Proton Spin at Small x

Yuri Kovchegov

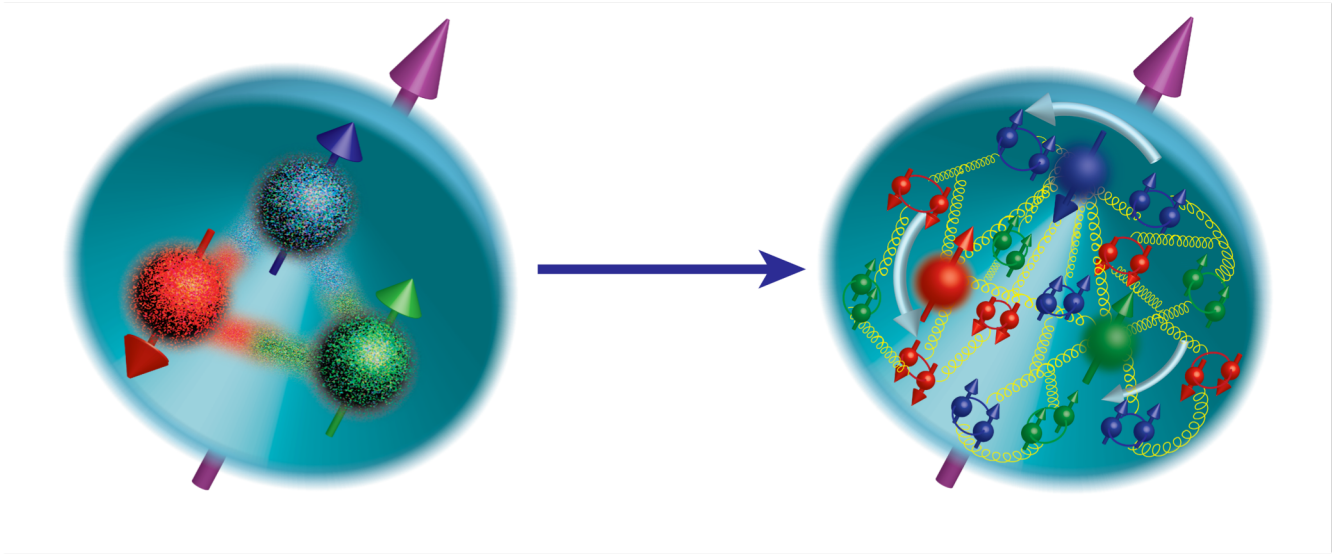
The Ohio State University

Outilne

- Goal: understanding the proton spin coming from helicities and OAM of small-x quarks and gluons.
- Brief introduction to small-x physics and evolution equations.
- Quark Helicity:
  - Quark helicity distribution at small x
  - Small-x evolution equations for quark helicity
  - Small-x asymptotics of quark helicity
- Gluon Helicity:
  - Gluon helicity distribution at small x
  - Small-x evolution equations for gluon helicity
  - Small-x asymptotics of quark helicity TMDs
- Quark and Gluon OAM at small x
- Preliminary phenomenology
- Conclusions

Goal: Proton Spin at Small x

Proton Spin



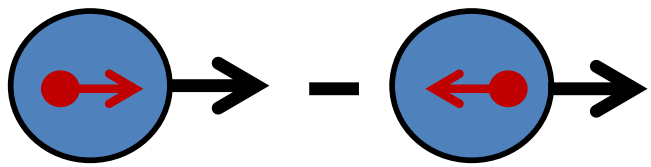
Our understanding of nucleon spin structure has evolved:

- In the 1980's the proton spin was thought of as a sum of constituent quark spins (left panel)

- Currently we believe that the proton spin is a sum of the spins of valence and sea quarks and of gluons, along with the orbital angular momenta of quarks and gluons (right panel)

Helicity Distributions

- To quantify the contributions of quarks and gluons to the proton spin on defines helicity distribution functions: number of quarks/gluons with spin parallel to the proton momentum minus the number of quarks/gluons with the spin opposite to the proton momentum:



- The helicity parton distributions are

$$\Delta f(x, Q^2) \equiv f^+(x, Q^2) - f^-(x, Q^2)$$

with the net quark helicity distribution

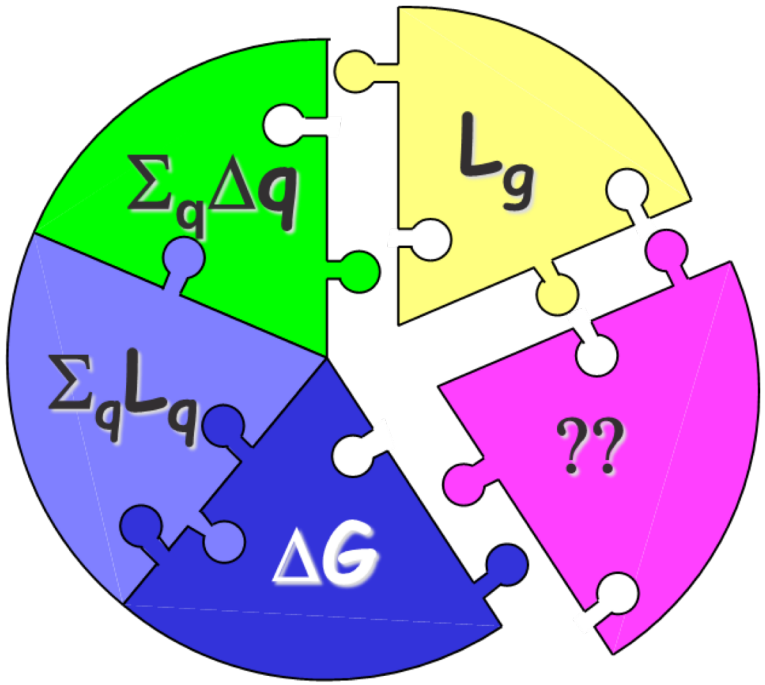
$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

and $\Delta G(x, Q^2)$ the gluon helicity distribution.

Proton Helicity Sum Rule

- Helicity sum rule (Jaffe-Manohar form):

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$



with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

- L_q and L_g are the quark and gluon orbital angular momenta

Proton Spin Puzzle

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2)$$

- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$\Delta\Sigma \approx 0.1 \div 0.2$$

- It appeared quarks do not carry all of the proton spin (which would have corresponded to $\Delta\Sigma = 1$).

- Missing spin can be

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

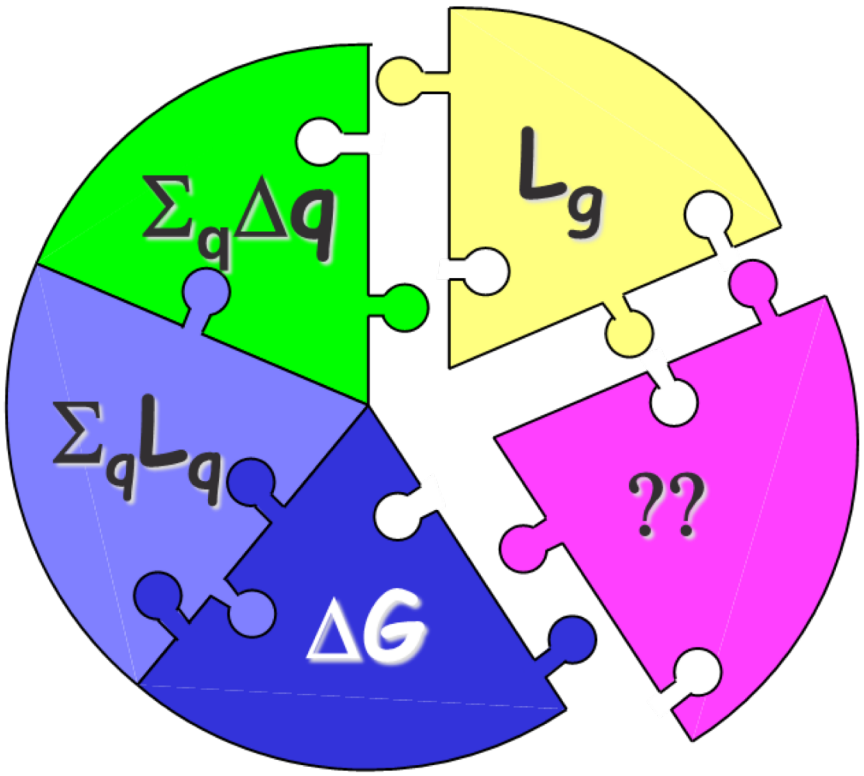
  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x:

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

Can't integrate down to zero, use x_{min} instead!

  - Or all of the above!

Current Knowledge of Proton Spin



- The proton spin carried by the quarks is estimated to be (for $0.001 < x < 1$)

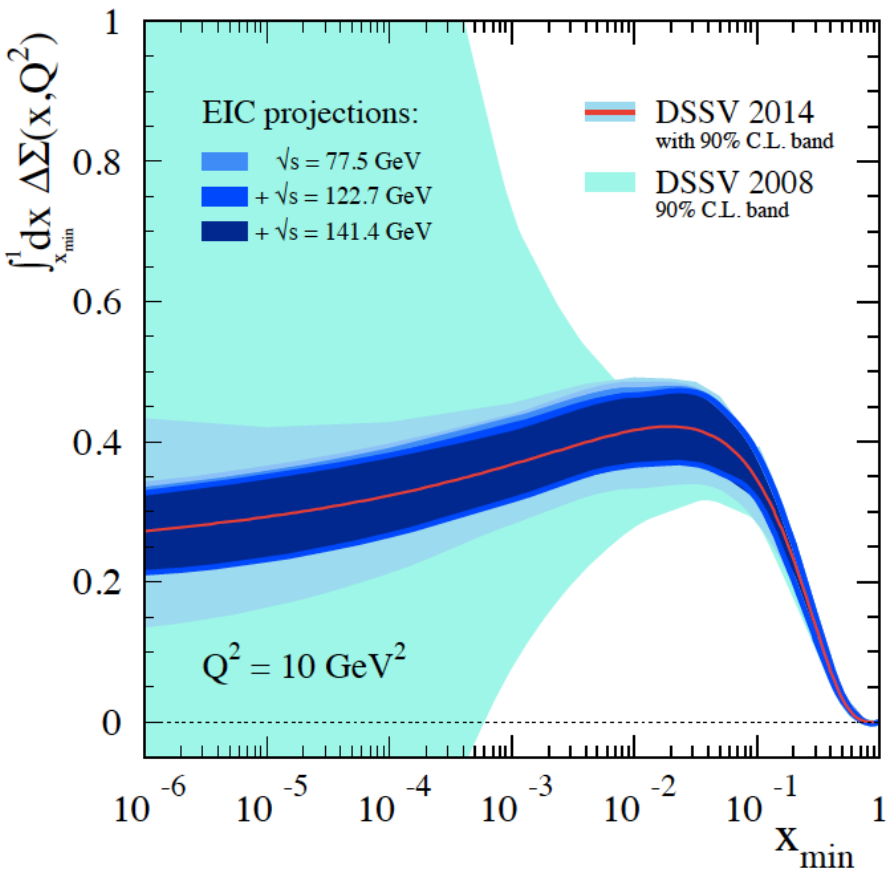
$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

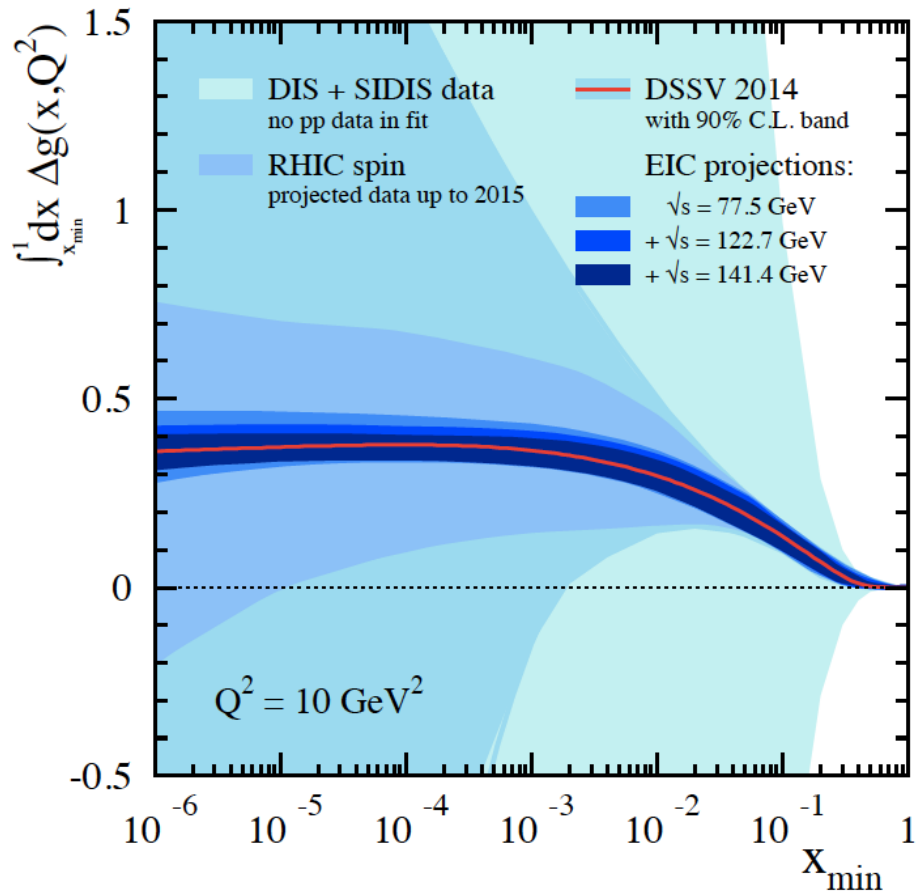
- The proton spin carried by the gluons is (for $0.05 < x < 1$)

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

- Unfortunately the uncertainties are large. Note also that the x-ranges are limited, with more helicity (positive or negative) possible at small x.

How much spin is at small x?





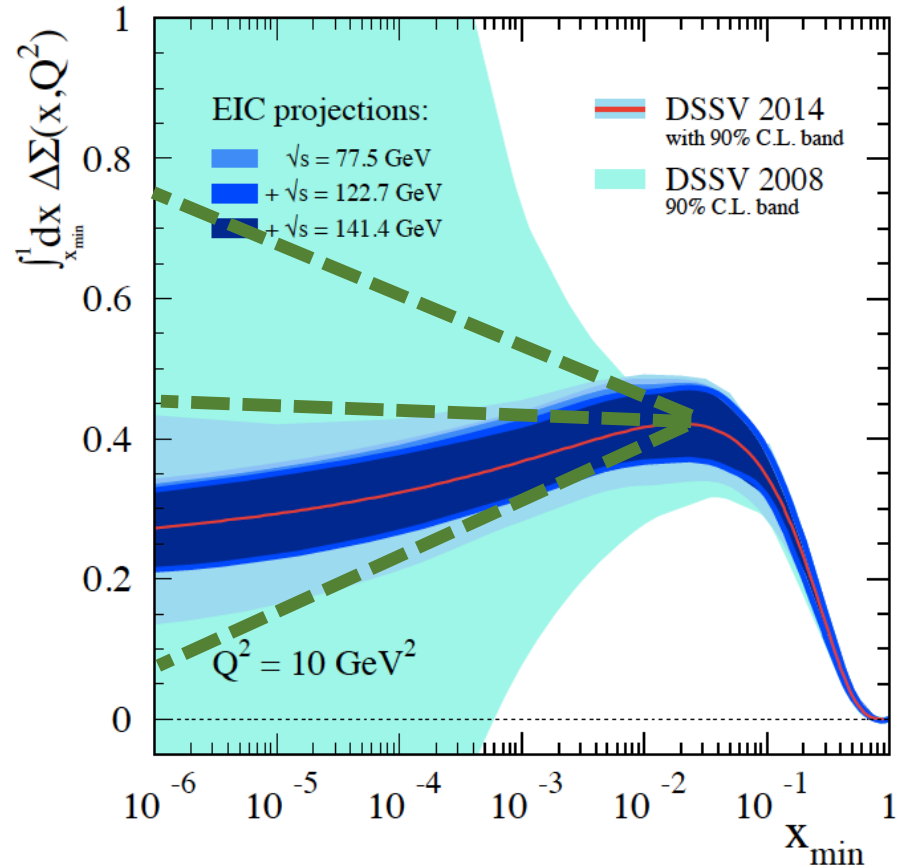
- E. Aschenaur et al, **arXiv:1509.06489** **[hep-ph]**, (DSSV = de Florian, Sassot, Stratmann, Vogelsang, helicity PDF parametrization)
- Uncertainties are very large at small x! (EIC may reduce them.)

Spin at small x

- The goal of this work is to provide theoretical understanding of helicity PDFs at very small x.

- Our work would provide guidance for future hPDFs parametrizations of the existing and new data (e.g., the data to be collected at EIC).

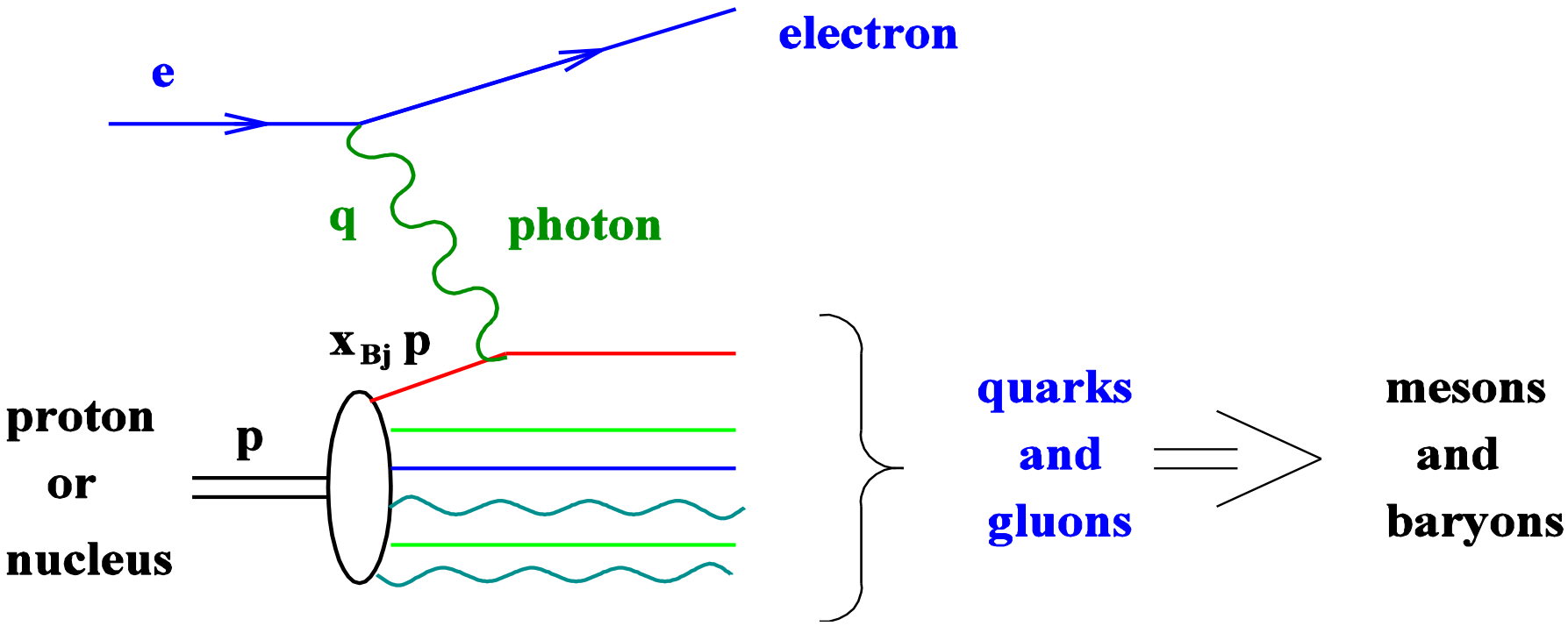
- Alternatively the data can be analyzed using our small-x evolution formalism.



Introduction: small-x physics

Quasi-classical approximation

Kinematics of DIS



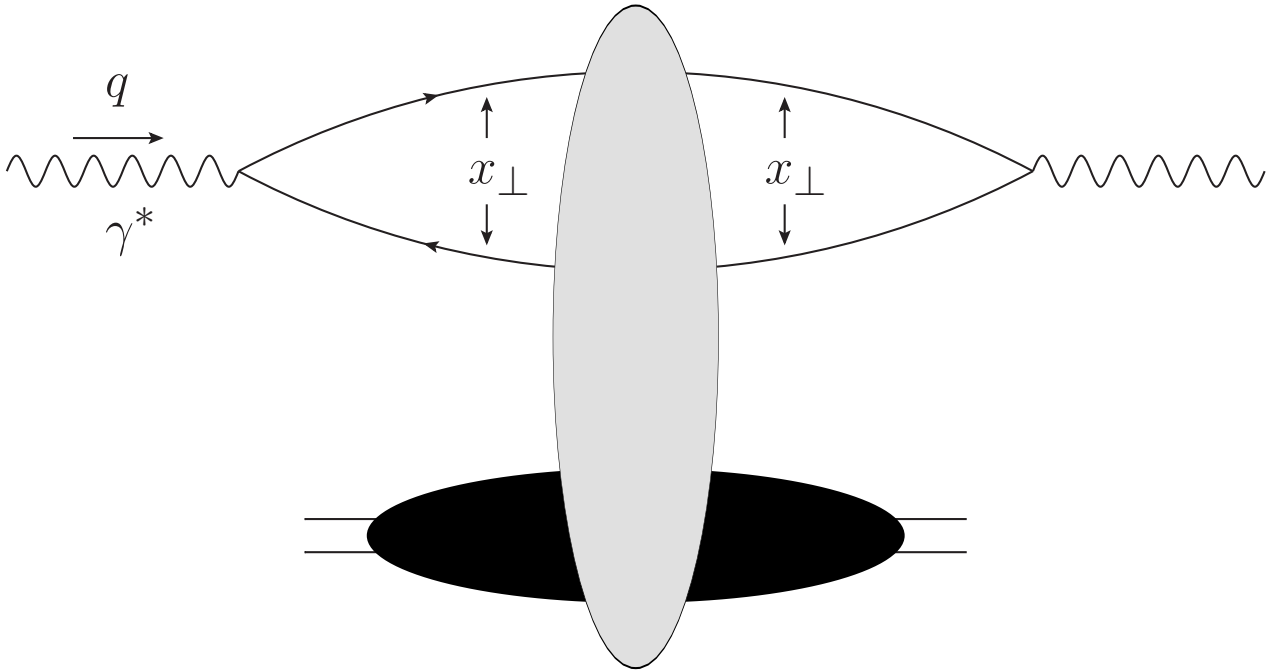
- Photon carries 4-momentum q_μ , its virtuality is

$$Q^2 = -q_\mu \, q^\mu$$

- Photon hits a quark in the proton carrying momentum $x_{Bj} p$ with p being the proton's momentum. Parameter x_{Bj} is the Bjorken x variable.

Dipole picture of DIS

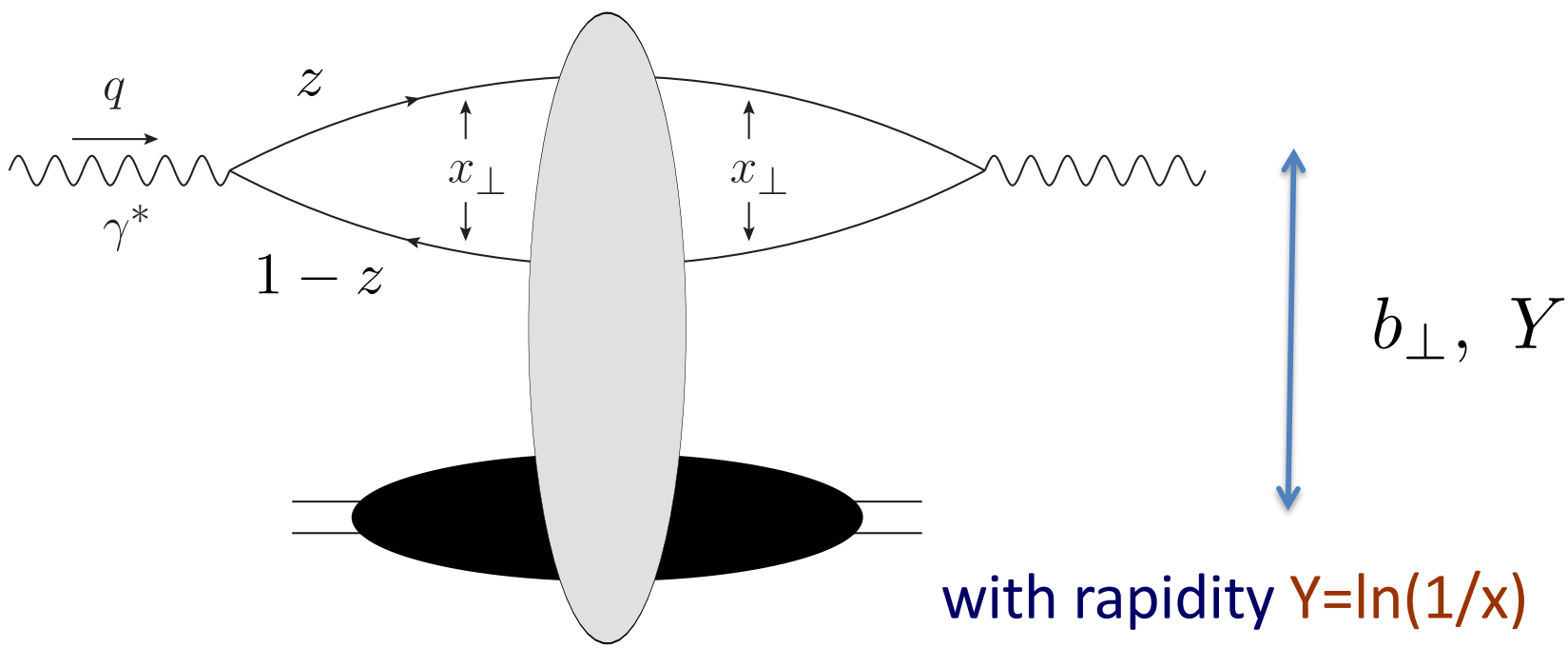
- In the dipole picture of DIS the virtual photon splits into a quark-antiquark pair, which then interacts with the target.
- The total DIS cross section and structure functions are calculated via:



Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

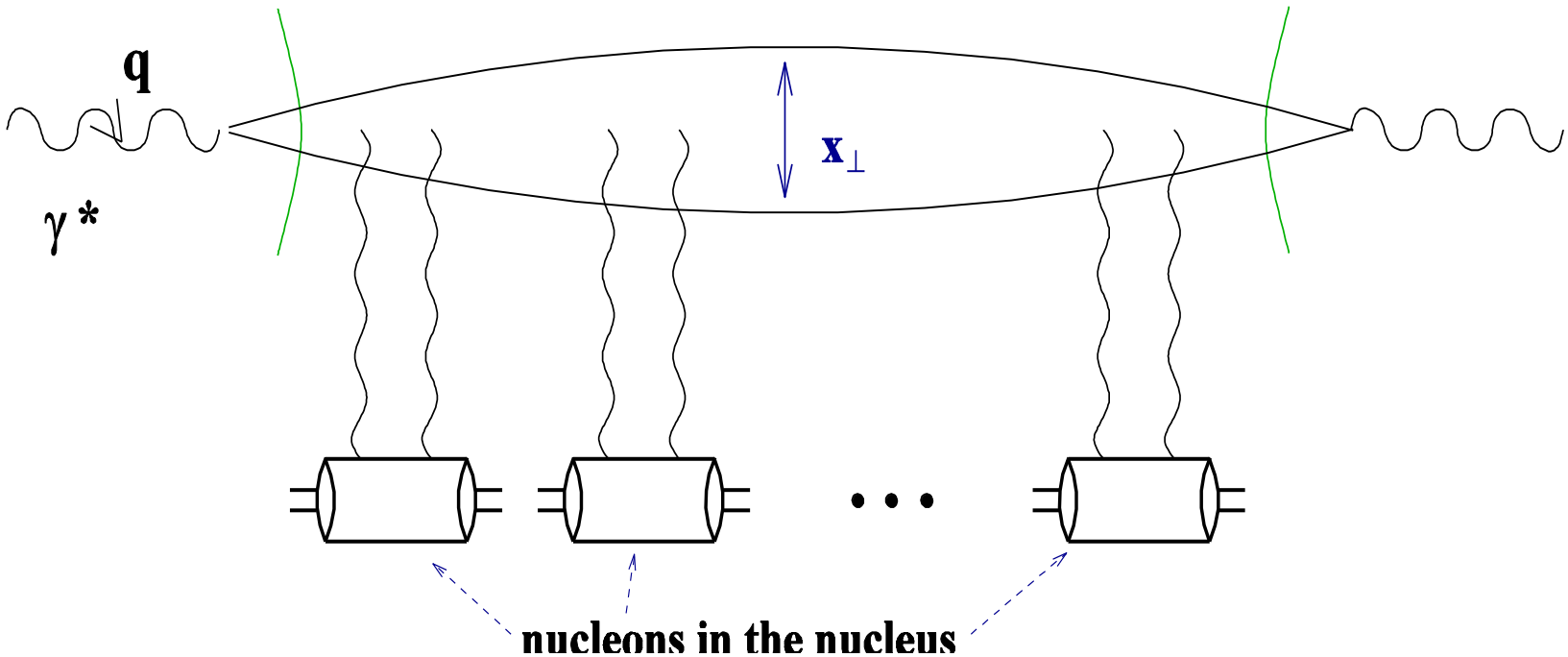
$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int_0^1 \frac{dz}{z\,(1-z)}\, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2\; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



with rapidity Y=ln(1/x)

DIS in the Classical Approximation

The DIS process in the rest frame of the target is shown below. It factorizes into



$$\sigma_{tot}^{\gamma * A}(x_{Bj}, Q^2) = |\Psi^{\gamma * \to q\bar{q}}|^2 \otimes N(x_\perp, Y = \ln 1/x_{Bj})$$

with rapidity Y=ln(1/x)

Dipole Amplitude

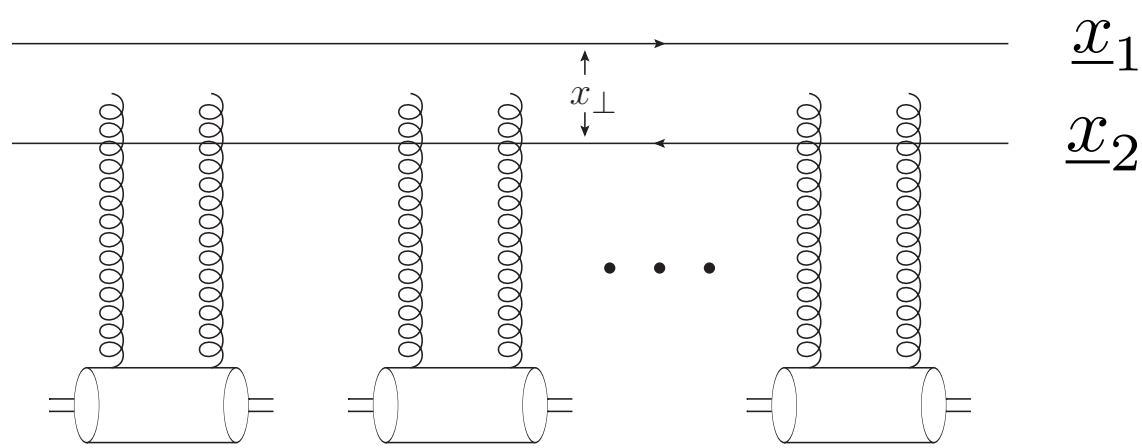
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \langle \mathrm{tr} \left[V(\underline{x}_1)\, V^\dagger(\underline{x}_2) \right] \rangle$$

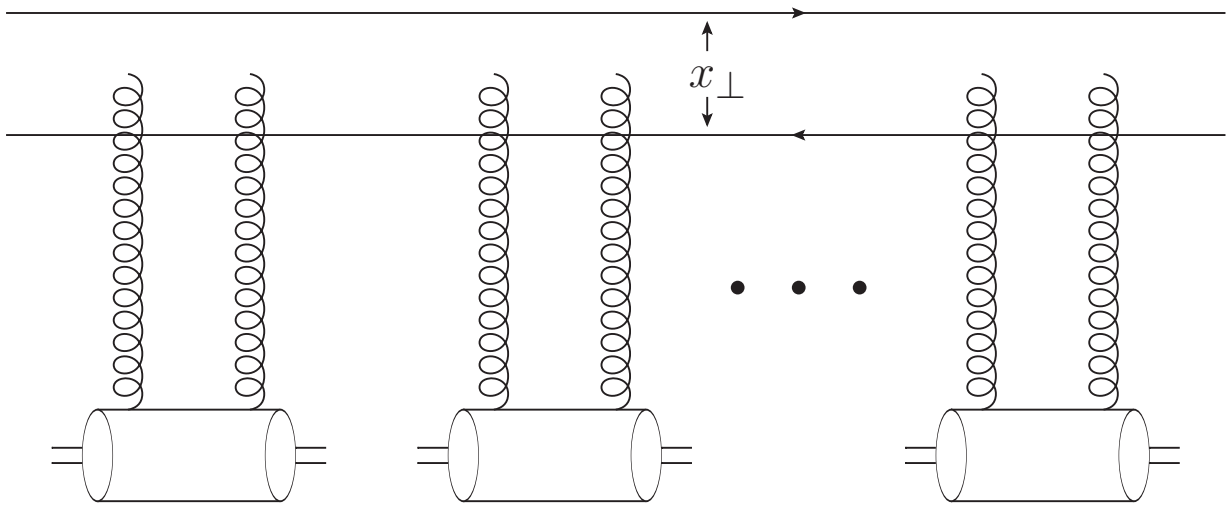
- Here we use the Wilson lines along the light-cone direction

$$V(\underline{x}) = \mathrm{P} \exp \left[i\, g \int_{-\infty}^{\infty} dx^+\, A^-(x^+, x^- = 0, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:

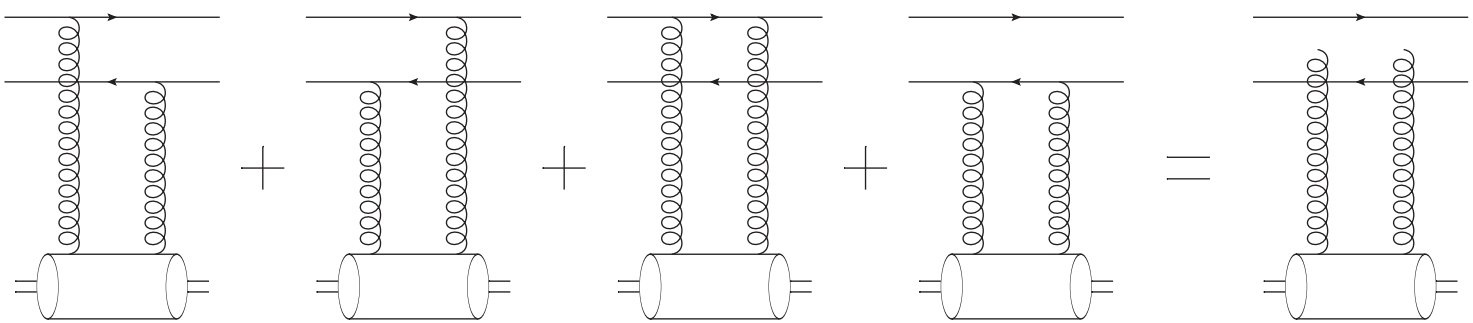


Quasi-classical dipole amplitude



A.H. Mueller, ‘90

Lowest-order interaction with each nucleon – two gluon exchange – the same resummation parameter as in the MV model: $\alpha_s^2 \, A^{1/3}$

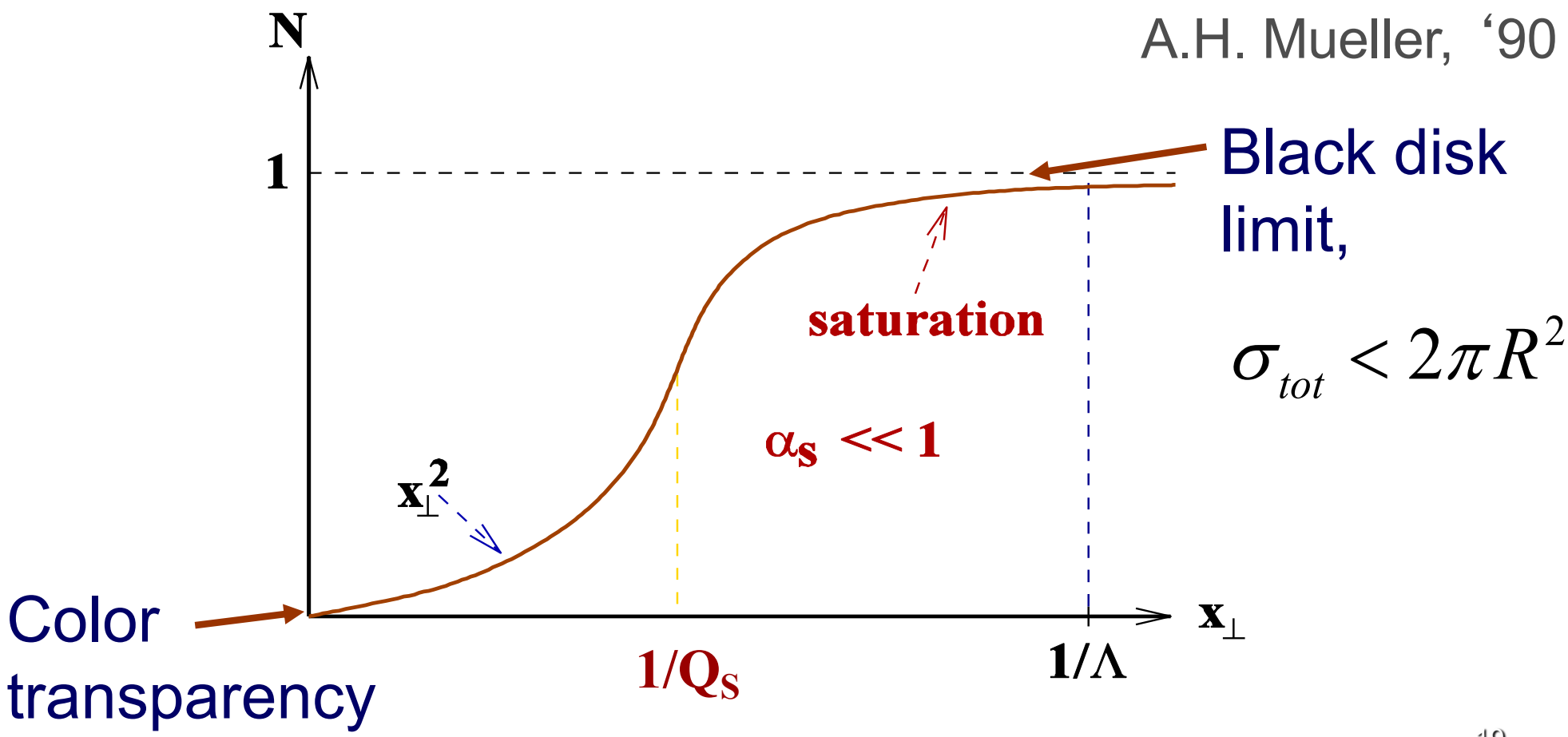


DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\, \ln \frac{1}{x_\perp\, \Lambda}\right]$$

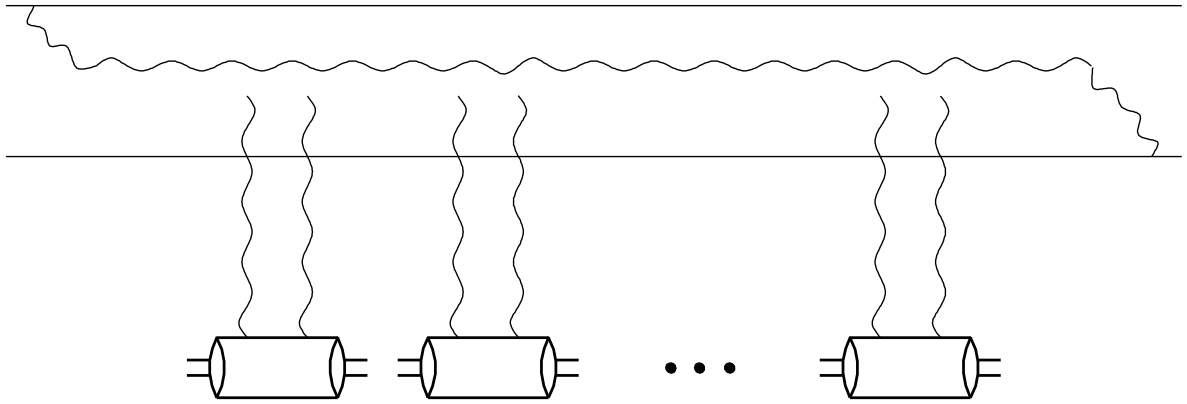
A.H. Mueller, '90



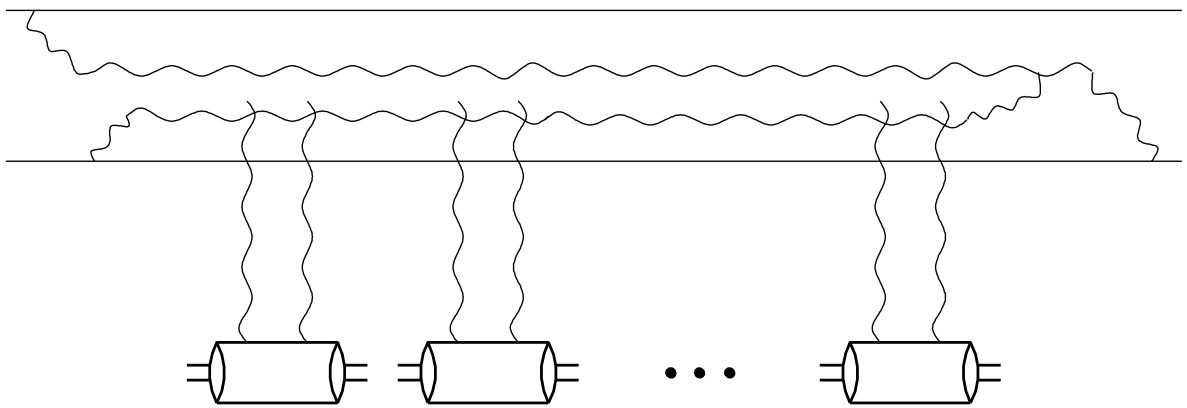
Small-x evolution equations

Quantum Evolution

- The energy (x) dependence comes in through nonlinear small-x BK/JIMWLK evolution, which resums the long-lived s-channel gluon corrections:



$$\alpha_s \ln \frac{1}{x} \sim \alpha_s Y \sim 1$$

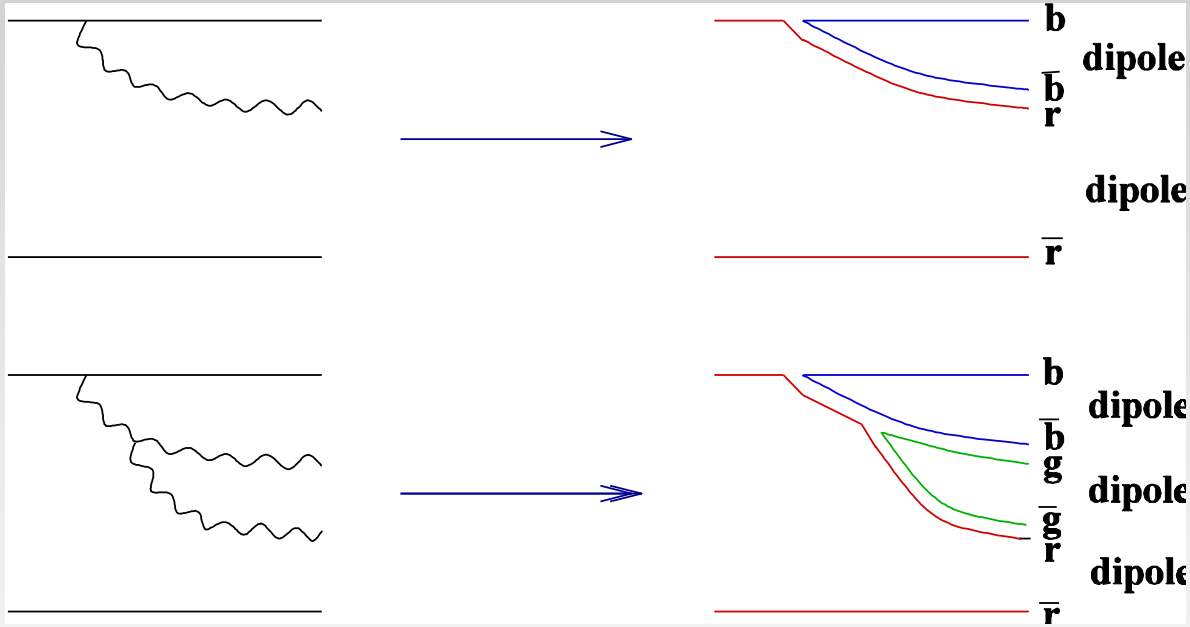


These extra gluons bring in powers of $\alpha_S \ln 1/x$, such that when $\alpha_S << 1$ and $\ln 1/x >> 1$ this parameter is $\alpha_S \ln 1/x \sim 1$ (leading logarithmic approximation, LLA).

Resumming Gluonic Cascade

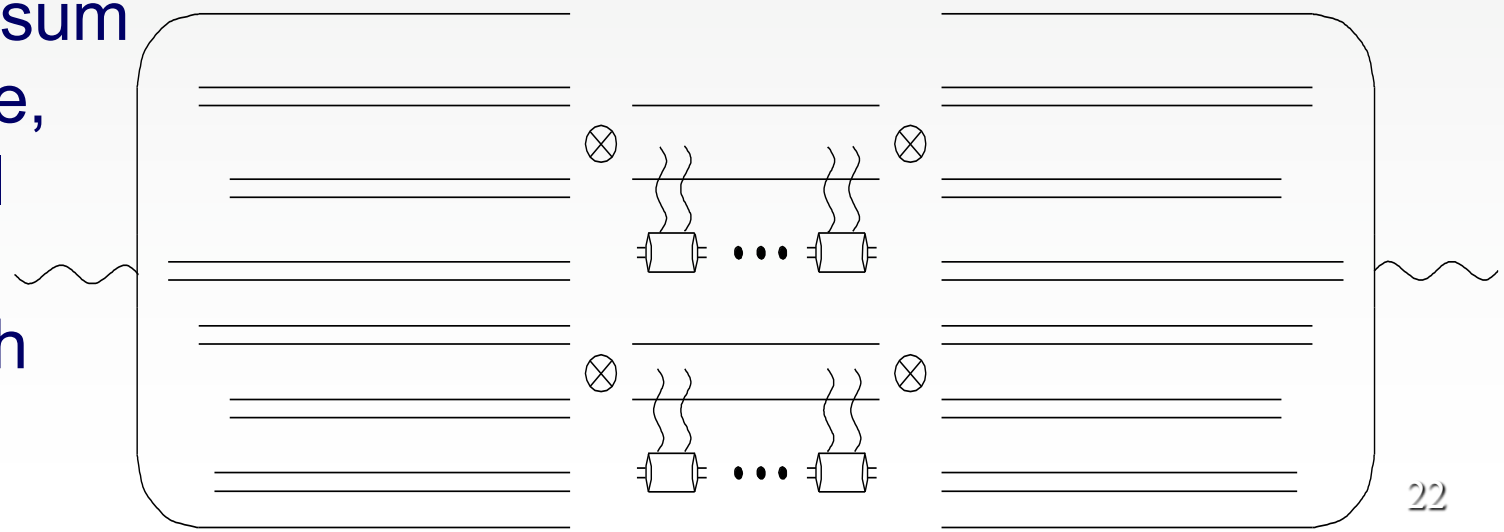
In the large-N_C limit of QCD the gluon corrections become color dipoles. Gluon cascade becomes a dipole cascade.

A. H. Mueller, ’93-’94

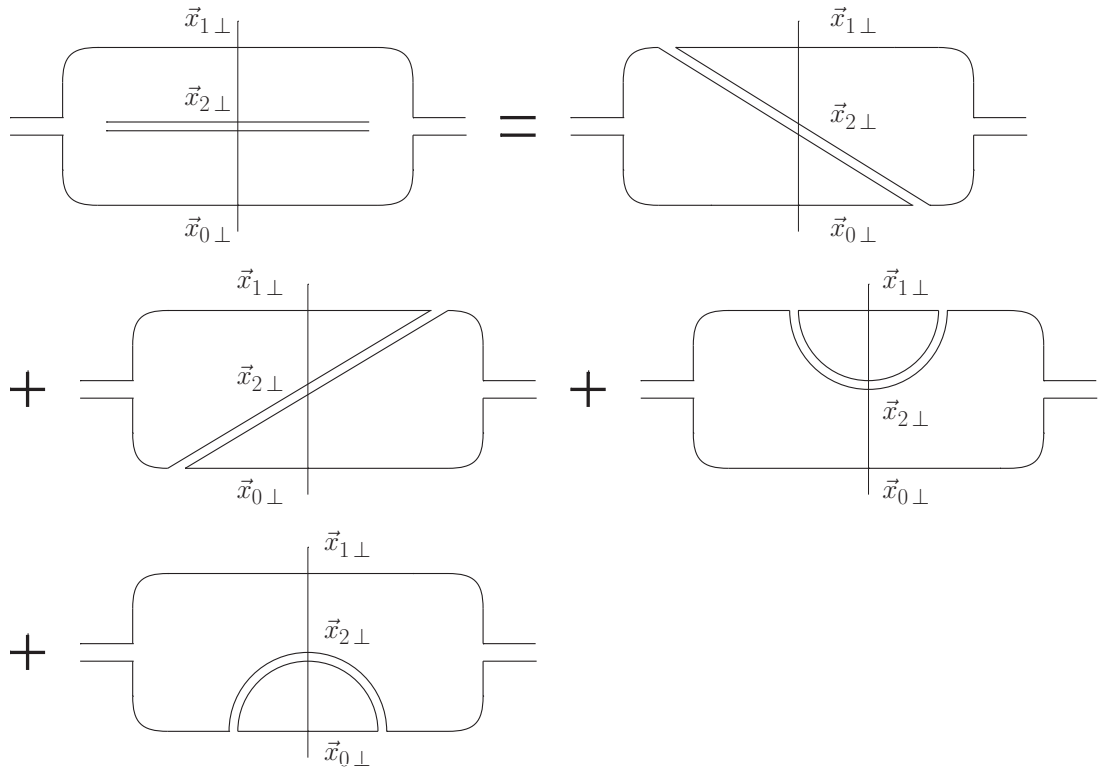


We need to resum dipole cascade, with each final state dipole interacting with the target.

Yu. K. ‘99



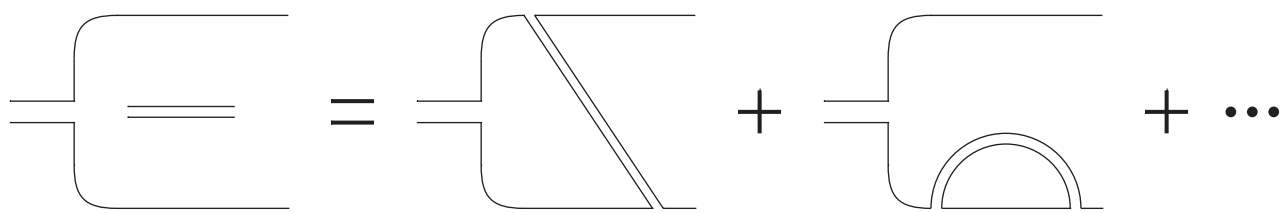
Notation (Large-N_C)



Real emissions in the amplitude squared

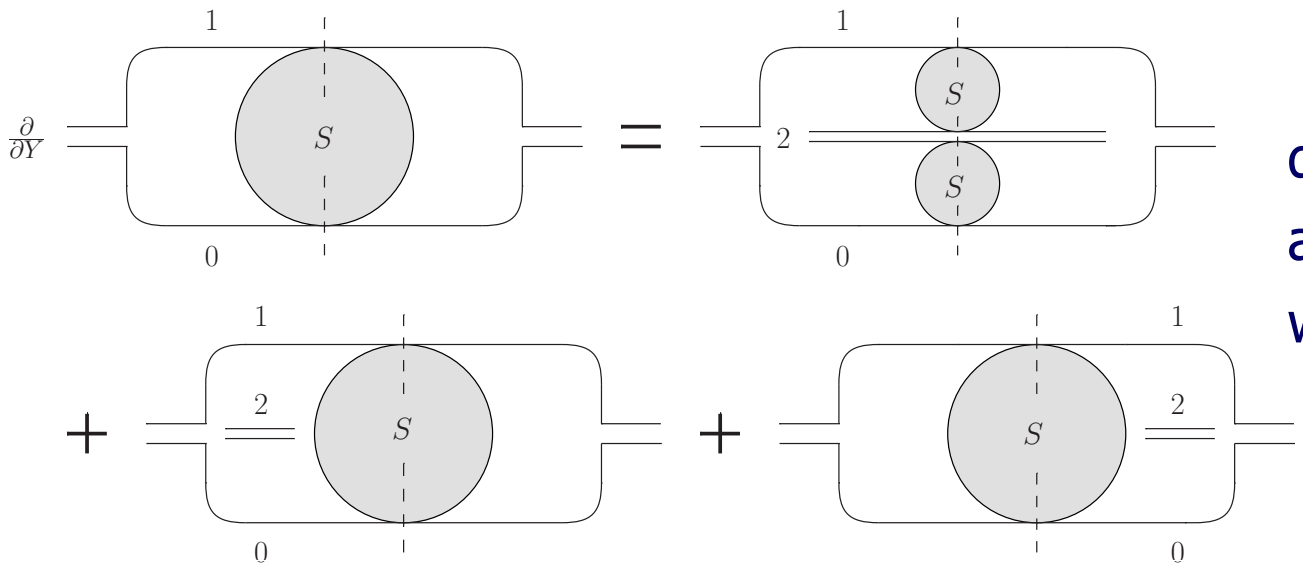
(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



Nonlinear Evolution

To sum up the gluon cascade at large-N_C we write the following equation for the dipole S-matrix:



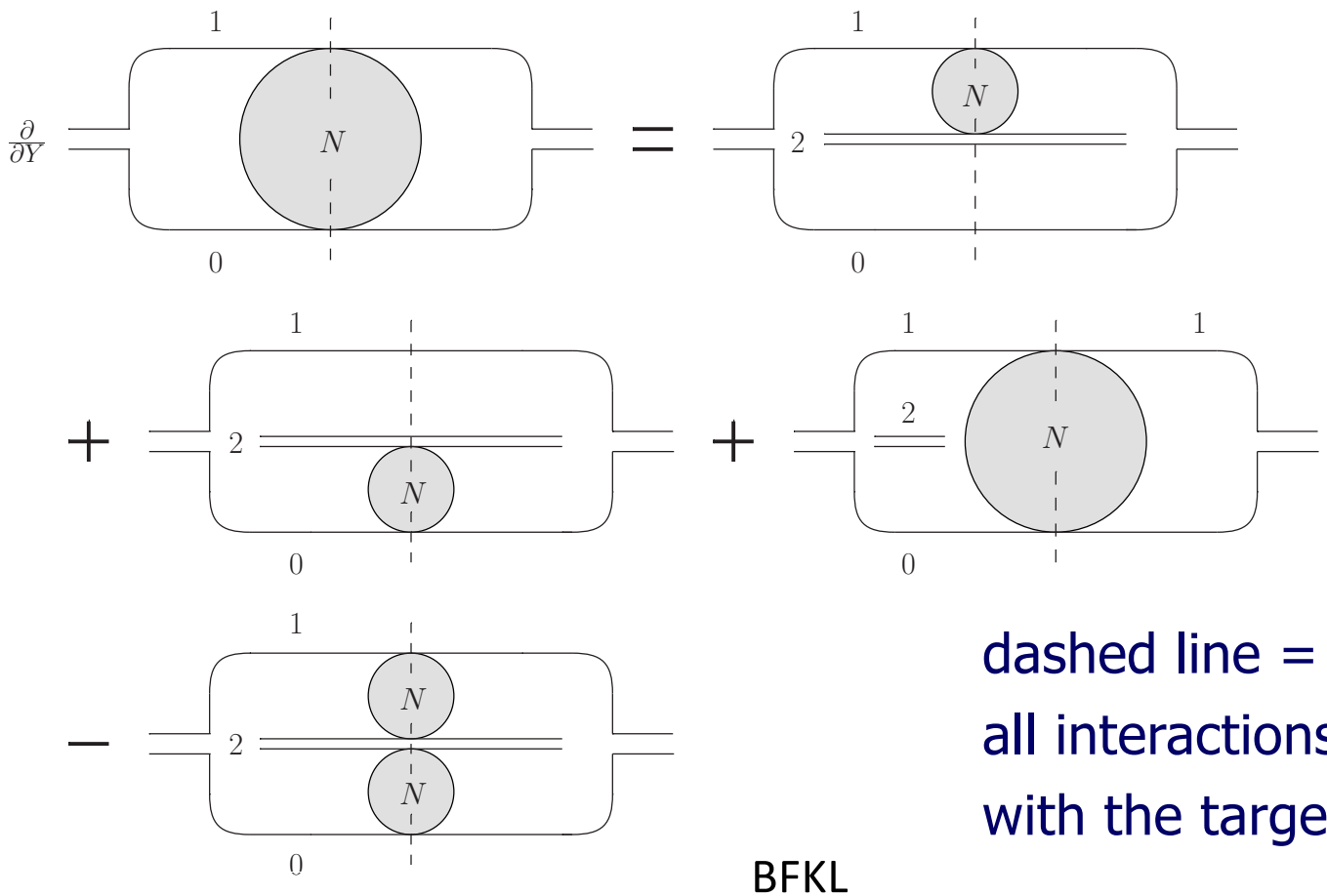
dashed line = all interactions with the target

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2\, x_{21}^2} \left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y) \right]$$

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.

Nonlinear evolution at large N_c

As N=1-S we write



dashed line = all interactions with the target

BFKL

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \, \frac{x_{01}^2}{x_{02}^2 \, x_{21}^2} \, [N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y) \, N_{\mathbf{x}_2,\mathbf{x}_1}(Y)]$$

Balitsky '96, Yu.K. '99

Beyond large-N_c: JIMWLK Equation

- JIMWLK evolution equation (1997-2002):

$$\partial_Y W_Y[\alpha] = \alpha_s \left\{ \frac{1}{2} \int d^2x_\perp \, d^2y_\perp \, \frac{\delta^2}{\delta\alpha^a(x^-, \vec{x}_\perp)\, \delta\alpha^b(y^-, \vec{y}_\perp)} \left[\eta^{ab}_{\vec{x}_\perp \vec{y}_\perp} W_Y[\alpha] \right] \right.$$
$$\left. - \int d^2x_\perp \, \frac{\delta}{\delta\alpha^a(x^-, \vec{x}_\perp)} \left[\nu^a_{\vec{x}_\perp} W_Y[\alpha] \right] \right\}$$

with

$$\eta^{ab}_{\vec{x}_{1\perp} \vec{x}_{0\perp}} = \frac{4}{g^2 \pi^2} \int d^2x_2 \, \frac{\vec{x}_{21} \cdot \vec{x}_{20}}{x_{21}^2 \, x_{20}^2} \left[\mathbf{1} - U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} + U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{0\perp}} \right]^{ab}$$

$$\nu^a_{\vec{x}_{1\perp}} = \frac{i}{g \, \pi^2} \int \frac{d^2x_2}{x_{21}^2} \, \mathrm{Tr} \left[T^a U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} \right]$$

- Here U is the adjoint Wilson line on a light cone,

$$U_{\vec{x}_\perp} = \mathrm{P} \exp \left\{ i \, g \int_{-\infty}^{\infty} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \vec{x}_\perp) \right\}$$

Resummation parameter

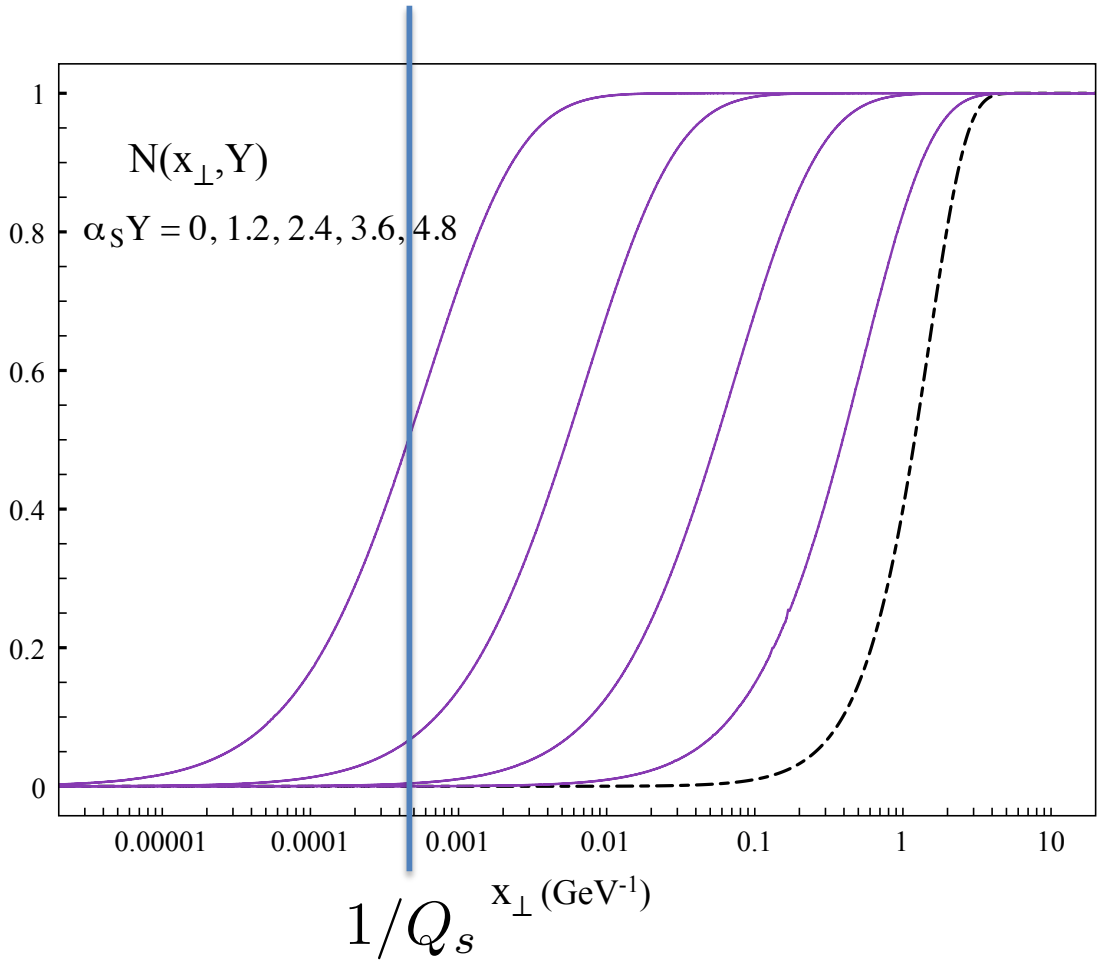
- BK equation resums powers of

$$\alpha_s \, N_c \, \ln \frac{1}{x}$$

- The Galuber-Mueller/McLerran-Venugopalan initial conditions resum powers of

$$\alpha_s^2 \, A^{1/3}$$

Solution of BK equation

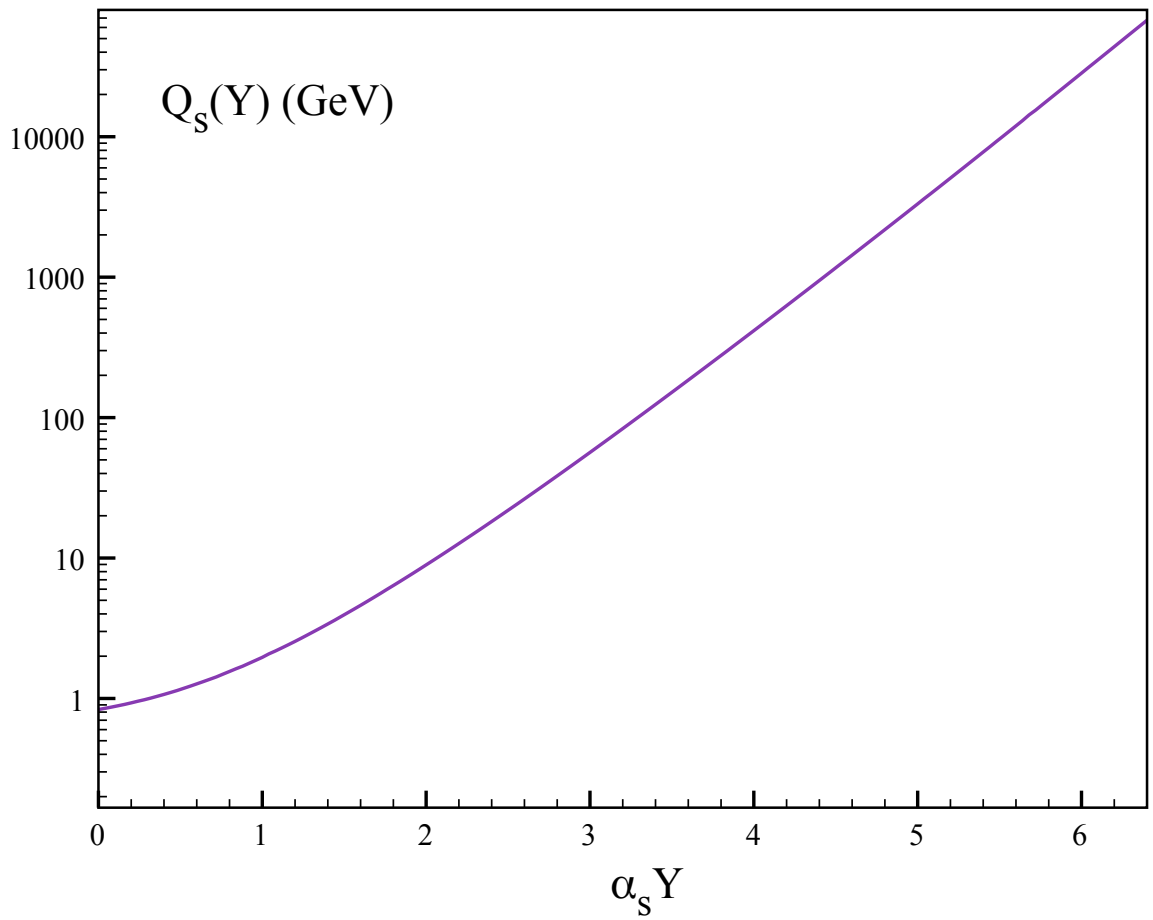


numerical solution
by J. Albacete '03
(earlier solutions were
found numerically by
Golec-Biernat, Motyka, Stasto,
by Braun and by Lublinsky et al
in '01)

BK solution preserves the black disk limit, N<1 always
(unlike the linear BFKL equation)

$$\sigma^{q\bar{q}A} = 2 \int d^2b \, N(x_\perp, b_\perp, Y)$$

Saturation scale



numerical solution by J. Albacete

Small-x Asymptotics

- BFKL solution gives (x<<1) $$N \sim \left(\frac{1}{x}\right)^{\alpha_P - 1}$$

  with

$$\alpha_P - 1 = \frac{4\,\alpha_s\,N_c}{\pi}\,\ln 2 \approx 2.77\,\frac{\alpha_s\,N_c}{\pi}$$

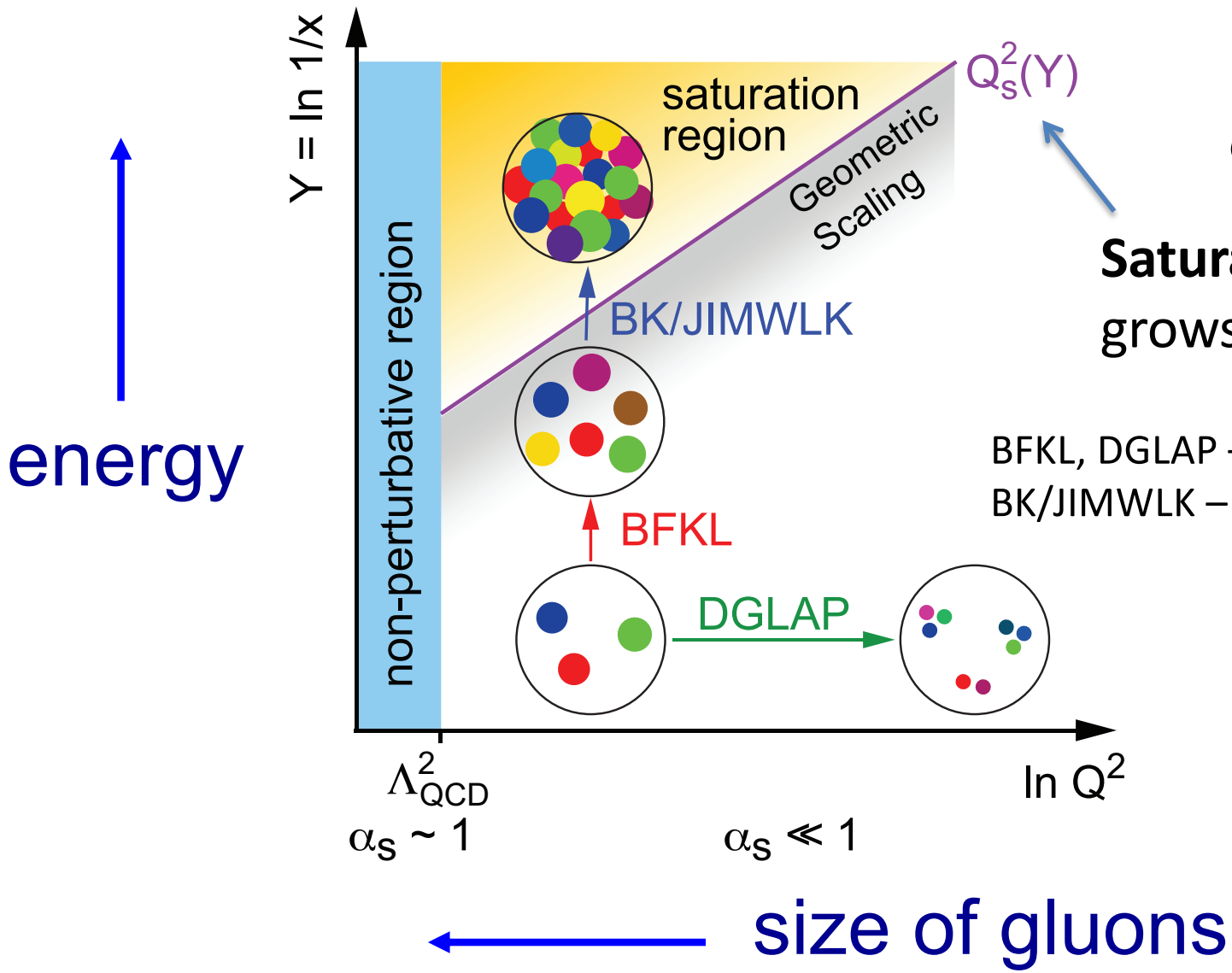
- NLO corrections are known (Fadin, Lipatov ’98; Ciafaloni, Camici ’98; Balitsky, Chirilli ‘07 for BK).

- Full BK equation solution also leads to saturation at very small x (x<<<1):

$$N \sim \mathrm{const}$$

- Below we will refer to the BFKL-like linear regime as the “small-x asymptotics” of TMDs. It should be understood that at even smaller x saturation is expected to come in and stop the small x evolution.

Map of High Energy QCD

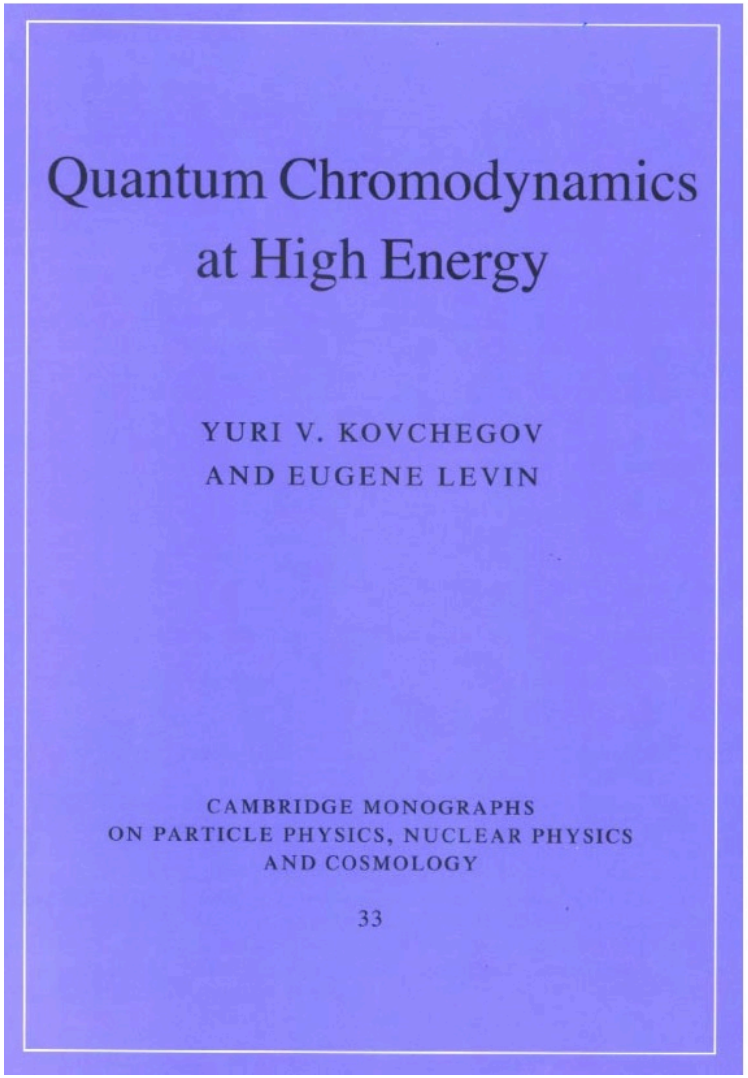


$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

Saturation Scale grows with energy

BFKL, DGLAP – linear equations
BK/JIMWLK – nonlinear

References



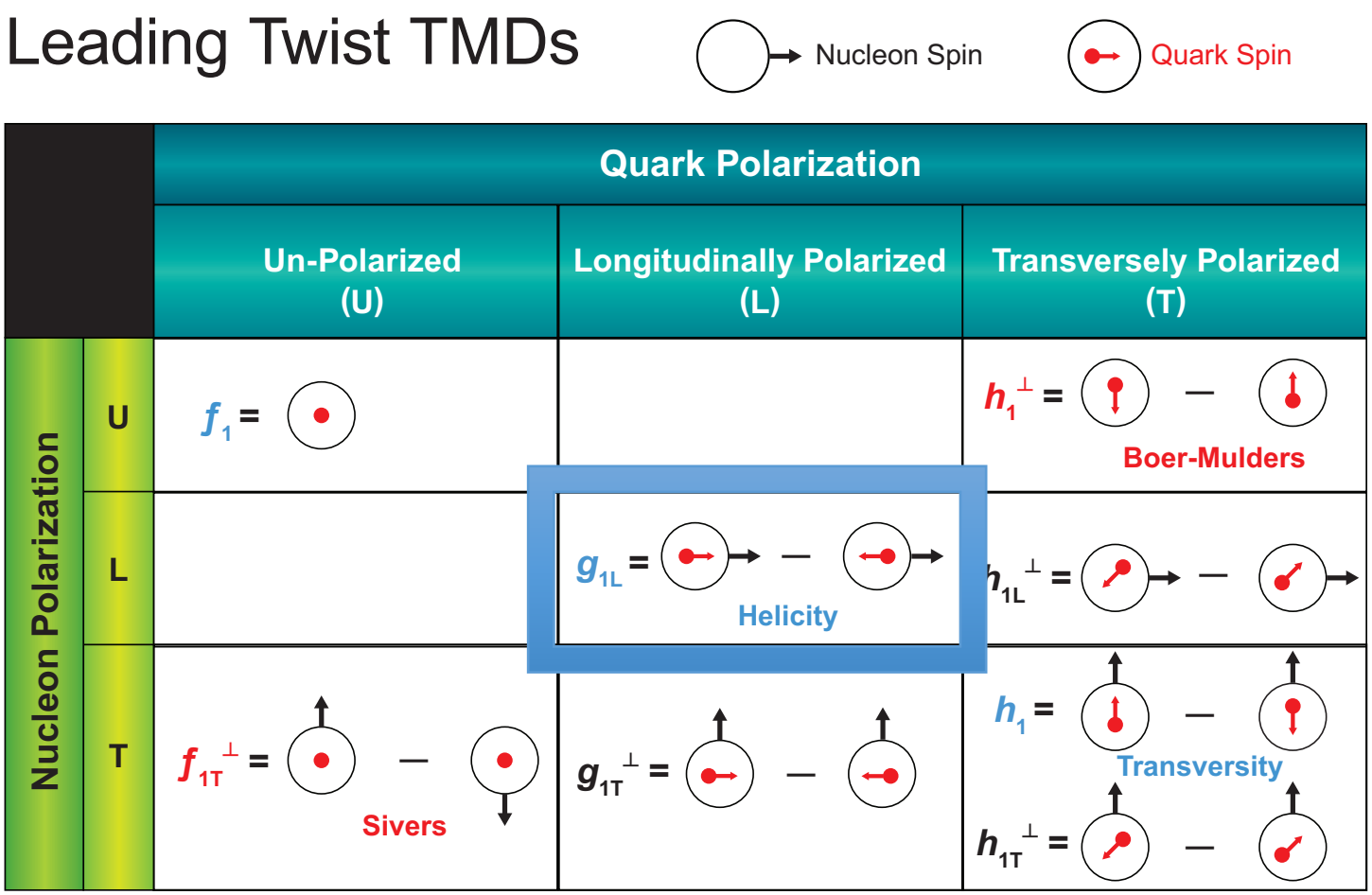
Published in September 2012
by Cambridge U Press

Quark Helicity at Small x (flavor-singlet case)

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph], arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph], arXiv:1703.05809 [hep-ph], arXiv:1808.09010 [hep-ph]

Observables

- We want to calculate quark helicity PDF and TMD at small x.



Quark Helicity TMD

- We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

$$g_{1L}^q(x,k_T^2) = \frac{1}{(2\pi)^3}\,\frac{1}{2}\sum_{S_L} S_L \int d^2r\, dr^-\, e^{ik\cdot r}\langle p,S_L|\bar{\psi}(0)\,\mathcal{U}[0,r]\,\frac{\gamma^+\gamma^5}{2}\,\psi(r)|p,S_L\rangle_{r^+=0}$$

- At small-x, in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in $A^-=0$ gauge for the + moving proton)

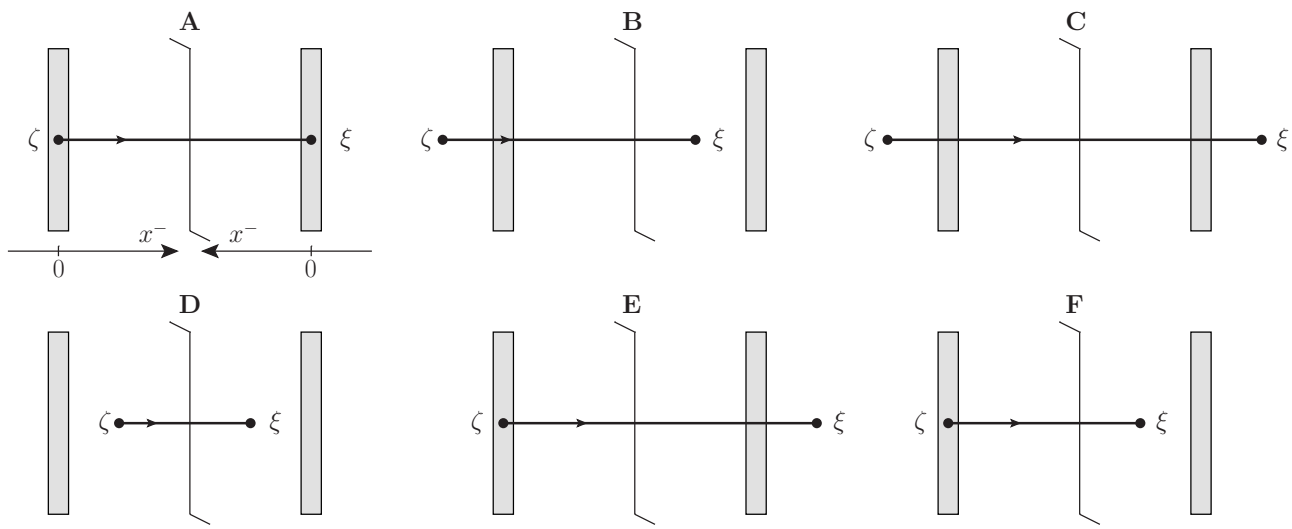
$$g_{1L}^q(x,k_T^2) = \frac{2p^+}{(2\pi)^3}\int d^2\zeta\, d\zeta^-\, d^2\xi\, d\xi^-\, e^{ik\cdot(\zeta-\xi)}\left(\frac{1}{2}\gamma^+\gamma^5\right)_{\alpha\beta}\left\langle\bar{\psi}_\alpha(\xi)\,V_{\underline{\xi}}[\xi^-,\infty]\;V_{\underline{\zeta}}[\infty,\zeta^-]\,\psi_\beta(\zeta)\right\rangle$$

where the fundamental light-cone Wilson line is

$$V_{\underline{x}}[b^-,a^-] = \mathrm{P}\exp\left\{ig\int_{a^-}^{b^-} dx^-\, A^+(x^-,\underline{x})\right\}$$

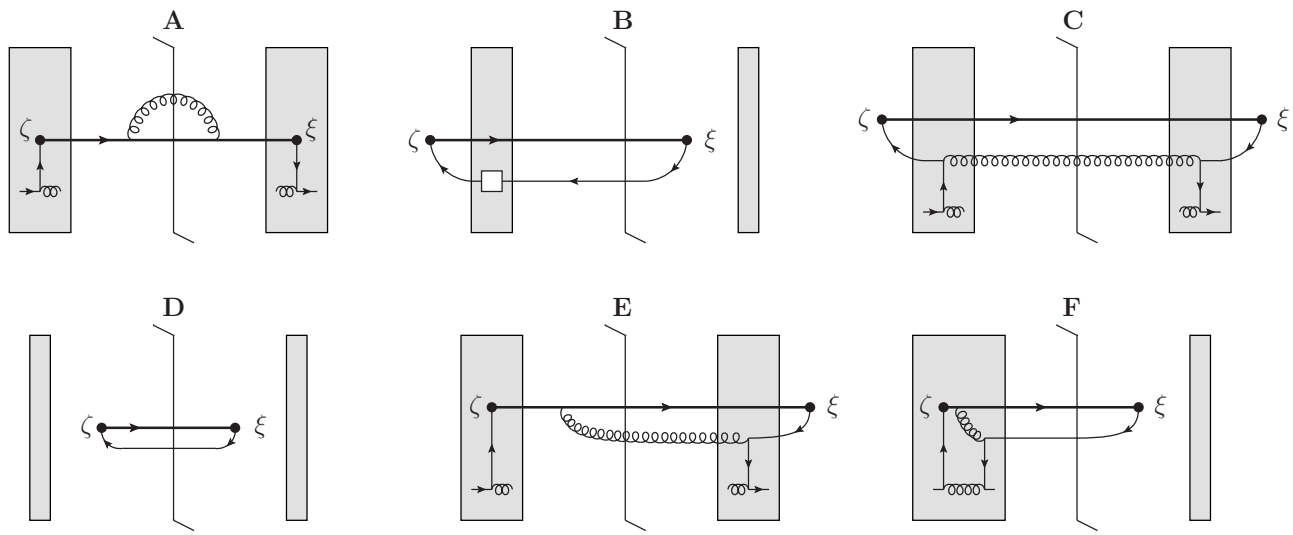
Quark Helicity TMD at Small x

- At high energy/small-x the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:



$$g_{1L}^q(x,k_T^2) = \frac{2p^+}{(2\pi)^3} \sum_X \int d^2\zeta\, d\zeta^-\, d^2\xi\, d\xi^-\, e^{ik\cdot(\zeta-\xi)} \left(\tfrac{1}{2}\gamma^+\gamma^5\right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi)\, V_{\underline{\xi}}[\xi^-,\infty]\, |X\rangle\ \langle X|\, V_{\underline{\zeta}}[\infty,\zeta^-]\, \psi_\beta(\zeta) \right\rangle$$

Quark Helicity TMD at Small x



- Diagram D does not transfer spin information from the target. Diagram C is canceled as we move t-channel quarks across the cut.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other at the leading (DLA) order (Ward identity).
- We are left with the diagram B.

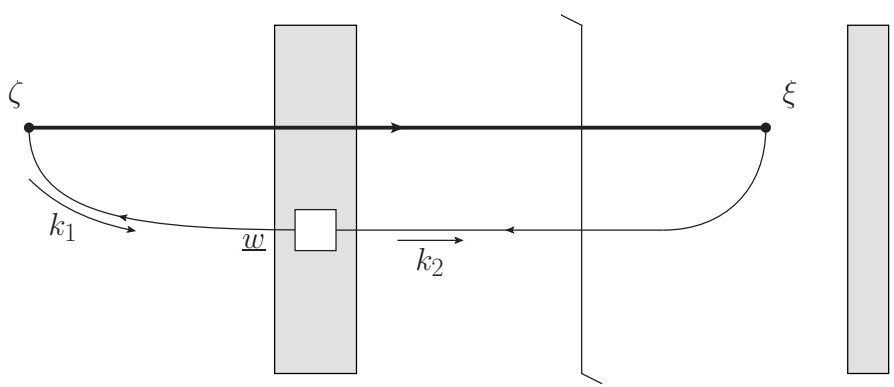
Quark Helicity TMD at Small x

- Evaluating diagram B we arrive at

$$g_{1L}^q(x,k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta\, d^2w\, d^2y\, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2} G_{\underline{w},\underline{\zeta}}(zs)$$

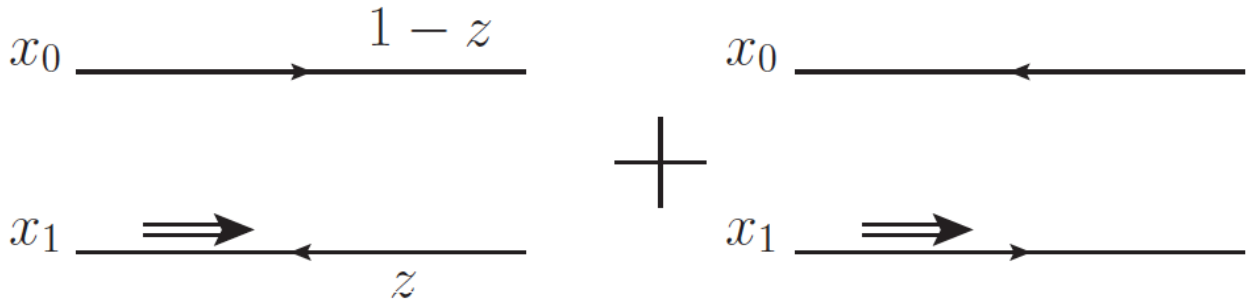
  where $G_{w\zeta}$ is the polarized dipole amplitude (defined on the next slide).

- Here s is the cms energy squared, Λ is some IR cutoff, underlining denotes transverse vectors, z = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark



Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \mathrm{Re} \left\langle\!\!\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

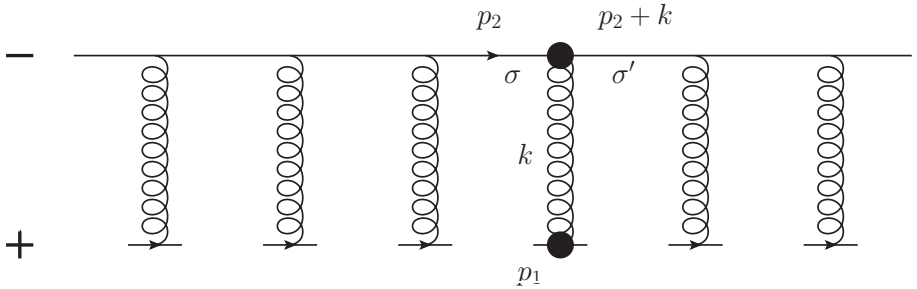
$$V_{\underline{x}} \equiv \mathcal{P} \exp\left[ig \int_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x})\right]$$

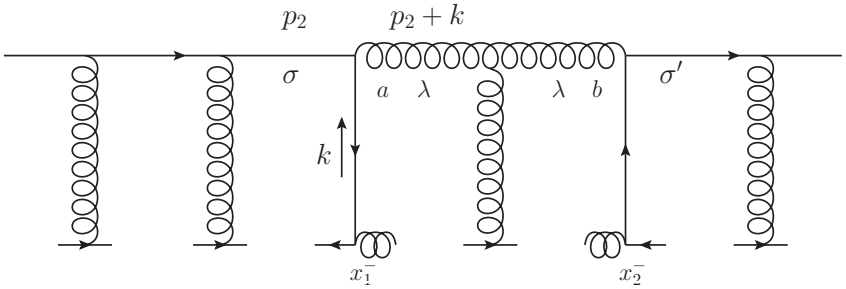
- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle (z) \equiv zs \left\langle \mathcal{O} \right\rangle (z)$$

Polarized fundamental “Wilson line”

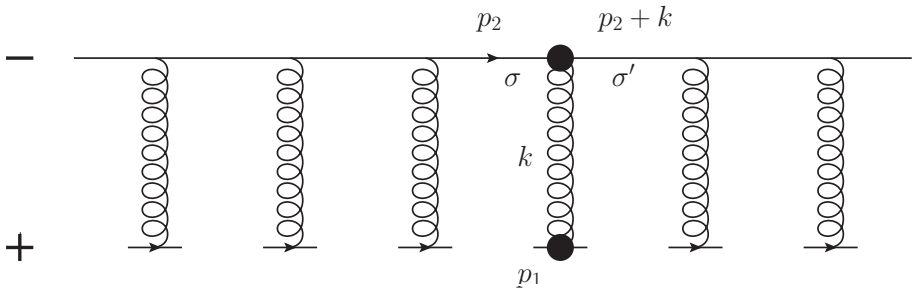
- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized “Wilson line” V^{pol}, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.

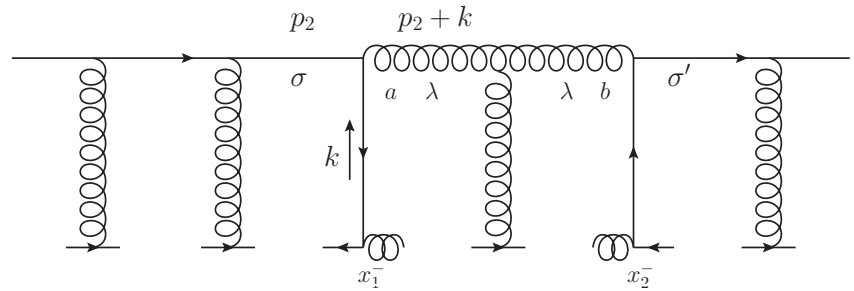




- At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.

Polarized fundamental “Wilson line”





- In the end one arrives at (KPS ‘17; YK, Sievert, ‘18; cf. Chirilli ‘18)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2}\gamma^+ \gamma^5\right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

- We have employed an adjoint light-cone Wilson line

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp\left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x})\right]$$

- Note the simple physical meaning of the first term:

$$-\vec{\mu} \cdot \vec{B} = -\mu_z \, B_z = \mu_z \, F^{12}$$

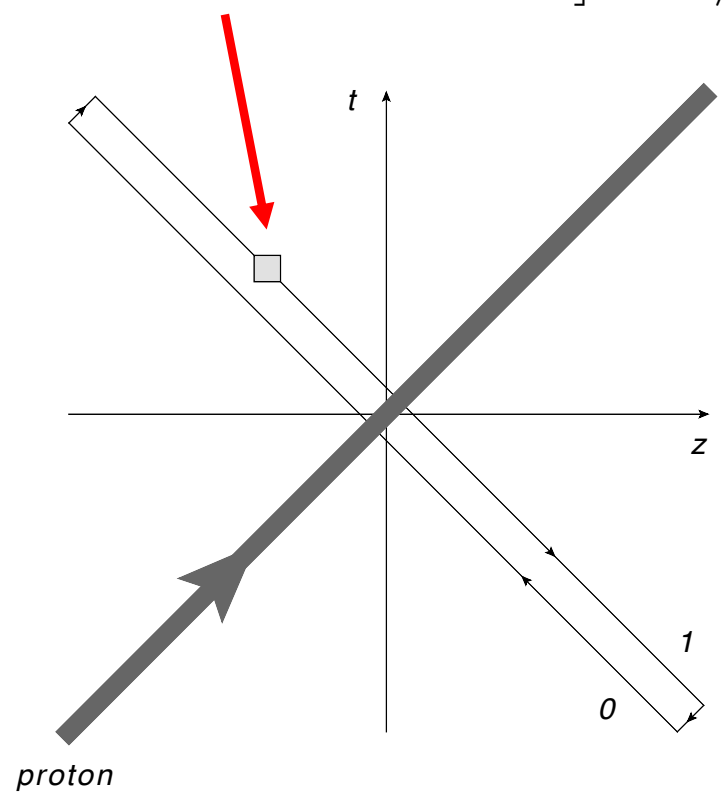
Polarized Dipole Amplitude

- The polarized dipole amplitude is then defined by

$$G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty,-\infty]V_{\underline{1}}[-\infty,x^-]\,(-ig)\,\underline{\nabla}\times\underline{\tilde{A}}(x^-,\underline{x})\,V_{\underline{1}}[x^-,\infty]\right] + \mathrm{c.c.}\right\rangle(z)$$

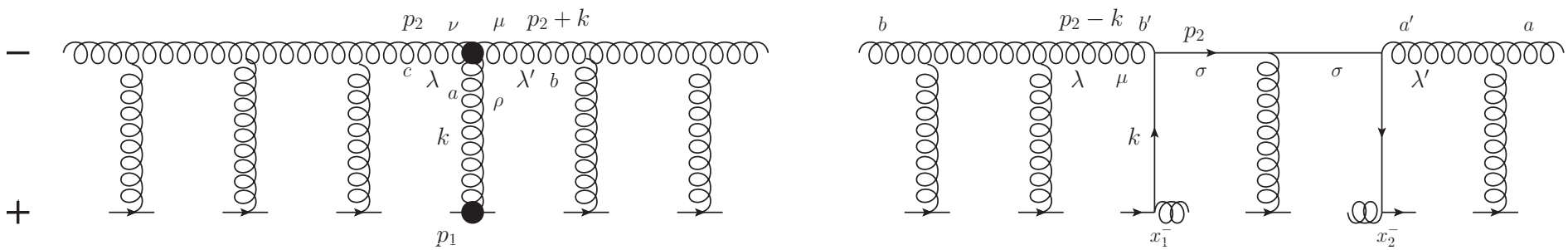
with the standard light-cone Wilson line

$$V_{\underline{x}}[b^-,a^-] = \mathrm{P}\exp\left\{ig\int_{a^-}^{b^-} dx^-\,A^+(x^-,\underline{x})\right\}$$



Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.



- The calculation is similar to the quark scattering case. It yields (cf. Chirilli ‘18)

$$(U^{pol}_{\underline{x}})^{ab} = \frac{2i\,g\,p_1^+}{s}\int\limits_{-\infty}^{+\infty} dx^- \;\left(U_{\underline{x}}[+\infty,x^-]\,\mathcal{F}^{12}(x^+=0,x^-,\underline{x})\,U_{\underline{x}}[x^-,-\infty]\right)^{ab}$$

$$-\frac{g^2\,p_1^+}{s}\int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^-\; U^{aa'}_{\underline{x}}[+\infty,x_2^-]\,\bar{\psi}(x_2^-,\underline{x})\,t^{a'}\,V_{\underline{x}}[x_2^-,x_1^-]\,\frac{1}{2}\,\gamma^+\gamma_5\,t^{b'}\,\psi(x_1^-,\underline{x})\,U^{b'b}_{\underline{x}}[x_1^-,-\infty] - \text{c.c.}$$

Small-x Evolution at large N_c

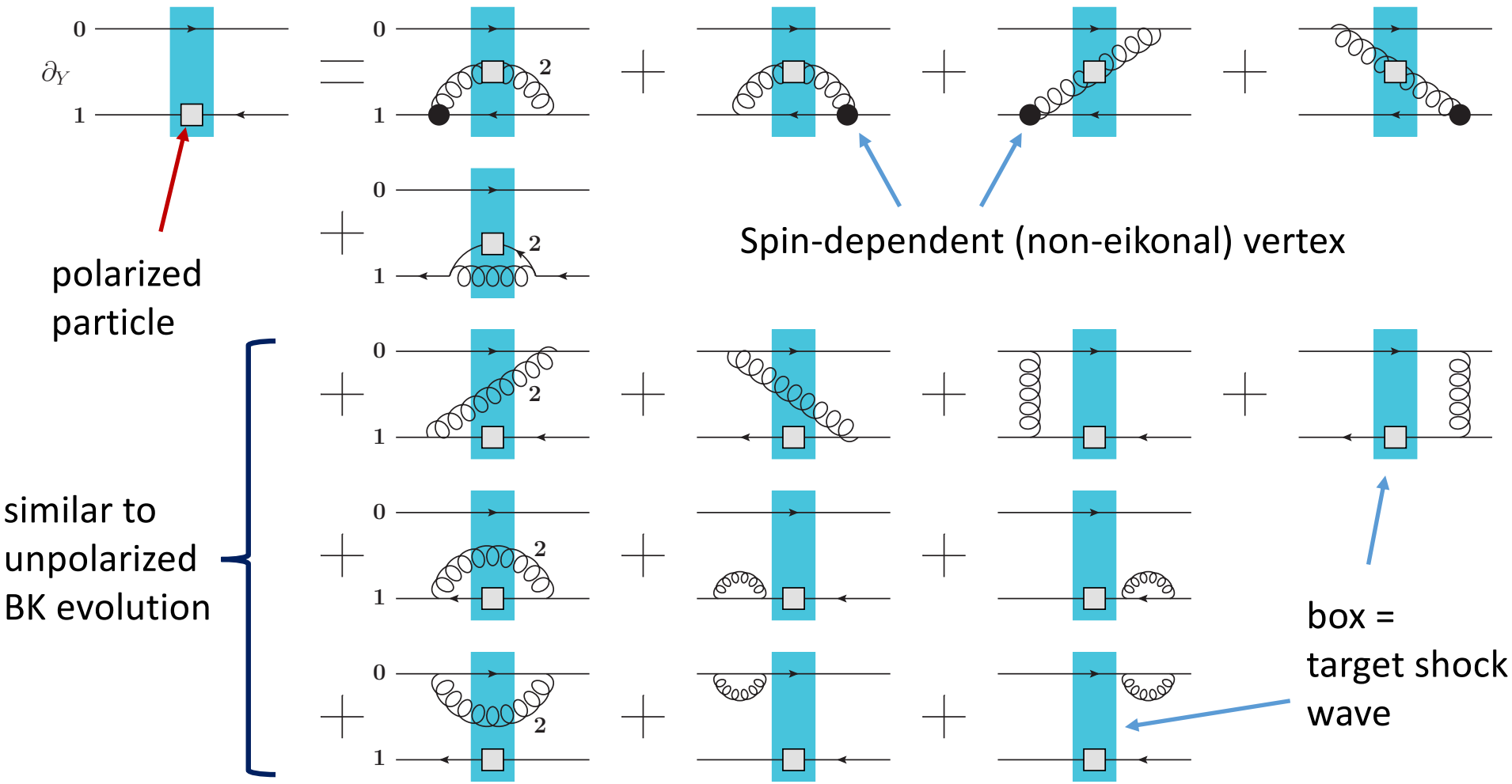
- At large N_c the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large N_c the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4\, G_{10}(z)$$

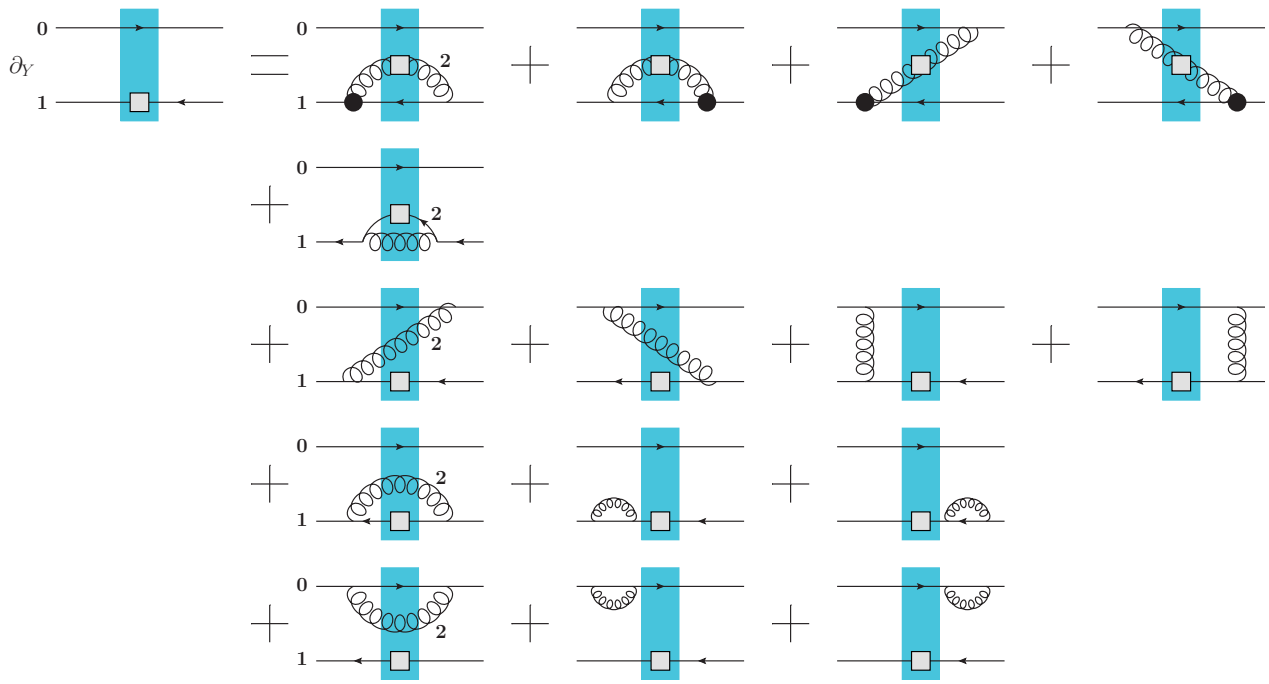
(Note that the factor is 4, not 2 like in the unpolarized dipole case.)

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\,s}\,\langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\,s}$$

$$\frac{1}{N_c}\,\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\,V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c}\,\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\,V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2}\int_{z_i}^{z}\frac{dz'}{z'}\int_{\rho'^2}\frac{d^2x_2}{x_{21}^2}$$

$$\times\left\{\theta(x_{10}-x_{21})\,\frac{2}{N_c}\,\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_0^{unp}\,t^a\,V_1^{unp\,\dagger}\right]\,U_2^{pol\,ba}\right\rangle\!\!\right\rangle(z')\right.$$

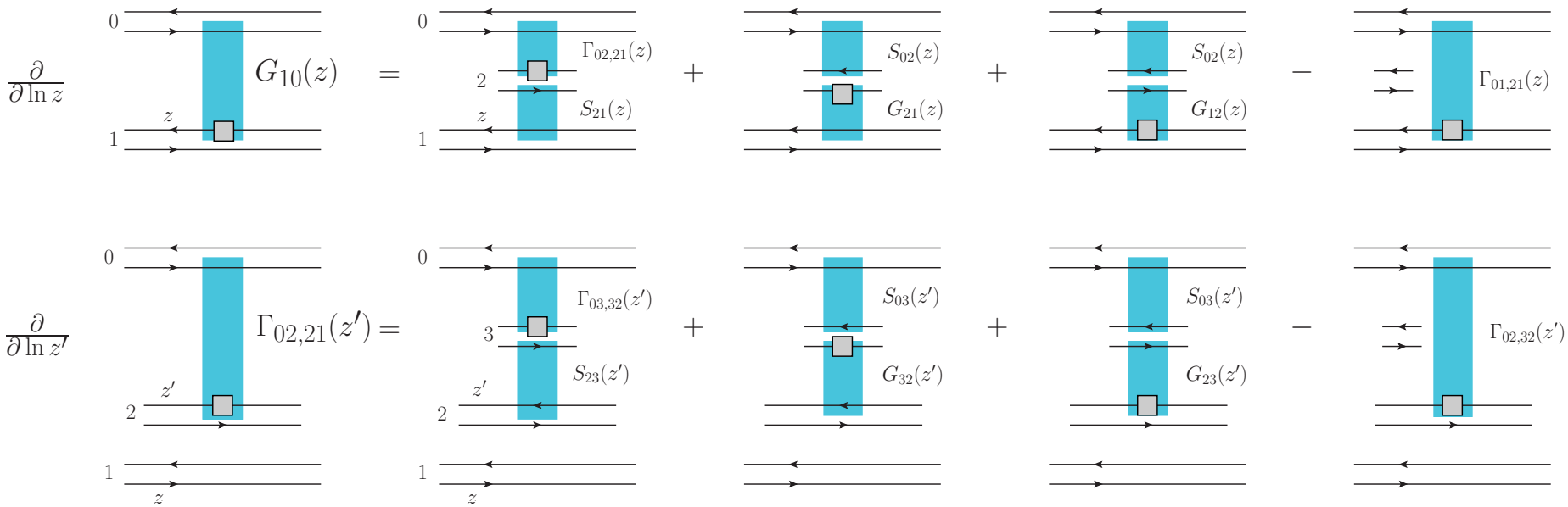
$$+\,\theta(x_{10}^2 z - x_{21}^2 z')\,\frac{1}{N_c}\,\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_0^{unp}\,t^a\,V_2^{pol\,\dagger}\right]\,U_1^{unp\,ba}\right\rangle\!\!\right\rangle(z')$$

$$\left.+\,\theta(x_{10}-x_{21})\,\frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\,V_2^{unp\,\dagger}\right]\,\mathrm{tr}\left[V_2^{unp}\,V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z') - N_c\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\,V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z')\right]\right\}$$

Equation does not close!

Polarized Dipole Evolution in the Large-N_c Limit

In the large-N_c limit the equations close, leading to a system of 2 equations:



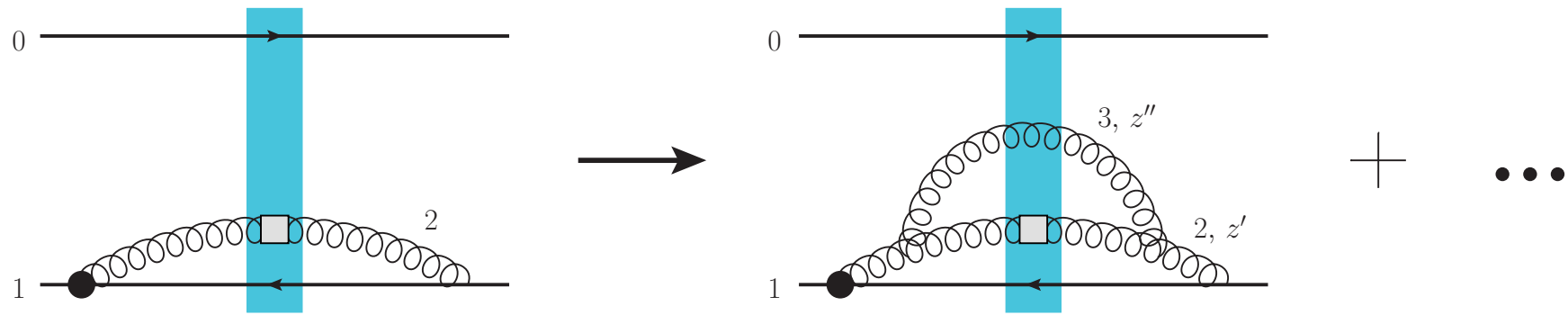
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^{z} \frac{dz'}{z'} \int_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\,S_{21}(z') + 2\,G_{21}(z')\,S_{02}(z') + G_{12}(z')\,S_{02}(z') - \Gamma_{01,\,21}(z')\right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{z_i}^{z'} \frac{dz''}{z''} \int_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\,S_{23}(z'') + 2\,G_{32}(z'')\,S_{03}(z'') + G_{23}(z'')\,S_{03}(z'') - \Gamma_{02,\,32}(z'')\right]$$

S = found from BK/JIMWLK, it is LLA

“Neighbor” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may ’know’ about another dipole:



$$x_{21}^2\, z' \gg x_{32}^2\, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

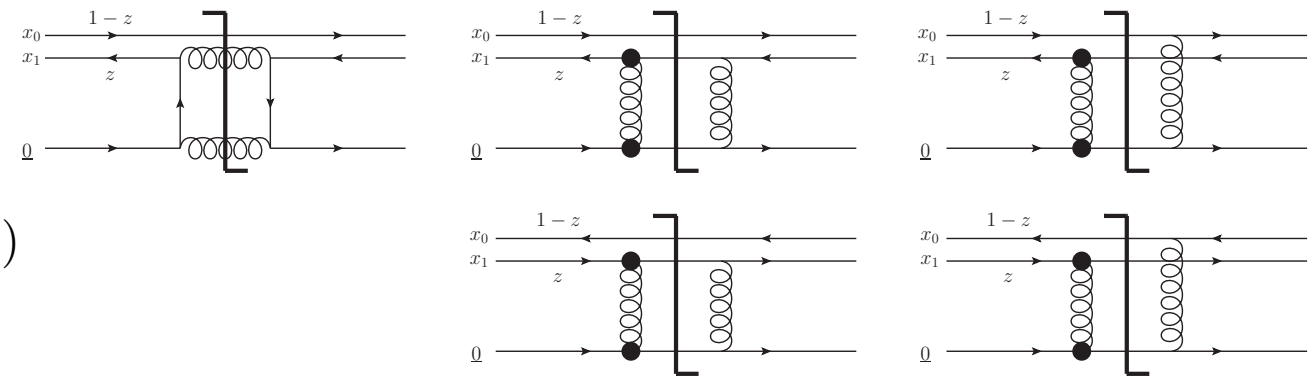
Large-N_c Evolution

- In the strict DLA limit (S=1) and at large N_c we get (here Γ is an auxiliary function we call the 'neighbour dipole amplitude') (KPS '15)

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z')\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'')\right]$$

- The initial conditions are given by the Born-level graphs



$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2\, \ln(zs\, x_{10}^2)\right]$$

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \, \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

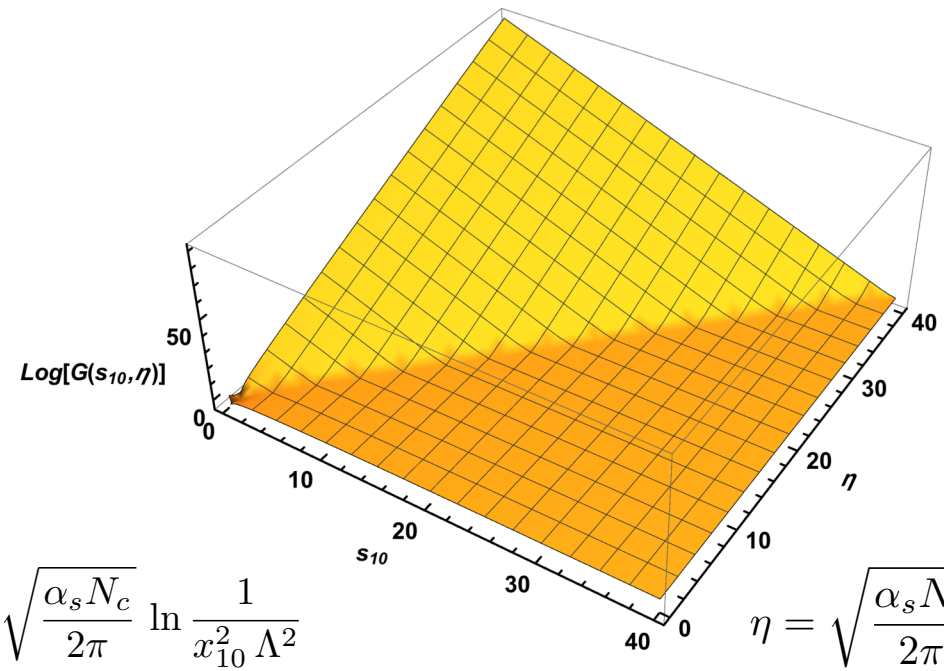
$$\alpha_s \, \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{1}{x_{10}^2\, \Lambda^2} \qquad \eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{zs}{\Lambda^2}$$

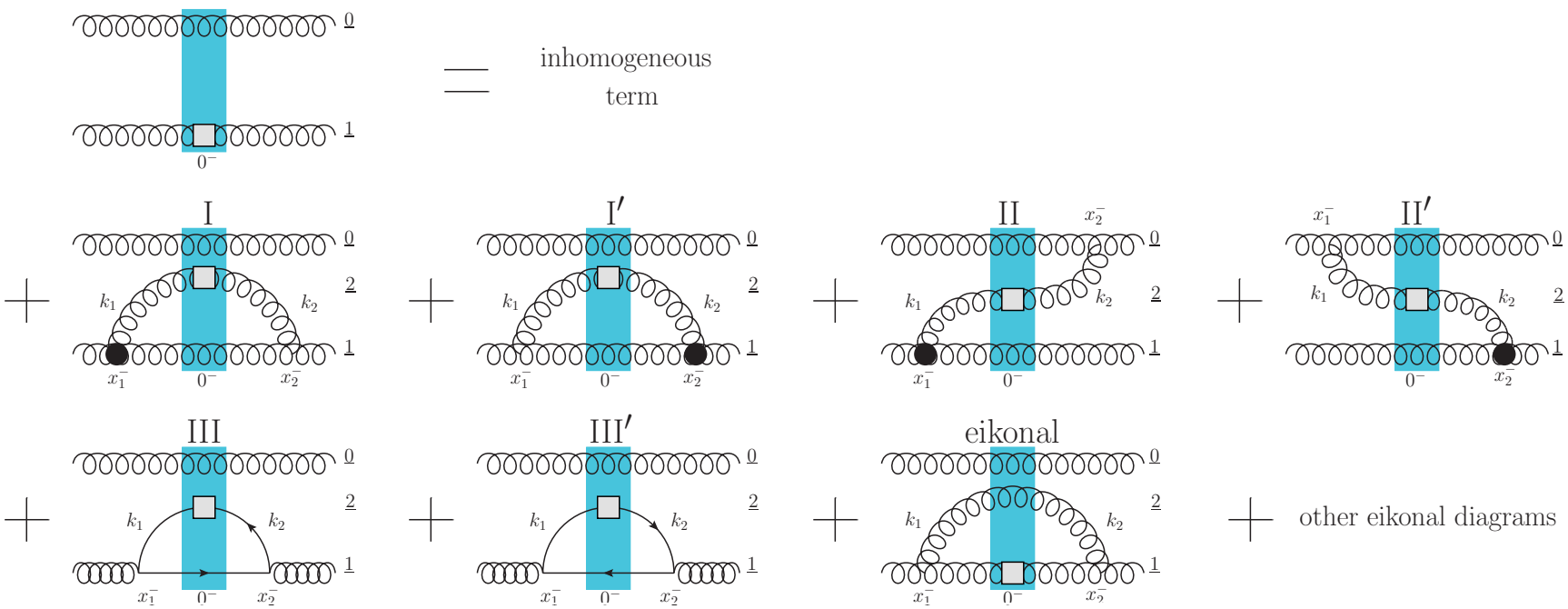
- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

Small-x Evolution at large N_c&N_f

- At large N_c&N_f there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

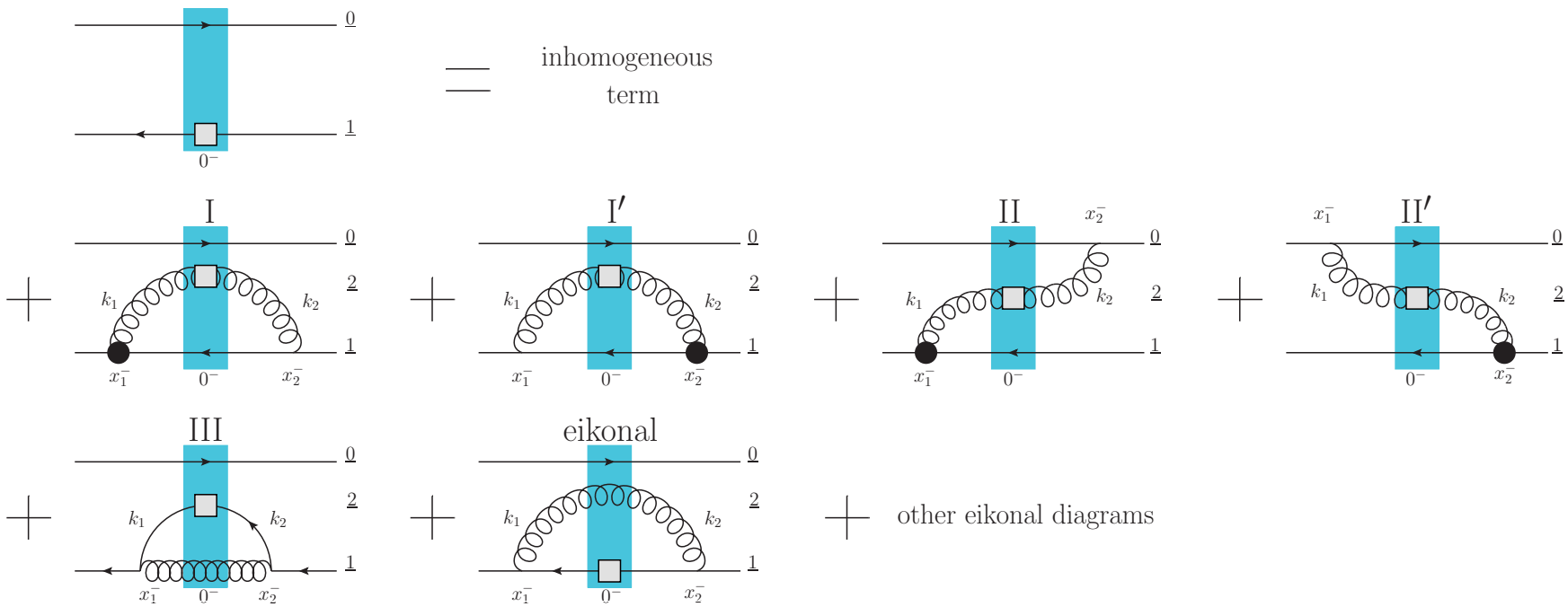
- Here's the adjoint dipole evolution:



Small-x Evolution at large N_c&N_f

- At large N_c&N_f there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

- Here's the fundamental dipole evolution:



Small-x Evolution at large N_c&N_f

- The resulting equations are

$$Q_{10}(zs) = Q_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left\{ \frac{1}{2}\,\Gamma_{02,21}^{adj}(z') + \frac{1}{2}\,G_{21}^{adj}(z') + Q_{12}(z') - \bar{\Gamma}_{01,21}(z') \right\}$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, Q_{21}(z'),$$

$$G_{10}^{adj}(z) = G_{10}^{adj\,(0)}(z) + \frac{\alpha_s\, N_c}{2\pi} \int_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int_{1/(z'\,s)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{10,21}^{adj}(z') + 3\,G_{21}^{adj}(z') \right]$$

$$- \frac{\alpha_s\, N_f}{2\,\pi} \int_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int_{1/(z'\,s)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, \bar{\Gamma}_{02;21}(z'),$$

These are yet to be solved.

$$\Gamma_{10,21}^{adj}(z') = \Gamma_{10,21}^{adj\,(0)}(z') + \frac{\alpha_s\, N_c}{2\pi} \int_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int_{1/(z''\,s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{10,32}^{adj}(z'') + 3\,G_{32}^{adj}(z'') \right]$$

$$- \frac{\alpha_s\, N_f}{2\,\pi} \int_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int_{1/(z''\,s)}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2}\, \bar{\Gamma}_{03;32}(z''),$$

$$\bar{\Gamma}_{10,21}(z') = \bar{\Gamma}_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left\{ \frac{1}{2}\,\Gamma_{03,32}^{adj}(z'') + \frac{1}{2}\,G_{32}^{adj}(z'') + Q_{32}(z'') - \bar{\Gamma}_{01,32}(z \right.$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{x_{21}^2 z/z'} \frac{dx_{32}^2}{x_{32}^2}\, Q_{32}(z'').$$

Gluon Helicity at Small x

Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

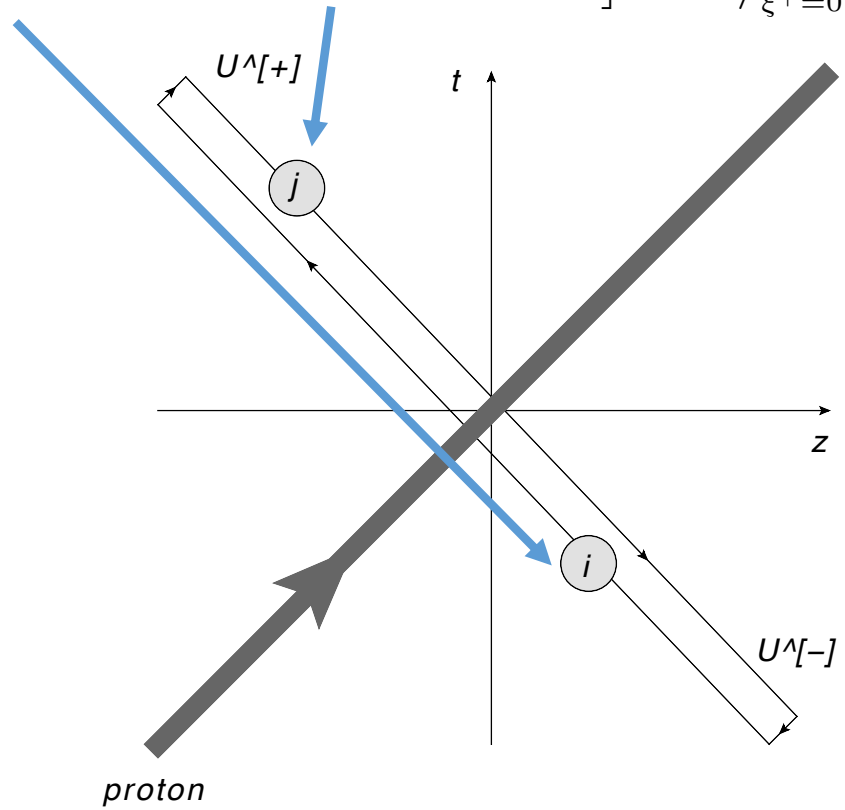
Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x,k_T^2) = \frac{-2i\,S_L}{x\,P^+}\int \frac{d\xi^-\,d^2\xi}{(2\pi)^3}\,e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}}\,\Big\langle P,S_L|\epsilon_T^{ij}\,\mathrm{tr}\Big[F^{+i}(0)\,\mathcal{U}^{[+]\dagger}[0,\xi]\,F^{+j}(\xi)\,\mathcal{U}^{[-]}[\xi,0]\Big]\,|P,S_L\Big\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past Wilson line staples (hence the name `dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

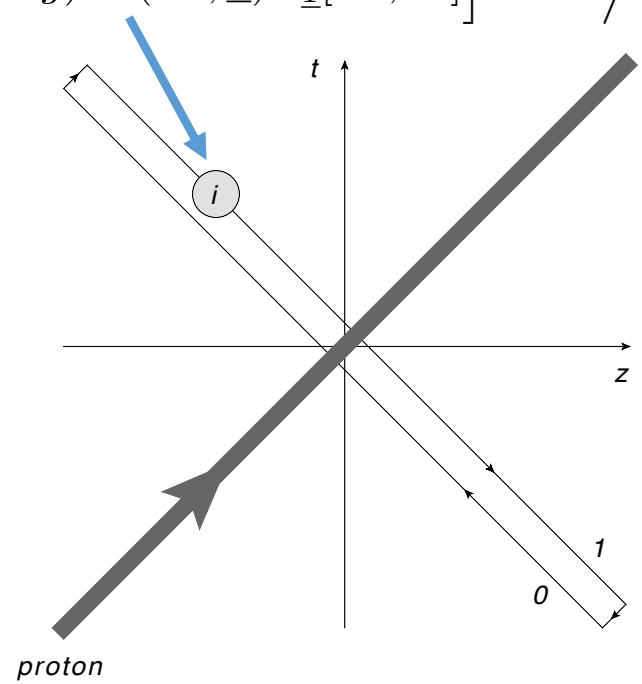
- At small x, the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G\,dip}(x, k_T^2) = \frac{8i\,N_c\,S_L}{g^2(2\pi)^3} \int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\, k_\perp^i\, \epsilon_T^{ij} \left[\int d^2b_{10}\, G_{10}^j(zs = \frac{Q^2}{x})\right]$$

- Here we obtain a new operator, which is a transverse vector (written here in $A^-=0$ gauge):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty,-\infty]V_{\underline{1}}[-\infty,x^-]\,(-ig)\,\tilde{A}^i(x^-,\underline{x})\,V_{\underline{1}}[x^-,\infty]\right] + \mathrm{c.c.}\right\rangle(z)$$

- Note that $k_\perp^i\,\epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-,\underline{x})$ -- different from the polarized dipole amplitude!



Dipole TMD vs dipole amplitude

- Note that the operator for the <u>dipole</u> gluon helicity TMD

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty,-\infty]V_{\underline{1}}[-\infty,x^-]\,(-ig)\,\tilde{A}^i(x^-,\underline{x})\,V_{\underline{1}}[x^-,\infty]\right] + \mathrm{c.c.}\right\rangle(z)$$

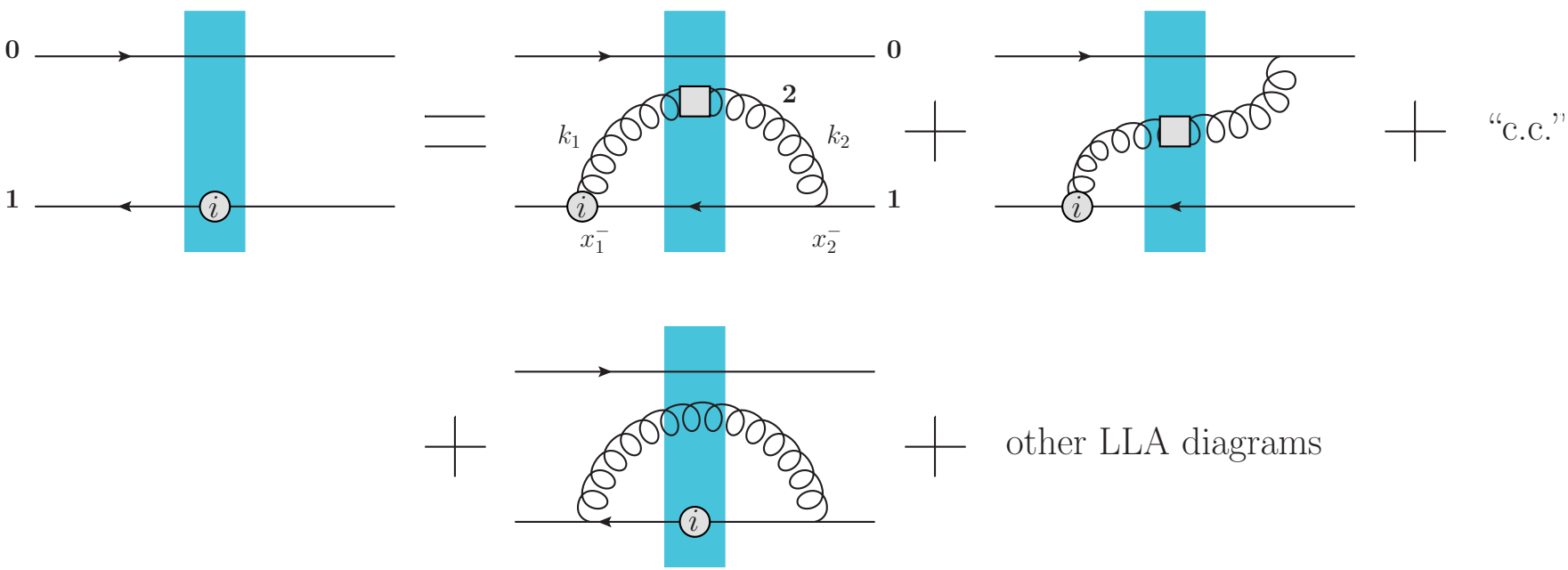
is different from the polarized <u>dipole</u> amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty,-\infty]V_{\underline{1}}[-\infty,x^-]\,(-ig)\,\underline{\nabla}\times\underline{\tilde{A}}(x^-,\underline{x})\,V_{\underline{1}}[x^-,\infty]\right] + \mathrm{c.c.}\right\rangle(z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the 'dipole' name may not even be valid for such TMDs.)
- This is different from the unpolarized gluon TMD case.

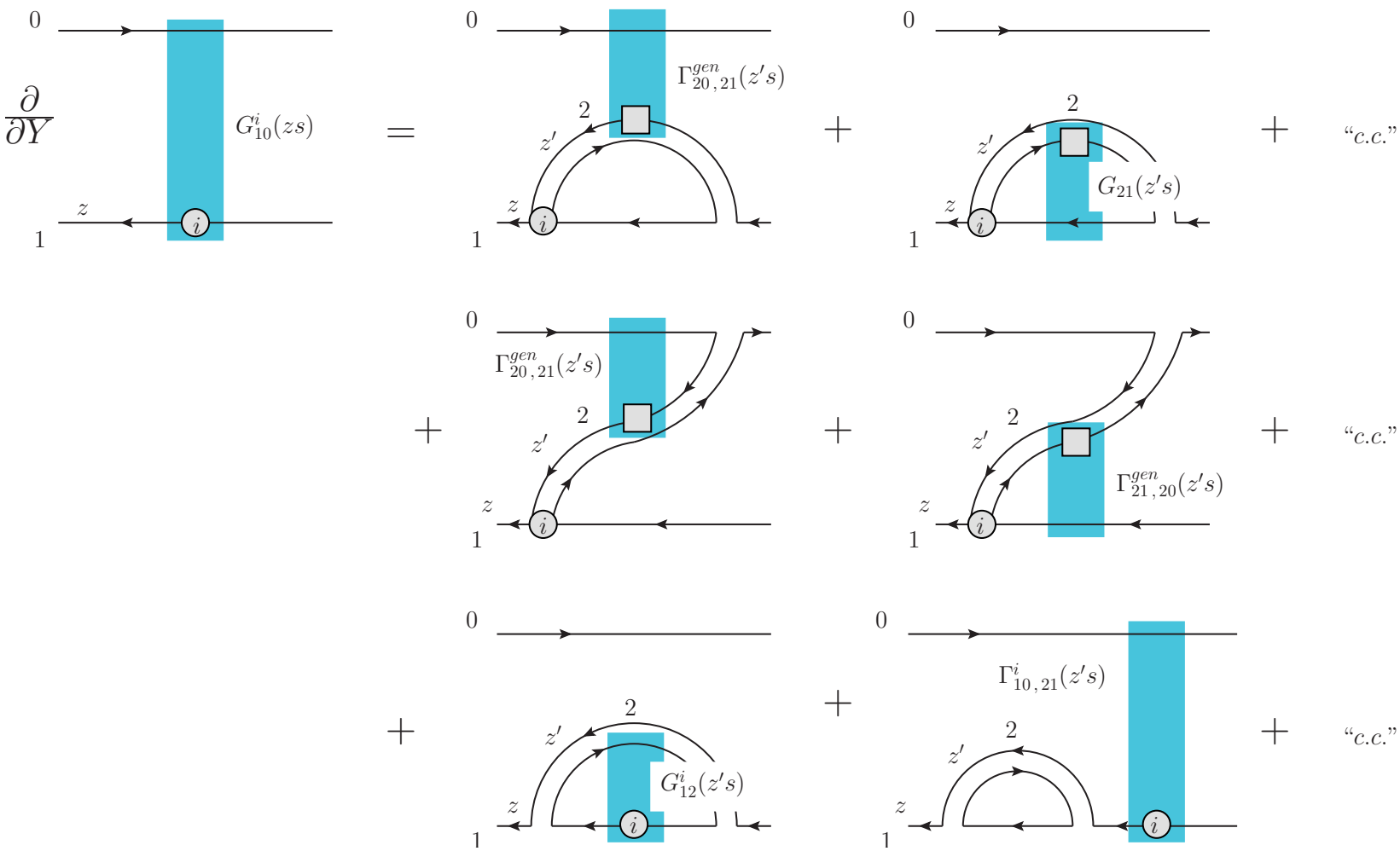
Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



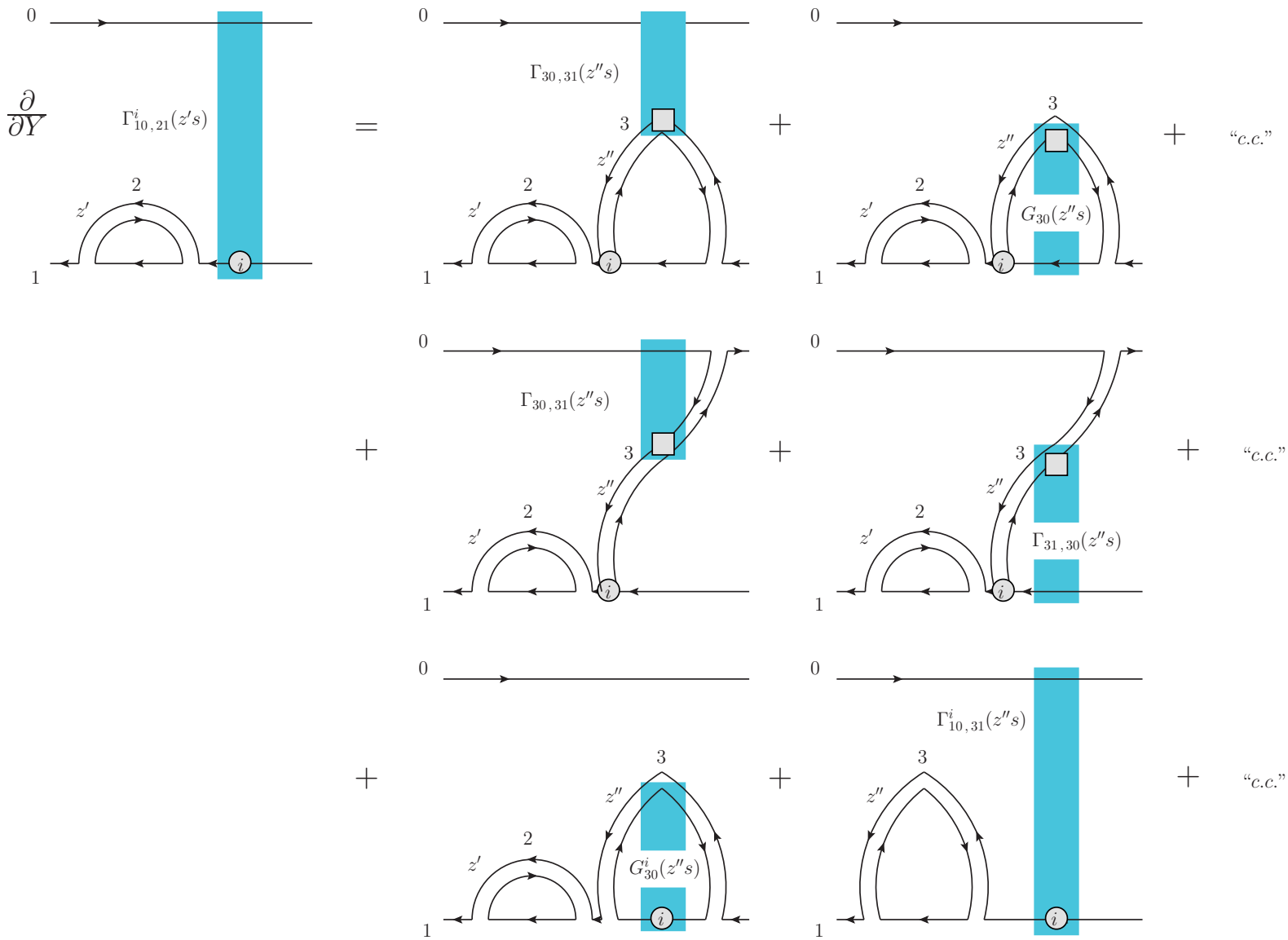
Large-N_c Evolution: Diagrams

- At large-N_c the equations are



Large-N_c Evolution: Diagrams

- and



Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{21})^j_\perp}{x^2_{21}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{20})^j_\perp}{x^2_{20}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x^2_{10}s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x^2_{10}} \frac{dx^2_{21}}{x^2_{21}} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{31})^j_\perp}{x^2_{31}} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{30})^j_\perp}{x^2_{30}} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x^2_{10}s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x^2_{10}\,,\,x^2_{21}\frac{z'}{z''}\right]} \frac{dx^2_{31}}{x^2_{31}} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s) \right].$$

Large-N_c Evolution: Equations

- Here

$$\Gamma^{gen}_{20,21}(z's) = \theta(x_{20} - x_{21})\,\Gamma_{20,21}(z's) + \theta(x_{21} - x_{20})\,G_{20}(z's)$$

is an object which we know from the quark helicity evolution, as the latter gives us G and Γ.

- Note that our evolution equations mix the gluon (G^i) and quark (G) small-x helicity evolution operators:

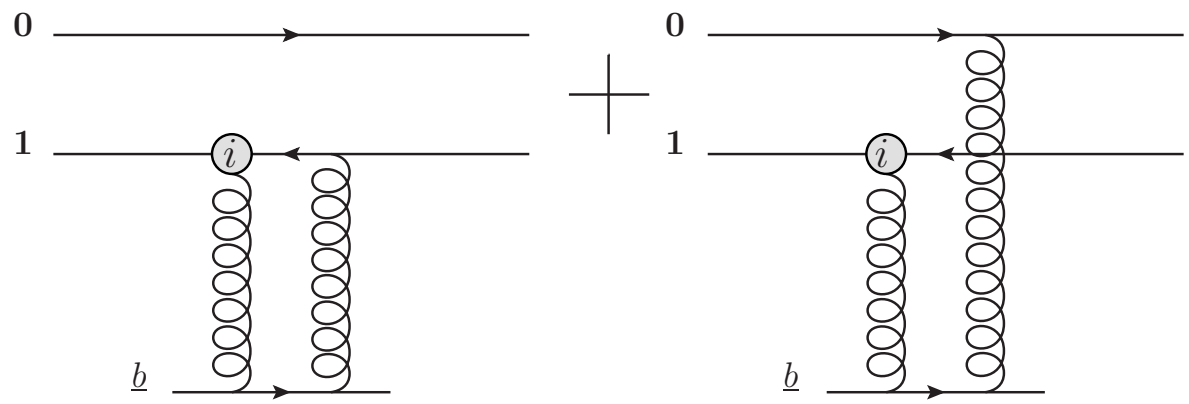
$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{21})^j_\perp}{x^2_{21}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon^{ij}_T \, (x_{20})^j_\perp}{x^2_{20}} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x^2_{10}s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x^2_{10}} \frac{dx^2_{21}}{x^2_{21}} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2b_{10}\, G_{10}^{i\,(0)}(zs) = \int d^2b_{10}\, \Gamma_{10,21}^{i\,(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\,\pi\,\epsilon^{ij}\,x_{10}^j\,\ln\frac{1}{x_{10}\,\Lambda}$$

- Note that these initial conditions have no ln s, unlike the initial conditions for the quark evolution:

$$\int d^2b_{10}\, G_{10}^{(0)}(zs) = \int d^2b_{10}\, \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\pi\,\ln(zs\,x_{10}^2)$$

Large-N_c Evolution: Power Counting

- The kernel mixing G^i or Γ^i with G and Γ is LLA:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

LLA

DLA

- But, the initial conditions for G and Γ have an extra $\ln s$ as compared to G^i and Γ^i, making the two terms comparable (order-α_S^2 in $\alpha_S \, ln^2 s \sim 1$ DLA power counting).

Large-N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small-x gluon helicity intercept

$$\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We obtain the small-x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G\,dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

Main Physics Results

- At large N_c we get for helicity

$$\Delta q(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \tfrac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 1.88\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

Orbital Angular Momentum

- The small-x asymptotics of the quark and gluon orbital angular momentum (OAM) in the Jaffe-Manohar decomposition shown above can be tackled similarly (YK, 2019).

- In the large-N_C limit we obtain

$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_C}{2\pi}}},$$

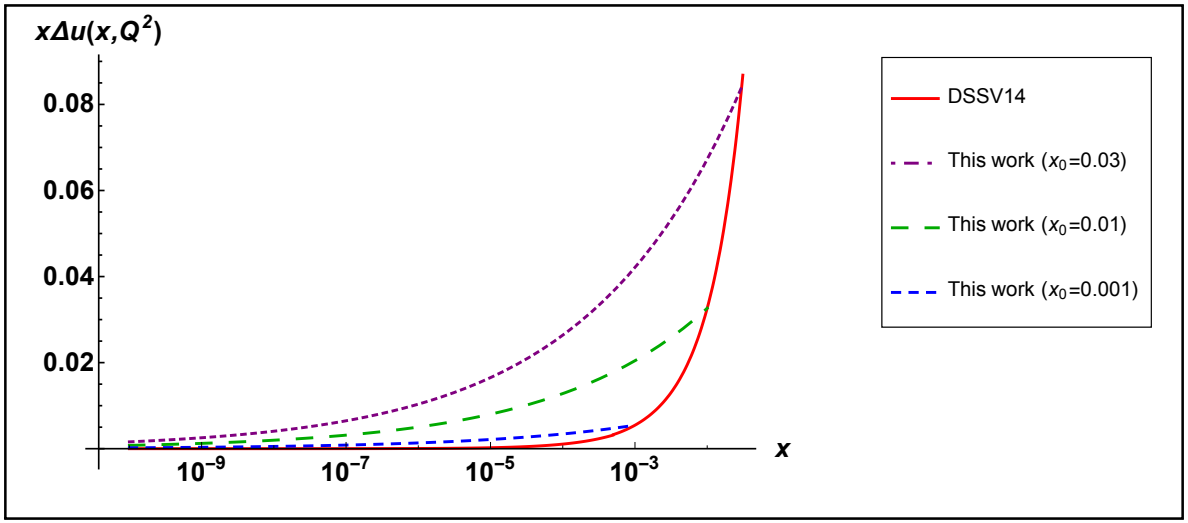
$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_C}{2\pi}}}$$

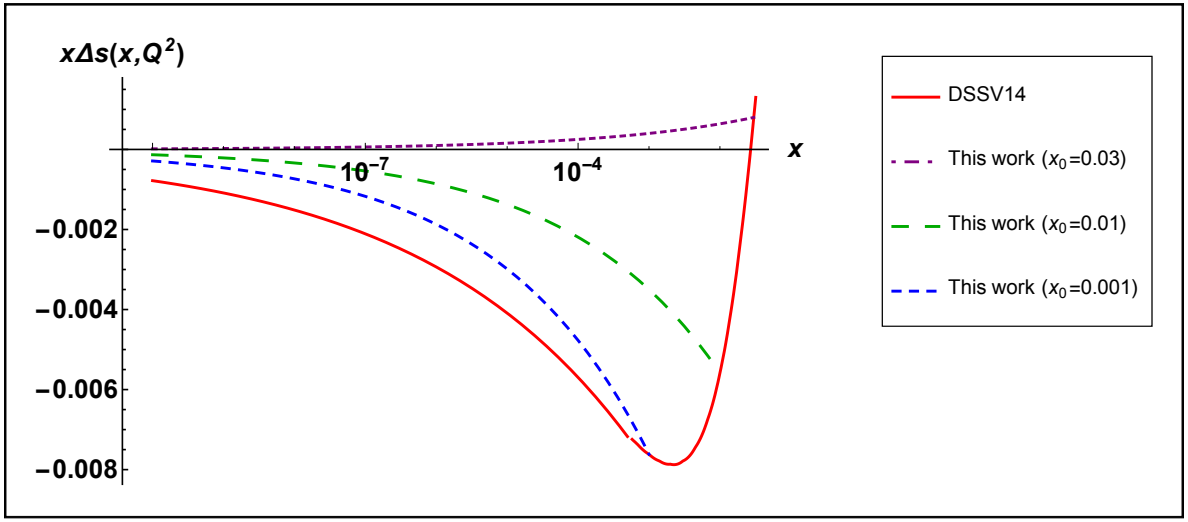
Preliminary Helicity Distributions Phenomenology at Small x

Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

Impact of our $\Delta\Sigma$ on the proton spin

- We have attached a $\Delta\tilde{\Sigma}(x, Q^2) = N\, x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :

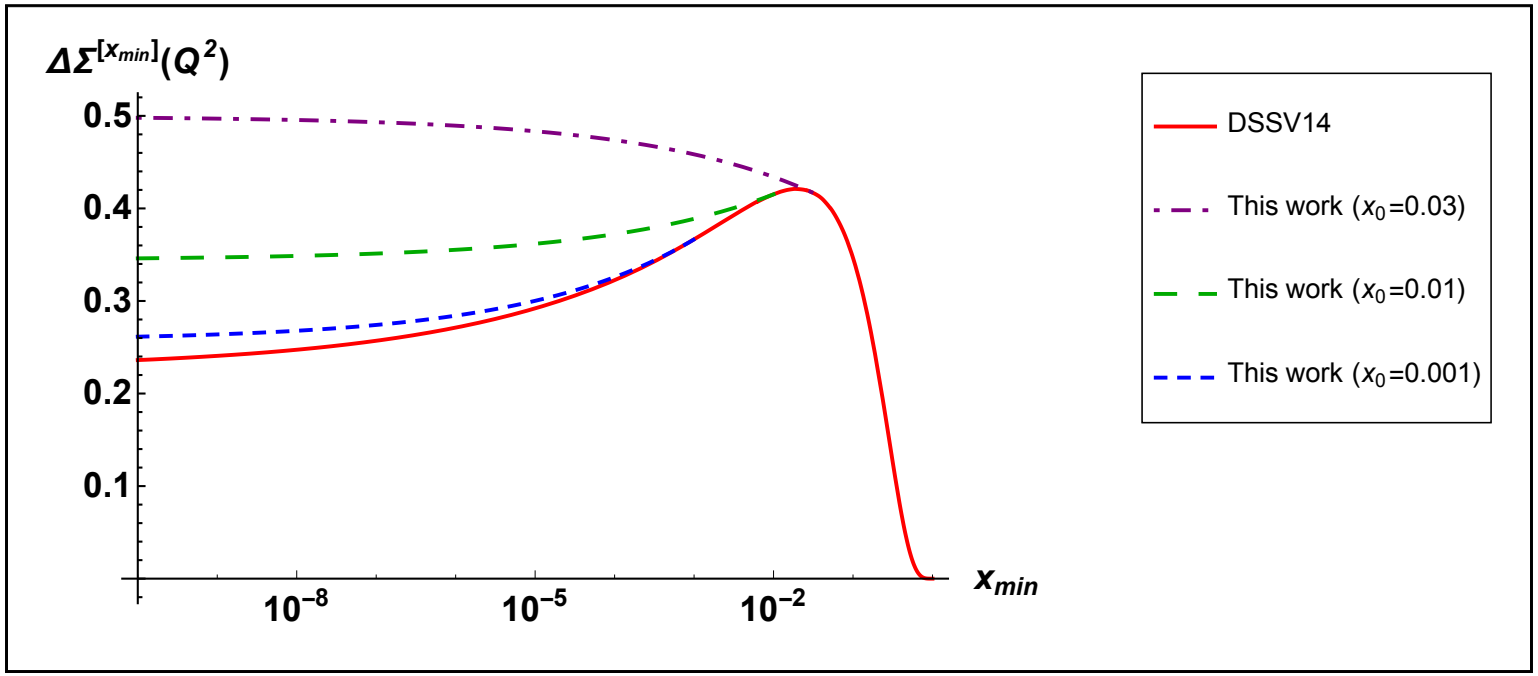




"ballpark" phenomenology

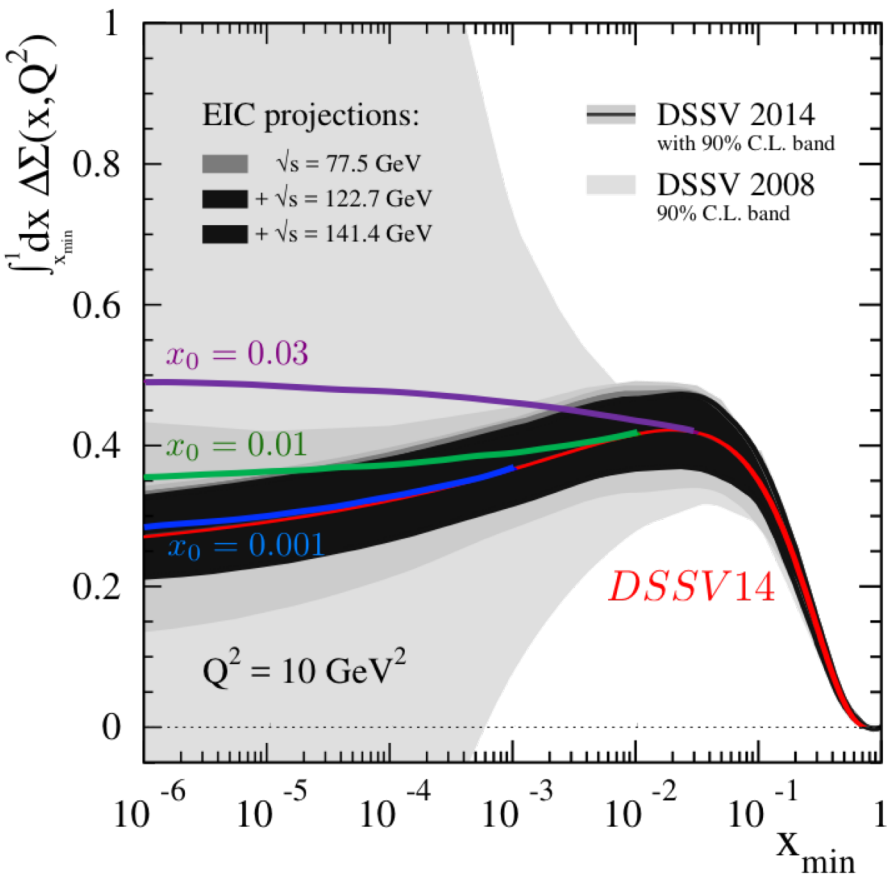
Impact of our $\Delta\Sigma$ on the proton spin

- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

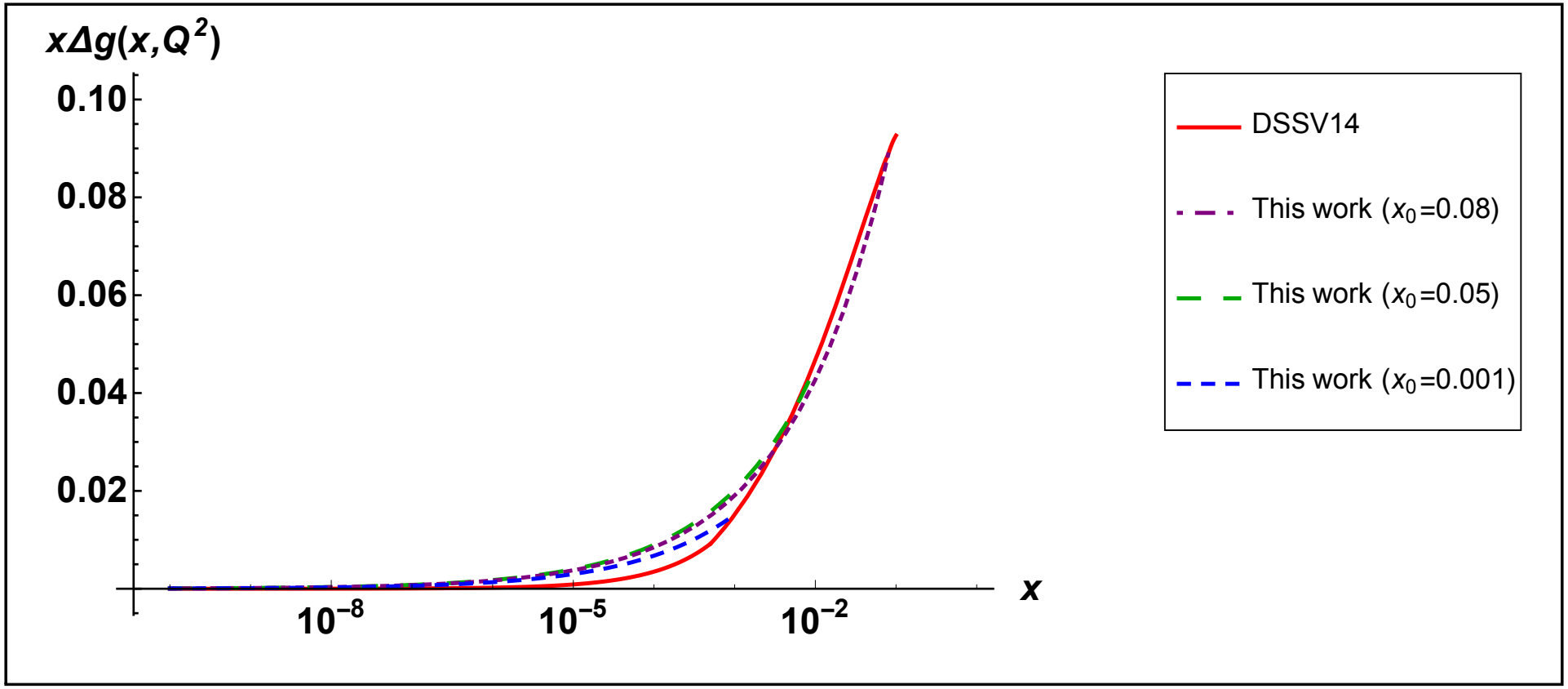
Impact on proton spin



- Here we compare our results for the total quark helicity with DSSV, now including their error band.
- We observe consistency of our lower two curves with DSSV.
- Our upper curve disagrees with DSSV, but agrees with NNPDF (Nocera, Santopinto, '16).
- Better phenomenology is needed. EIC would definitely play a role.

Impact of our ΔG on the proton spin

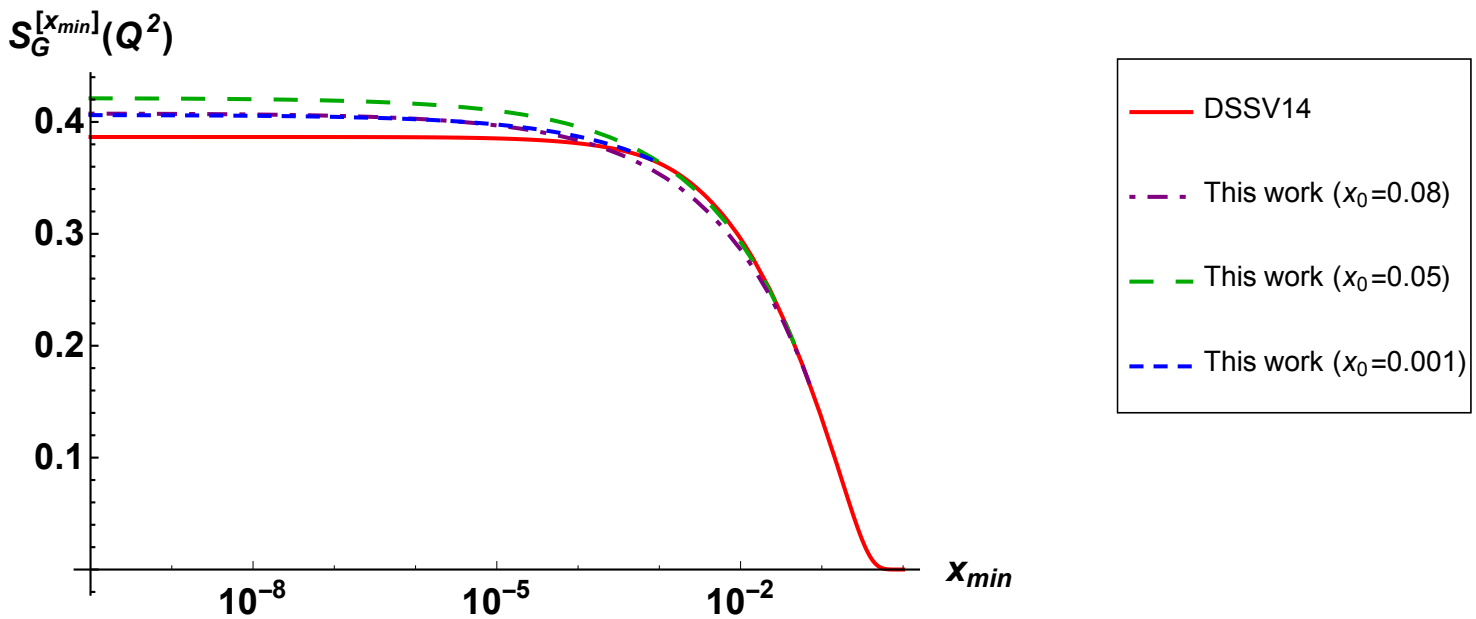
- We have attached a $\Delta\tilde{G}(x,Q^2) = N\,x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



"ballpark" phenomenology

Impact of our ΔG on the proton spin

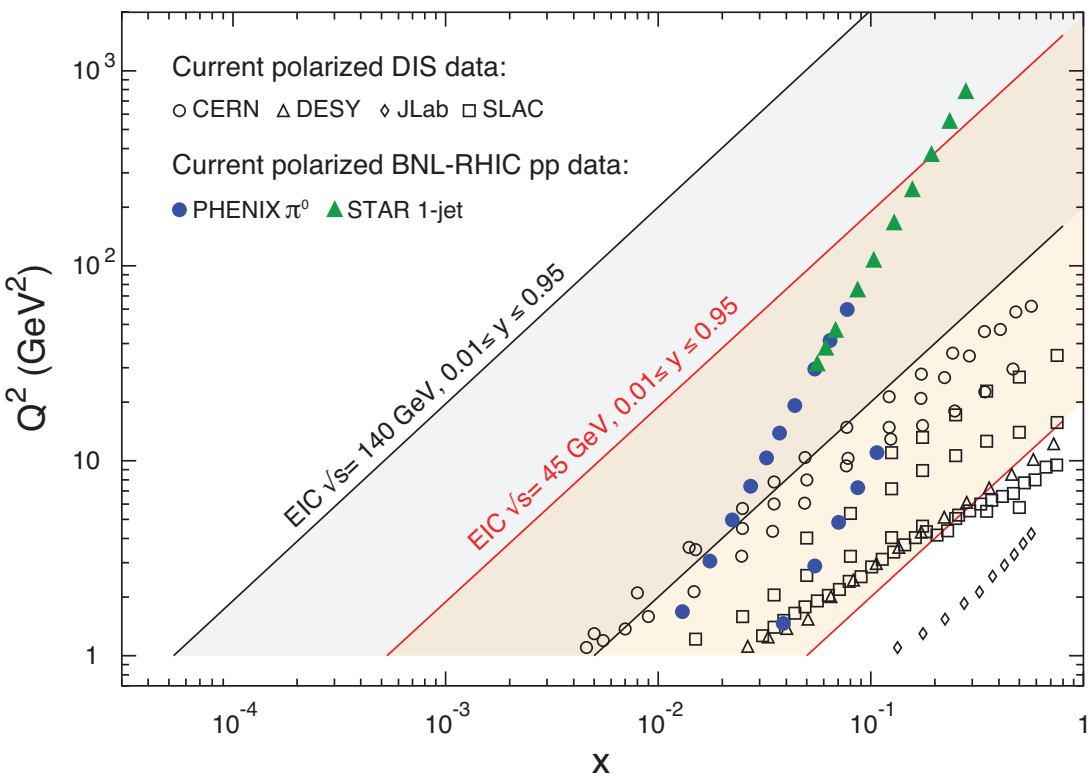
- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta G(x, Q^2)$ we plot it for x_0=0.08, 0.05, 0.001:

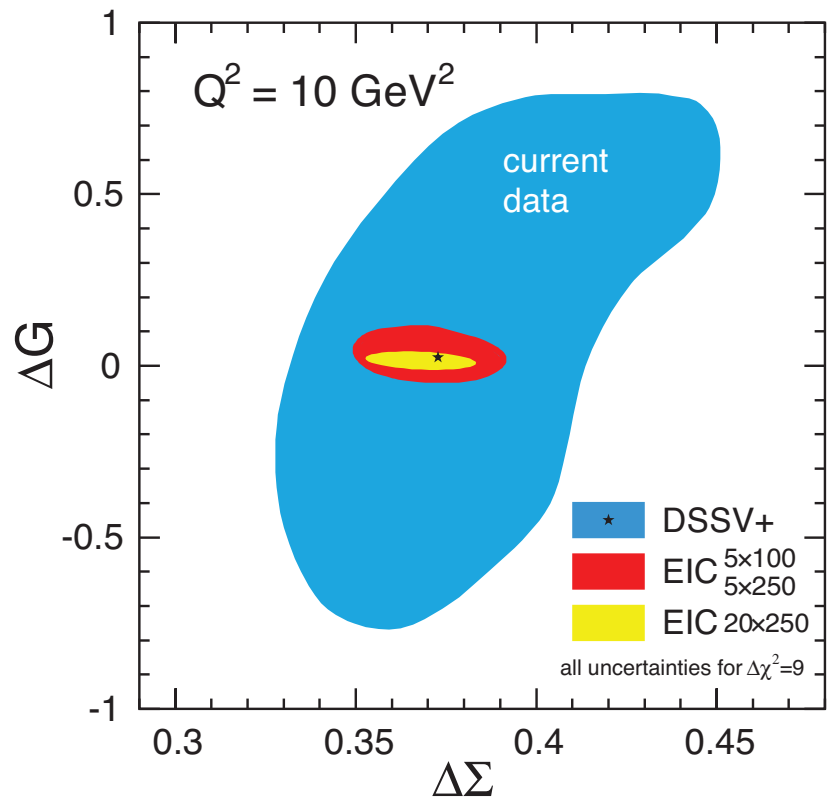


- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.

EIC & Spin Puzzle

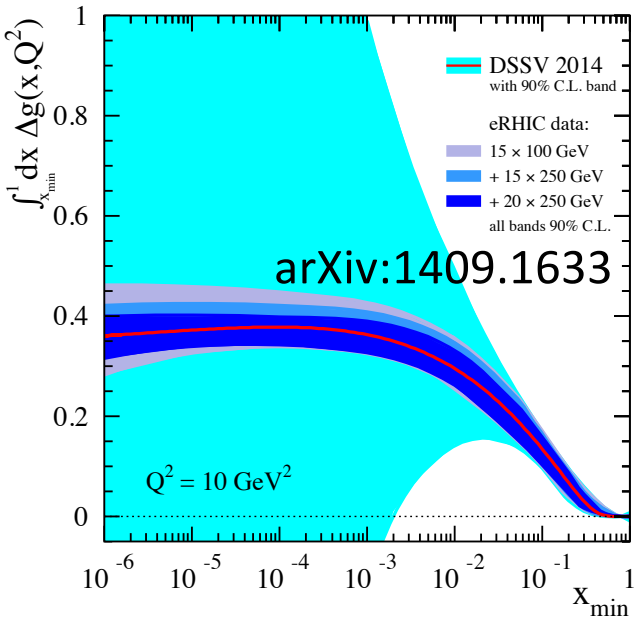
- Parton helicity distributions are sensitive to low-x physics.
- EIC would have an unprecedented low-x reach for a polarized DIS experiment, allowing to pinpoint the values of quark and gluon contributions to proton's spin:

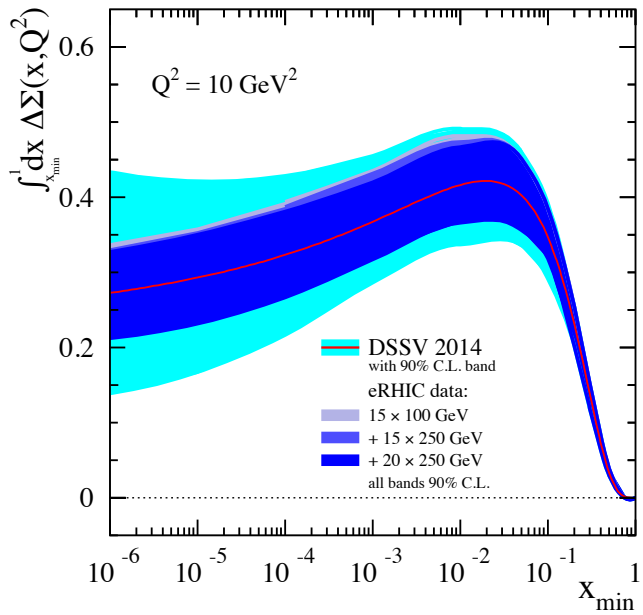


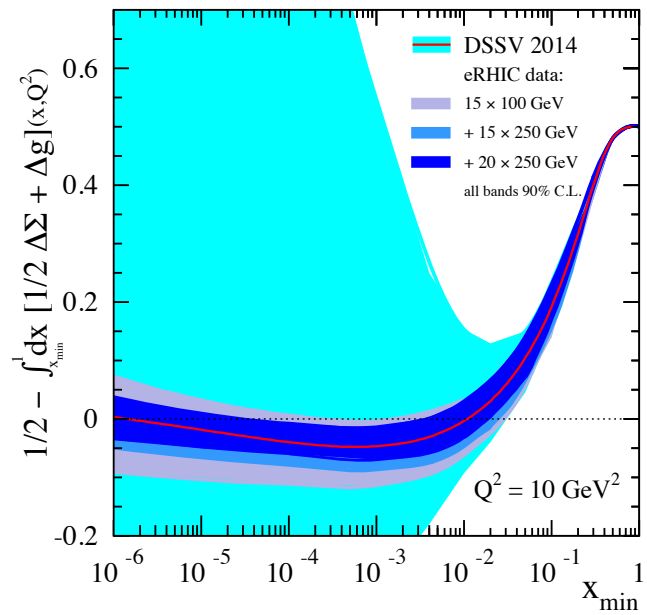


- ΔG and $\Delta\Sigma$ are integrated over x in the $0.001 < x < 1$ interval.

EIC: Solving the Spin Puzzle







1/2 - Gluon - Quarks = orbital angular momentum

- Above plot shows the running integral of $\Delta g(x,Q^2)$ from x_{min} to 1 as a function of x_{min}
- Large reduction in uncertainty on ΔG from EIC can be seen
- EIC will also reduce the uncertainty on the quark contribution to the proton spin

Constraints on gluon and quark contributions will provide information on the orbital angular momentum component of proton spin

E. Aschenaur et al, **arXiv:1409.1633 [hep-ph]**

Conclusions

- At large N_c we have obtained the following small-x asymptotics:

$$\Delta q(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \tfrac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 1.88\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}},$$

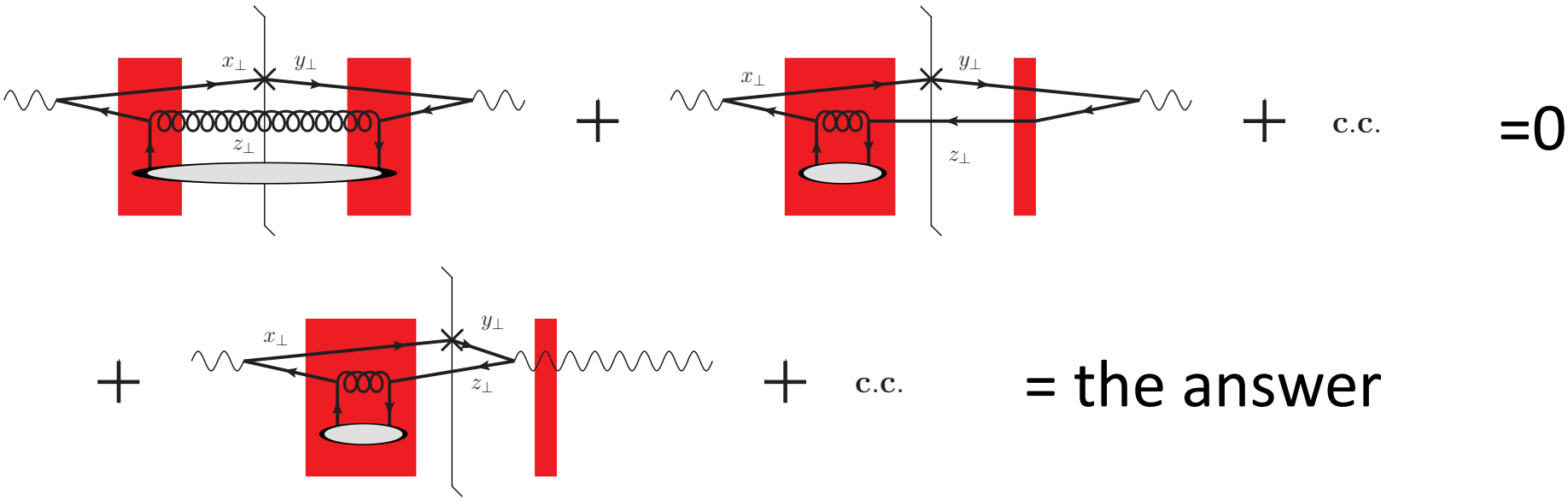
$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}}$$

- Future helicity and OAM work will involve + solving the large-N_C&N_f equations + including running coupling corrections + LLA corrections + phenomenology to constrain the spin+OAM coming from small-x quarks and gluons.
- EIC should be able to measure helicity TMDs with high precision and down to fairly small x. We may also be able to learn something about OAM.

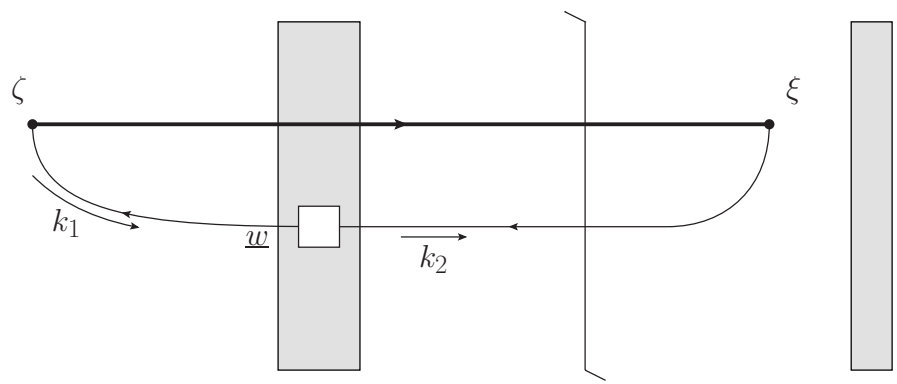
Backup Slides

Quark Helicity TMD at Small x

- Dominance of diagram B can also be obtained by applying crossing symmetry to the SIDIS process (KS '15):



- Compare the last line to the diagram B: reflecting the cc amplitude into the amplitude reduces the above diagram to the one on the right.



Quark and Gluon OAM at Small x

Quark OAM: Definition

- We begin by writing the (Jaffe-Manohar) quark OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k,b)$$

with the quark SIDIS Wigner distribution

$$W^{q,SIDIS}(k,b) = 2 \sum_X \int d^2 r \, dr^- \; e^{ik\cdot r} \left\langle \bar{\psi}_\alpha \left(b - \tfrac{1}{2} r\right) \, V_{\underline{b} - \frac{1}{2}\underline{r}} \left[b^- - \tfrac{1}{2} r^-, \infty\right] |X\rangle \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \right.$$
$$\left. \times \; \langle X| \, V_{\underline{b} + \frac{1}{2}\underline{r}} \left[\infty, b^- + \tfrac{1}{2} r^-\right] \; \psi_\beta \left(b + \tfrac{1}{2} r\right) \right\rangle$$

- Here, and above, the angle brackets denote ”CGC averaging” in the (polarized) proton target:

$$\left\langle \hat{\mathcal{O}}(b,r) \right\rangle = \frac{1}{2P^+} \int \frac{d^2\Delta \, d\Delta^+}{(2\pi)^3} \, e^{ib\cdot\Delta} \left\langle P + \frac{\Delta}{2} \middle| \hat{\mathcal{O}}(0,r) \middle| P - \frac{\Delta}{2} \right\rangle$$

Quark OAM: small-x simplifications

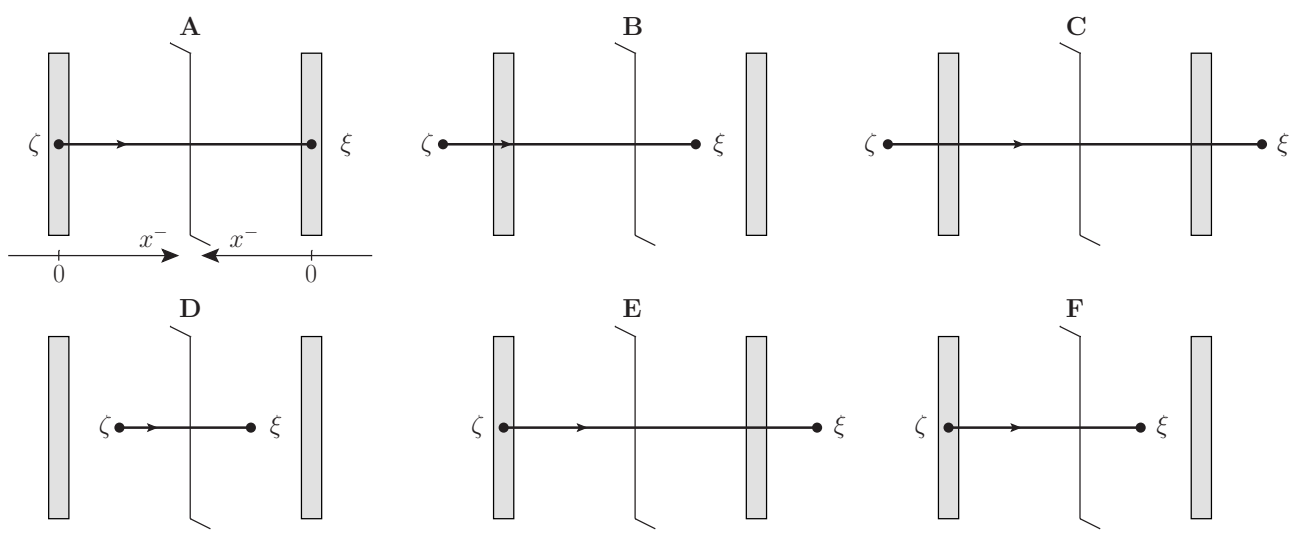
- The resulting quark OAM "PDF" is

$$L_q(x,Q^2) = \frac{2P^+}{(2\pi)^3} \sum_X \int d^2k_\perp\, d^2\zeta\, d\zeta^-\, d^2\xi\, d\xi^-\, e^{ik\cdot(\zeta-\xi)} \left(\frac{\underline{\zeta}+\underline{\xi}}{2} \times \underline{k}\right) \Big\langle \bar\psi_\alpha(\xi)\, V_{\underline{\xi}}[\xi^-,\infty]\, |X\rangle\, \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \times \langle X|\, V_{\underline{\zeta}}[\infty,\zeta^-]\, \psi_\beta(\zeta)\Big\rangle$$

- This can be compared to quark helicity,

$$g_{1L}^q(x,k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta\, d\zeta^-\, d^2\xi\, d\xi^-\, e^{ik\cdot(\zeta-\xi)} \left(\frac{1}{2}\gamma^+\gamma^5\right)_{\alpha\beta} \Big\langle \bar\psi_\alpha(\xi)\, V_{\underline{\xi}}[\xi^-,\infty]\; V_{\underline{\zeta}}[\infty,\zeta^-]\, \psi_\beta(\zeta)\Big\rangle$$

- The operators are different, but the structure is similar. The quark OAM can be evaluated in the same way as the quark helicity operator: only diagram B survives.



Quark OAM: small-x expression

- After some algebra we arrive at the following small-x expression for quark OAM:

$$L_{q+\bar{q}}(x,Q^2) = \frac{8N_c}{(2\pi)^5} \int d^2k_\perp \, d^2x_{10} \, d^2x_1 \, e^{i\underline{k}\cdot\underline{x}_{10}} \, \frac{\underline{x}_{10}}{x_{10}^2} \times \frac{\underline{k}}{\underline{k}^2} \, \underline{x}_1 \times \underline{k} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, G_{10}(zs) - \sum_f [\Delta q^f(x,Q^2) + \Delta\bar{q}^f(x,Q^2)]$$

- The result is written in terms of the polarized dipole amplitude $G_{10}(z)$. It seems we are done, right?
- This is almost correct. The remaining minor technicality is that the above quark OAM depends on the "first moment" of the polarized dipole amplitude

$$I^k(\underline{x}_{10}, zs) = \int d^2x_1 \, x_1^k \, G_{10}(zs)$$

while all our earlier results for the quark helicity were derived for the "zeroth moment", the impact-parameter integrated polarized dipole amplitude

$$G(x_{10}^2, zs) = \int d^2x_1 \, G_{10}(zs)$$

Quark OAM: small-x asymptotics

- It turns out that the "first moment" of the polarized amplitude is subleading. It grows with energy as a smaller power of energy

$$I^k(\underline{x}_{10}, zs) \sim \left(zsx_{10}^2\right)^{2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

than the flavor-singlet quark helicity distribution

$$\Delta\Sigma(x,Q^2) = \sum_f [\Delta q^f(x,Q^2) + \Delta\bar{q}^f(x,Q^2)] \sim \left(\frac{1}{x}\right)^{\alpha_h^q} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \approx \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Since 2.31 > 2, we get (cf. Y. Hatta & D.-J. Yang, 2018)

$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that this is not a complete cancellation, the contribution to the proton spin is

$$\frac{1}{2}\,\Delta\Sigma(x,Q^2) + L_{q+\bar{q}}(x,Q^2) = -\frac{1}{2}\,\Delta\Sigma(x,Q^2)$$

Gluon OAM: definition

- The gluon OAM story is similar. We start with the Wigner distribution definition

$$L_z = \int \frac{d^2b_\perp db^- \, d^2k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k,b)$$

with the dipole Wigner distribution for gluons

$$W^{G\,dip}(k,b) = \frac{4}{xP^+} \int d\xi^- \, d^2\xi_\perp \; e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}}$$
$$\times \left\langle \mathrm{tr}\left[F^{+i}(b - \tfrac{1}{2}\xi)\, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi)\, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

- We obtain the following expression for the gluon OAM "PDF" (cf. Hatta et al, 2016)

$$L_G(x,Q^2) = \frac{4}{(2\pi)^3\, x} \int d^2b_\perp db^- \, d^2k_\perp \, d\xi^- \, d^2\xi_\perp \; (\underline{b} \times \underline{k}) \; e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}}$$
$$\times \left\langle \mathrm{tr}\left[F^{+i}(b - \tfrac{1}{2}\xi)\, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi)\, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

Gluon OAM: small-x expression

- Gluon OAM at small x can (similarly to the quark OAM) be rewritten in terms of the "moment" of the polarized dipole amplitude G_{10}^i for the gluon helicity TMD. This object is different from the polarized amplitude for the quark.
- We get

$$L_G(x,Q^2) = -\frac{8iN_c}{g^2\,(2\pi)^3}\int d^2x_{10}\,d^2k_\perp\,e^{i\underline{k}\cdot\underline{x}_{10}}\,(\underline{k}\cdot\underline{x}_{10})\,G_5\left(x_{10}^2, zs = \frac{Q^2}{x}\right)$$

where

$$\int d^2x_1\;x_1^j\,\nabla_{10}^i\,G_{10}^i(zs) = x_{10}^j\,G_4(x_{10}^2, zs) + \epsilon^{jk}x_{10}^k\,G_5(x_{10}^2, zs)$$

- We write down and solve the equations for G_5.

Gluon OAM: small-x asymptotics

- We arrive at the following relation

$$L_G(x,Q^2) = \left(\frac{\alpha_h^q}{4}\,\ln\frac{Q^2}{\Lambda^2}\right)\,\Delta G(x,Q^2)$$

where

$$\alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\,N_c}{2\pi}}$$

- We conclude that

$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\,N_c}{2\pi}}} \sim \left(\frac{1}{x}\right)^{1.88\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

- Note that with the DLA accuracy we could also simply conclude that

$$|L_G| \ll |\Delta G|$$

Conclusions for OAM asymptotics

- We have constructed the small-x asymptotics of the quark and gluon OAM (in the Jaffe-Manohar decomposition).

- In the large-N_C limit we obtain

$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}},$$

$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}}$$

Helicity PDFs at Small-x

- Theoretical calculations by Bartels, Ermolaev and Ryskin (BER, 1996) and by YK, Pitonyak and Sievert (2015-17, KPS).
- KPS results (large-N_C, this talk):

$$\Delta q(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 1.88\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

- BER results at large-N_C (for both quark and gluon helicity distributions):

$$\alpha_h^{BER} = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

For finite N_C and N_f=4 BER have 3.66 → 3.45.

Helicity PDFs at Small-x

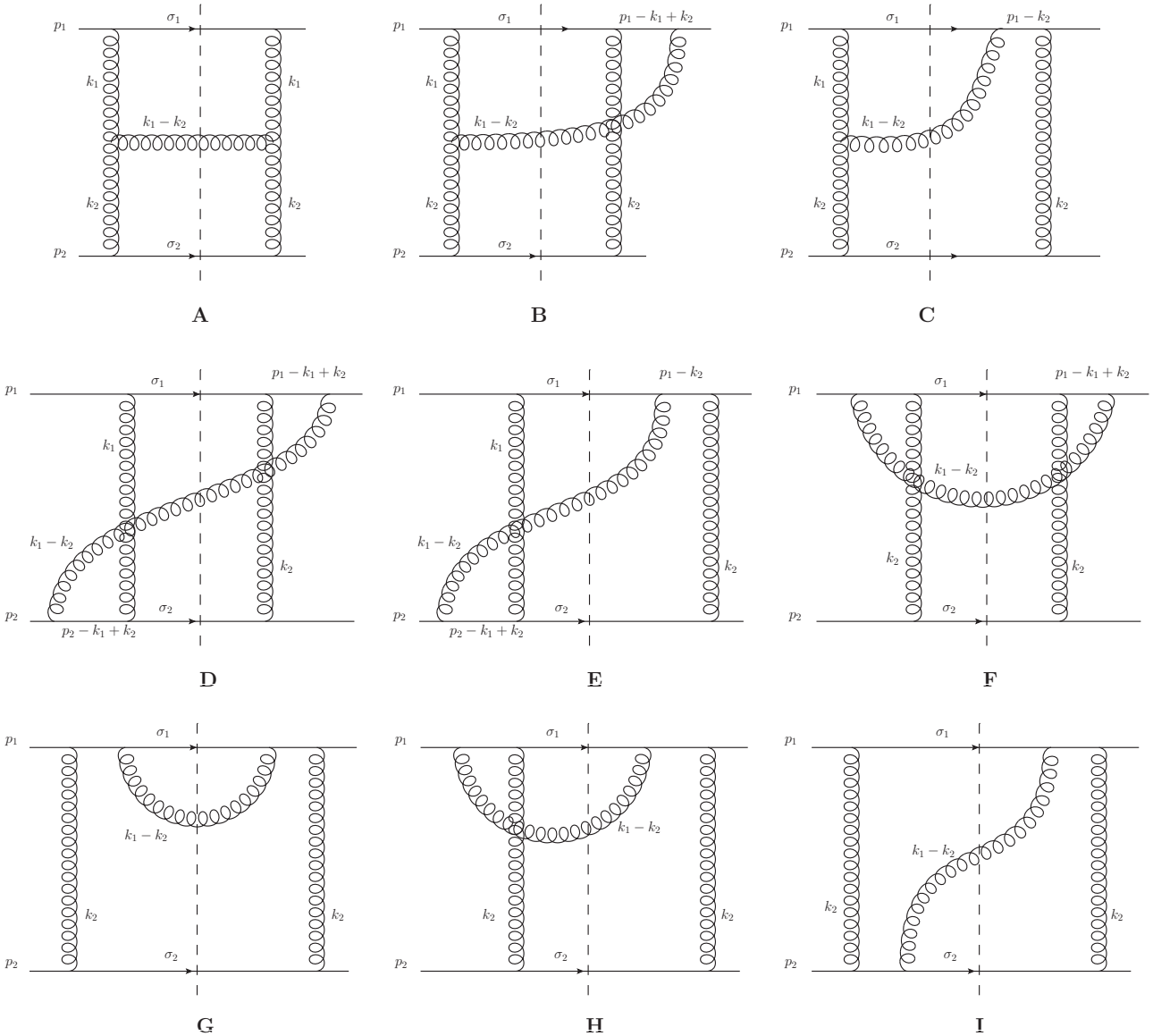
$$PDF(x) \sim \left(\frac{1}{x}\right)^{\alpha}$$

- The summary of the existing powers of x is

Observable	Evolution	Intercept	$Q^2 = 3$ GeV^2 $\alpha_s = 0.343$	$Q^2 = 10$ GeV^2 $\alpha_s = 0.249$	$Q^2 = 87$ GeV^2 $\alpha_s = 0.18$
Unpolarized flavor singlet structure function F_2	LO BFKL Pomeron	$1 + \frac{\alpha_s N_c}{\pi} 4 \ln 2$	1.908	1.659	1.477
Unpolarized flavor non-singlet structure function F_2	Reggeon	$\sqrt{\frac{2\,\alpha_s C_F}{\pi}}$	0.540	0.460	0.391
Flavor singlet structure function g_1^S	us (Pure Glue)	$2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$	0.936	0.797	0.678
	BER (Pure Glue)	$3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.481	1.262	1.073
	BER ($N_f = 4$)	$3.45\sqrt{\frac{\alpha_s N_c}{2\pi}}$	1.400	1.190	1.011
Flavor non-singlet structure function g_1^{NS}	BER and us (large-N_c)	$\sqrt{\frac{\ }{\pi}}$	0.572	0.488	0.415

- Similar disagreement exists for OAM distributions at small x.
- However, even the smaller KPS powers lead to potentially important contributions to the proton spin coming from small x. (next)

Comparison with BER



To better understand BER work, we tried calculating one (real) step of DLA helicity evolution for the qq->qq scattering.

It appears that we have identified the $k_2 >> k_1$ (or $k_1 >> k_2$) regime in which diagrams A, B, C, D, E, I are DLA, which was not considered by BER for B, C, … I.