From the Structure of Hadrons to the Muon Anomalous Magnetic Moment

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It was incomplete at least in two ways.

It conjectured but did not prove the model was renormalizable.

It was just a theory of leptons, not hadrons, including protons, neutrons, pions, kaons.
The Standard Model

“Contribution of neutral pseudoscalar mesons to $a_\mu^{HLbL}$ within a Schwinger-Dyson equations approach to QCD”,
K. Raya, AB, P. Roig,

“Pion and Kaon box contribution to $a_\mu^{HLbL}$”
A. Miramontes, AB, K. Raya, P. Roig,

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"Prospects for precise predictions of $a_\mu$", G. Colangelo et. al. Contribution to: 2022 Snowmass Summer Study,
e-Print: 2203.15810 [hep-ph].

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A. Miramontes, et. al., in progress
A muon with spin $s$ has a magnetic moment: $\mu = g \frac{e}{2m} s$

The factor $g$ is called the gyro-magnetic factor. The Dirac equation for a charged elementary fermion with spin 1/2 implies $g = 2$.

The **anomalous magnetic moment** is the deviation from $g = 2$, parameterized by $a_\mu = (g-2)/2$.

It appears due to radiative corrections. Renormalization of QED was established in 1943 and 1947-1948 by Tomonaga, Schwinger and Feynman.

The leading contribution to $a_\mu$, calculated by Schwinger in 1949, is:

The amplitude of a muon scattering off an external electromagnetic field $A$ is: $(q=p_2-p_1)$:

\[
M = -ie \langle \mu_{p_2} | J^\mu (0) | \mu_{p_1} \rangle A_\mu (q)
\]
The QED Contributions

The complete result for $a_\mu$ is

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{EW}}.$$ 

The pure QED contribution is by far the largest and has been evaluated up to $\mathcal{O}(\alpha^5)$ with negligible numerical uncertainty.

$$a_\mu^{\text{QED}} = A_1 + A_2(m_\mu/m_e) + A_2(m_\mu/m_\tau) + A_3(m_\mu/m_e, m_\mu/m_\tau)$$

$$A_i = \left(\frac{\alpha}{\pi}\right) A_i^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A_i^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A_i^{(6)} + \cdots, \text{ for } i = 1, 2, 3$$

Summing all the terms in the perturbative QED expansion up to $\mathcal{O}(\alpha^5)$, the complete QED contribution to the muon anomalous magnetic moment is as follows:

$$a_\mu^{\text{QED}}(\alpha(Cs)) = 116 \, 584 \, 718.931(7)(17)(6)(100)(23)(104) \times 10^{-11}$$

$$a_\mu^{\text{QED}}(\alpha(a_c)) = 116 \, 584 \, 718.842(7)(17)(6)(100)(28)(106) \times 10^{-11}$$

The EW Contributions

These contributions are defined as all SM contributions that are not contained in QED or QCD. The **EW contributions** are strongly suppressed by the heavy masses of the EW bosons; they contribute, numerically, at the same order as the HLbL correction.

The one-loop diagrams were evaluated as early as 1972 by **Weinberg** and **Jackiw**.

R. Jackiw, Steven Weinberg, Phys. Rev. D 5 (1972) 2396-2398

\[ a^\text{EW(1)}_\mu \propto \frac{\alpha}{4\pi s_w^2} \frac{m_\mu^2}{M_W^2}, \quad m_\mu^2/M_W^2 \approx 10^{-6} \]

A similar suppression happens in new physics models with new heavy particles of mass \( M_{\text{NP}} \). Typically such particles contribute terms \( \sim \alpha (m_\mu/M)^2 \).

Their contribution only shows up at the level of the **seventh significant digit**. It has been evaluated up to two loops and is known to better than one percent.
The EW Contributions

It has been evaluated up to 2 loops and leading logarithms at the 3 loop level. One loop result is:

\[ a_{\mu}^{\text{EW}(1)} = 194.79(1) \times 10^{-11} \]
\[ a_{\mu;\text{bos}}^{\text{EW}(2)} = -19.96(1) \times 10^{-11} \]

Large leading logs at two loops

Summing up the numerical results of the one-loop contributions, the two-loop computations and the leading three-loop logarithms, we have the full weak interactions contribution:

\[ a_{\mu}^{\text{EW}} = 153.6(1.0) \times 10^{-11} \]

Sample bosonic two-loop Feynman diagrams contributing to \( a_{\mu}^{\text{EW}} \)

\[ a_{\mu}^{\text{QED}} = 116 \text{, } 584 \text{, } 718.931 \times 10^{-11} \]

Sample fermionic two-loop Feynman diagrams contributing to \( a_{\mu}^{\text{EW}} \)
QCD - Hadronic Vacuum Polarization

Hadronic contributions are the most difficult to calculate and are responsible for almost all theoretical uncertainty.

The leading hadronic contribution appears at $O(\alpha^2)$ and is due to hadronic vacuum polarization.

Higher-order insertions of HVP at NLO. The gray blobs refer to HVP, the white one to leptonic VP.


$Lattice world average:
\begin{align*}
    a_{\mu}^{HVP, \ LO} &= 6931(40) \times 10^{-11} ,
    \\
    a_{\mu}^{HVP, \ NLO} &= -98.3(7) \times 10^{-11} ,
    \\
    a_{\mu}^{HVP, \ NNLO} &= 12.4(1) \times 10^{-11}.
\end{align*}$
The **hadronic light-by-light** scattering contribution appears at $\mathcal{O}(\alpha^3)$.

\[
q^H_{\mu L} = 92(19) \times 10^{-11}
\]

(phenomenology + lattice QCD)
QCD - Hadronic Light by Light Scattering

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QCD: Facts and Challenges

QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).

- Quarks and gluons do not reach detectors.
- Formation of color-singlet bound states: “Hadrons” mesons, baryons, tetraquarks, molecules
- Emergence of hadron masses (EHM) from QCD dynamics

![Diagram showing Higgs mechanism and dynamics of gluons]

- Higgs mechanism: Quark Mass ~ 3 MeV
- Dynamics of gluons: Proton Mass = 938.27 MeV
  - ~ 1% of proton mass
  - ~ 99% of proton mass
**QCD: Current Understanding and Challenges**

Origins of **confinement** and **dynamical mass generation** can be traced back to the Green functions of **quarks** and **gluons**.

These emergent phenomena of **QCD**, non-existent in perturbation theory are naturally linked to the infrared enhancement of the **strong running coupling**.

The effects of the pattern of **dynamical mass generation** are traceable in the **form factors** of mesons which contribute to the **anomalous magnetic moment** of the **muon**.

\[
S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma \cdot p + M_f(p^2))
\]
QCD – Schwinger-Dyson Equations

Gauge Technique – Non Perturbative Solutions

➢ A. Salam, R. Delbourgo, Phys. Rev. 135 (1964) 6, B1398-B1427.

DCSB – Non-perturbative QED


DCSB – Spectrum of PS-Mesons


DCSB – MT Model - Vector Mesons


Quark propagator:

$$S(p^2, \mu^2) = \frac{Z(p^2, \mu^2)}{i \gamma \cdot p + M(p^2)}$$

Renormalizable Electrodynamics of Scalar and Vector Mesons. II

Abdus Salam*
Imperial College, London, England

Robert Delbourgo†
Imperial College, London, England

The “gauge technique” for solving field theories introduced in an earlier paper is applied to scalar and vector electrodynamics.

A. Dyson-Schwinger Set;

For a typical 3-field (e.g., electron-photon) interaction the well-known Dyson equations

$$S^{-1} = Z_2 S_0^{-1} + Z_4 e^2 \int \Gamma S T \phi D$$  \hspace{1cm} (I.1)

$$D^{-1} = Z_2 D_0^{-1} + Z_4 e^2 \int \Gamma S T \phi S$$  \hspace{1cm} (I.2)

SDE: electron propagator
Pions: Bound States and Goldstone Bosons

The pattern of **dynamical chiral symmetry breaking** and the Bethe-Salpeter amplitude to be computed by solving the Bethe-Salpeter equation.

\[
\Gamma_\pi(k, P) = \gamma_5 \left[ iE_\pi(k; P) + \gamma \cdot PF_\pi(k; P) + \gamma \cdot k k \cdot PG_\pi(k; P) + \sigma_{\mu\nu} k_\mu P_\nu H_\pi(k; P) \right]
\]
In studying the **elastic** or **transition form factors**, it is the **photon** which probes the dressed quarks inside the **bound states**, highlighting the importance of the **quark-photon vertex**.

Gauge covariance (WTI, TTI, LKFT), kinematic singularities, perturbation theory, multiplicative renormalizability.

\[
\Gamma_T^{\mu}(p, k, q) = \sum_{i=1}^{8} \tau_i(p^2, k^2, q^2) T_{\mu}^{ni}(p, k)
\]

\[
T_1^{\mu} = p_\mu (k \cdot q) - k_\mu (p \cdot q),
\]

\[
T_2^{\mu} = [p_\mu (k \cdot q) - k_\mu (p \cdot q)] (\not{q} + \not{k}),
\]

\[
T_3^{\mu} = q^2 \gamma_\mu - q_\mu \not{q},
\]

\[
T_4^{\mu} = q^2 \left[ \gamma_\mu (k + \not{p}) - (k + p)_\mu \right]
+ 2(k - p)_\mu \sigma_{\nu \lambda} p^\nu k^\lambda,
\]

\[
T_5^{\mu} = -\sigma_{\mu \nu} q^\nu,
\]

\[
T_6^{\mu} = \gamma_\mu (p^2 - k^2) + (p + k)_\mu \not{q},
\]

\[
T_7^{\mu} = \frac{1}{2} (p^2 - k^2) \left[ \gamma_\mu (\not{q} + \not{k}) - (p + k)_\mu \right]
- (p + k)_\mu \sigma_{\nu \lambda} p^\nu k^\lambda,
\]

\[
T_8^{\mu} = \gamma_\mu \sigma_{\nu \lambda} p^\nu k^\lambda - p_\mu \not{k} + k_\mu \not{q}.
\]
\[ \pi^0 \rightarrow \gamma^* \gamma \] Transition Form Factor

The pattern of dynamical chiral symmetry breaking dictates the \( Q^2 \) evolution of the transition form factor. Experiment and asymptotic QCD for largest \( Q^2 \) provide verifications.

\[ M(p^2) \sim (p^2)^{-\alpha} \]

\( \alpha = 0 \quad \alpha = 1 \)

BES III ?

Expected error size for \( Q^2 \)-dependent (stat. and a part of sys. errors) component at Belle II
\[ \pi^0 \rightarrow \gamma^*\gamma \] Transition Form Factor

Satisfies abelian anomaly and agrees with the prediction of asymptotic QCD.
Agrees well with experiment from low to intermediate range of momenta and favors Belle results.
The distribution amplitude (PDA) is broad and concave at the hadronic scale.
Within an error band stemming from the construction of the PDA from the lattice moments, it is in good agreement.

\( \eta_c, \eta_b \rightarrow \gamma^*\gamma \) Transition Form Factor

\[ G(Q^2) = \int_0^1 dx \frac{\phi_{\eta_c}(x) + \phi_{\eta_b}(x)}{x} \]

\( \eta_c \) transition form factor agrees with the only experimental results available.

It slightly disagrees with the results of the \textit{nrQCD}. It suggests that the \textit{nrQCD} is perhaps not good enough for \( \eta_c \).

\( \eta_b \): It agrees with the band provided by the \textit{nrQCD}.

\( \eta_c \): The agreement is noticeably better when NNLO contributions are added.

PDA is narrower. We expect asymptotic QCD limit to be reached at much higher \( Q^2 \).


\( \eta, \eta' \rightarrow \gamma^* \gamma \) Transition Form Factor

\( \eta \) and \( \eta' \): In the distribution amplitude, the dotted curve is the conformal limit

\( \eta \): The dot-dashed curve is the DA at 2GeV.

\( \eta \) and \( \eta' \): The solid blue curve is the u-d DA.

\( \eta \) and \( \eta' \): The dashed green curve is the s DA.

The electromagnetic form factor of $\pi^\pm$ is central to the JLab research.


Our computation is consistent with the valence quark distribution amplitude (PDA) which is broad and concave at the hadronic scale.

The results agree with earlier computation of the electromagnetic form factors of $\pi^\pm$.

The **electromagnetic form factors** of \( \pi^\pm \) and \( K^\pm \) can be measured till \( Q^2 \sim 10 \text{ Gev}^2 \) and \( 5 \text{ GeV}^2 \) respectively in the 12 GeV upgrade of JLab.


The results obtained from the **Schwinger-Dyson** equations appear robust under the calculational scheme employed and ingredients used as in the computation of the **transition form factors** and the DA of the kaon.

A. Miramontes et. al., Phys. Rev. D 105 (2022) 7, 074013

Neutral Pseudoscalar Pole Contributions

The transition form factor $\pi^0 \rightarrow \gamma^*\gamma$ extended to $\pi^0 \rightarrow \gamma^*\gamma^*$.

Dispersive methods:
- $a^\pi^0_{\mu}-pole = 63.6(2.7) \times 10^{-11}$
- $a^\eta_{\mu}-pole = 16.3(1.4) \times 10^{-11}$
- $a^{\eta'}_{\mu}-pole = 14.5(1.9) \times 10^{-11}$

Lattice:
- $a^{\pi^0}_{\mu}-pole = 59.7(3.6) \times 10^{-11}$

\( \pi^\pm \) and \( K^\pm \) Box Contributions

\[ k = p' - p \]

\[ \mu^- (p) \quad \mu^- (p') \quad = \quad \pi^0 + \ldots + \quad \Box \quad + \ldots + \]

Leading order \( P \)-box contributions to \( a_{\mu}^{\text{Eeln.}} \), where the corresponding \( P \) meson EFFs are highlighted by the purple filled circles.

A. Miramontes, AB, K. Raya, P. Roig, Phys. Rev. D 105 (2022) 7, 074013

Radial excitations of \( \pi \) and \( K \):

A. Miramontes, et. al. In preparation (Preliminary)

\[ a_{\mu}^{\pi^+ \text{-box}} = -(15.6 \pm 0.2) \times 10^{-11} \]

\[ a_{\mu}^{\pi^- \text{-box}} = -15.9(2) \times 10^{-11} \]

\[ a_{\mu}^{K^+ \text{-box, VMD}} = -0.50 \times 10^{-11} \]

Summary and Outlook

We have calculated the light ($\pi^0$, $\eta$, $\eta'$) and heavy pseudoscalar ($\eta_c$, $\eta_b$) pole contributions to the anomalous magnetic moment of the muon $a_\mu$.

We have also computed the $\pi^\pm$ and $K^\pm$ box diagram contributions to $a_\mu$.

We have 1st results for the box diagram contributions for the first radial excitations $\pi_1$ and $K_1$ to $a_\mu$.

In this formalism we also compute pion/kaon electromagnetic form factors, pseudoscalar transition form factors, distribution amplitudes, distribution functions, generalized distribution functions, satisfying axial vector WTI and connecting with asymptotic QCD and computing transition form factors of nucleons to their excited states.

"Timelike electromagnetic kaon form factor"
A.S. Miramontes, AB, Phys.Rev.D 107 (2023) 1, 014016 • e-Print: 2212.10800 [hep-ph]


What next? Axial vectors, scalars, hadron vacuum polarization contributions.
The complete result for $a_\mu$ is

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{EW}}.$$
FNL is analyzing data from run2 and run3. run4 is finished; run5 is planned in future.

To be able to meet the final precision $\Delta a_\mu(E989)=16\times10^{-11}$ projected for the Fermilab experiment is the challenge and goal for theory.


JPARC data taking will begin in 2025. First results are expected in 2027.

The tension between experiment and theory provides exciting years of testing the standard model through $a_\mu$ precision measurement and calculations.

It may provide indications of physics beyond the standard model of Salam, Glashow, Weinberg.