning

Imagine that every packet on a network gets a random 80-bit number (the “hash”) based on its contents and the timestamp, and that new packets are being generated at 1 megahertz ($2^{20}$ per second). How long, on average, before two packets have been generated with a common hash?

a. 1 second ($2^0$ seconds)

b. 10 days ($2^{20}$ seconds or $2^{40}$ packets)

c. $3 \times 10^4$ years ($2^{40}$ seconds)

d. $3 \times 10^{10}$ years ($2^{60}$ seconds)
This scaling analysis demystifies the birthday paradox

365 possible hashes

\[ p_{\text{no shared birthday}} \approx \left(1 - \frac{1}{365}\right)^{n^2/2} \approx \exp\left(-\frac{n^2}{2 \times 365}\right). \]

\( p_{\text{no shared birthday}} \) drops to 1/e when

\[ n \sim \sqrt{2 \times 365} \approx 27. \]
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Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
Fluid mechanics is difficult, so we need insight

What is the fuel efficiency of a 747?
The brute-force method is hopelessly difficult

Equations of fluid mechanics

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}
\]

\[\nabla \cdot \mathbf{v} = 0\]

where

\(\rho\) = air density

\(p\) = pressure

\(\mathbf{v}\) = velocity

\(\nu\) = (kinematic) viscosity

\(t\) = time
A more insightful analysis uses proportional reasoning

What is the approximate ratio of the fall speeds $v_{\text{big}} / v_{\text{small}}$?

a. 2 : 1

b. 1 : 1

c. 1 : 2
A more insightful analysis uses proportional reasoning

What is the approximate ratio of the fall speeds $v_{\text{big}}/v_{\text{small}}$?

a. $2 : 1$

b. $1 : 1$ Drag force is proportional to area!

c. $1 : 2$
We also need a symmetry principle: dimensional analysis

\[
\text{drag force} \sim \frac{\text{area} \times \text{density} \times \text{speed} \times \text{viscosity}}{\text{kilograms} \times \text{meters}^2 \times \text{meter} \times \text{second}^2}
\]
We also need a symmetry principle: dimensional analysis

\[
\text{drag force} \sim \text{area} \times \text{density} \times \text{speed? viscosity?}
\]

\[
\frac{\text{kilograms} \times \text{meters}}{\text{second}^2} \sim \frac{\text{meters}^2}{\text{kilograms}} \times \frac{\text{meters}^2}{\text{meter}^3} \times \frac{\text{meters}^2}{\text{second}^2}
\]
We also need a symmetry principle: dimensional analysis

\[ \text{drag force} \sim \text{area} \times \text{density} \times \text{speed}^2 \]

\[
\begin{align*}
\text{drag force} & \quad \text{area} \quad \text{density} \quad \text{speed}^2 \\
\text{kilograms} \times \text{meters} & \quad \text{meters}^2 \\
\text{second}^2 & \quad \text{kilograms} \quad \text{meter}^3 \\
\end{align*}
\]
We use proportional reasoning again

Fuel consumption is proportional to the drag force, and

\[ \text{drag force} \sim \text{area} \times \text{density} \times \text{speed}^2. \]

The ratio of plane-to-car fuel consumptions is therefore

\[
\frac{\text{plane consumption}}{\text{car consumption}} \sim \frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}} \times \frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}} \times \frac{\text{speed}_{\text{plane}}^2}{\text{speed}_{\text{car}}^2}
\]
We use proportional reasoning again.

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We use proportional reasoning again

Fuel consumption is proportional to the drag force, and

drag force $\sim$ area $\times$ density $\times$ $\text{speed}^2$.

The ratio of plane-to-car fuel consumptions is therefore

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We use proportional reasoning again

Fuel consumption is proportional to the drag force, and

\[ \text{drag force} \sim \text{area} \times \text{density} \times \text{speed}^2. \]

The ratio of plane-to-car fuel consumptions is therefore

\[
\frac{\text{plane consumption}}{\text{car consumption}} \sim \frac{\text{area}_{\text{plane}}}{\text{area}_{\text{car}}} \times \frac{\text{density}_{\text{plane}}}{\text{density}_{\text{car}}} \times \frac{\text{speed}^2_{\text{plane}}}{\text{speed}^2_{\text{car}}} \sim 300
\]
We use proportional reasoning again

Fuel consumption is proportional to the drag force, and

$$\text{drag force} \sim \text{area} \times \text{density} \times \text{speed}^2.$$  

The ratio of plane-to-car fuel consumptions is therefore

$$\frac{\text{plane consumption}}{\text{car consumption}} \sim \frac{\text{area}_\text{plane}}{\text{area}_\text{car}} \times \frac{\text{density}_\text{plane}}{\text{density}_\text{car}} \times \frac{\text{speed}_{\text{plane}}^2}{\text{speed}_{\text{car}}^2} \sim 300.$$  

But 300 passengers on a plane flight; only 1 passenger in a car.

Planes and cars are equally fuel efficient!
The connection between falling cones and flying planes helps us estimate the cost of a plane ticket.

A Boston–Los Angeles roundtrip is roughly 5000 miles.

\[
5000 \text{ miles} \times \frac{1 \text{ gallon}}{25 \text{ miles}} \times \frac{2 \text{ dollars}}{1 \text{ gallon}} \sim 500 \text{ dollars}.
\]
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
We can understand acoustics without calculating (too much)

Doubling the block’s thickness changes the note frequency by what factor?

a. 4  
b. 2  
c. $\sqrt{2}$  
d. no change  
e. $1/\sqrt{2}$  
f. $1/2$  
g. $1/4$
We can understand acoustics without calculating (too much)

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   d. no change  
   e. $1/\sqrt{2}$  
   f. 1/2  
   g. 1/4

Let’s try it!
A spring model of wood explains the doubling in frequency

Compare the stored energies for the same deflection $y$:
A spring model of wood explains the doubling in frequency

Compare the stored energies for the same deflection $y$: 
A spring model of wood explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

$\times 4$ the energy per spring
A spring model of wood explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

4× the energy per spring
2× the number of springs
A spring model of wood explains the doubling in frequency

Compare the stored energies for the same deflection $y$:

$4 \times$ the energy per spring
$2 \times$ the number of springs
$
\underbrace{8 \times \text{the stored energy}}_{\text{stiffness} \times y^2}$
A spring model of wood explains the doubling in frequency.

Compare the stored energies for the same deflection $y$:

- $4\times$ the energy per spring
- $2\times$ the number of springs
- $8\times$ the stored energy

$$\text{stiffness} \times y^2$$

(bending) frequency $\sim \sqrt{\frac{\text{stiffness}}{\text{mass}}} = \sqrt{\frac{8\times}{2\times}} = 2\times$. 
Frequency is proportional to thickness

\[ f \propto h^1. \]
What if we vary the length (rather than the thickness)?

\[ f \propto l^x. \]

What is the scaling exponent \( x \)?
Dimensional analysis tells us the exponent

The frequency depends on the block’s three dimensions $l, w, \text{ and } h$ as well as a characteristic of the material, the speed of sound $c_s$.

$$f \sim h^1 l^x w^y c_s^\beta.$$
Dimensional analysis tells us the exponent

The frequency depends on the block’s three dimensions $l, w,$ and $h$ as well as a characteristic of the material, the speed of sound $c_s$.

$$f \sim h^1 l^x w^y c_s^1$$
Dimensional analysis tells us the exponent

The frequency depends on the block’s three dimensions $l, w, \text{ and } h$ as well as a characteristic of the material, the speed of sound $c_s$.

\[ f \sim h^1 l^x w^0 c_s^1 \]
Dimensional analysis tells us the exponent

The frequency depends on the block’s three dimensions \( l, w, \) and \( h \) as well as a characteristic of the material, the speed of sound \( c_s \).

\[
f \sim h^1 l^{-2} w^0 c_s^1
\]

Thus,

\[
f \propto l^{-2}
\]

or

\[
l \propto f^{-1/2}.
\]
We can test this prediction experimentally.
We can test this prediction experimentally.
The data closely match the scaling prediction.
Insight helps us develop new understanding

1. Need for insight

2. Examples
   a. factorials
   b. coincidences
   c. drag
   d. musical notes
Without insight we cannot develop new understanding.
Insight helps us develop new understanding

invertible methods of analysis

system / theory

behavior / data

design / invention
Insight helps us develop new understanding

The goal [of teaching] should be, not to implant in the students’ mind every fact that the teacher knows now; but rather to implant a way of thinking that enables the student, in the future, to learn in one year what the teacher learned in two years.

Only in that way can we continue to advance from one generation to the next.

Never make a calculation until you already know the answer

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Produced with free software:
Emacs, LuaTEX, ConTEXt, Asymptote, and MetaPost

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