Color Glass Condensate Density Matrix: Lindblad evolution, entanglement entropy, Wigner functional and all that

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Motivation.

LHC - high energy hadronic collisions interesting correlations are observed in particle production (i.e. "ridge" correlations) in p-p, p-Pb and Pb-Pb collisions.

Do they arise due to strong final state interactions?

Do they arise due to nontrivial initial state structure?

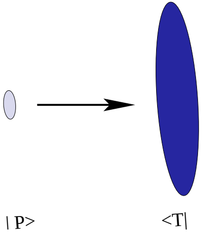
The question is not quite settled yet, partly because we do not have a robust understanding of the hadronic wave function.

In particular - there must be correlation between density fluctuations and transverse momentum fluctuations in the wave function.

Main motivation - initiate approach which can give us a handle on understanding this type of initial state correlations.

Color Glass Condensate - hadronic scattering at high energy.

At high energy hadronic scattering is relatively simple.
CGC "paradigm":
A "projectile" hadron $|P\rangle$ and a "target" hadron $|T\rangle$ at high energy scatter eikonally



$|P\rangle$ - a distribution of (possibly large) color charge density $j^a(x)$.
$|T\rangle$- an ensemble of (possibly strong) color fields $\alpha^a(x)$.

Not all is relevant

The scattering matrix is eikonal:

$$\hat{S} = \exp\{i \int d^2x_\perp j^a_P(x_\perp)\alpha^a_T(x_\perp)\}$$

In the extreme high energy limit the color field looks like a shock wave - has infinitesimal with in the longitudinal direction.

Longitudinal structure becomes irrelevant.

Projectile color charge density and target color charge fields effectively depend only on transverse coordinates.

$$j^a_P(x_\perp) = \int_{x^-} j^a_P(x_\perp, x^-), \qquad \alpha^a_T(x_\perp) = \int dx^+ \alpha_T(x_\perp, x^+)$$

CGC as EFT

At high energy most degrees of freedom become irrelevant.

The scattering matrix knows only about the total (transverse) color charge density in the projectile
In the CGC setup one "integrates out" all gluonic DoF save total color charge density $j^a(x_\perp)$.

Any observable that depends only on $j^a(x_\perp)$ is calculated as

$$\langle O(j)\rangle = \int DjW[j]O(j)$$

Strictly $j^a(x_\perp)$ are not mutually commuting, so some modifications are necessary. But when $j^a \sim 1/\alpha_s$, this is irrelevant.

Energy dependence.

The "probability density" $W[j]$ depends on energy.

Why?

Boost the projectile - new gluons are emitted into the wave function. These gluons contribute additional color charge density - so the distribution changes with energy.

The leading order in QCD coupling (but to all orders in the charge density!) the energy evolution with energy has been known for a while.

Gribov-Levin-Ryskin (GLR 1983); Balitsky hierarchy (1996); Kovchegov equation (1999); JIMWLK Hamiltonian(1997-2001)

JIMWLK evolution.

The "probability density" $W[j]$ depends on energy (rapidity) via the JIMWLK equation:

$$\frac{d}{d\eta} W[j] = H^{JIMWLK} W[j]$$

Define a unitary matrix $S(x) = P \exp\{i \int_{x^-} T^a A^{+a}(x^-, x)\}$ (single gluon scattering amplitude)
It is related to the color charge by classical Yang-Mills equations:

$$\frac{i}{g} \partial_i [S^\dagger \partial_i S] = j$$

Then

$$H^{JIMWLK} = \frac{\alpha_s}{2\pi^2} \int d^2z \, Q_i^a(z) \, Q_i^a(z)$$

the Hermitian amplitude $Q_i^a(z)$ is the "single inclusive gluon emission amplitude"

$$Q_i^a(z) = \int d^2x \frac{(x-z)_i}{(x-z)^2} [S^{ab}(z) - S^{ab}(x)] J_R^b(x) \, .$$

the generators of color rotation J_R

$$J_R^a(x) = -\mathrm{tr} \left\{ S(x) T^a \frac{\delta}{\delta S^\dagger(x)} \right\}$$

But is that all there is?

Not all interesting operators depend only on j!

Simple example: scattering matrix on a dense target.

$$\hat{S}^\dagger \hat{j}(x_\perp)\hat{S} = V(x_\perp)\hat{j}(x_\perp)\,,$$

For dense target V is a unitary matrix which is far away from unity.
Clearly S does not commute with j!

Another type of observables which are not diagonal in the j basis - see later.
To accommodate such observables one needs to consider a nondiagonal density matrix

$$\hat{\rho}: \qquad \langle j|\hat{\rho}|j'\rangle \neq \delta(j-j')$$

The averages are now calculated as

$$\langle O\rangle = \mathrm{Tr}[O\hat{\rho}] = \int DjDj'\langle j|O|j'\rangle\langle j'|\hat{\rho}|j\rangle$$

How does $\hat\rho$ evolve?

No proper derivation - so treat it as "intelligent guess" (but work in progress with Ming Li)

$$\frac{d}{dy}\rho[j,j'] = \int \frac{d^2z_\perp}{2\pi}\left[Q_i^a[z_\perp,j] + Q_i^a[z_\perp,j']\right]^2 \rho[j,j']\,,$$

This can be rewritten in the operator form

$$\frac{d}{dy}\hat\rho = \int \frac{d^2z_\perp}{2\pi}\left[\hat Q_i^a[z_\perp],\left[\hat Q_i^a[z_\perp],\hat\rho\right]\right]\,,$$

Where is JIMWLK?

Interesting property: for any function $F[j,j']$

$$\int \frac{d^2z_\perp}{2\pi}\left[Q_i^a[z_\perp,j] + Q_i^a[z_\perp,j']\right]^2 F[j,j']|_{j=j'} = \int \frac{d^2z_\perp}{2\pi}\left[Q_i^a[z_\perp,j]\right]^2 F[j,j]$$

Diagonal matrix elements of $\hat\rho$: $W[j] = \langle j|\hat\rho|j\rangle$ evolve autonomously according to JIMWLK!

Kossakowski-Lindblad evolution

(Almost) General form consistent with Markovian evolution (no memory)

$$\frac{d}{dt}\hat{\rho} = i[H,\hat{\rho}] + [Q_i,[Q_i,\hat{\rho}]]$$

Preserves:

1. Positivity of eigenvalues of $\hat{\rho}$,
2. Trace of $\hat{\rho}$.

Thus preserves properties of $\hat{\rho}$ as a density matrix.

Q_i - Lindblad or "jump operator" - related with the jump of the "bath" into a different quantum state. In our case indeed Q - the soft gluon production amplitude, so the soft gluon "bath" changes state.

Process is Markovian if the "integrated" degrees of freedom are very fast - then the memory of their state is lost within one time step of the evolution.

Note: our evolution is not in time, but it still turns out to be Lindblad.

Exploring properties of $\hat{\rho}$.

What do we expect from $\hat{\rho}$ at high energy?

Basically we expect it to become more and more diagonal, since the rapidity evolution has to lead to more and more decoherence.

Does the (ever decreasing) nondiagonality matter?

Will study this in a simple Gaussian approximation.

Gaussian density matrix.

Let us take a simple ansatz:

$$\rho[\alpha,\alpha'] = \mathcal{N}\exp\left\{\int d^2x_\perp d^2y_\perp \mathrm{tr_c}\right.$$

$$\left[-(\alpha(x_\perp)+\alpha'(x_\perp))\mu_y^{-2}(x_\perp,y_\perp)(\alpha(y_\perp)+\alpha'(y_\perp))\right.$$

$$\left.\left.-(\alpha(x_\perp)-\alpha'(x_\perp))\lambda_y^{-2}(x_\perp,y_\perp)(\alpha(y_\perp)-\alpha'(y_\perp))\right]\right\}.$$

With $\partial^2\alpha^a(x) = j^a(x)$

Large λ^{-2} - strongly mixed $\hat{\rho}$

Can include one more term, but it does not play any role in the following - so drop it.

Will first study "dilute" or BFKL regime.

Gaussian evolution in dilute regime 1

Approximate $\hat{\rho}$ by a Gaussian at any stage of evolution.
Take the set of operators:

$$\hat{O}_i = \left\{ \alpha^a(x_{1\perp})\alpha^a(x_{2\perp}) \quad , \frac{\delta}{\delta\alpha^a(x_{1\perp})}\frac{\delta}{\delta\alpha^a(x_{2\perp})} \right\} .$$

For each O_i we calculate the expectation value

$$\langle \hat{O}_i \rangle_{(\mu_y,\lambda_y)} \equiv \mathrm{Tr}[\hat{O}_i \hat{\rho}]$$

and then take its derivative with respect to rapidity

$$\frac{d}{dy}\langle \hat{O}_i \rangle = \frac{\partial \langle \hat{O}_i \rangle}{\partial \mu_y}\frac{d\mu_y}{dy} + \frac{\partial \langle \hat{O}_i \rangle}{\partial \lambda_y}\frac{d\lambda_y}{dy} .$$

Then calculate:

$$Tr[\hat{O}_i \frac{d}{dy}\hat{\rho}] = \int \frac{d^2 z_\perp}{2\pi} \mathrm{Tr}\left\{ \hat{\rho}[\hat{Q}^a_i(z_\perp), [\hat{Q}^a_i(z_\perp), \hat{O}_i]] \right\}$$

And equate:

$$\frac{\partial \langle \hat{O}_i \rangle}{\partial \mu_y}\frac{d\mu_y}{dy} + \frac{\partial \langle \hat{O}_i \rangle}{\partial \lambda_y}\frac{d\lambda_y}{dy} = \int D\alpha D\alpha' \left\{ O_i(\alpha',\alpha) \int \frac{d^2 z_\perp}{2\pi} \left[Q^a_k(z_\perp,\alpha) + Q^a_k(z_\perp,\alpha') \right]^2 \rho(\alpha,\alpha') \right\} .$$

Gaussian evolution in dilute regime 2

In dilute regime:

$$Q_i^a[z_\perp, \alpha] \approx -\frac{g^2}{2\pi} \int d^2x_\perp \frac{(x_\perp - z_\perp)_i}{(x_\perp - z_\perp)^2} T_{ab}^d \left(\alpha^d(z_\perp) - \alpha^d(x_\perp)\right) \frac{\delta}{\delta\alpha^b(x_\perp)},$$

When all said and done:

$$\frac{\partial}{\partial y}\bar{\mu}_y^2(x_{1\perp}, x_{2\perp}) = \frac{N_c}{2\pi}\left(\frac{g^2}{2\pi}\right)^2 \int d^2z_\perp \Bigg\{ - \frac{(x_{1\perp} - x_{2\perp})^2}{(x_{1\perp} - z_\perp)^2(x_{2\perp} - z_\perp)^2}$$

$$\times \left[\bar{\mu}_y^2(x_{1\perp}, x_{2\perp}) + \bar{\mu}_y^2(z_\perp, z_\perp) - \bar{\mu}_y^2(z_\perp, x_{2\perp}) - \bar{\mu}_y^2(x_{1\perp}, z_\perp)\right] \Bigg\}.$$

$$\frac{\partial}{\partial y}\lambda_y^{-2}(x_{1\perp}, x_{2\perp}) = \frac{N_c}{2\pi}\left(\frac{g^2}{2\pi}\right)^2 \Bigg\{ -\delta^{(2)}(x_{1\perp} - x_{2\perp}) \int d^2x_\perp d^2y_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - x_{1\perp})^2(y - x_{1\perp})^2}\lambda_y^{-2}(x_\perp, y_\perp)$$

$$+ \int d^2z_\perp \Bigg[\frac{(z_\perp - x_{1\perp})^2}{(z_\perp - x_{2\perp})^2(x_{1\perp} - x_{2\perp})^2}\lambda_y^{-2}(z_\perp, x_{1\perp}) + \frac{(z_\perp - x_{2\perp})^2}{(z_\perp - x_{1\perp})^2(x_{2\perp} - x_{1\perp})^2}\lambda_y^{-2}(z_\perp, x_{2\perp})$$

$$- \frac{(x_{2\perp} - x_{1\perp})^2}{(x_{2\perp} - z_\perp)^2(x_{1\perp} - z_\perp)^2}\lambda_y^{-2}(x_{1\perp}, x_{2\perp})\Bigg]\Bigg\}$$

Both eqs. for μ^2 and λ^{-2} are forms of BFKL equation!

Gaussian evolution in dilute regime 3

Both grow with leading BFKL exponent:

$$\gamma = \frac{4\alpha_s N_c}{\pi} \ln 2 \, .$$

$\mu^2 \sim e^{\gamma\eta}$ grows: the charge density in the wave function grows.

$\lambda^{-2} \sim e^{\gamma\eta}$ grows: the density matrix becomes more and more mixed!

$$\rho[\alpha, \alpha'] \to \delta(\alpha - \alpha')$$

Gaussian evolution in dilute regime: entropy

Entanglement entropy - generally calculable for a Gaussian density matrix:

$$S_e = \frac{1}{2}\mathrm{tr}\left[\ln\left(\frac{\mu_y^2\lambda_y^{-2}-1}{4}\right)+\sqrt{\mu_y^2\lambda_y^{-2}}\,\mathrm{acosh}\left(\frac{\mu_y^2\lambda_y^{-2}+1}{\mu_y^2\lambda_y^{-2}-1}\right)\right]$$

Assuming $\mu^2\lambda^{-2}\gg 1$:

$$\frac{dS_e}{dy}\approx\frac{1}{2}\mathrm{tr}\left[\mu_y^{-2}\frac{\partial\mu_y^2}{\partial y}+\lambda_y^2\frac{\partial\lambda_y^{-2}}{\partial y}\right]$$

In the BFKL regime for large enough energy

$$\frac{dS_e}{dy}\approx\gamma$$

Half of the entropy growth is contributed by the growth of the number of particles (μ^2) and half by continued decoherence as the energy grows (λ^{-2}).

Close to saturation: the Levin-Tuchin regime 1

Close to saturation we consider the evolution of the "dipole" and the "Pomeron"

$$d(x_{1\perp}, x_{2\perp}) \equiv \frac{1}{N_c} \mathrm{tr}_c [S^\dagger(x_{1\perp}) S(x_{2\perp})]$$

$$P^\dagger(x_{1\perp}, x_{2\perp}) = J_R^a(x_{1\perp}) J_R^a(x_{2\perp}).$$

Quite generally we can calculate the evolution of these two operators without reference to the exact form of the density matrix.

$$\frac{d}{dy}\hat{O} = \int \frac{d^2 z_\perp}{2\pi} [\hat{Q}_i^a(z_\perp), [\hat{Q}_i^a(z_\perp), \hat{O}]$$

Close to saturation: the Levin-Tuchin regime 2

And we obtain

$$\frac{d}{dy} d(x_{1\perp}, x_{2\perp}) = \frac{\alpha_s N_c}{\pi}$$

$$\int \frac{d^2 z_\perp}{2\pi} \frac{(x_{1\perp} - z_\perp) \cdot (x_{2\perp} - z_\perp)}{(x_{1\perp} - z_\perp)^2 (x_{2\perp} - z_\perp)^2} \left[d(x_{1\perp}, z_\perp) d(z_\perp, x_{2\perp}) - d(x_{1\perp}, x_{2\perp}) \right]$$

No surprises - the Balitsky - Kovchegov equation.

But also

$$\frac{d}{dy} P^\dagger(x_{1\perp}, x_{2\perp}) = -\frac{g^2 N_c}{(2\pi)^3} \Bigg[\delta(x_{1\perp} - x_{2\perp})$$

$$\int d^2 x_\perp d^2 y_\perp \frac{(x_\perp - x_{1\perp}) \cdot (y - x_{2\perp})}{(x_\perp - x_{1\perp})^2 (y - x_{2\perp})^2} P^\dagger(x_\perp, y_\perp)$$

$$+ \int d^2 x_\perp \left[\frac{(x_\perp - x_{1\perp}) \cdot (x_{2\perp} - x_{1\perp})}{(x_\perp - x_{1\perp})^2 (x_{2\perp} - x_{1\perp})^2} - \frac{1}{(x_\perp - x_{1\perp})^2} \right] P^\dagger(x, x_{2\perp})$$

$$- \int d^2 z_\perp \frac{(x_{1\perp} - z_\perp) \cdot (x_{2\perp} - z_\perp)}{(x_{1\perp} - z_\perp)^2 (x_{2\perp} - z_\perp)^2} - \frac{1}{(x_{1\perp} - z_\perp)^2} \Bigg] P^\dagger(x_{1\perp}, x_{2\perp})$$

$$- \int d^2 x_\perp \frac{(x_{1\perp} - x_{2\perp}) \cdot (x_\perp - x_{2\perp})}{(x_{1\perp} - x_{2\perp})^2 (x_\perp - x_{2\perp})^2} P^\dagger(x_{1\perp}, x_\perp) \Bigg] + (x_{1\perp} \leftrightarrow x_{2\perp})$$

This is again the BFKL equation (even in the dense regime!)

Close to saturation: the Levin-Tuchin regime 3

Is the BFKL in dense regime surprising?

May seem so, but not really. $P^{\dagger}$ has the meaning of charge density correlator in the target.

JIMLWK regime: dense target, but dilute projectile. So it should be BFKL.

But you will never get it if you use naive JIMWLK and not Kossakowski-Linblad evolution!

Close to saturation: the Levin-Tuchin regime 4

The ansatz for density matrix:

$$\rho(S,\bar{S}) = \mathcal{N}\exp\Big\{-\mathrm{tr_c}\int d^2x_\perp d^2y_\perp$$

$$\Big[\frac{\bar{\mu}_y^{-2}(x_\perp,y_\perp)}{4}[S^\dagger(x_\perp)+\bar{S}^\dagger(x_\perp)][S(y_\perp)+\bar{S}(y_\perp)]$$

$$+\bar{\lambda}_y^{-2}(x_\perp,y_\perp)[S^\dagger(x_\perp)-\bar{S}^\dagger(x_\perp)][S(y_\perp)-\bar{S}(y_\perp)]\Big]\Big\},$$

In dilute limit reduces to a Gaussian matrix.

Would like to follow the evolution of averages like in the Gaussian case. Don't know quite how to calculate averages. Will use the factorizable ansatz

$$\langle SSSS...\rangle = \Sigma\langle SS\rangle_{color\ \ singlet}\langle SS\rangle_{color\ \ singlet}\cdots$$

Good for extracting terms leading in the projectile area.

Close to saturation: the Levin-Tuchin regime 5

The Levin-Tuchin scaling:

$$\bar{\mu}_y^2(x_{1\perp}, x_{2\perp}) = \exp\{-\xi \ln^2[(x_{1\perp} - x_{2\perp})^2 Q_s^2]\}$$

And we find:

$$\bar{\lambda}_y^{-2}(x_{1\perp}, x_{2\perp})\bar{\mu}_y^2(x_{1\perp}, x_{2\perp}) \approx \bar{\lambda}_0 \exp(\gamma y)$$

And for the entropy we find

$$\frac{dS_e}{dy} \approx \frac{1}{2}\gamma$$

Makes sense: the density almost does not grow anymore, but the decoherence still increases. All the leading contribution to entropy is from the growth of decoherence.

Wigner functional

Our real motivation: are there correlations between density and current in the hadronic state?

Should be there - and especially important for rare configurations.

In QM - Wigner function gives (approximately) joint probability distribution for momentum and coordinate.

$$\mathcal{W}(x,p) = \int dy e^{iyp} \left\langle x + \frac{y}{2} \middle| \hat{\rho} \middle| x - \frac{y}{2} \right\rangle$$

We define:

$$\mathcal{W}[j,\Phi] = \int Dj' \exp\left[i \int d^2x_\perp \Phi^a(x_\perp) j'^a(x_\perp) \right] \rho \left[j + \frac{j'}{2}, j - \frac{j'}{2} \right]$$

Current density.

Current density?

Start from the commutation relation:

$$[j^a(y_\perp), j_i^b(x_\perp)] = if^{abc} j_i^c(x_\perp)\delta^{(2)}(x_\perp - y_\perp) + if^{abc} j^c(x_\perp)\partial_i^y \delta^{(2)}(x_\perp - y_\perp)$$

In the effective theory?

$$\Phi = -i\frac{\delta}{\delta j}$$

And

$$j_i^a(x_\perp) = f^{abc} j^b(x_\perp)\partial_i \Phi^c(x_\perp)$$

Current density

Take Gaussian density matrix

$$\langle j_i^a(x_\perp) j_j^b(y_\perp)\rangle \equiv \int DjD\Phi\, \mathcal{W}_G[j,\Phi] j_i^a(x_\perp) j_j^b(y_\perp)$$

$$= N_c \delta_{a,b}\, \mu^2(x_\perp, y_\perp)\partial_i^{x_\perp}\partial_j^{y_\perp}\lambda^{-2}(x_\perp, y_\perp)$$

$$\langle j^a(x_\perp) j^a(y_\perp) j_i^b(z_\perp) j_i^b(w_\perp)\rangle - \langle j^a(x_\perp) j^a(y_\perp)\rangle\langle j_i^b(z_\perp) j_i^b(w_\perp)\rangle = \frac{N_c(N_c^2-1)}{4}$$

$$\times\left[\mu^2(x_\perp, z_\perp)\mu^2(y_\perp, w_\perp) + \mu^2(x_\perp, w_\perp)\mu^2(y_\perp, z_\perp)\right]\partial_i^{z_\perp}\partial_i^{w_\perp}\lambda^{-2}(z_\perp, w_\perp)$$

Interesting:

$\langle j_i j_j\rangle$ grows with twice the BFKL exponent, like diffractive cross section.

$\langle jjj_i j_j\rangle$ - connected part has the same energy dependence as disconnected.

Current correlations grow fast, charge-current correlations does not factorize even at high energy.

CONCLUSIONS

It's the beginning.

It will hopefully have continuation.

Need to think...