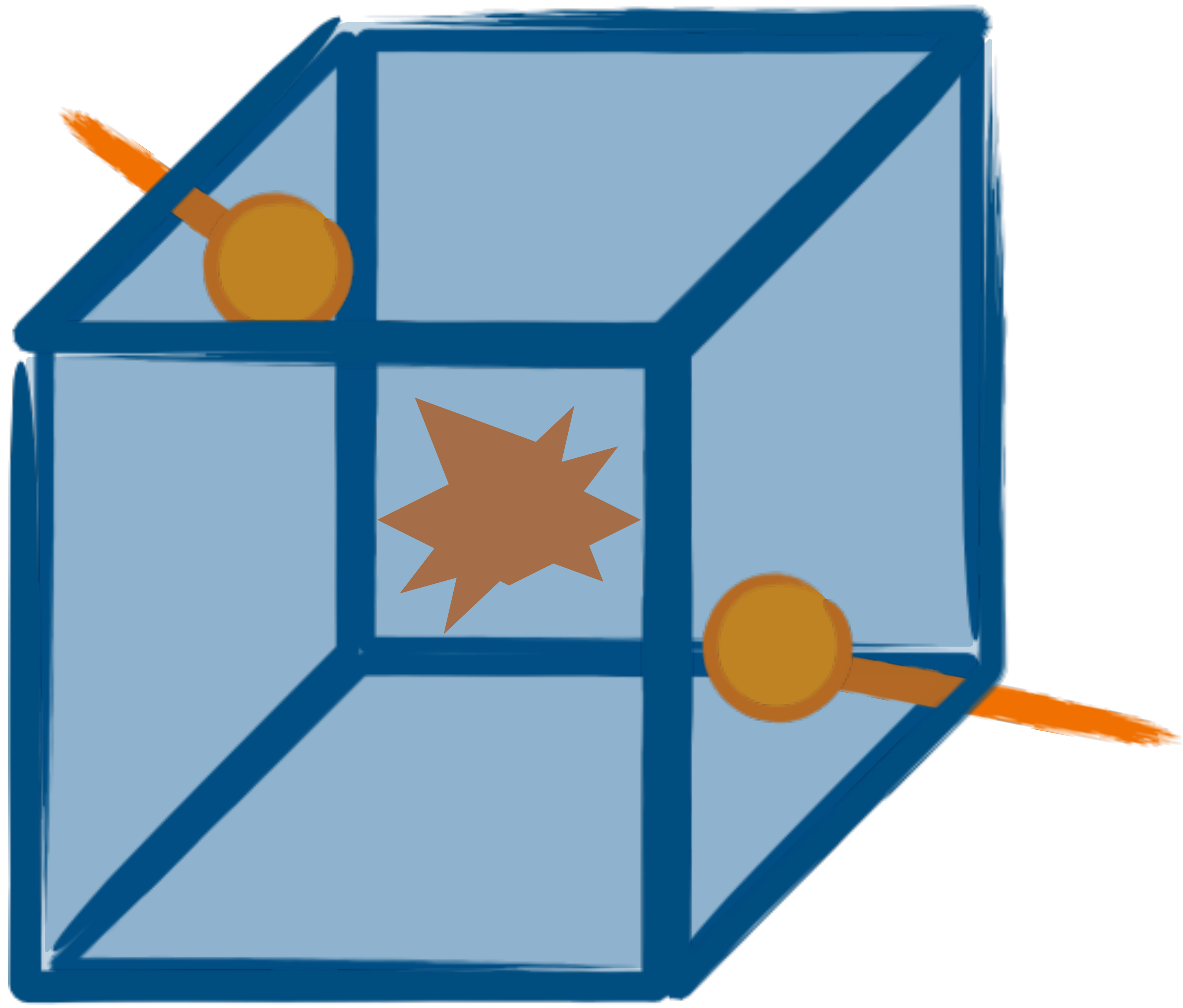
Two Troubled Hadrons in a Box

Addressing left-hand cut issues in the Lüscher scattering formalism

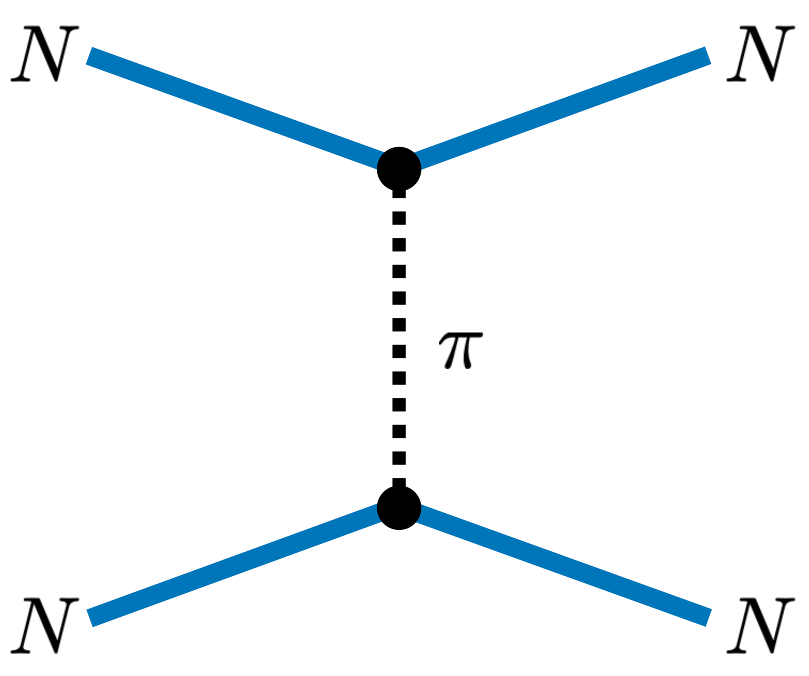


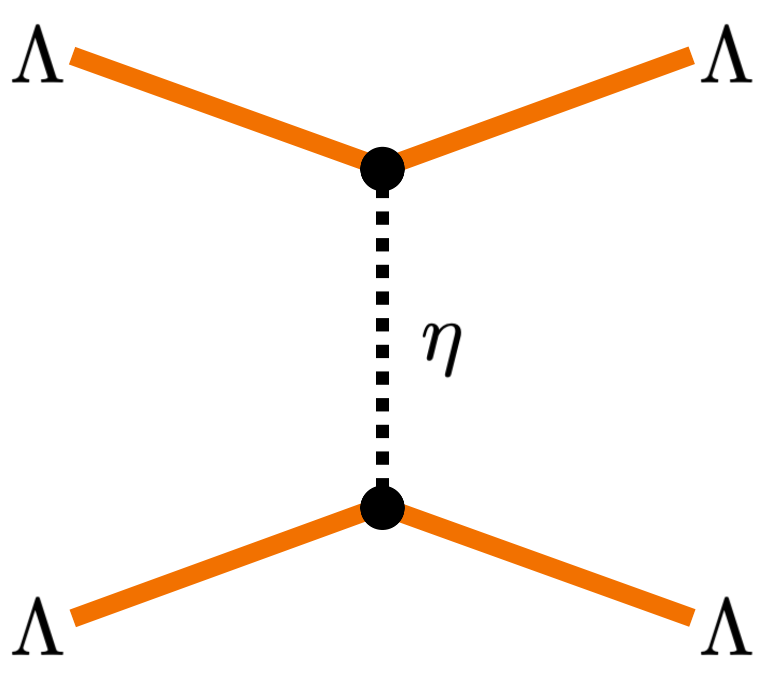
André Baião Raposo

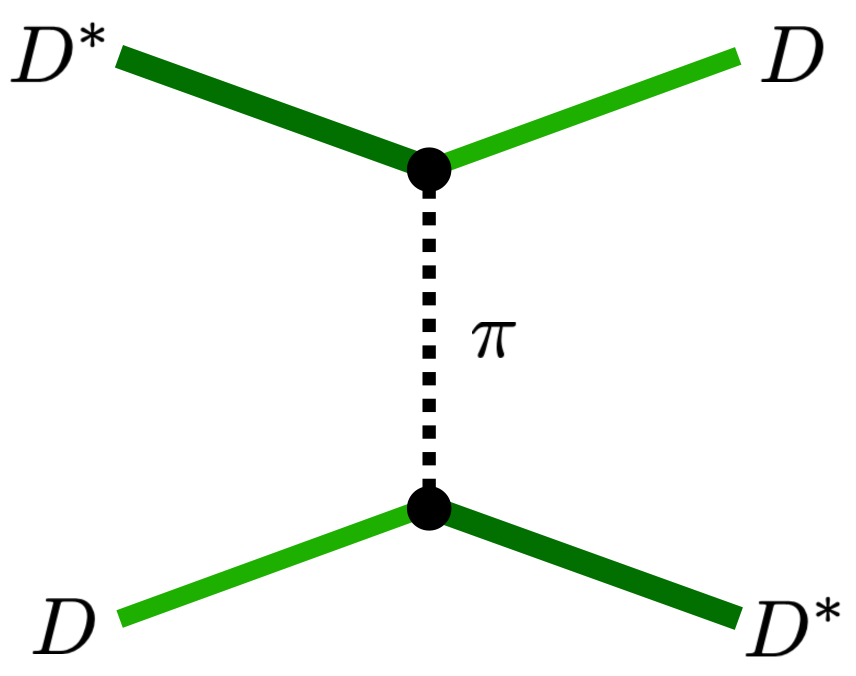
based on work with Max T. Hansen

0. Brief motivation and overview

- studying scattering using Lattice QCD requires indirect methods, such as the ***Lüscher method*** for 2-to-2 scattering
- recent lattice calculations of baryon-baryon and meson-meson scattering have encountered issues when applying standard formalism
- processes considered have ***left-hand cuts*** in the angular momentum projected scattering amplitudes
- cuts due to ***single exchanges of lighter mesons***
- application of standard formalism at energies on the cut leads to inconsistencies: we predict a real amplitude predicted but amplitude should be complex!



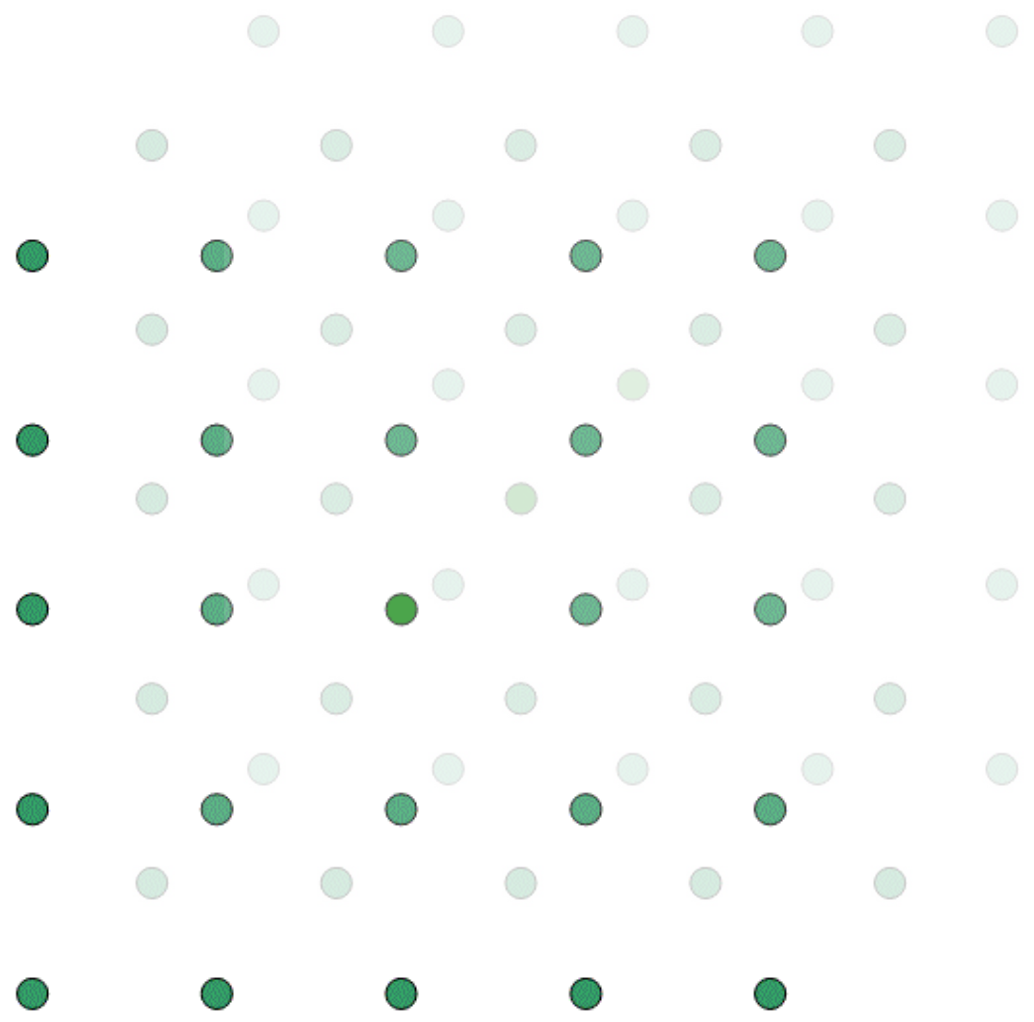




We revisit the derivation of the standard formalism and propose a solution in the form of a ***modified quantisation condition***

0. Lattice QCD

- computational method allowing non-perturbative calculations of QCD
- QCD path integral implemented in finite and discretised Euclidean spacetime – the ***lattice***
- field configurations sampled using Monte Carlo methods, weighed by the Euclidean action
- observables obtained by averaging over field configurations
- infinite-volume and continuum extrapolations often necessary for meaningful predictions



$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, \mathcal{D}U \;\; f[\psi, \overline{\psi}, U]\, e^{-S^E[\psi,\overline{\psi},U]}$$

fermion fields

gauge fields

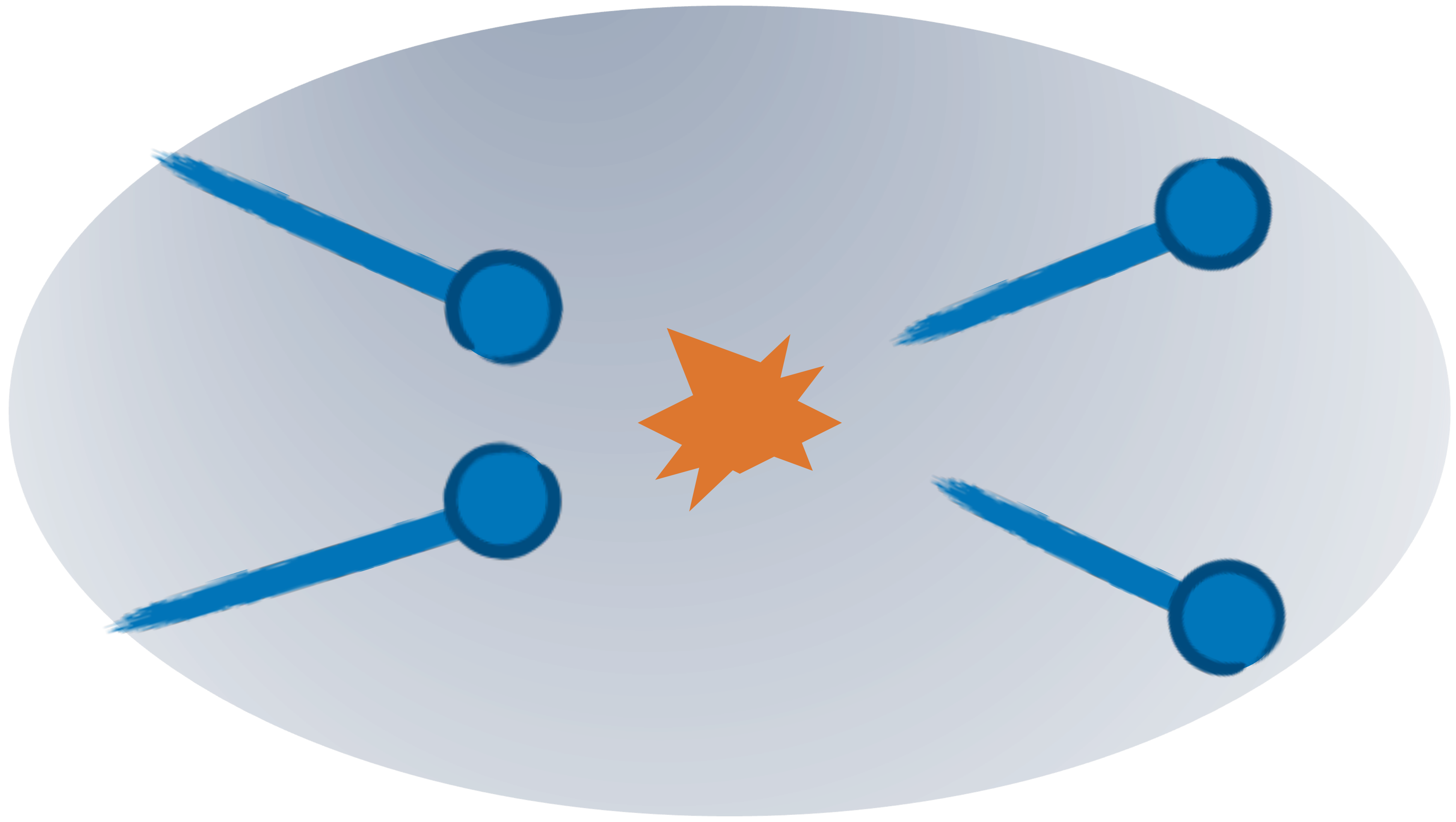
$$\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, \mathcal{D}U \;\; e^{-S^E[\psi,\overline{\psi},U]}$$

partition function

Euclidean action

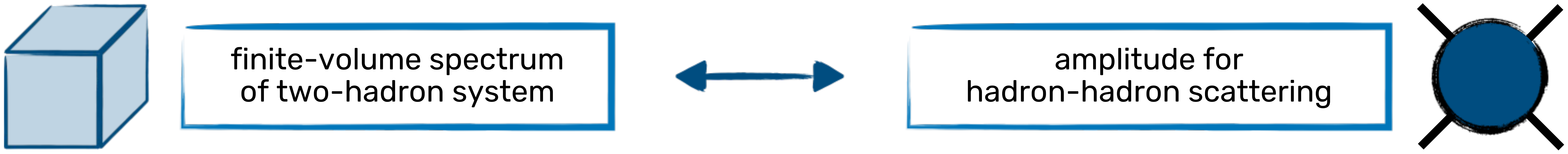
0. What about scattering?

- direct study not possible on the lattice
  (effects of finite-volume, Euclidean signature,…)
- need indirect methods:
  - finite-volume methods
  - (spectral functions…)



Finite-volume methods: exploit the volume-dependence to extract scattering information

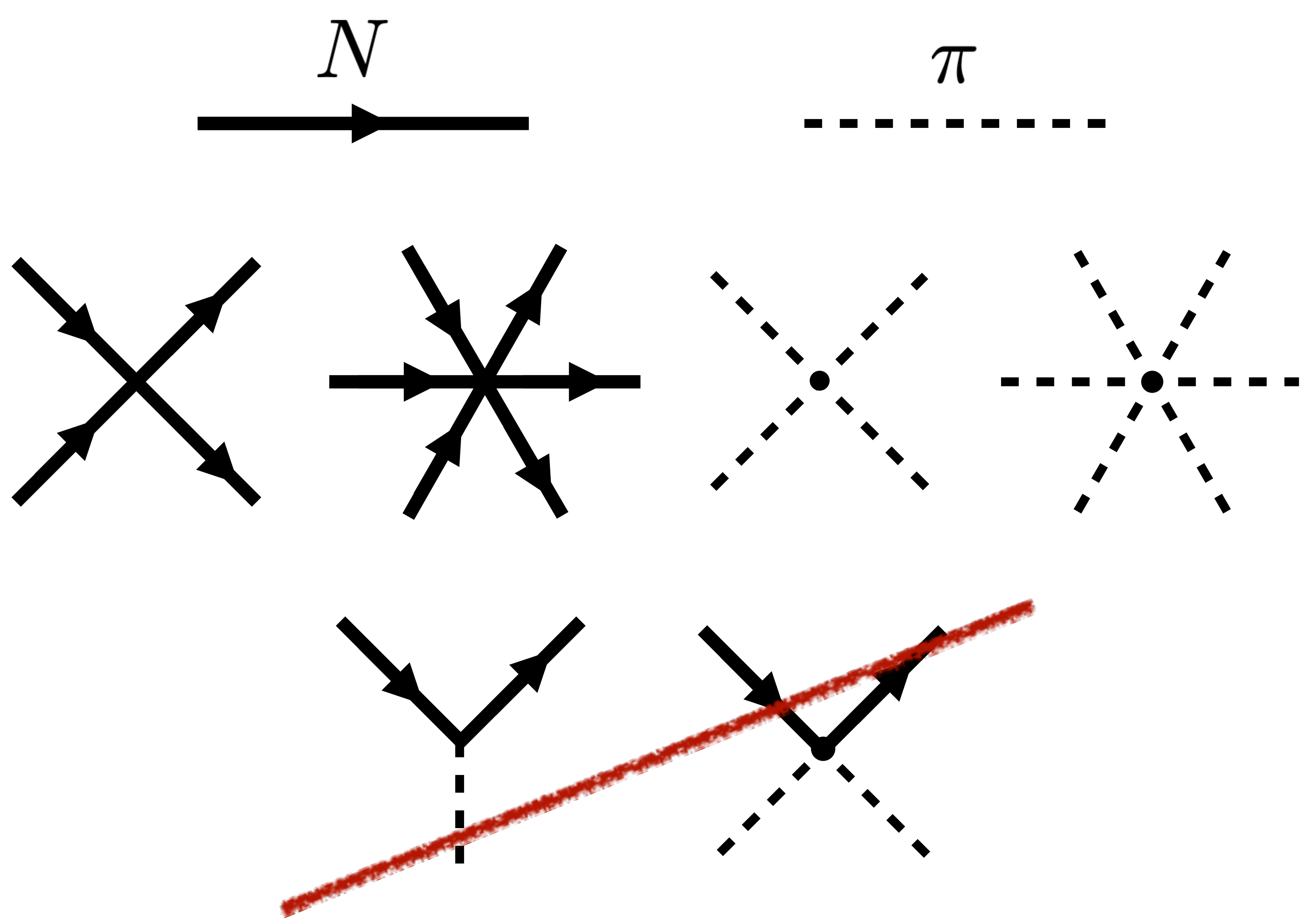
Leading method is ***Lüscher formalism*** for 2-to-2 scattering (and its numerous extensions)



[Lüscher 1986] and
many others

1. A detour into infinite-volume scattering

- degrees of freedom of QCD at low energies: QCD-stable hadrons $|\pi\rangle, |K\rangle, |N\rangle$, …
- study a toy model EFT of scalar "nucleons" and "pions", of masses M_N and M_π respectively, with $M_\pi < M_N$
- no assumptions on the form of the interactions, but baryon number is conserved
- ***for now, assume N and π are not coupled***



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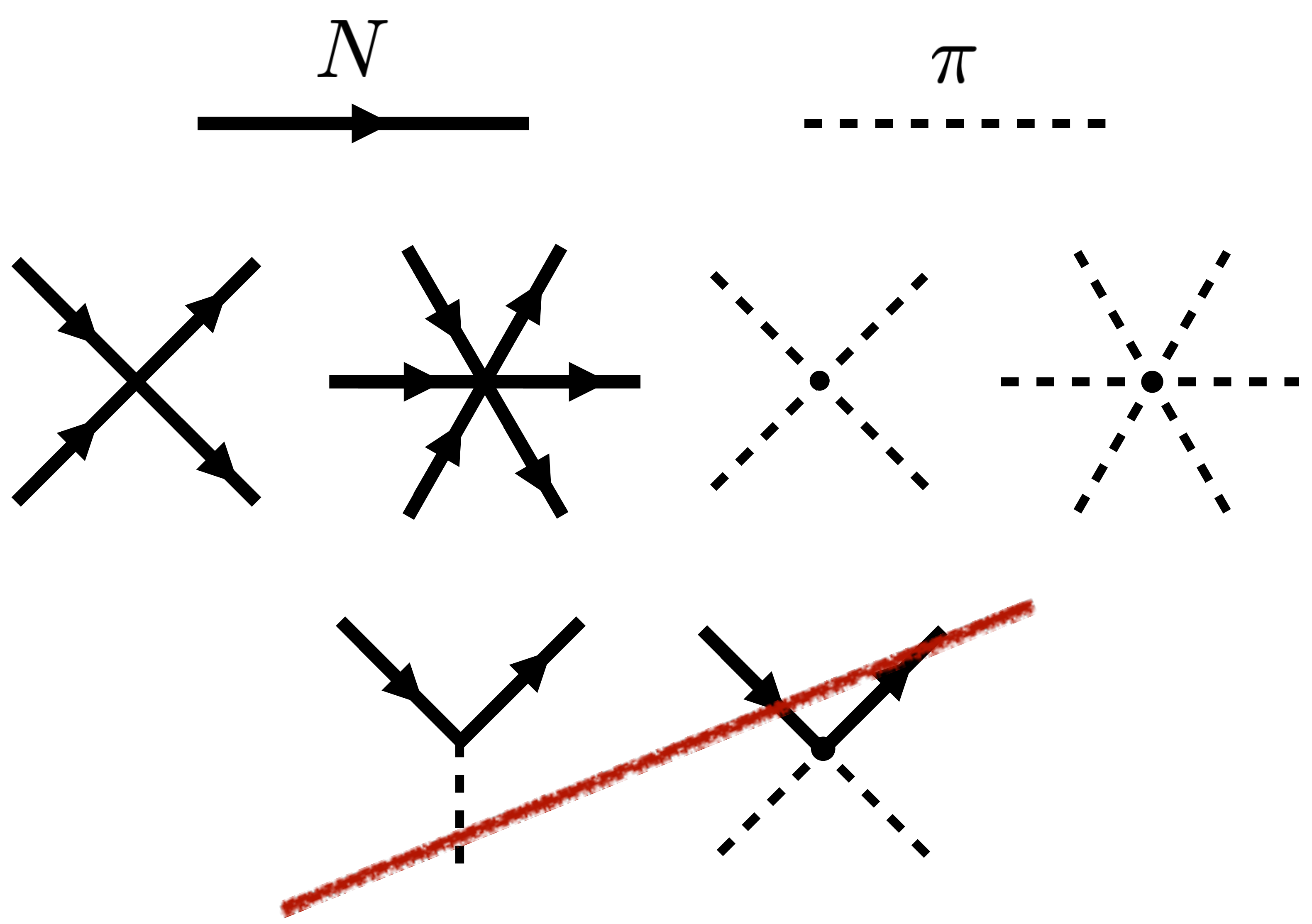
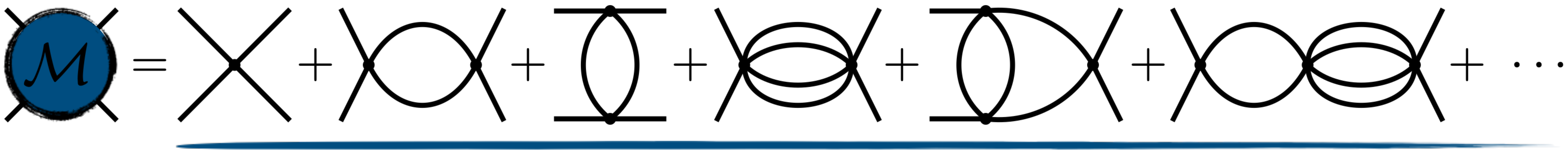


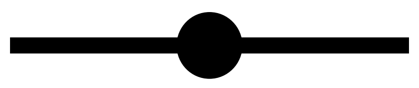
The ***scattering amplitude*** for NN elastic scattering given by the infinite sum:



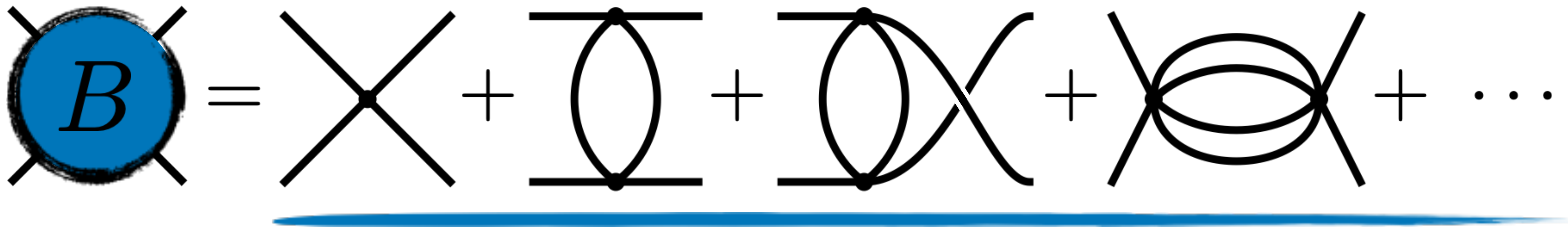
all amputated $NN \to NN$ diagrams

1. A detour into infinite-volume scattering

Dressed propagator

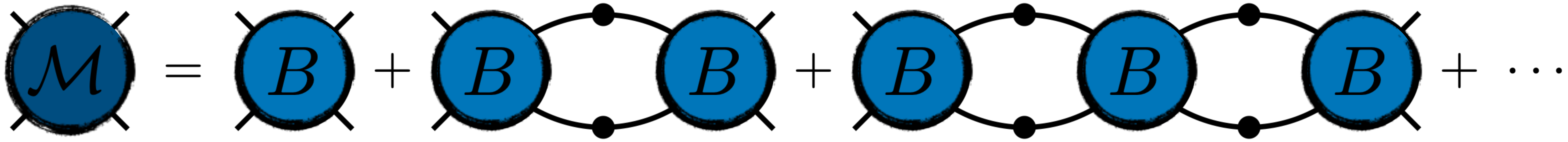


Bethe-Salpeter kernel



all amputated $NN \to NN$ diagrams which are 2-particle irreducible in the s-channel

We can write the ***amplitude*** as:



Can also project to definite angular momentum using a partial-wave expansion:

$$\mathcal{M}(s, \theta_{\text{cm}}) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, P_\ell(\cos\theta_{\text{cm}})\, \mathcal{M}_\ell(s)$$

CM scattering angle

Legendre polynomial

partial-wave amplitudes

1. Structure of the scattering amplitude

We will want to study the scattering amplitude when projected to specific angular momenta:

$$\mathcal{M}(s,\theta_{\text{cm}}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\theta_{\text{cm}})\,\mathcal{M}_\ell(s)$$

CM scattering angle — Legendre polynomial — partial-wave amplitudes

Using the ***optical theorem*** in elastic regime $(2M_N)^2 < s = E_{\text{cm}}^2 < (4M_N)^2$:

$$\operatorname{Im}\mathcal{M}_\ell(s) = \rho(s)\,|\mathcal{M}_\ell(s)|^2$$

phase space factor: $$\rho(s) = \frac{1}{32\pi}\sqrt{1-\frac{4M_N^2}{s}}$$

square root cut

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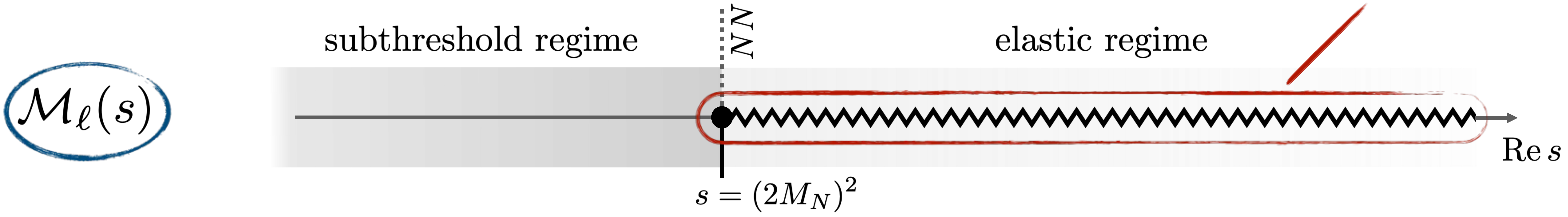

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$$\rho(s) = \frac{1}{32\pi}\sqrt{1 - \frac{4M_N^2}{s}}$$

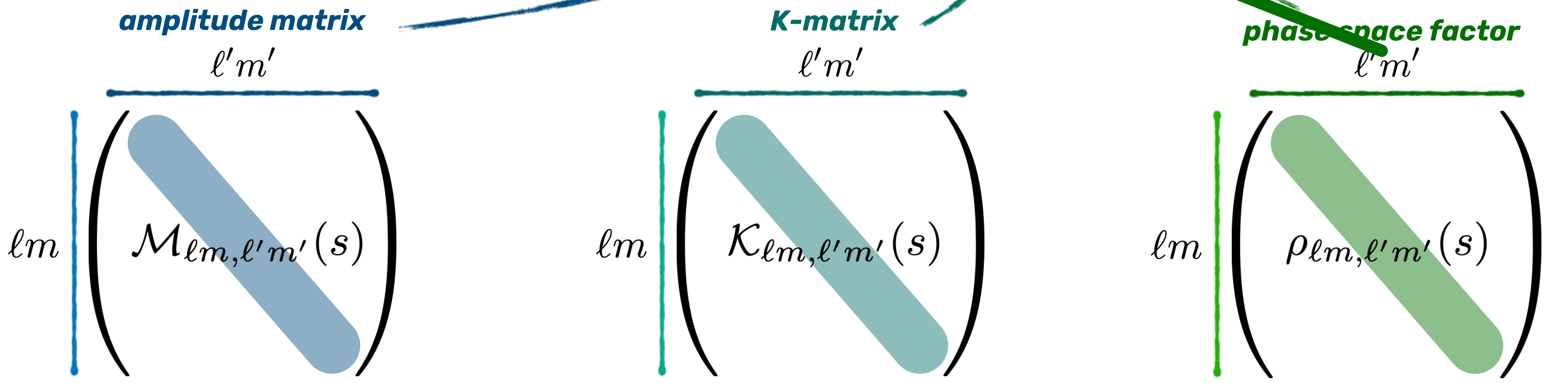
phase space factor

…which we can solve by introducing the ***K-matrix***:

(real, contains no branch cuts)

$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

matrices in angular momentum index space $\ell m, \ell' m'$

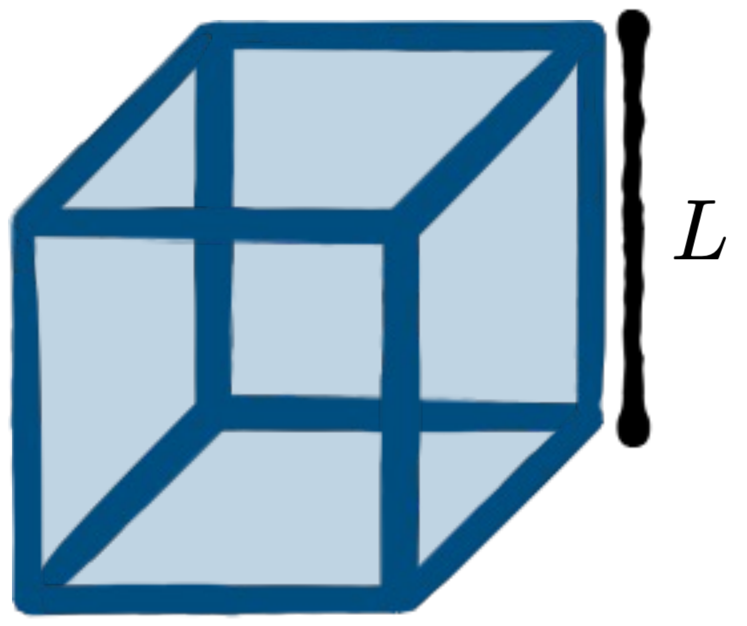


$$\mathcal{M}_{\ell m, \ell' m'}(s) = \delta_{\ell \ell'} \delta_{m m'} \mathcal{M}_\ell(s)$$

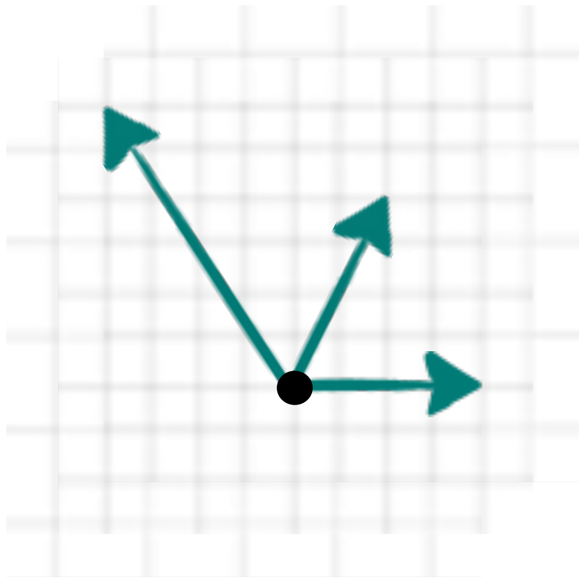
$$\mathcal{K}_{\ell m, \ell' m'}(s) = \delta_{\ell \ell'} \delta_{m m'} \mathcal{K}_\ell(s)$$

$$\rho_{\ell m, \ell' m}(s) = \delta_{\ell \ell'} \delta_{m m'} \rho(s)$$

2. Going to a finite volume

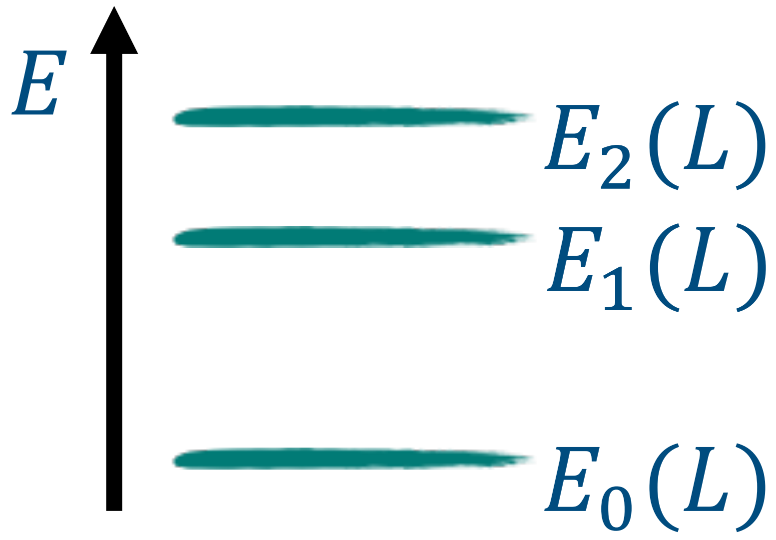


- periodic cubic spatial volume of side L
- finite but large time extent T
- L large enough to neglect $\mathcal{O}(e^{-M_\pi L})$ effects
- neglect discretisation effects

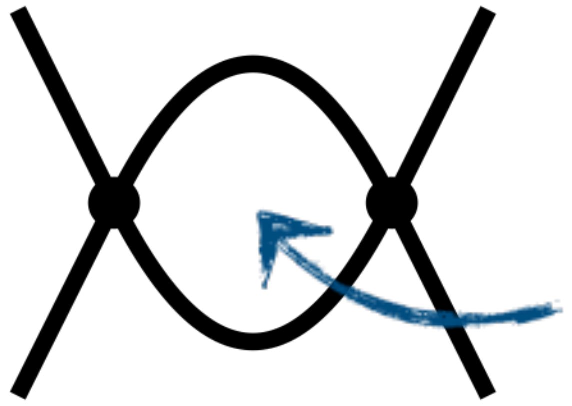


$$\boldsymbol{k} \in \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$$

discretised momenta



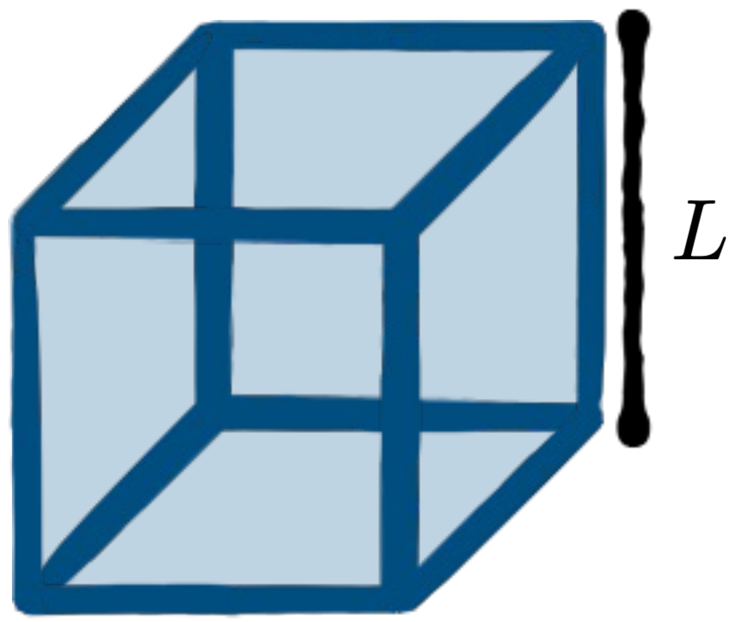
discretised spectrum



$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \longrightarrow \frac{1}{L^3}\sum_{\boldsymbol{k}\in\frac{2\pi}{L}\mathbb{Z}^3}$$

summed spatial loop momenta

2. Going to a finite volume



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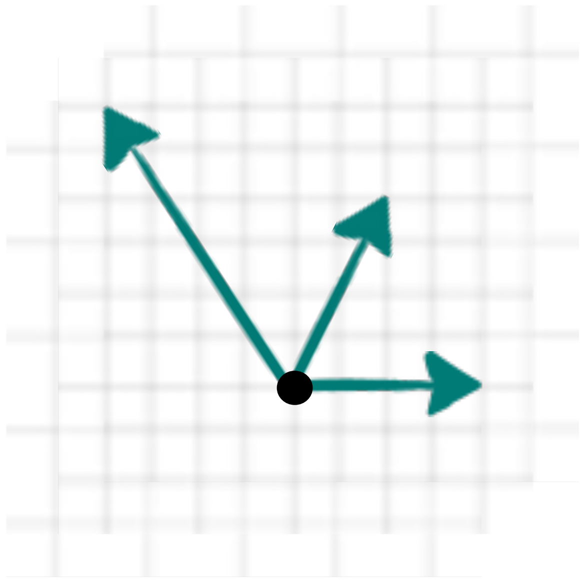

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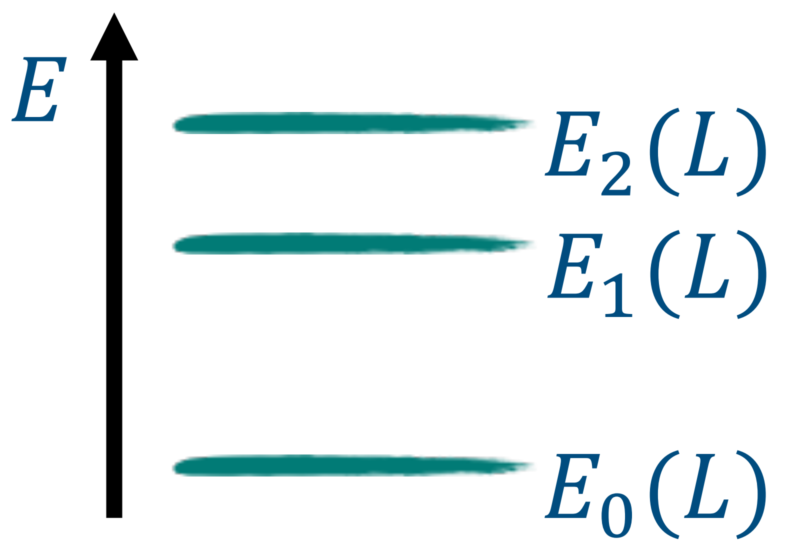
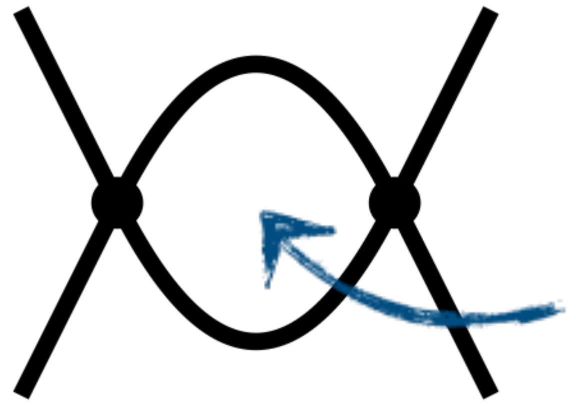


discretised spectrum



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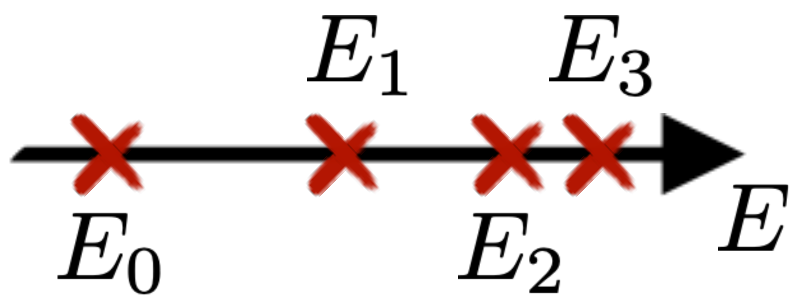
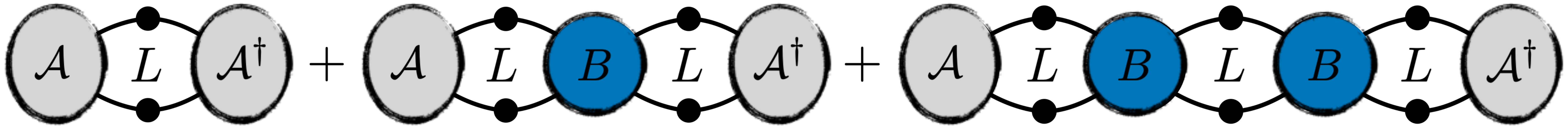
summed spatial loop momenta

Our main tools are ***finite-volume correlators*** $C_L(P)$:

- operators with appropriate quantum numbers
- poles at FV energies of the system

$$C_L(P) = \int d^4x\; e^{-iP\cdot x}\langle \mathcal{A}(x)\mathcal{A}^\dagger(0)\rangle_L$$



$=$  $+ \cdots$

2. Tracking the volume dependence

How do we deal with FV loops? → study the difference between FV and IV loops

$$\textcircled{L} = \textcircled{\infty} + \left[\textcircled{L} - \textcircled{\infty} \right]$$

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \qquad \int_{\boldsymbol{k}} \qquad \boxed{\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}}$$

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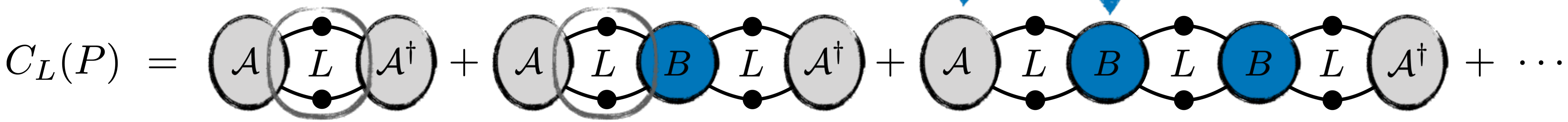
$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \qquad \int_{\boldsymbol{k}} \qquad \frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}$$

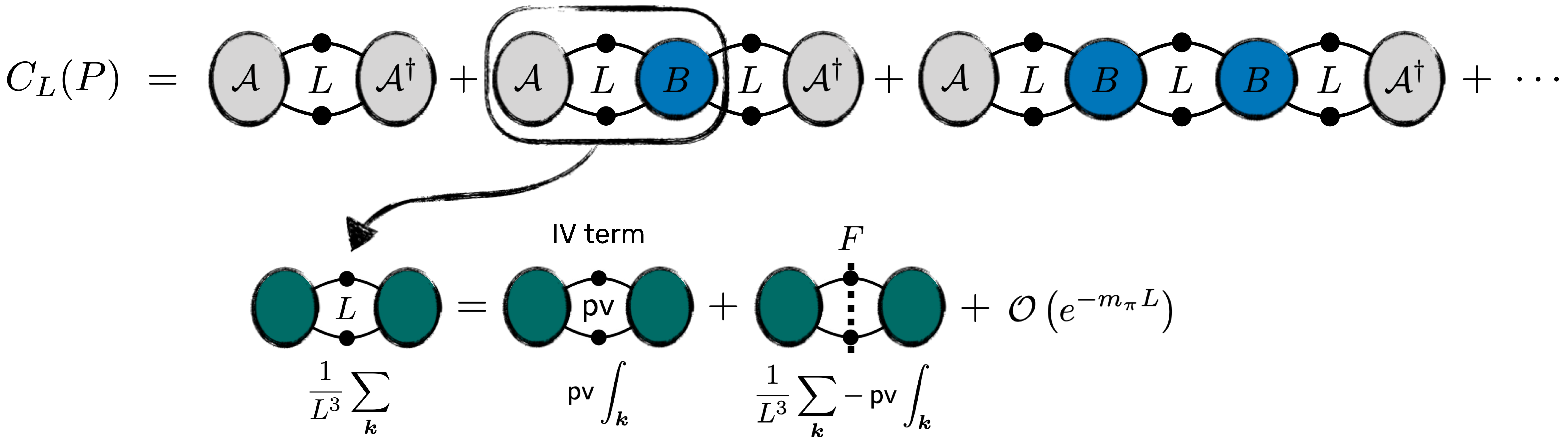
on-shell NN intermediate states: power-like suppression $\mathcal{O}(L^{-n})$

exponentially suppressed volume corrections $\mathcal{O}(e^{-M_\pi L})$ for other loops



replace end-caps and kernels with corresponding infinite-volume objects – neglect $\mathcal{O}(e^{-M_\pi L})$ effects

2. Tracking the volume dependence



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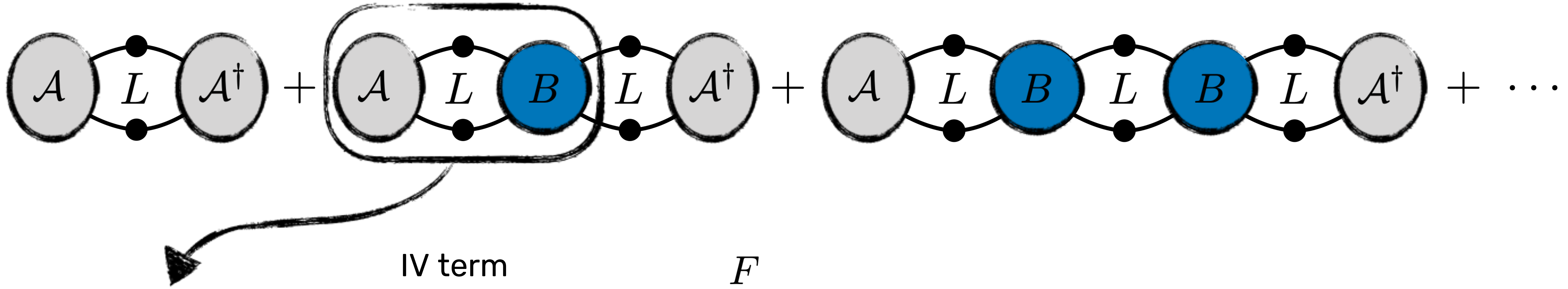
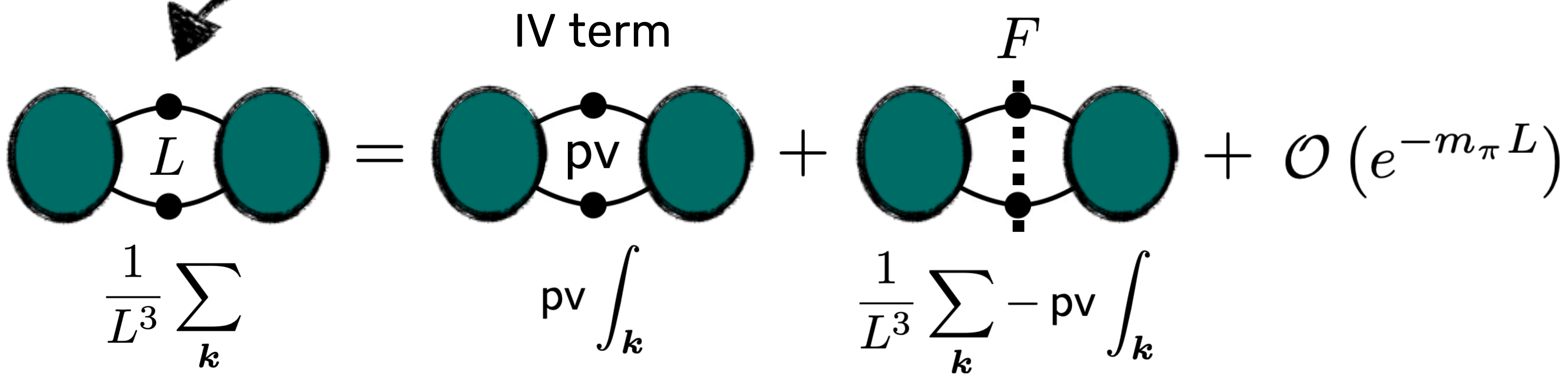
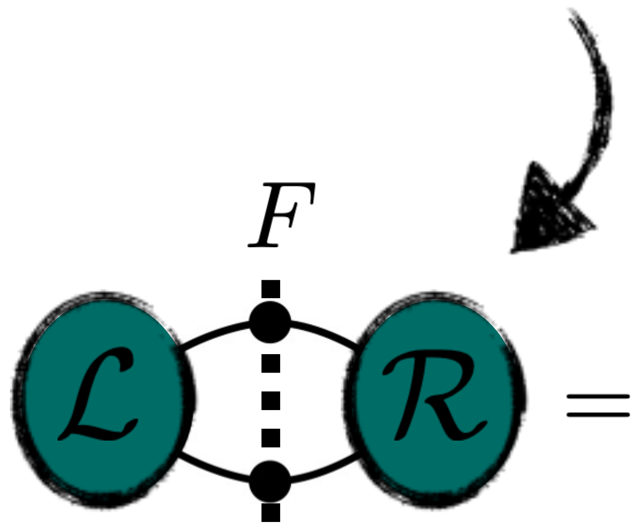




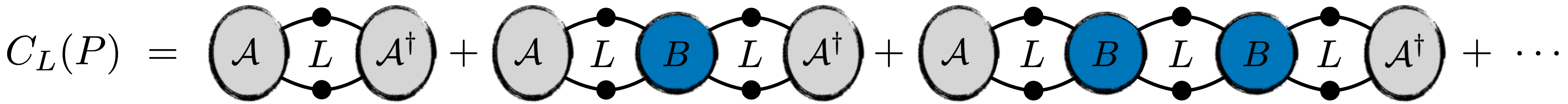


$$= \mathcal{L}^{\text{os}}_{\ell m}(s)\, iF_{\ell m,\ell' m'}(P,L)\, R^{\text{os}}_{\ell' m'}(s)$$

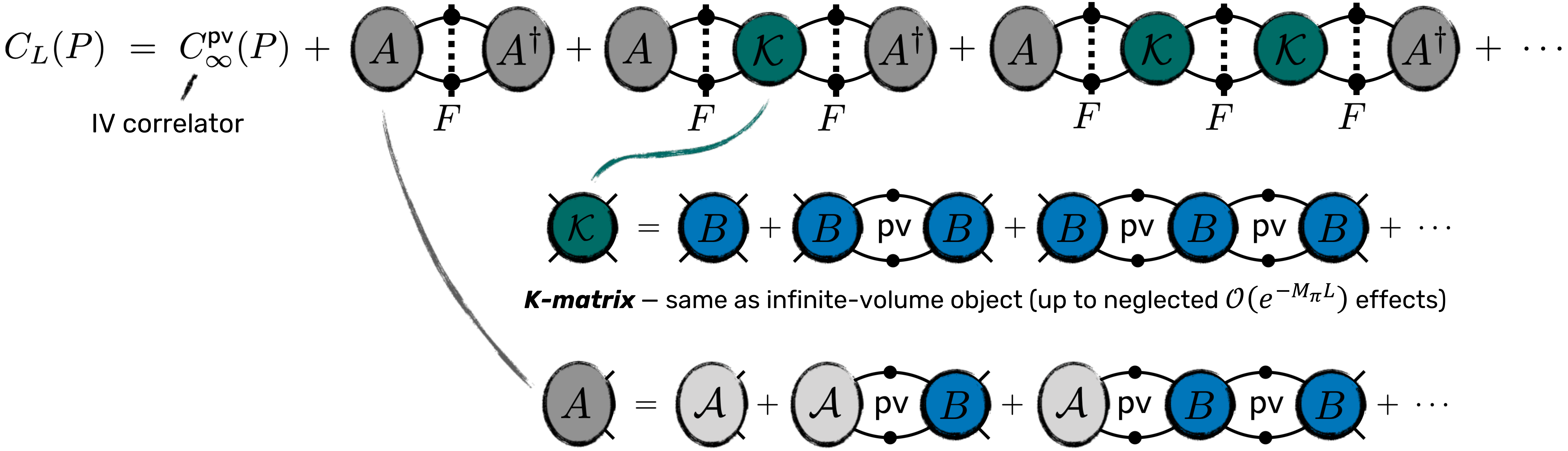
"F-cut" term: - tracks $\mathcal{O}(L^{-n})$ effects
- places neighbours on shell

$$F_{\ell m,\ell' m'}(P,L) = \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int_{\boldsymbol{k}}\right] \frac{1}{2}\frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]}\left(\frac{|\boldsymbol{k}^\star|}{k^\star_{\text{os}}}\right)^{\ell+\ell'}$$

2. Tracking the volume dependence



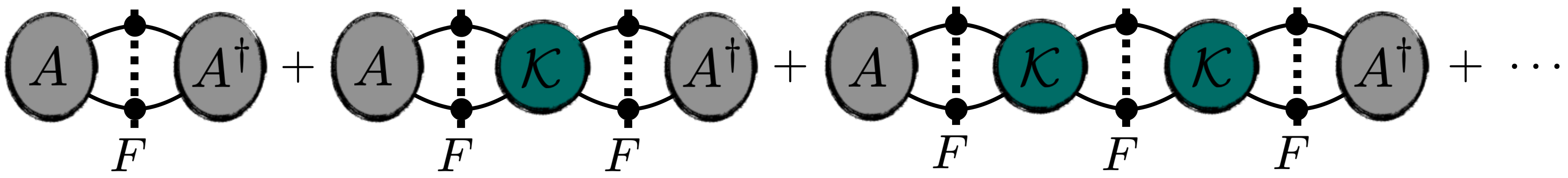
Apply separation to all 2-particle loops, re-organise:



K-matrix – same as infinite-volume object (up to neglected $\mathcal{O}(e^{-M_\pi L})$ effects)

2. Lüscher quantisation condition

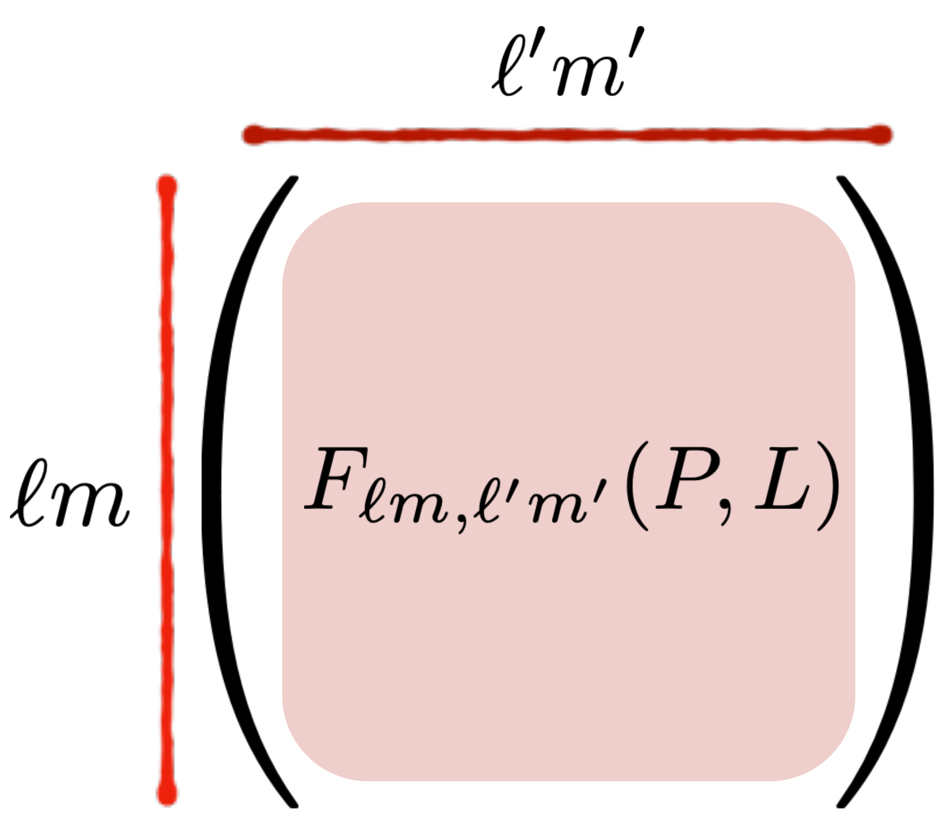
$C_L(P) = C_\infty^{\text{pv}}(P) +$



$$= C_\infty^{\text{pv}}(P) + A(P)\, iF(P,L)\, A^\dagger(P) + A(P)\, iF(P,L)\, i\mathcal{K}(s)\, iF(P,L)\, A^\dagger(P) + \cdots$$

$$= C_\infty^{\text{pv}}(P) + A(P) \frac{i}{F(P,L)^{-1} + \mathcal{K}(s)} A^\dagger(P)$$

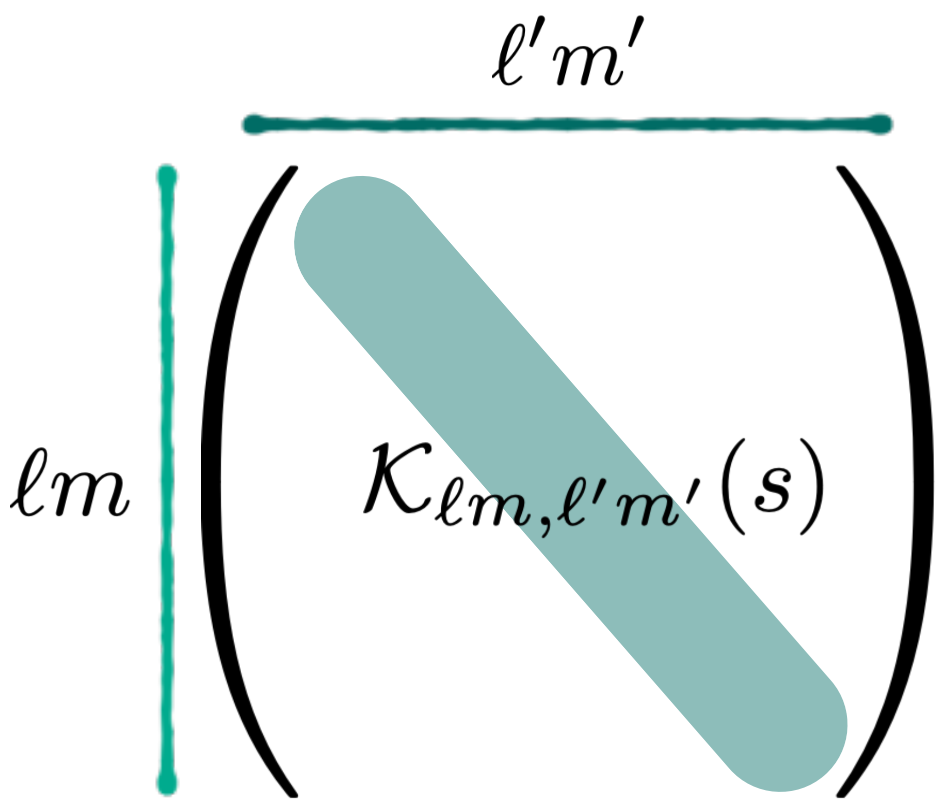
poles at the FV energies



***F* matrix**

of known functions

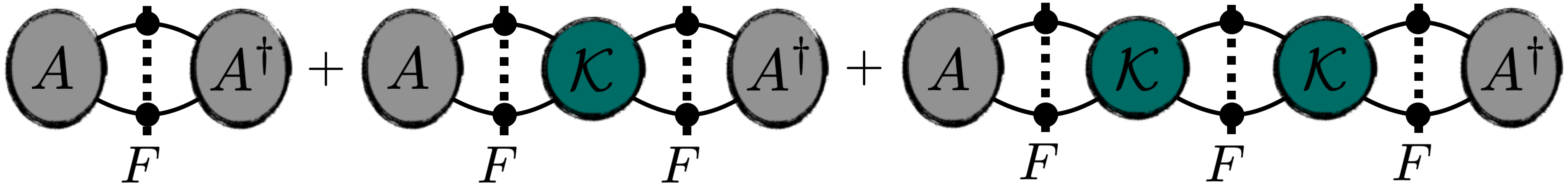
– encodes the FV effects



***K*-matrix**

– encodes IV physics

2. Lüscher quantisation condition

$C_L(P) \;=\; C_\infty^{\mathrm{pv}}(P) +$  $+ \cdots$

$$= C_\infty^{\mathrm{pv}}(P) + A(P)\, iF(P,L)\, A^\dagger(P) + A(P)\, iF(P,L)\, i\mathcal{K}(s)\, iF(P,L)\, A^\dagger(P) + \cdots$$

$$= C_\infty^{\mathrm{pv}}(P) + A(P) \frac{i}{F(P,L)^{-1} + \mathcal{K}(s)} A^\dagger(P)$$

poles at the FV energies

$$\det\left[F(P,L)^{-1} + \mathcal{K}(s)\right] = 0 \text{ at the FV energies}$$

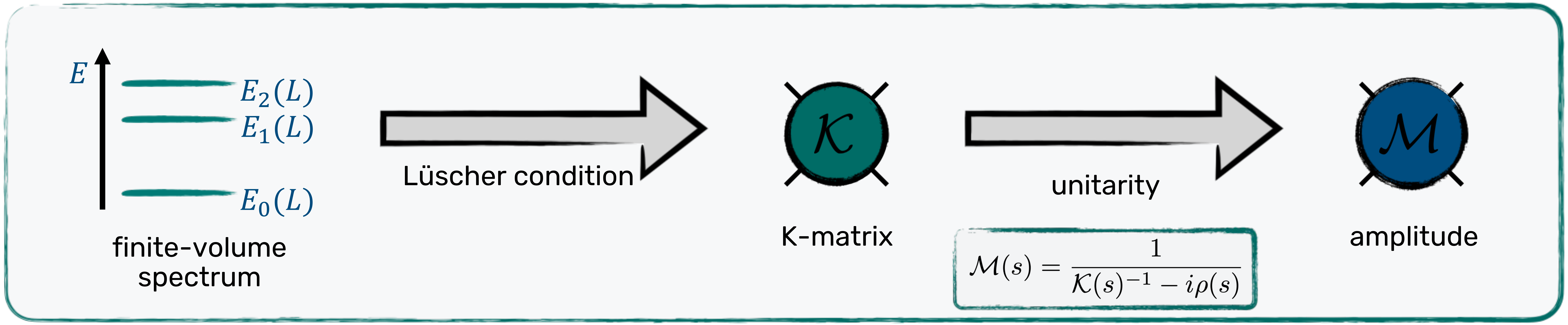
Lüscher quantisation condition
[Lüscher 1986] and many extensions

- original derivation for identical particle scattering, zero total momentum
- extended to non-identical particles, different masses, arbitrary spins, etc. by later work
- derivation outlined here follows [Kim, Sachrajda, Sharpe 2005]

2. Lüscher quantisation condition

$$\det\left[F(P,L)^{-1} + \mathcal{K}(s)\right] = 0 \qquad \text{at the FV energies}$$

Why is it helpful?

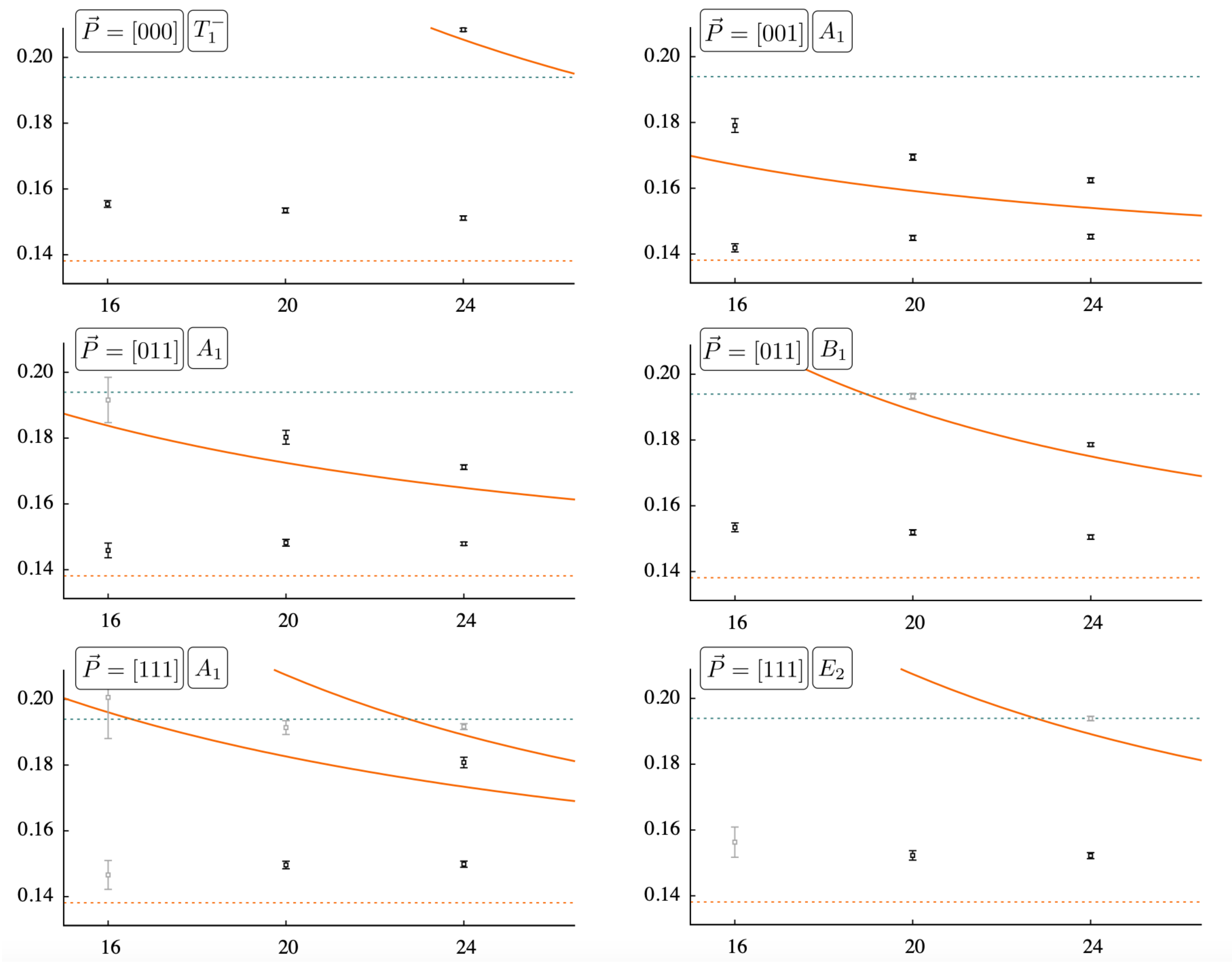


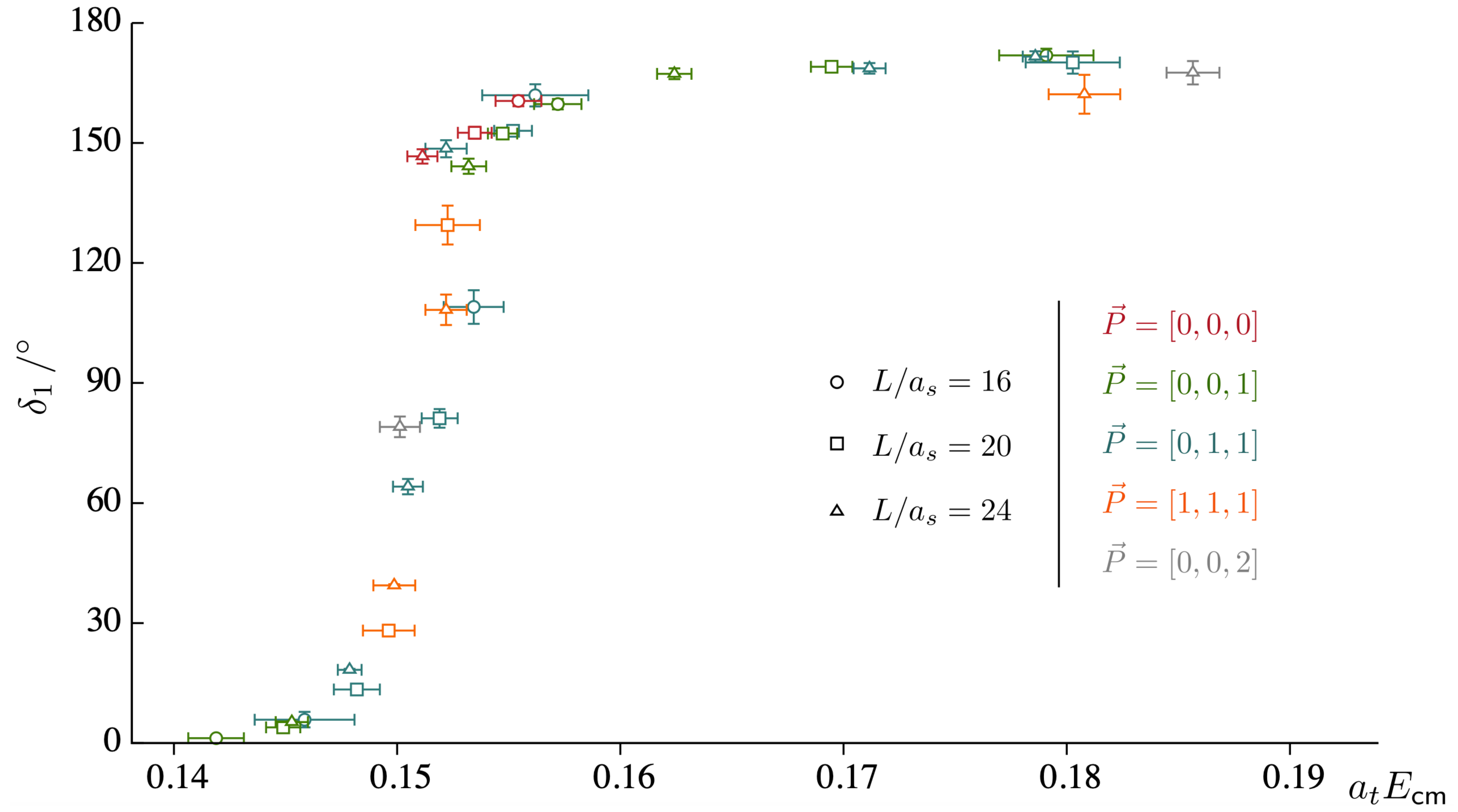
Workflow:

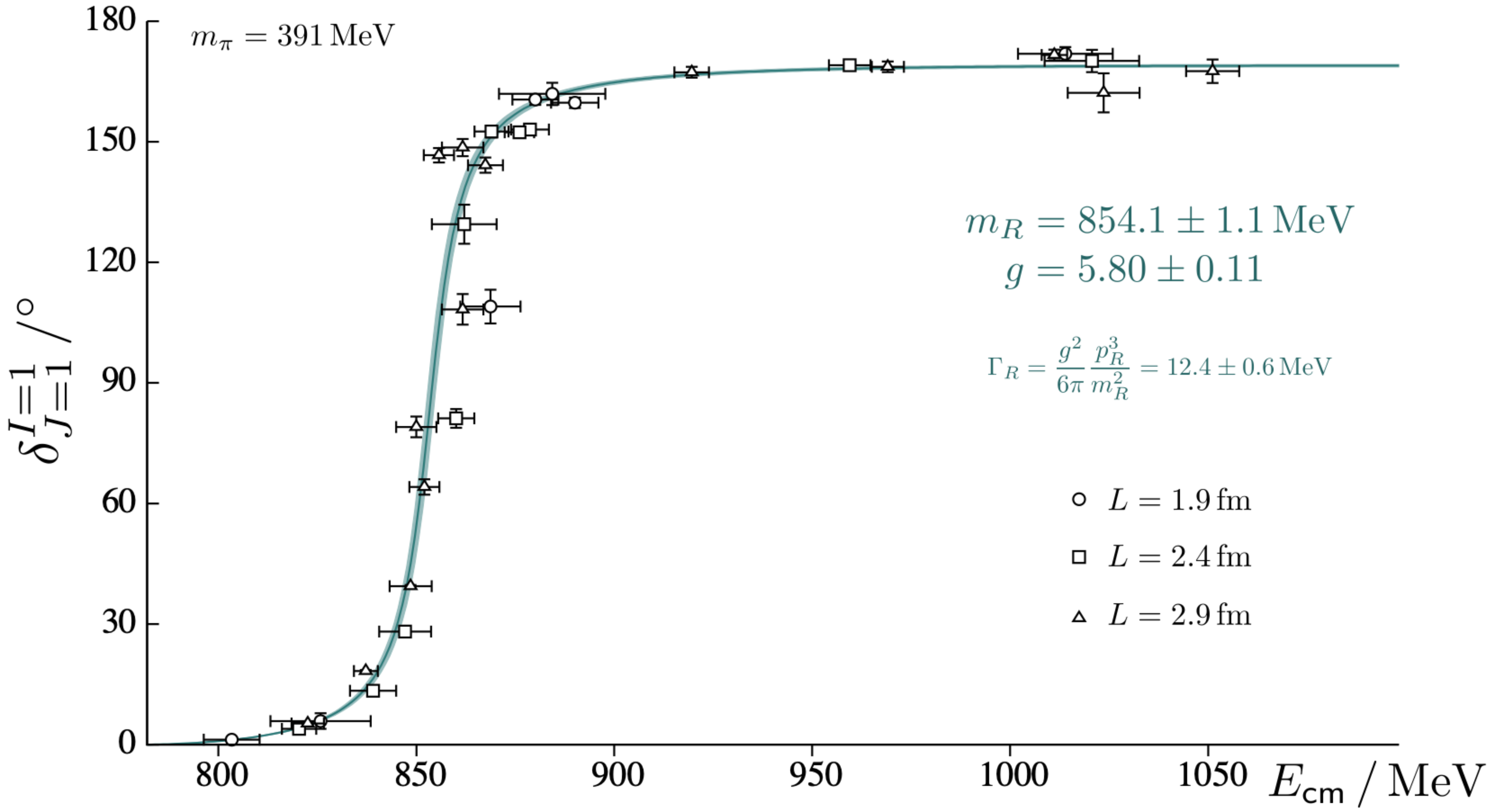
- finite-volume spectrum determined using lattice QCD
- Lüscher condition applied to get K-matrix
- apply the elastic unitarity relation to obtain amplitude

2. An example

$$\mathcal{K}(s) = 16\pi\sqrt{s}\,\frac{1}{p\cot\delta}$$





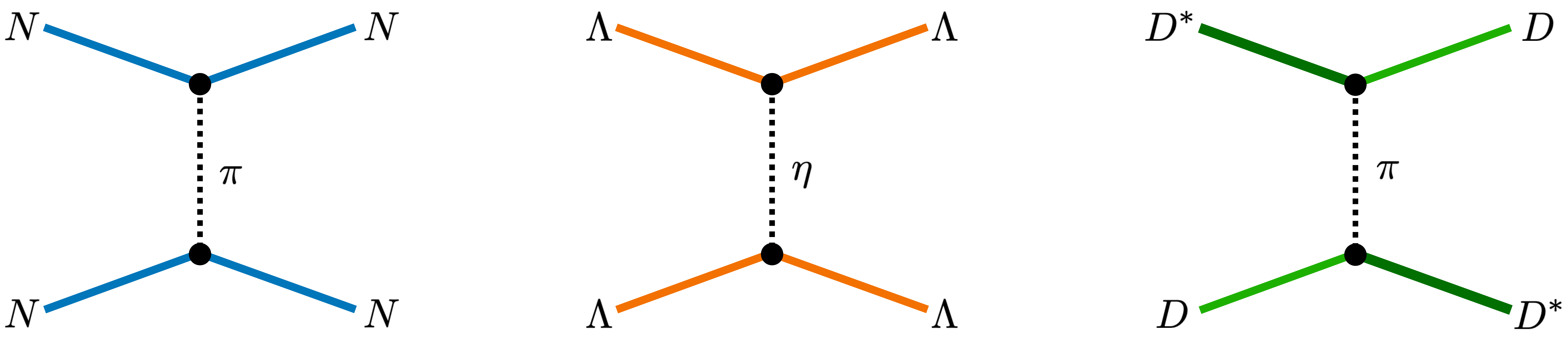


Example of P-wave in $\pi\pi$ scattering with $I = 1$ (adapted from [Dudek et al., 2013])

3. Running into trouble

- recent lattice calculations of baryon-baryon and meson-meson scattering have encountered some issues
- finite-volume energies extracted on top of left-hand cuts
- applying Lüscher formalism leads to inconsistencies

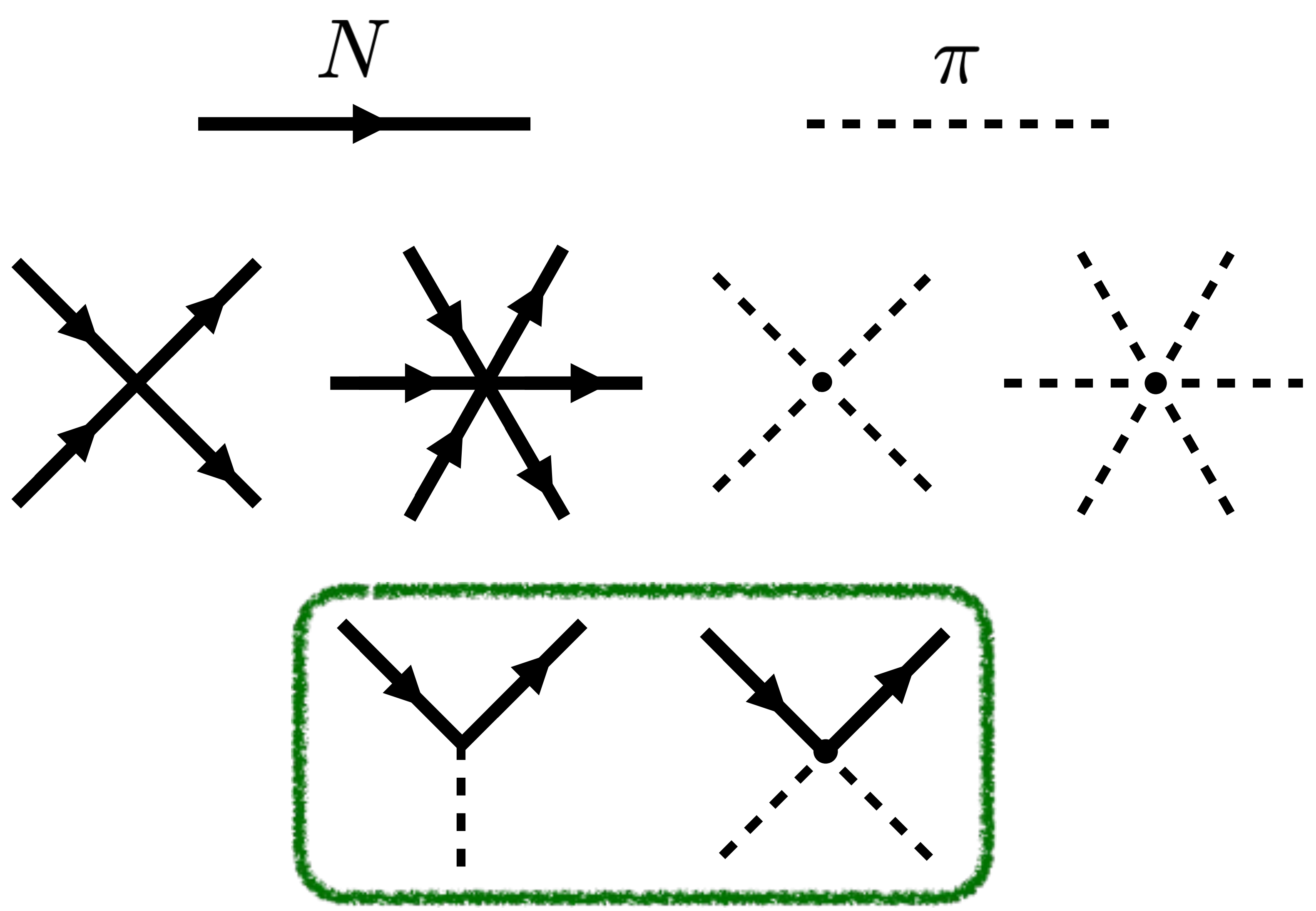
What are ***left-hand cuts***?



these processes involve exchanges of lighter mesons

3. A detour into infinite-volume scattering II

- study a toy model EFT of scalar "nucleons" and "pions", of masses M_N and M_π respectively, with $M_\pi < M_N$
- no assumptions on the form of the interactions, but baryon number is conserved
- N ***and*** π ***now coupled***



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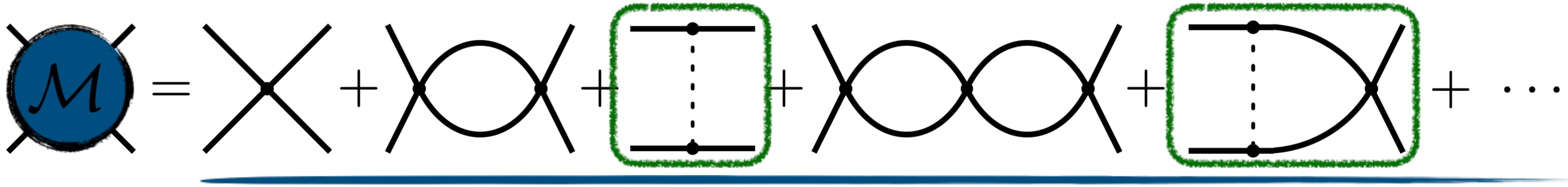


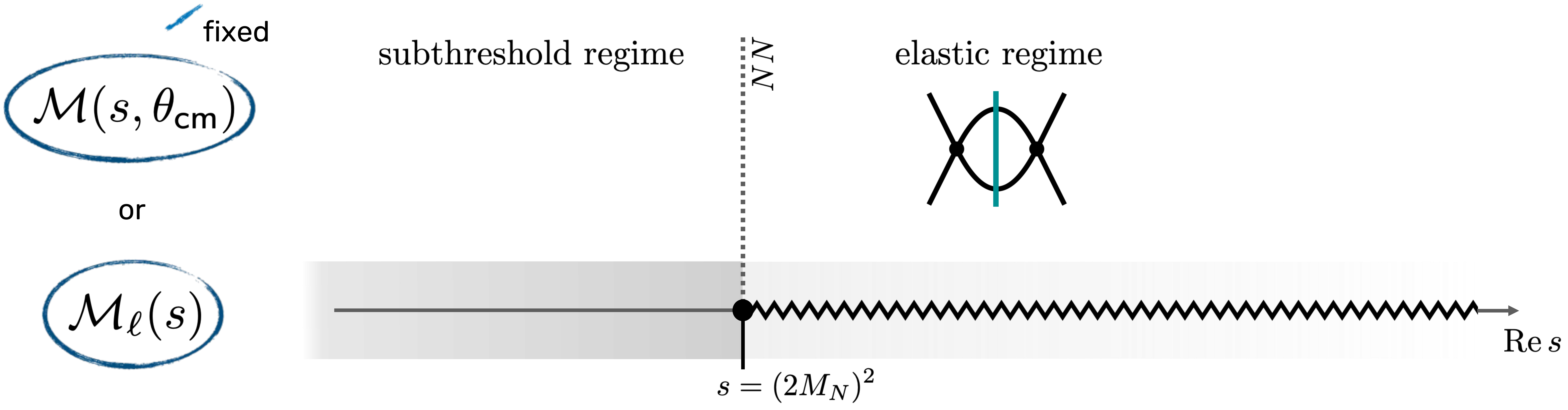
all amputated $NN \to NN$ diagrams

3. Structure of the amplitude with no pions

What is the analytic structure of the amplitude in the s plane for fixed CM scattering angle with no coupling between N and π?

- ***right-hand two-particle cut in elastic regime***

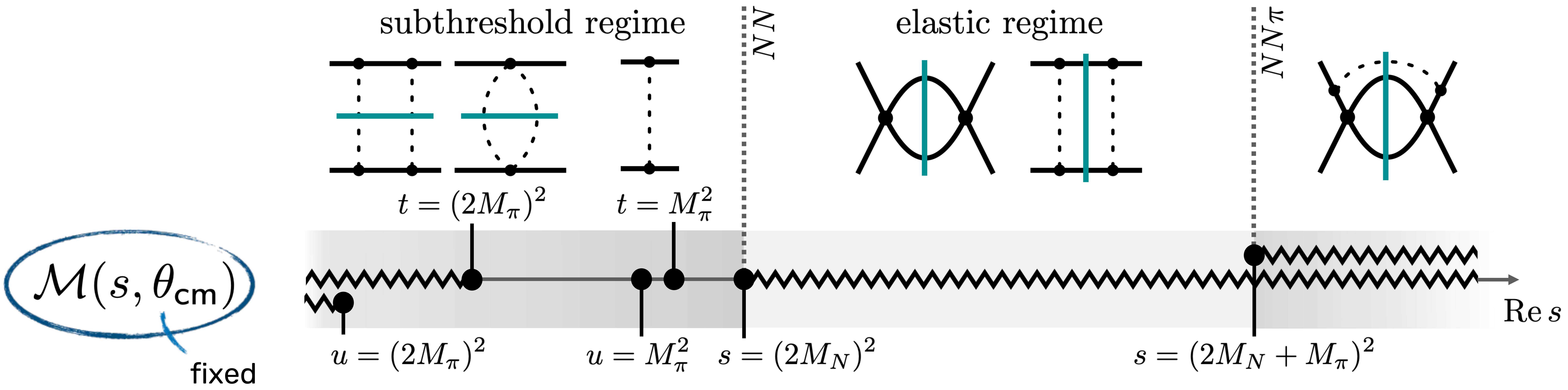
Same picture for the partial-wave amplitudes!



3. Structure of the amplitude with pions

What is the analytic structure of the amplitude in the s plane for fixed CM scattering angle when including pions?

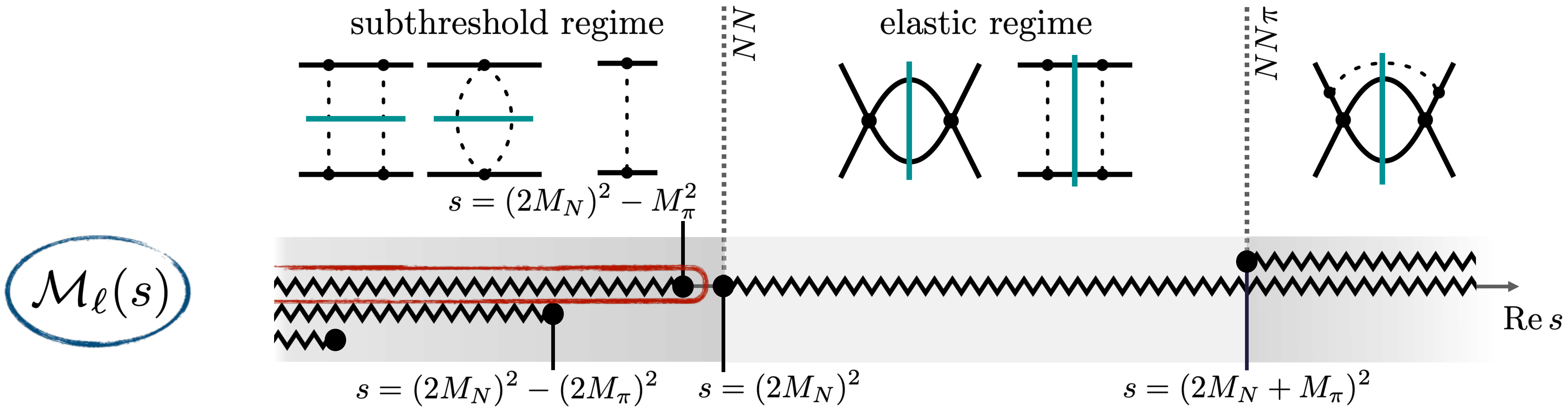
- right-hand two-particle cut in elastic regime
- three-particle cut above $NN\pi$ threshold
- ***sub-threshold poles due to single π exchanges***
- ***lower cuts due to multiple π exchanges***



3. Structure of the amplitude with pions

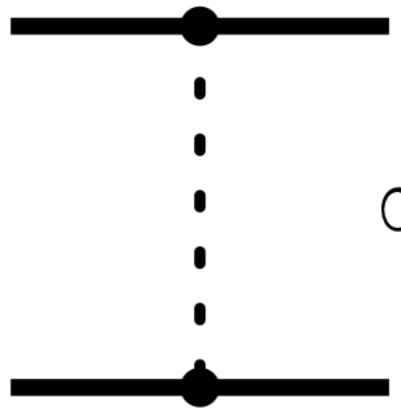
What is the analytic structure of the partial-wave amplitudes in the s plane?

- right-hand two-particle cut in elastic regime
- three-particle cut above $NN\pi$ threshold
- ***sub-threshold poles become left-hand cut***
- lower cuts due to multiple π exchanges

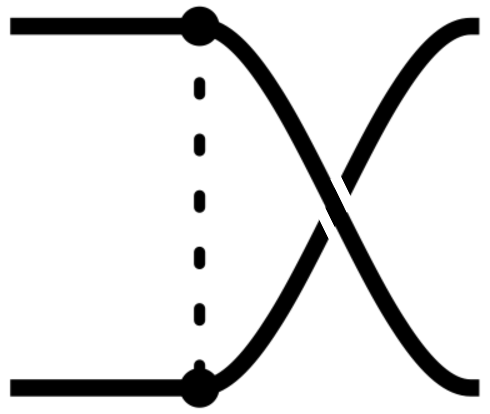


3. Origin of the left-hand cut: a closer look

- the nearest cut arises due to the π exchanges:



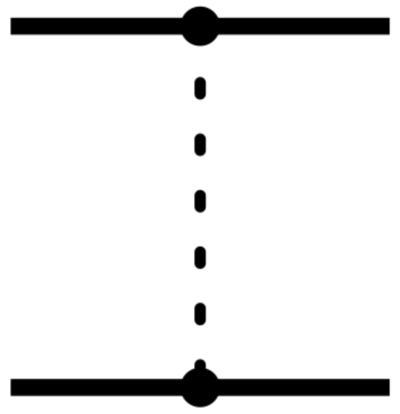
$$\propto \frac{1}{t - M_\pi^2 + i\epsilon}$$



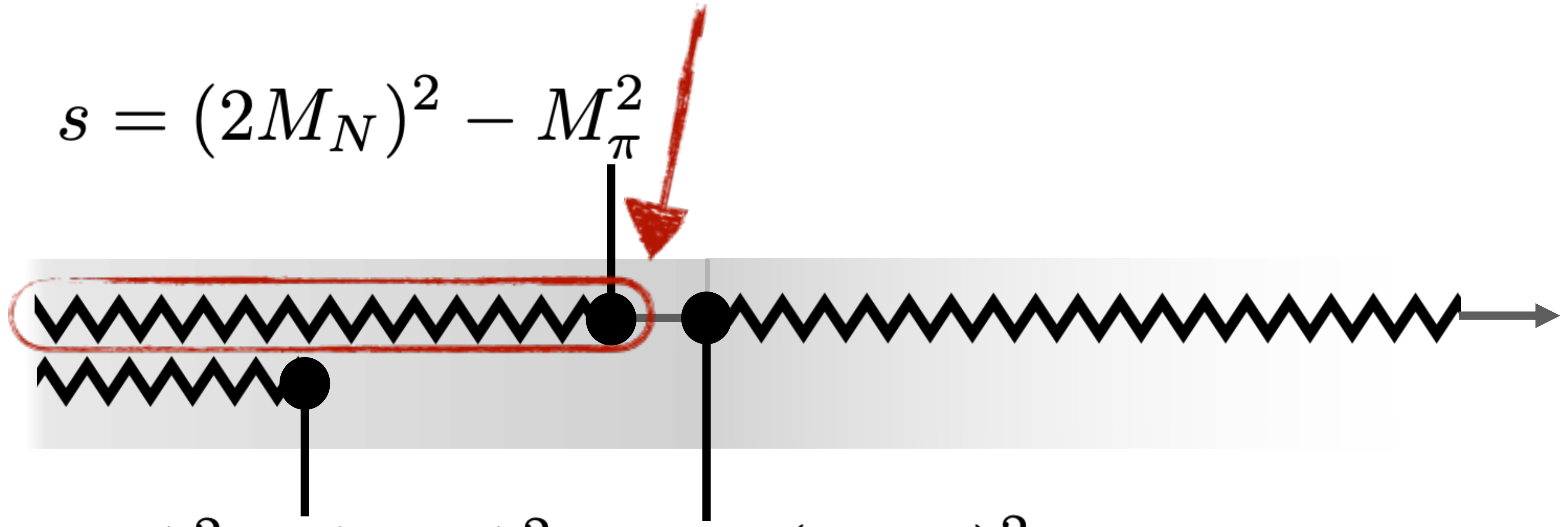
$$\propto \frac{1}{u - M_\pi^2 + i\epsilon}$$

- projecting to definite AM and with on-shell arguments, e.g. to $\ell = 0$:

$$\int d\cos\theta_{\text{cm}}$$



$$\propto \frac{1}{s - 4M_N^2} \log\left(\frac{s - 4M_N^2 + M_\pi^2}{M_\pi^2} - i\epsilon\right)$$



3. Running into trouble

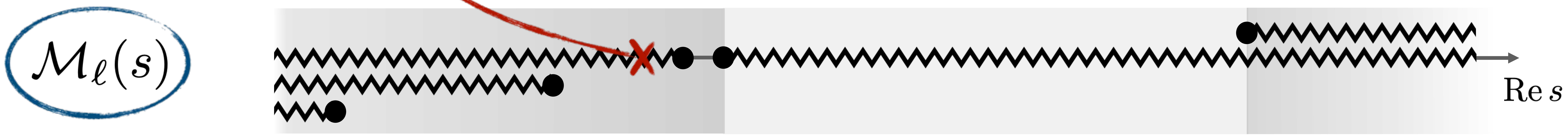
- recent lattice calculations of baryon-baryon and meson-meson scattering have encountered some issues
- finite-volume energies extracted on top of left-hand cuts
- applying Lüscher formalism leads to inconsistencies

What are ***left-hand cuts***? What happens there?

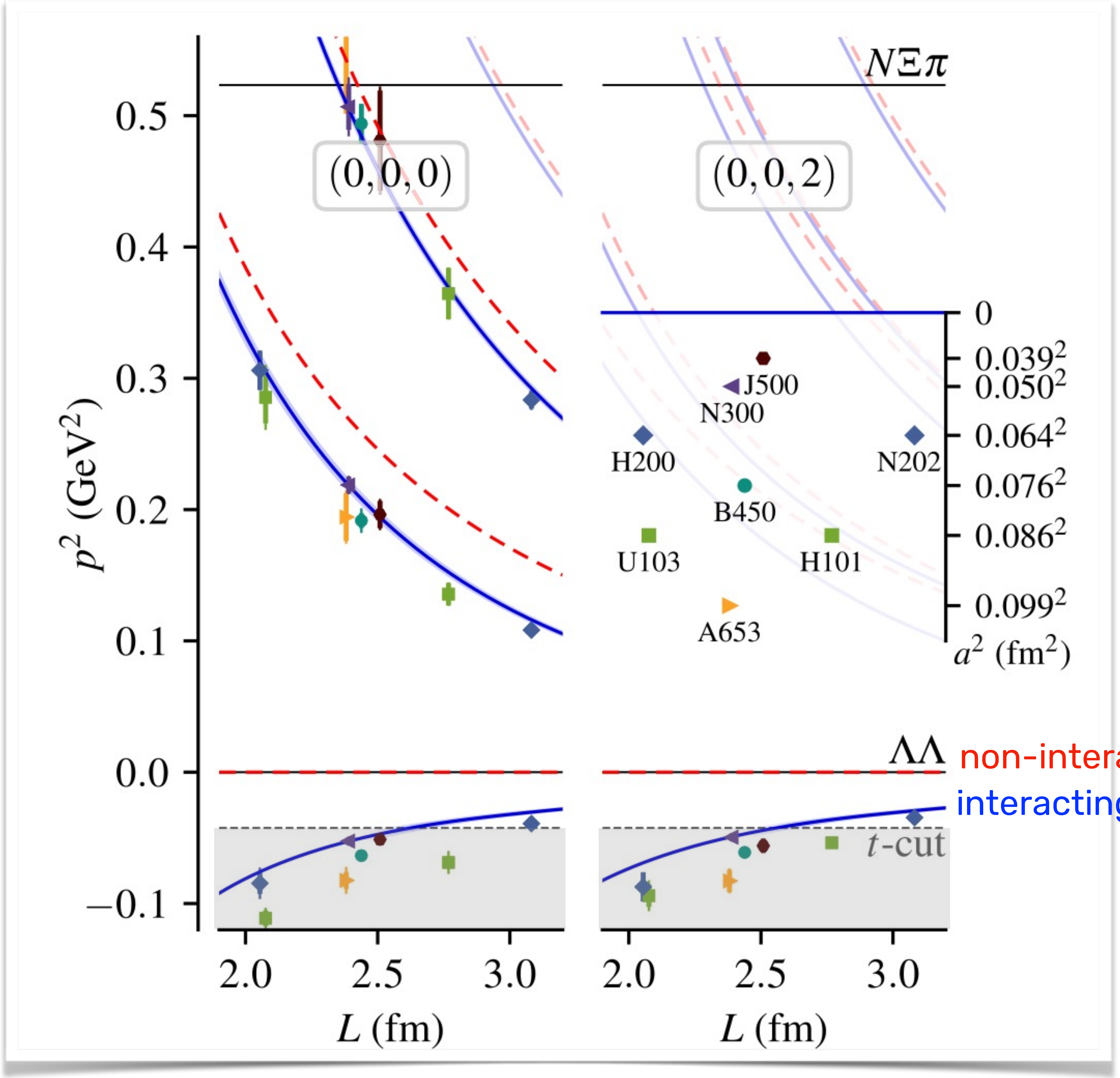
Lüscher condition

$$\det\left[F(P,L)^{-1}+\mathcal{K}(s)\right]=0$$

- $F(P,L)$ is real, therefore solutions for $\mathcal{K}(s)$ are real
- however, $\mathcal{K}(s)$ ***should be complex on the cut!***



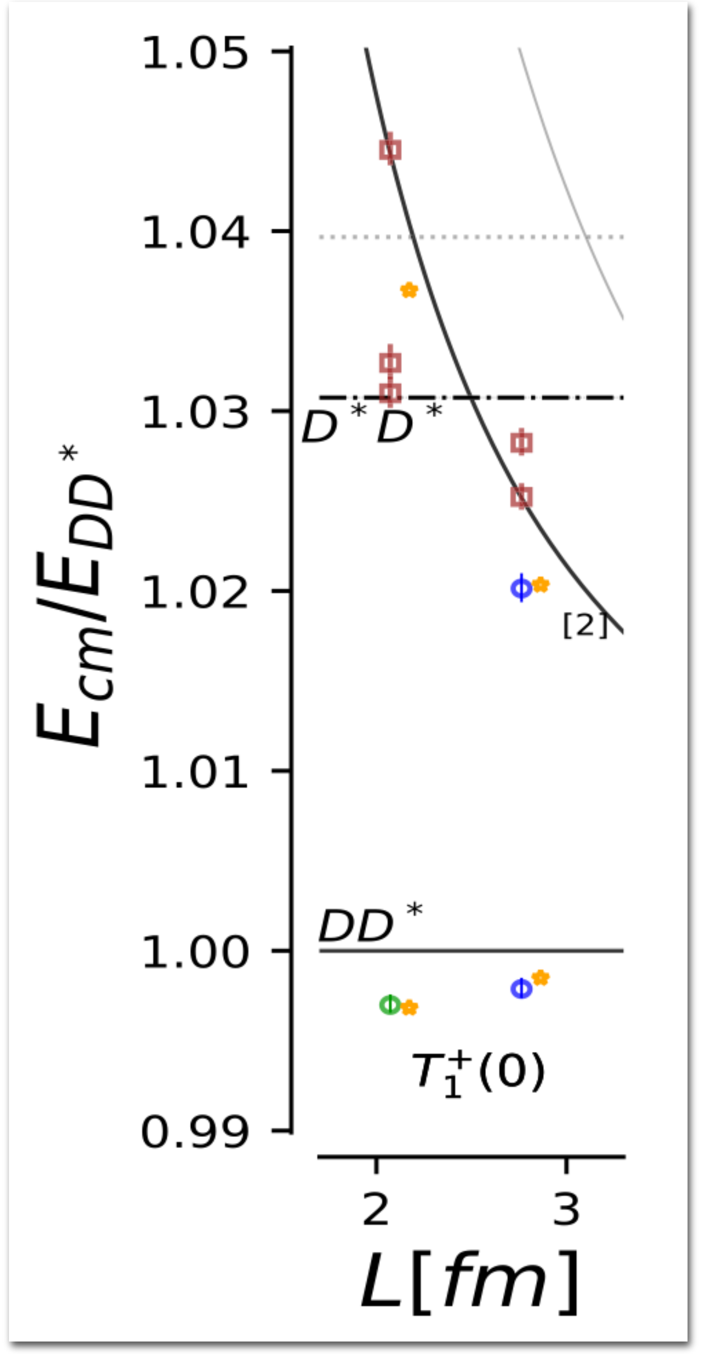
3. Running into trouble



non-interacting levels

interacting levels

$\Lambda\Lambda$ finite-volume spectra,
adapted from [Green, Hanlon, Junnarkar, Wittig 2021]



Spectra for DD^* system,
adapted from [Padmanath, Prelovsek 2022]

Role of left-hand cut contributions on pole extractions from lattice data: Case study for $T_{cc}(3875)^+$

Meng-Lin Du,[1] Arseniy Filin,[2] Vadim Baru,[2] Xiang-Kun Dong,[3,4] Evgeny Epelbaum,[2] Feng-Kun Guo,[3,4,5] Christoph Hanhart,[6] Alexey Nefediev,[7,8] Juan Nieves,[9] and Qian Wang[10,11,12]

We discuss recent lattice data for the $T_{cc}(3875)^+$ state to stress, for the first time, a potentially strong impact of left-hand cuts from the one-pion exchange on the pole extraction for near-threshold exotic states. In particular, if the left-hand cut is located close to the two-particle threshold, which happens naturally in the DD^* system for the pion mass exceeding its physical value, the effective-range expansion is valid only in a very limited energy range up to the cut and as such is of little use to reliably extract the poles. Then, an accurate extraction of the pole locations requires the

4. Where did we go wrong?

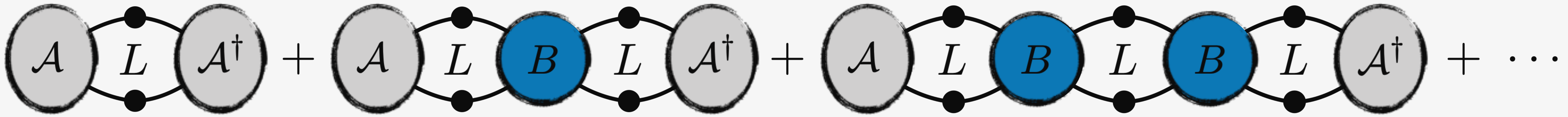
Why does the left-hand cut cut change things?

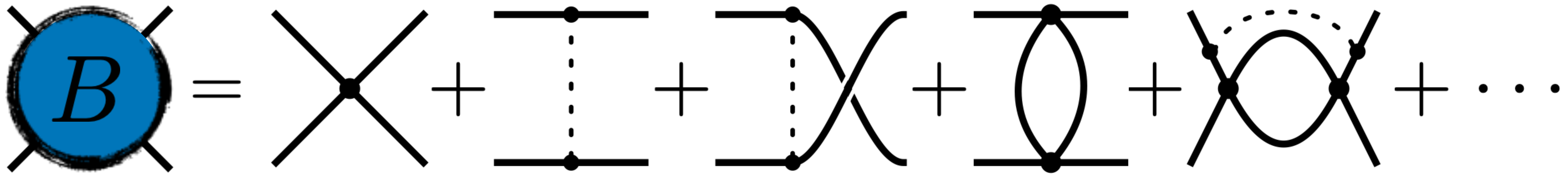
Apart from minor adjustments, our derivation set-up from before seems fine:

Our main tools are ***finite-volume correlators*** $C_L(P)$:

- operators with appropriate quantum numbers
- poles at FV energies of the system

$$C_L(P) = \int d^4x \; e^{-iP\cdot x} \langle \mathcal{A}(x)\mathcal{A}^\dagger(0)\rangle_L$$

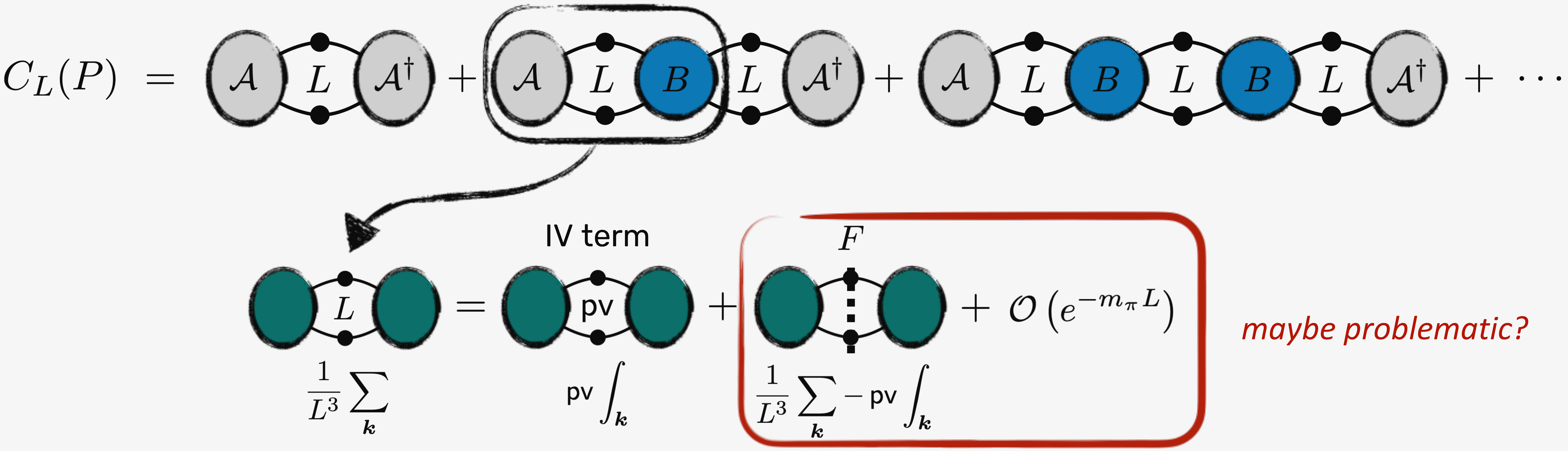
$=$ 

Bethe-Salpeter kernel now includes extra diagrams: 

we must re-analyse subsequent steps!

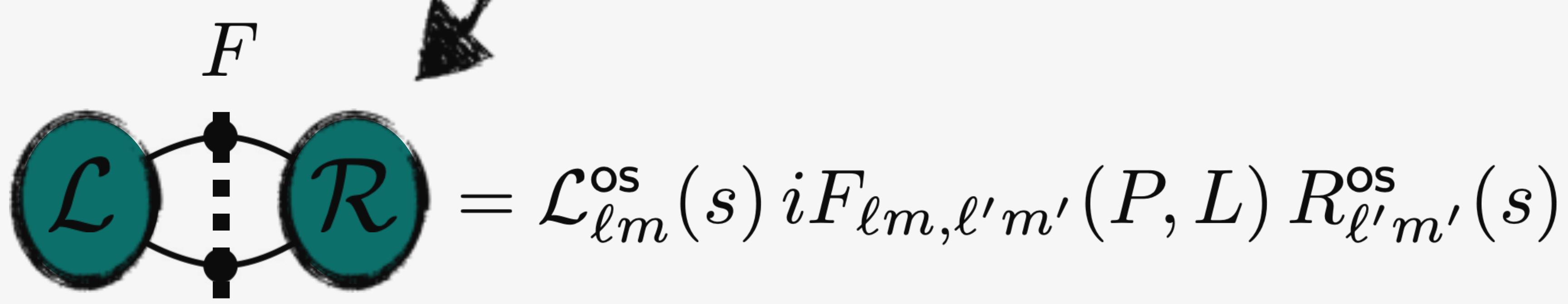
4. Where did we go wrong? Recall...

Tracking the volume dependence



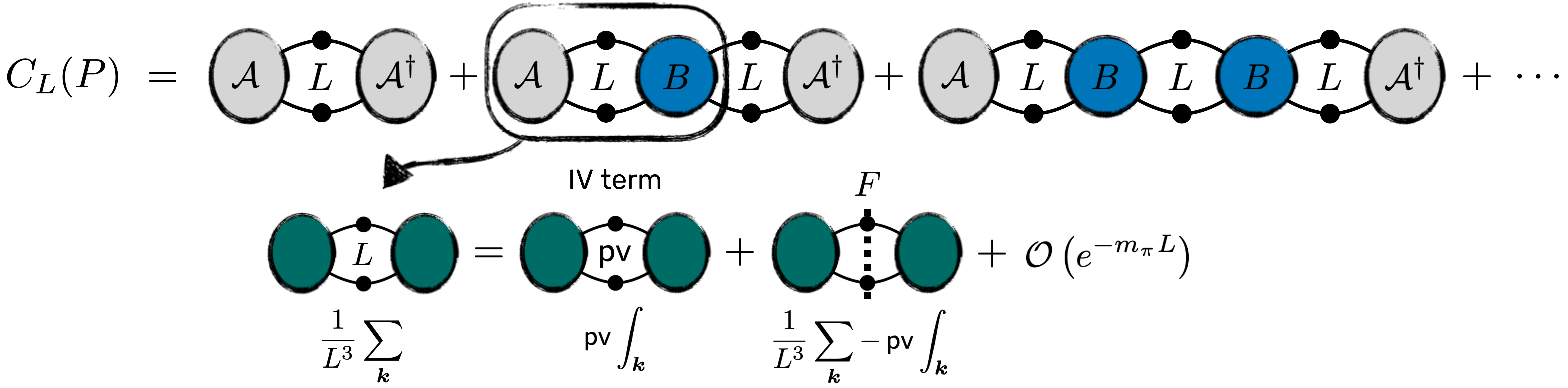
maybe problematic?

"F-cut" term: - tracks $\mathcal{O}(L^{-n})$ effects
- places neighbours on shell

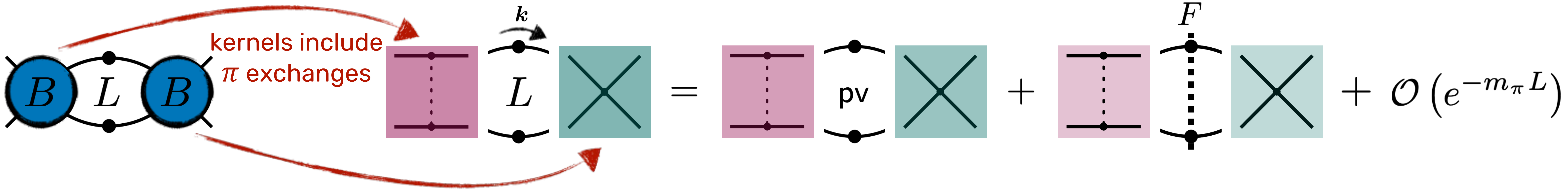


$$F_{\ell m,\ell' m'}(P,L) = \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int_{\boldsymbol{k}}\right] \frac{1}{2}\frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]}\left(\frac{|\boldsymbol{k}^\star|}{k^\star_{\text{os}}}\right)^{\ell+\ell'}$$

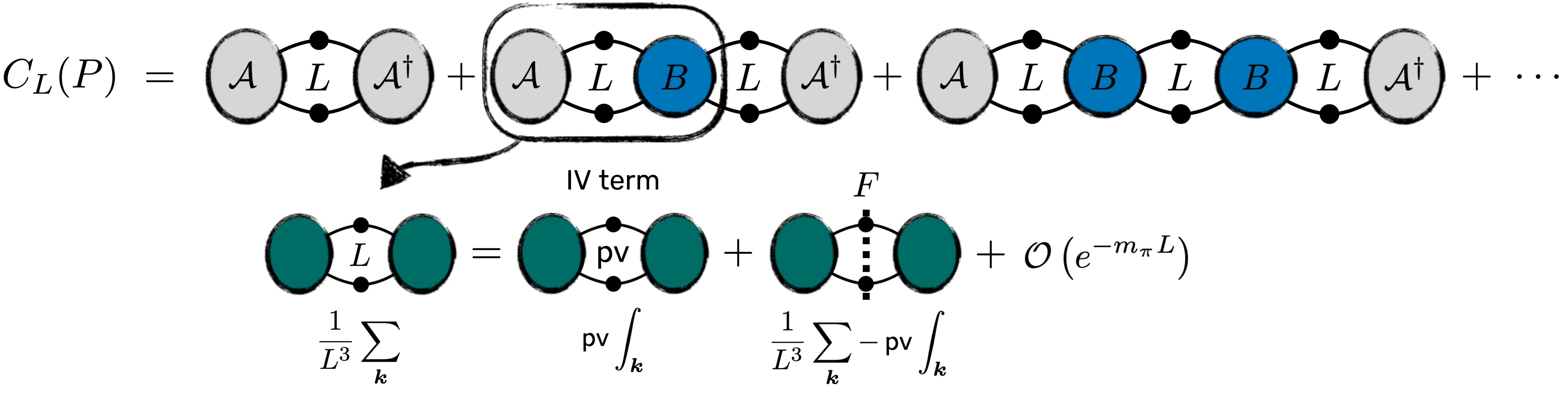
4. Where did we go wrong?



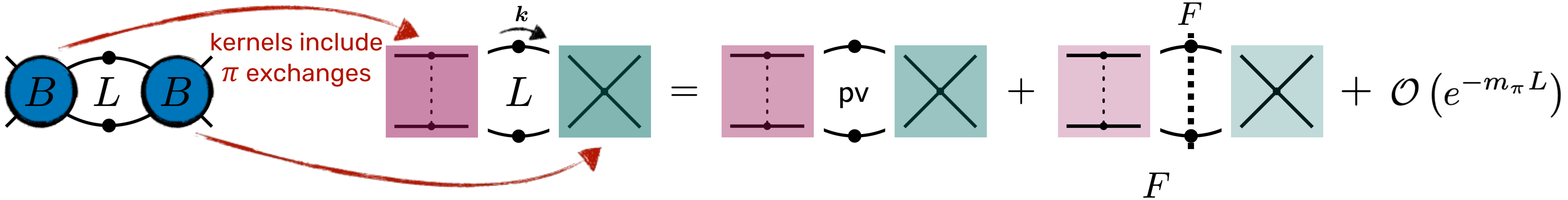
Apply to loop with two BS kernels:



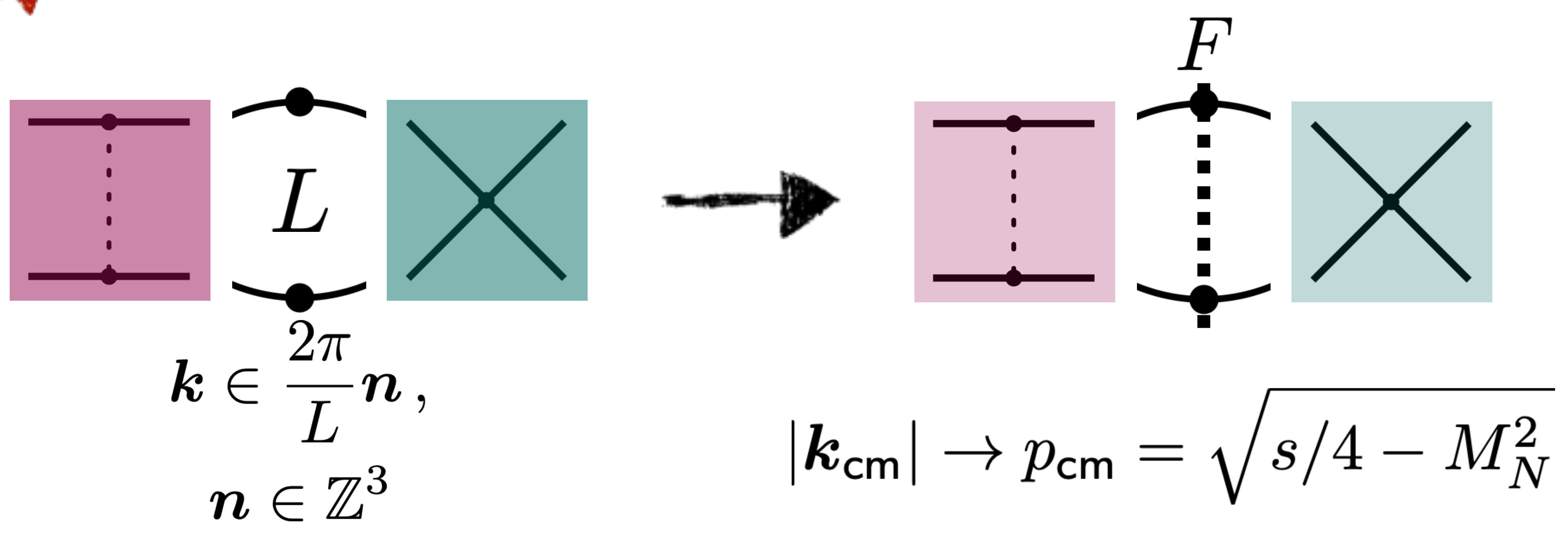
4. Where did we go wrong?



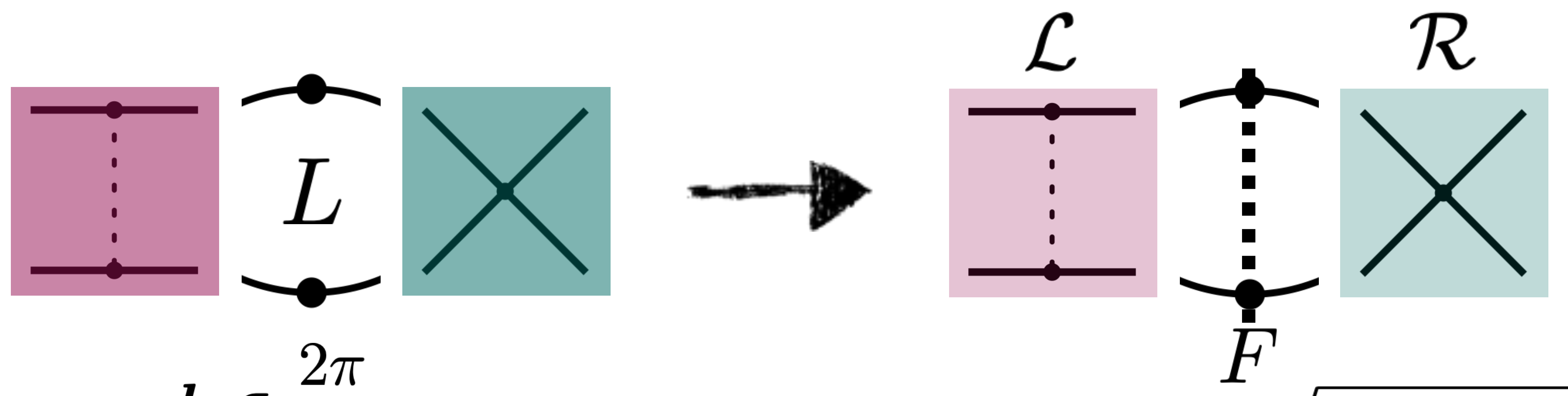
Apply to loop with two BS kernels:



F operation places neighbouring subdiagrams on-shell:



4. Where did we go wrong?



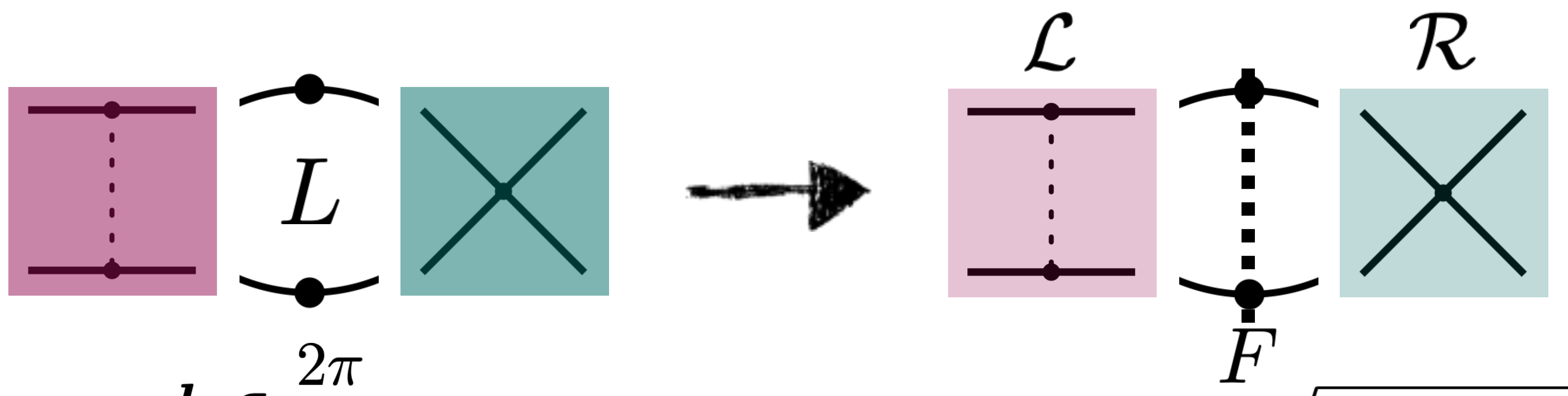
$$\boldsymbol{k} \in \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$$

$$|\boldsymbol{k}_{\text{cm}}| \to p_{\text{cm}} = \sqrt{s/4 - M_N^2}$$

- on-shell placement relies on on-shell off-shell difference being exponentially suppressed with the volume and therefore negligible

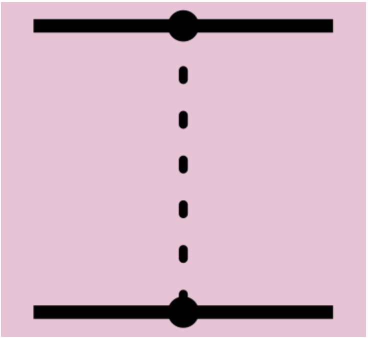
$$\propto \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}\right] \frac{\mathcal{L}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{R}(\boldsymbol{k}_{\text{cm}}^2) - \mathcal{L}(p_{\text{cm}}^2)\mathcal{R}(p_{\text{cm}}^2)}{\omega_N(\boldsymbol{k})[\boldsymbol{k}_{\text{cm}}^2 - p_{\text{cm}}^2]} \sim e^{-M_\pi L}$$

4. Where did we go wrong?



$\boldsymbol{k} \in \frac{2\pi}{L}\boldsymbol{n}\,,$
$\boldsymbol{n} \in \mathbb{Z}^3$

$|\boldsymbol{k}_{\text{cm}}| \to p_{\text{cm}} = \sqrt{s/4 - M_N^2}$

$$\int d\cos\theta_{\text{cm}}$$  $$\propto \frac{1}{s-4M_N^2}\log\left(\frac{s-4M_N^2+M_\pi^2}{M_\pi^2} - i\epsilon\right)$$

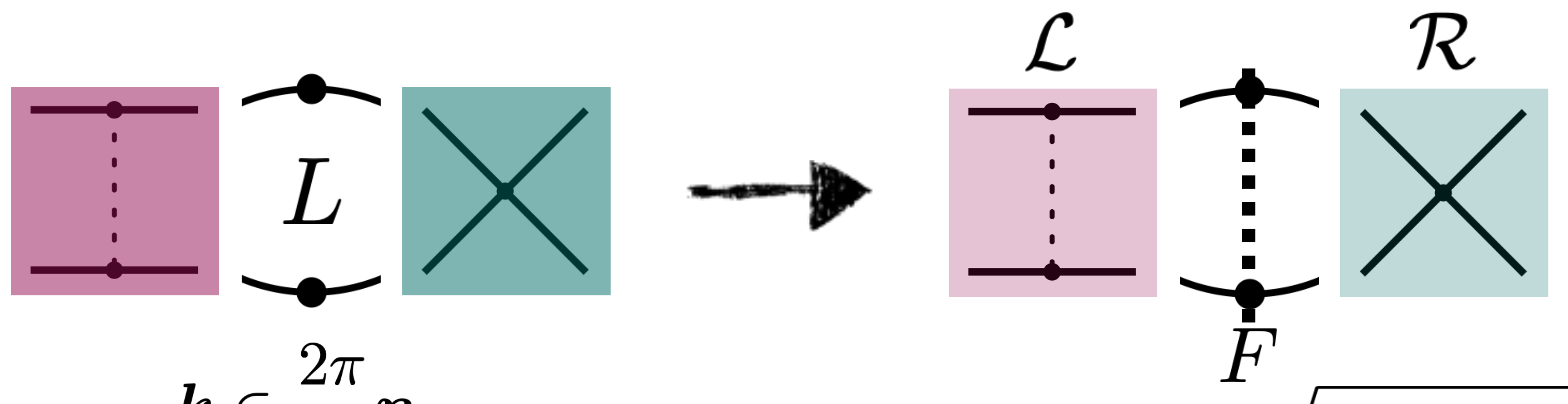
π exchange projected to $\ell =$ 0 and with on-shell kinematics

- on-shell placement relies on on-shell off-shell difference being exponentially suppressed with the volume and therefore negligible

$$\propto \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}\right] \frac{\mathcal{L}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{R}(\boldsymbol{k}_{\text{cm}}^2) - \mathcal{L}(p_{\text{cm}}^2)\mathcal{R}(p_{\text{cm}}^2)}{\omega_N(\boldsymbol{k})[\boldsymbol{k}_{\text{cm}}^2 - p_{\text{cm}}^2]} \sim e^{-M_\pi L}$$

- all fine above elastic threshold and nearest left-hand cut
- this breaks when we hit the cut (and just above): ***potentially large volume effects neglected if dropped***

4. Where did we go wrong?



$\boldsymbol{k} \in \frac{2\pi}{L}\boldsymbol{n}\,,$

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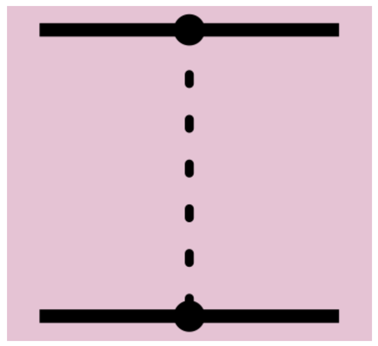
$\int d\cos\theta_{\text{cm}}$  $\propto \frac{1}{s - 4M_N^2}\log\left(\frac{s - 4M_N^2 + M_\pi^2}{M_\pi^2} - i\epsilon\right)$

π exchange projected to $\ell = 0$ and with on-shell kinematics

- all fine above elastic threshold and nearest left-hand cut
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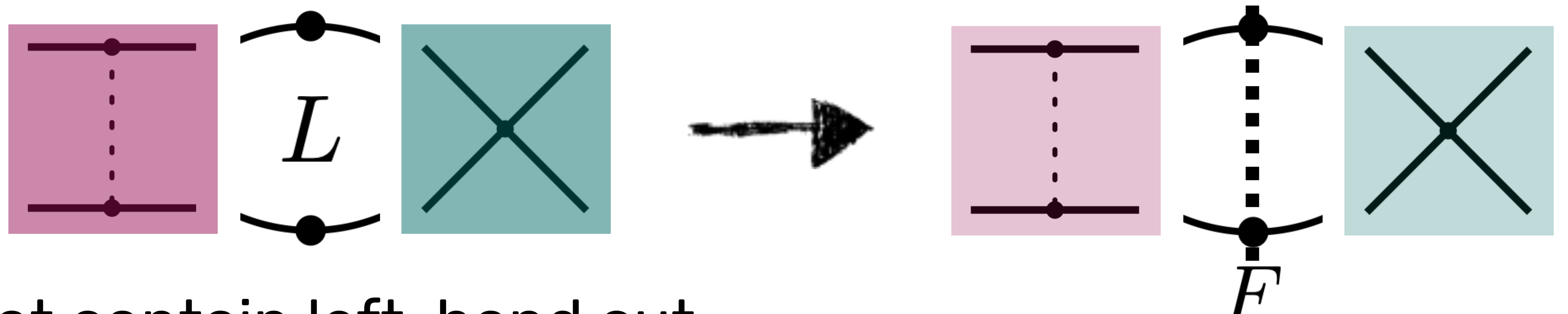
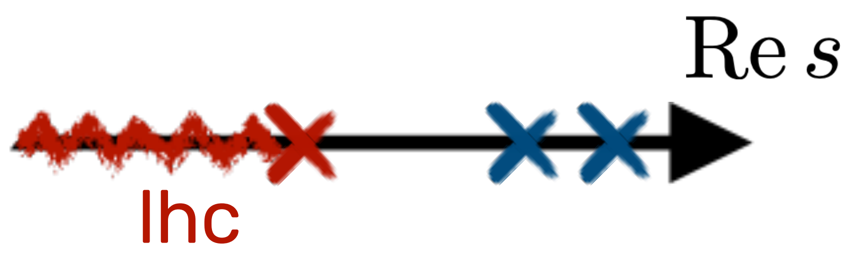


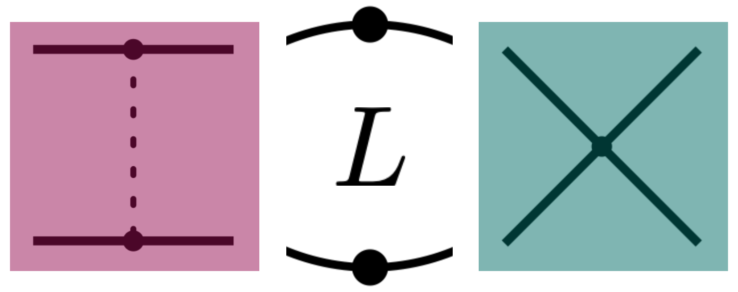
does not contain left-hand cut
not singular at branch point

$\mathcal{L}(p_{\text{cm}}^2)\, iF(P, L)\, \mathcal{R}(p_{\text{cm}}^2)$

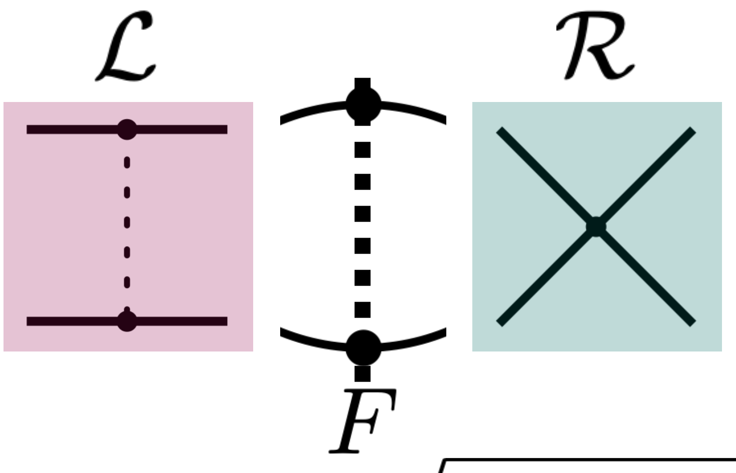
singular at branch point



4. On-shellness as the issue

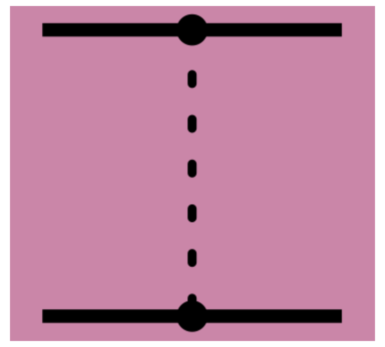


$$\boldsymbol{k} \in \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$$



$$|\boldsymbol{k}_{\text{cm}}| \to p_{\text{cm}} = \sqrt{s/4 - M_N^2}$$

$$\int d\cos\theta_{\text{cm}}$$



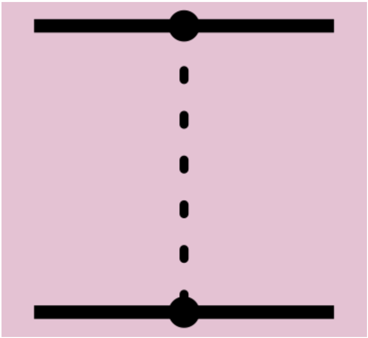
$$\propto \frac{1}{\boldsymbol{k}_{\text{cm}}^2} \log\left(\frac{4\boldsymbol{k}_{\text{cm}}^2 + M_\pi^2}{M_\pi^2} - i\epsilon\right)$$

partially on-shell kinematics

safe on the cut
(no energy dependence)

- loop momentum k is individually on mass shell $k \to (\omega_N(\boldsymbol{k}), \boldsymbol{k})$
- $P - k$ is not on shell

$$\int d\cos\theta_{\text{cm}}$$



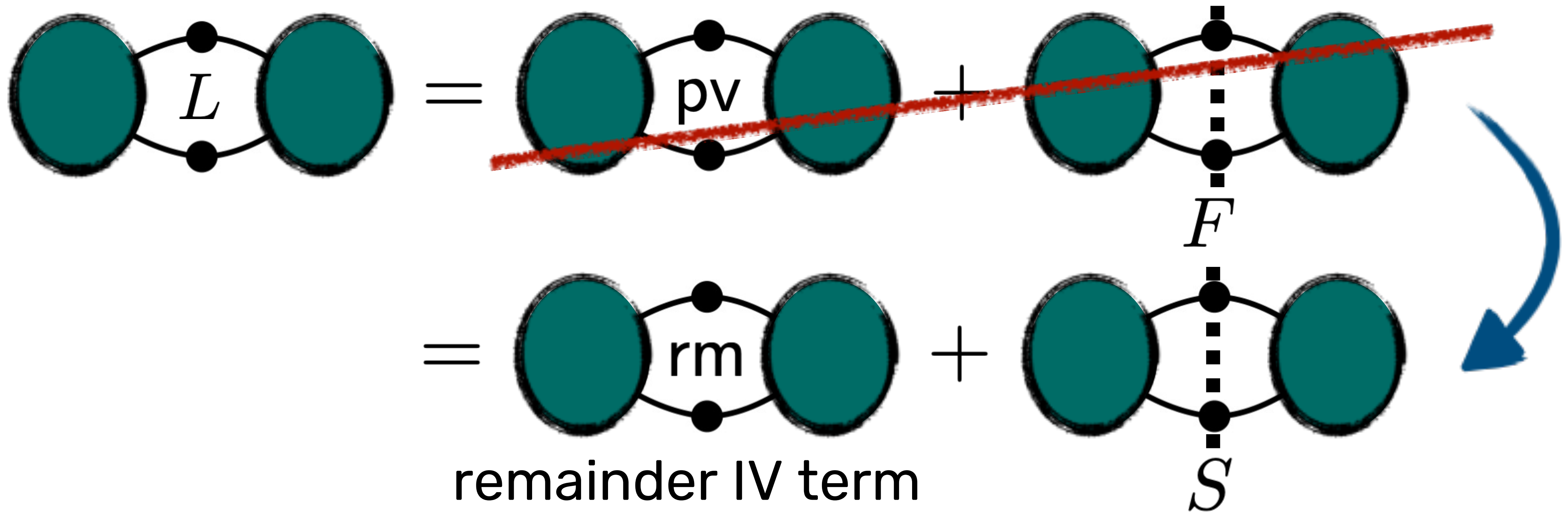
$$\propto \frac{1}{s - 4M_N^2} \log\left(\frac{s - 4M_N^2 + M_\pi^2}{M_\pi^2} - i\epsilon\right)$$

on-shell kinematics

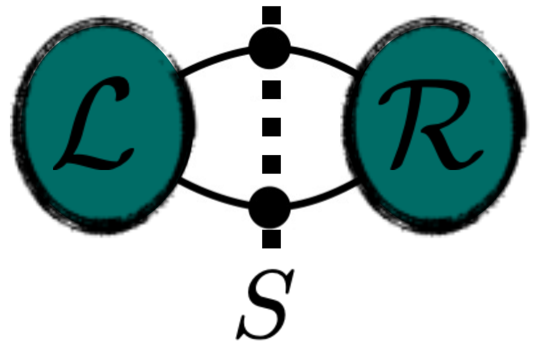
- momenta k and $P - k$ both on shell
- NN intermediate state on shell

5. Proposed formalism

- on-shellness of π exchanges seems to create the issues
- AM projection seems to be safe
- modify loop splitting procedure



change cutting to keep neighbours partially off shell



$$= \widetilde{\mathcal{L}}_{\boldsymbol{k}^\star \ell m}(P)\, iS_{\boldsymbol{k}^\star \ell m, \boldsymbol{k}'^\star \ell' m'}(P, L)\, \widetilde{\mathcal{R}}^*_{\boldsymbol{k}'^\star \ell' m'}(P)$$

sum over repeated indices

sum over spatial loop momentum → sum over repeated $\boldsymbol{k}^\star$ index

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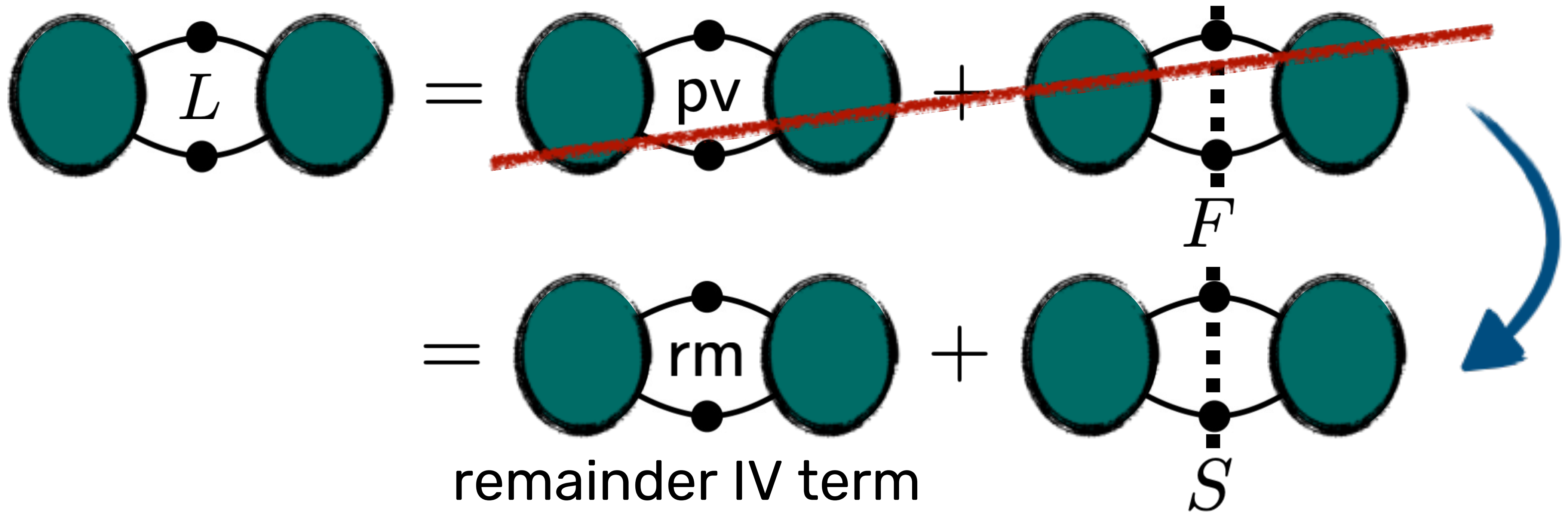

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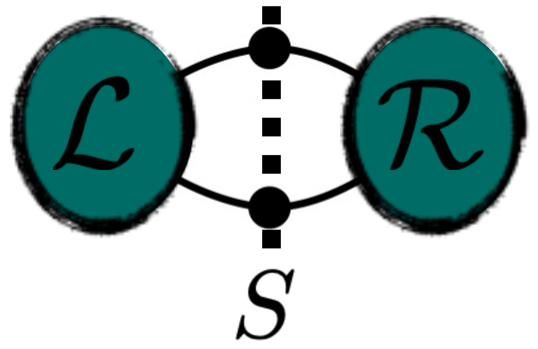

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sum over repeated indices

sum over spatial loop momentum → sum over repeated $\boldsymbol{k}^\star$ index

elements of S given by

$$S_{\boldsymbol{k}^\star \ell m, \boldsymbol{k}'^\star \ell' m'}(P,L) = \frac{1}{2L^3} \frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)\, \delta_{\boldsymbol{k}^\star \boldsymbol{k}'^\star}\, |\boldsymbol{k}^\star|^{\ell+\ell'}\, H(\boldsymbol{k}^\star)}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]}$$

compare with

$$F_{\ell m, \ell' m'}(P,L) = \left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int_{\boldsymbol{k}}\right] \frac{1}{2} \frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]} \left(\frac{|\boldsymbol{k}^\star|}{k^\star_{\text{os}}}\right)^{\ell+\ell'}$$

5. Proposed formalism

- on-shellness of π exchanges seems to create the issues
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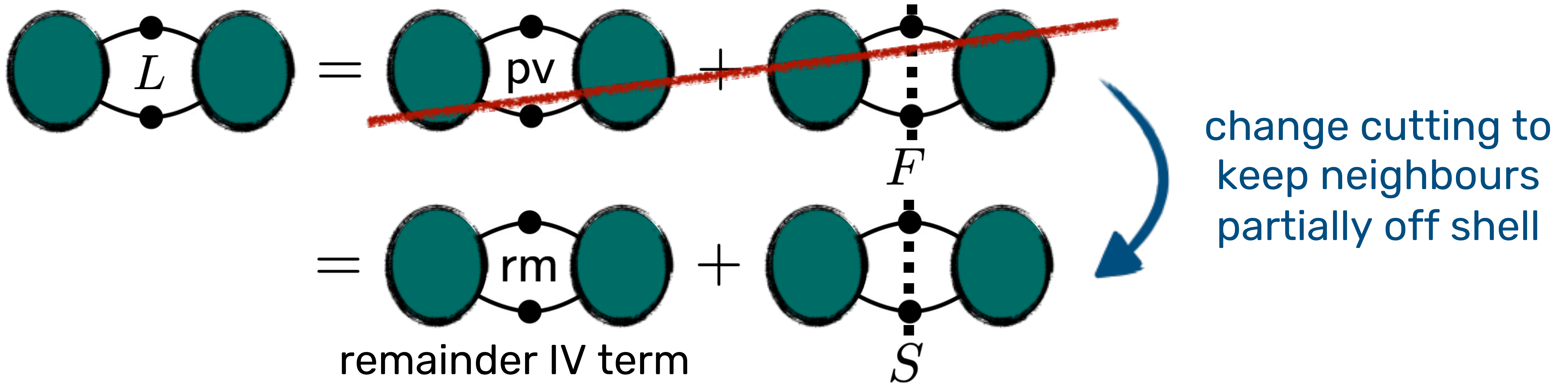
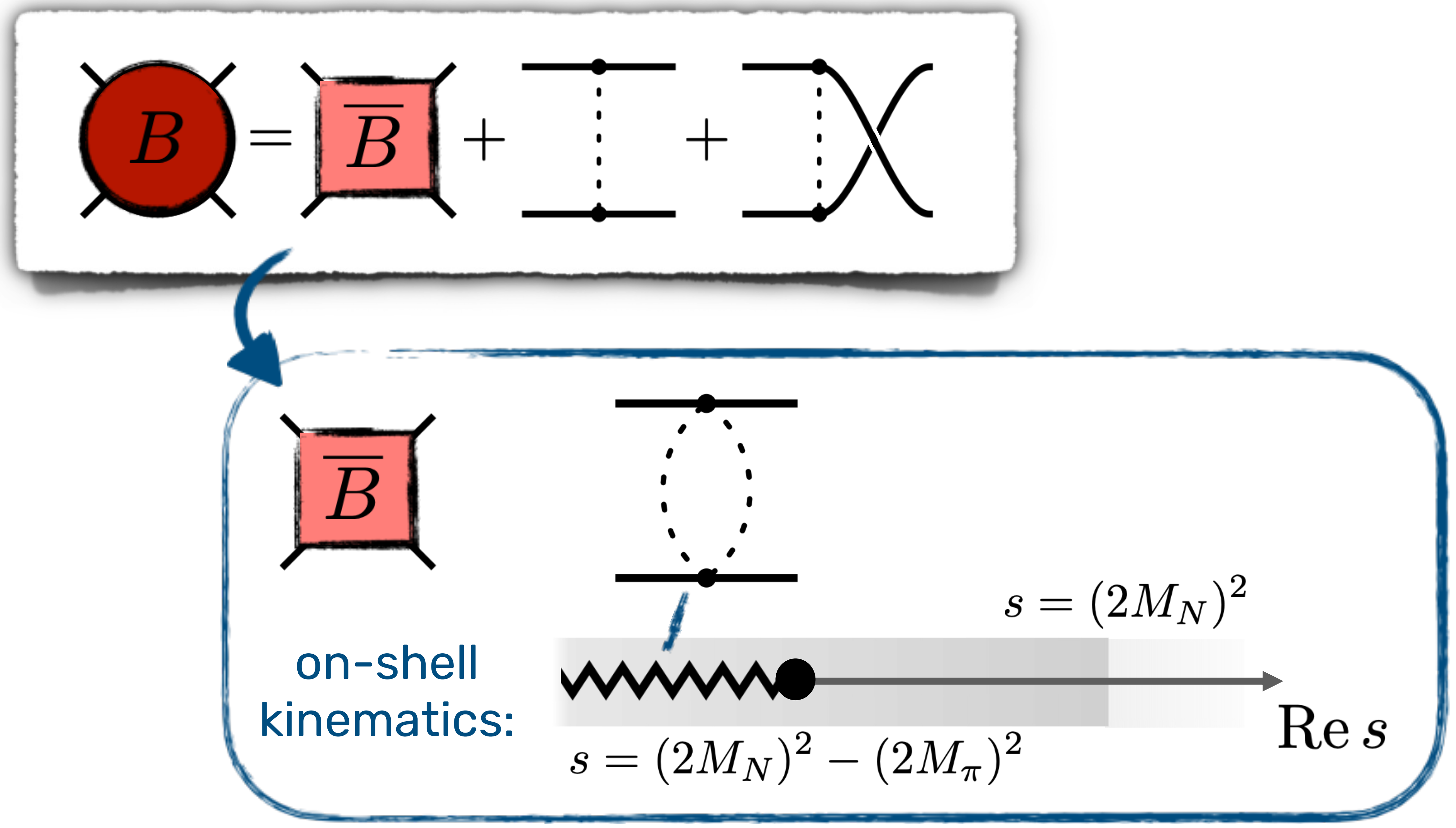
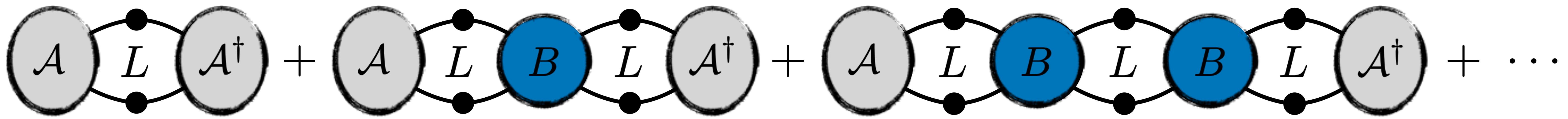


Main ingredients:

- index space extended from ℓm to CM loop momentum $\otimes$ angular momentum indices $\boldsymbol{k}_{\text{cm}}\ell m$ to keep neighbours off-shell
- define a modified kernel $\overline{B}$
- π exchanges kept off shell
- $\overline{B}$ safe down to second left-hand cut when on shell

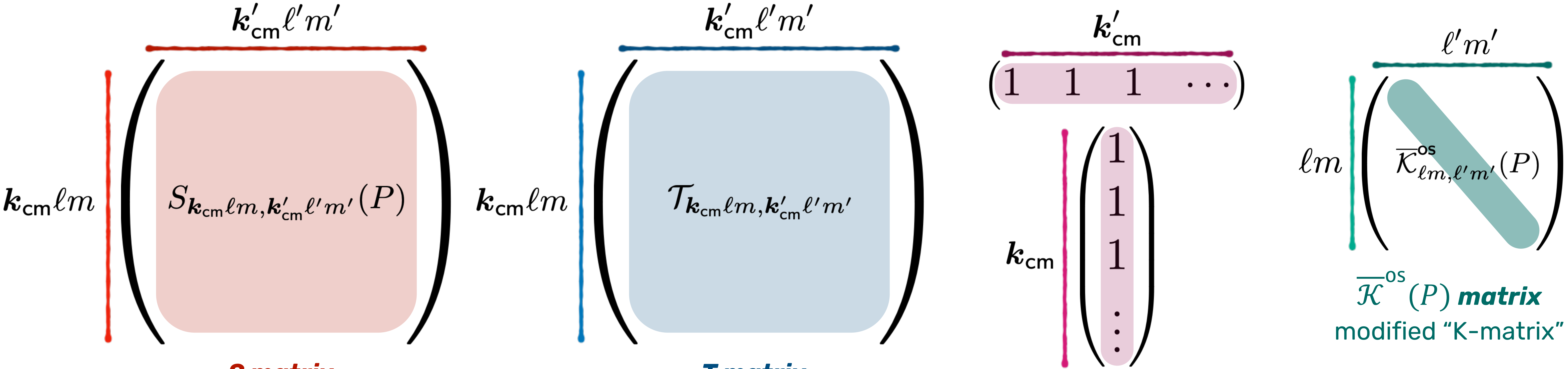


5. Adapted quantisation condition

$C_L(P) =$ 

$$\rightarrow \quad \det_{\boldsymbol{k}_{\text{cm}}\ell m}\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}^{\text{os}}(P)\xi + 2g^2\mathcal{T}\right] = 0$$

QC can be used to constrain $\overline{\mathcal{K}}^{\text{os}}(P)$ from the FV spectrum



S matrix
of known functions
– encodes the FV effects

T matrix
of known off-shell
logarithms

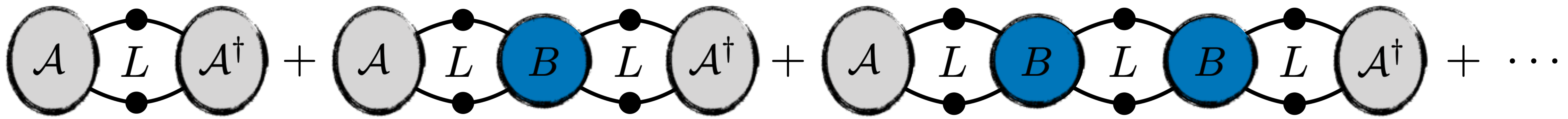
$\xi, \xi^\dagger$ trivial vector

$\overline{\mathcal{K}}^{\text{os}}(P)$ ***matrix***
modified "K-matrix"

g $NN\pi$ effective coupling

5. Adapted quantisation condition

$C_L(P) =$



$$\det_{\boldsymbol{k}_{\text{cm}}\ell m}\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}^{\text{os}}(P)\xi + 2g^2\mathcal{T}\right] = 0$$

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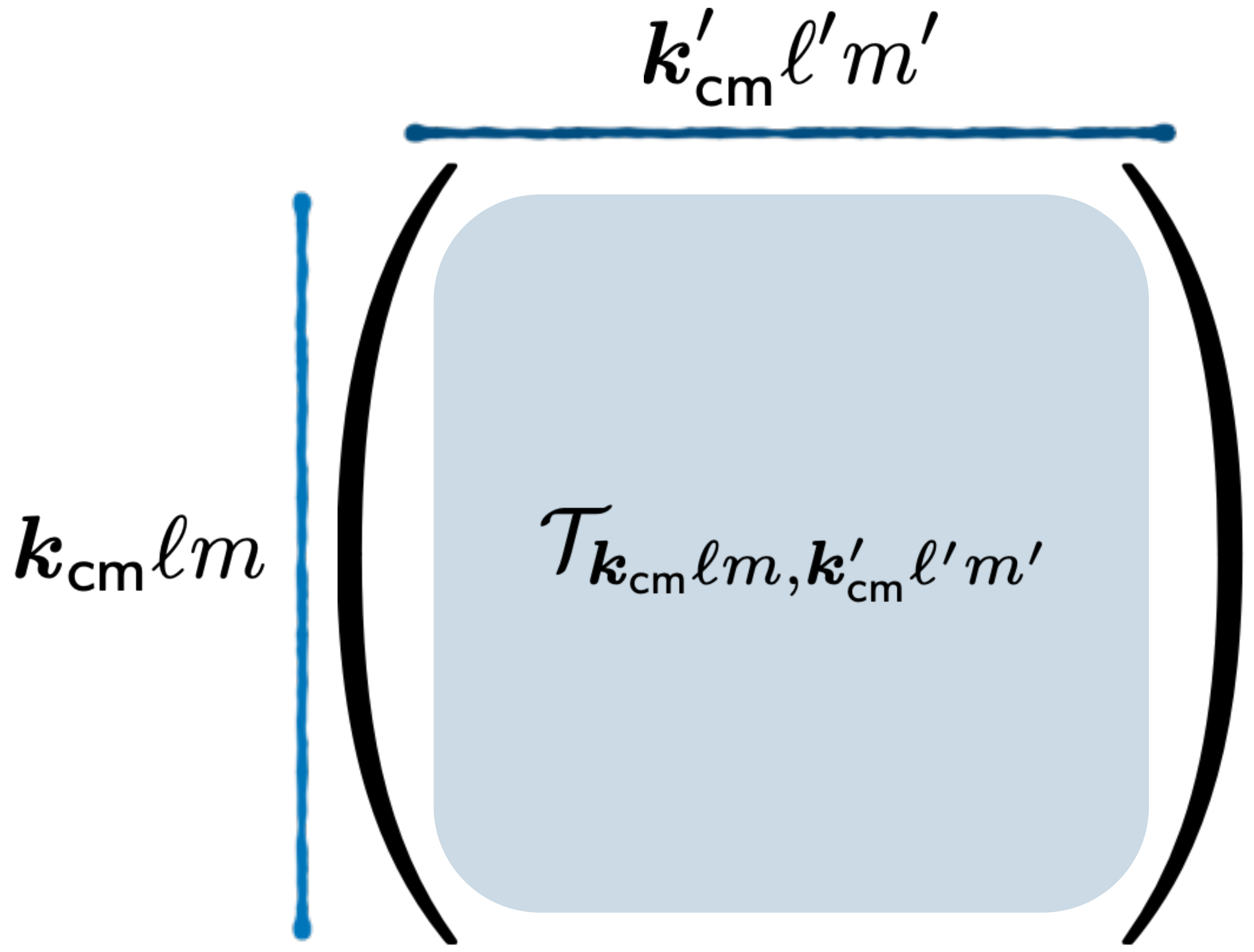


T matrix

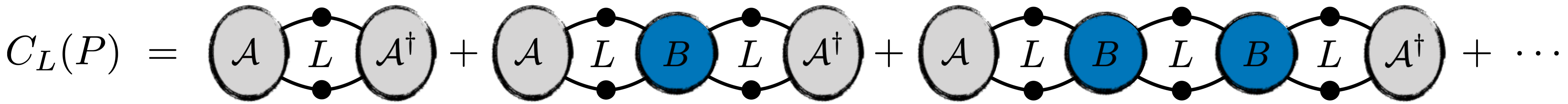
of known off-shell logarithms

e.g. S-wave result

$$\mathcal{T}_{\boldsymbol{k}_{\text{cm}}00,\boldsymbol{k}'_{\text{cm}}00} = \frac{1}{4|\boldsymbol{k}_{\text{cm}}||\boldsymbol{k}'_{\text{cm}}|}\log\left(\frac{2\omega_N(\boldsymbol{k}_{\text{cm}})\omega_N(\boldsymbol{k}'_{\text{cm}}) + 2|\boldsymbol{k}_{\text{cm}}||\boldsymbol{k}'_{\text{cm}}| - 2M_N^2 + M_\pi^2 - i\epsilon}{2\omega_N(\boldsymbol{k}_{\text{cm}})\omega_N(\boldsymbol{k}'_{\text{cm}}) - 2|\boldsymbol{k}_{\text{cm}}||\boldsymbol{k}'_{\text{cm}}| - 2M_N^2 + M_\pi^2 - i\epsilon}\right)$$

$$\omega_N(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + M_N^2}$$

5. Adapted quantisation condition

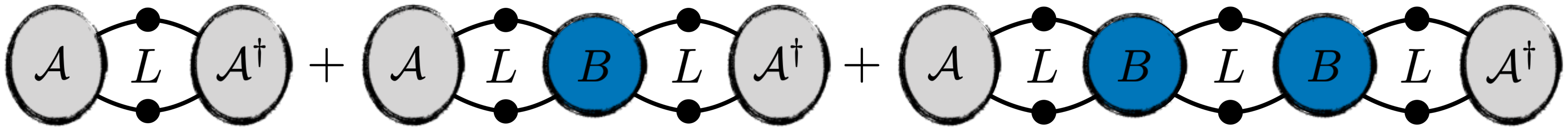


$$\rightarrow \quad \det_{\boldsymbol{k}_{\mathrm{cm}}\ell m}\left[S(P,L)^{-1} + \xi^{\dagger}\overline{\mathcal{K}}^{\mathrm{os}}(P)\xi + 2g^{2}\mathcal{T}\right] = 0$$

QC can be used to constrain $\overline{\mathcal{K}}^{\mathrm{os}}(P)$ from the FV spectrum

- inclusion of spin relatively straightforward: index space expanded to include spin state labels
- modified quantisation condition inspired by three-particle formalism work (Blanton, Briceño, Döring, Draper, Mai, Meißner, Müller, Hammer, Hansen, Pang, Romero-López, Rusetsky, Sharpe…)

5. Adapted quantisation condition

$C_L(P) =$  $+ \cdots$

$$\det_{\boldsymbol{k}_{\text{cm}}\ell m} \left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}^{\text{os}}(P)\xi + 2g^2 \mathcal{T} \right] = 0$$

QC can be used to constrain $\overline{\mathcal{K}}^{\text{os}}(P)$ from the FV spectrum

- inclusion of spin relatively straightforward: index space expanded to include spin state labels
- modified quantisation condition inspired by three-particle formalism work (Blanton, Briceño, Döring, Draper, Mai, Meißner, Müller, Hammer, Hansen, Pang, Romero-López, Rusetsky, Sharpe...)
- potentially more practical alternative re-writings of QC under investigation:

$$\det_{\ell m} \left[\overline{\mathcal{K}}^{\text{os}}(P_j)^{-1} + F^{\mathcal{T}}(P_j, L) \right] = 0$$

just in $\ell m, \ell' m'$ index space

$$F^{\mathcal{T}}(P,L) = \xi S(P,L) \frac{1}{1 + 2g^2 \mathcal{T}(P) S(P,L)} \xi^\dagger$$

extra momentum index
hiding inside F matrix

5. Extracting the amplitude

An extra step is needed to connect K-bar to the amplitude:

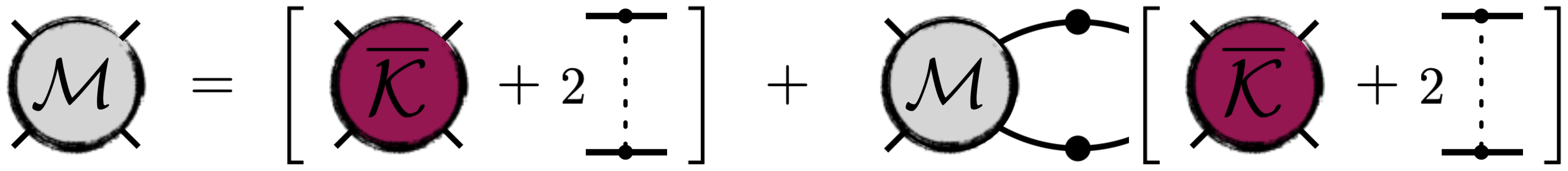
5. Extracting the amplitude

An extra step is needed to connect K-bar to the amplitude:

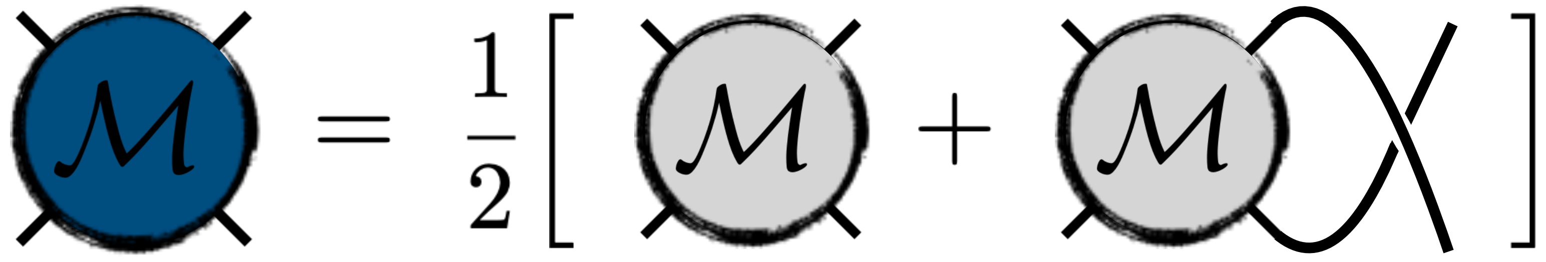
We need to solve integral equations of the type

$$\mathcal{M}^{\text{aux}}(P,p,p') = \mathcal{K}^{\mathcal{T}}(P,p,p') - \frac{1}{2}\int \frac{d^3\boldsymbol{k}^\star}{(2\pi)^3} \frac{\mathcal{M}^{\text{aux}}(P,p,k)\, H(\boldsymbol{k}^\star)\, \mathcal{K}^{\mathcal{T}}(P,k,p')}{4\omega_N(\boldsymbol{k}^\star)\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2 + i\epsilon\right]}$$

$$\mathcal{K}^{\mathcal{T}}(P,p,p') = \overline{\mathcal{K}}^{\text{os}}(P,p,p') + 2g^2\mathcal{T}(P,p,p')$$



solve for auxiliary amplitude



symmetrize to get amplitude

6. Summary

- left-hand cut issues arise from combination of infinite-volume effect + angular momentum projection + on-shell projection
- we have presented a method that extends the Lüscher formalism to the left-hand cut, accounting for both t- and u-channels and also spin
- full workflow including the solving of integral equations allows extraction of the amplitude
- modified procedure has been shown to be equivalent to standard Lüscher method when the latter is applicable
- paper is already up on the arXiv! [ABR and Hansen 2023]

6. Outlook

- extensions of formalism (e.g. non-identical particles, different masses, lower energy range) currently being investigated (towards applications such as DD^* scattering)
- comparison with proposed EFT-based alternative approaches [Meng, Baru, Epelbaum, Filin, Gasparyan 2023]
- implementation in the form of a Python library
- taking advantage of progress in solving integral equations in the three-particle RFT formalism to implement algorithms to extract the amplitude from K-bar matrix
- clarifying and exploring connections and consistency with three-particle formalism (e.g. this method as a limiting case?) — see recent work by [Hansen, Romero-López, Sharpe 2024]
- exploring potential connections to dispersive methods

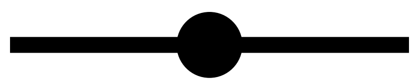
Thank you for your attention!

... any questions?

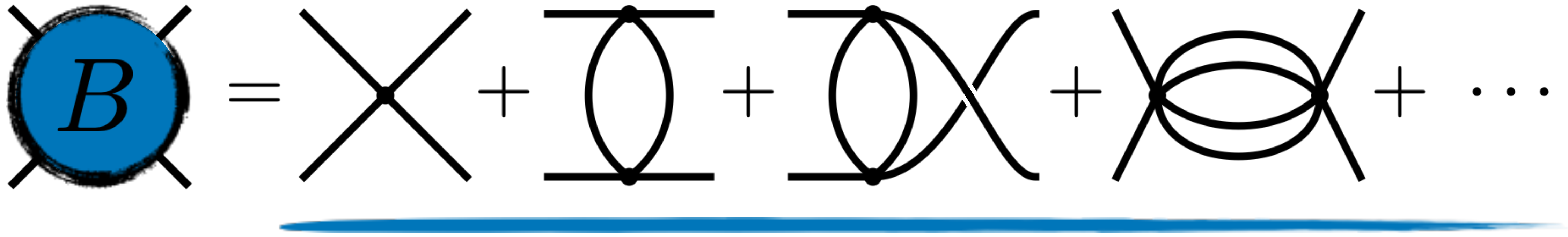
Back-up slides…

Structure of the scattering amplitude

dressed propagator

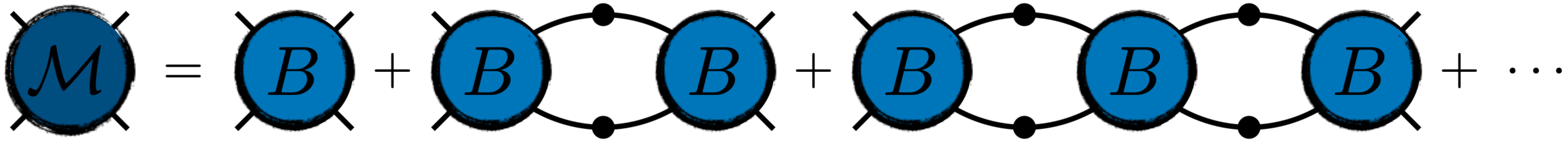


Bethe-Salpeter kernel

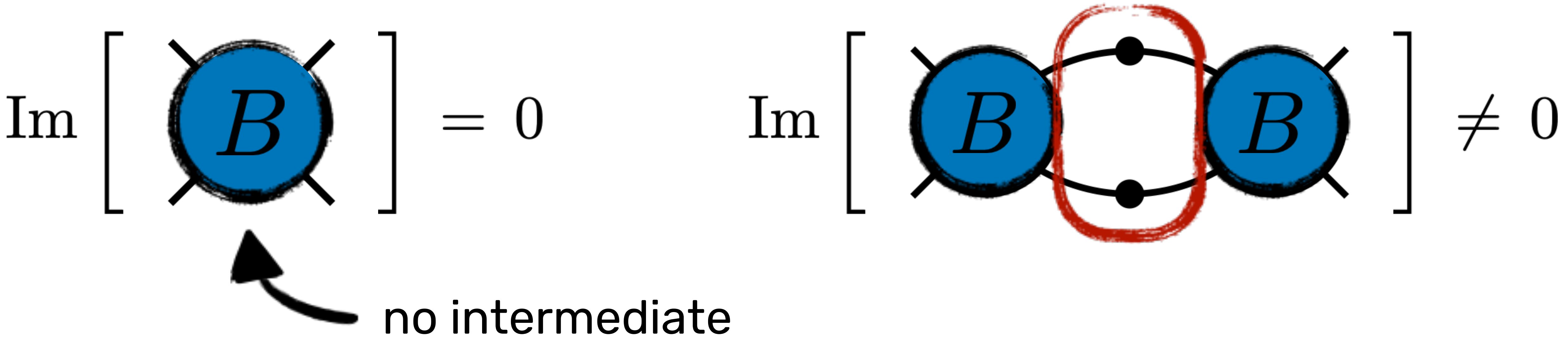


all amputated $NN \to NN$ diagrams which are 2-particle irreducible in the s-channel

We can write the ***amplitude*** as:

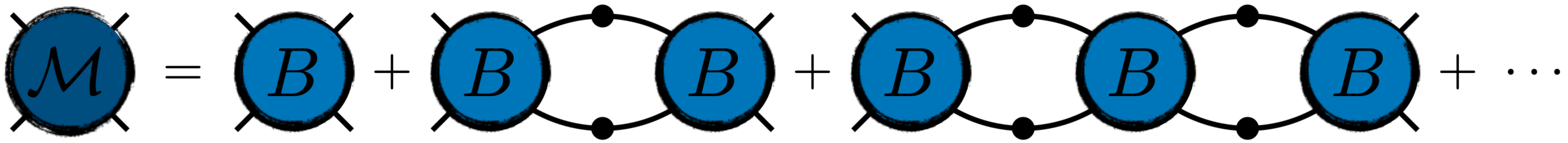


In the elastic regime, ***only two-particle (***NN***) states can go on shell***:

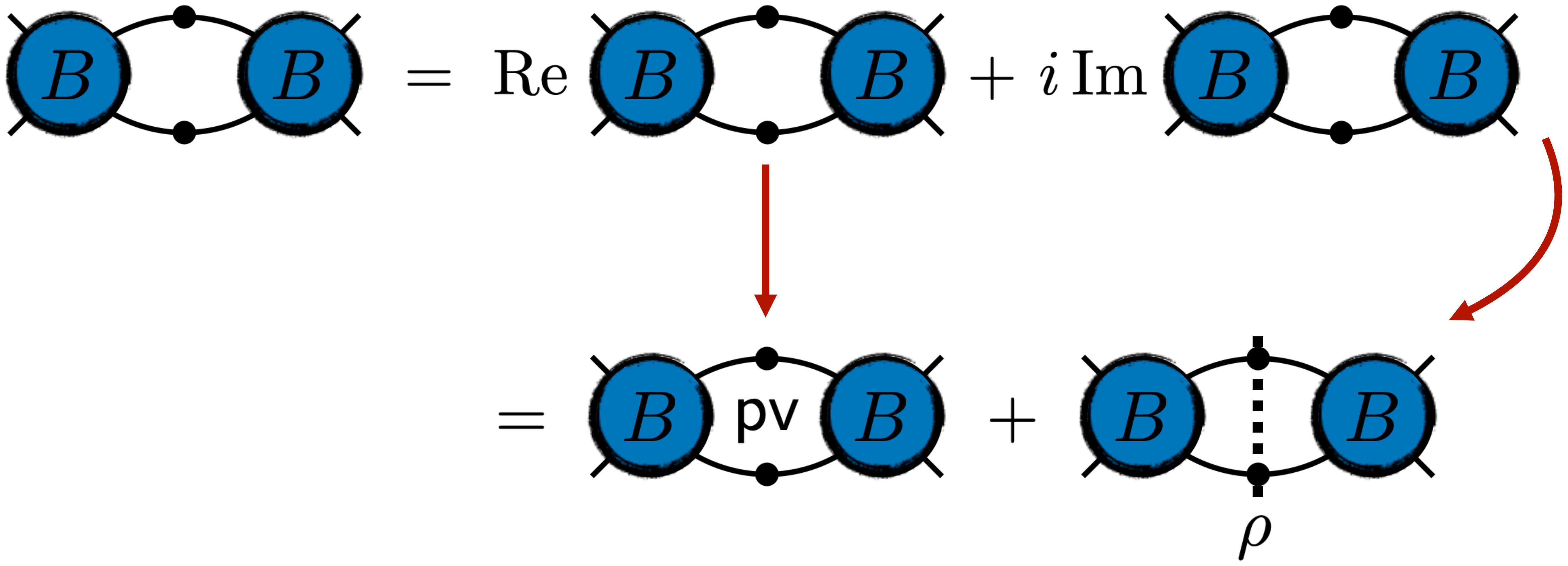


no intermediate NN states

Structure of the scattering amplitude



Split two-particle loops into real and imaginary parts:



imaginary part of the loop: delta functions put neighbouring kernels on shell

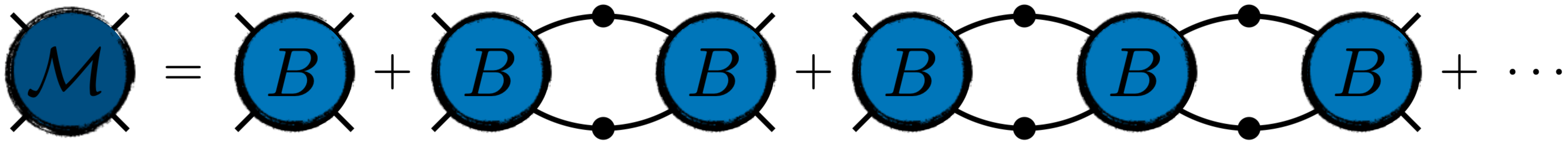
$$\int_k B(P,p,k)\,\delta\big(k^2-M_N^2\big)\,\delta\big((P-k)^2-M_N^2\big)\,B(P,k,p')$$

$$B(s)\,i\rho(s)\,B(s)$$

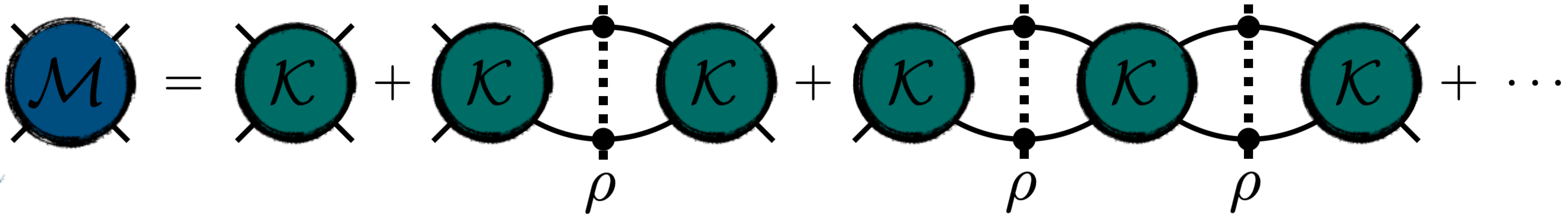
$$\rho_{\ell m,\ell' m'}(s)=\delta_{\ell\ell'}\delta_{mm'}\,\frac{1}{32\pi}\sqrt{1-\frac{4M_N^2}{s}}$$

apply this separation to all two-particle loops

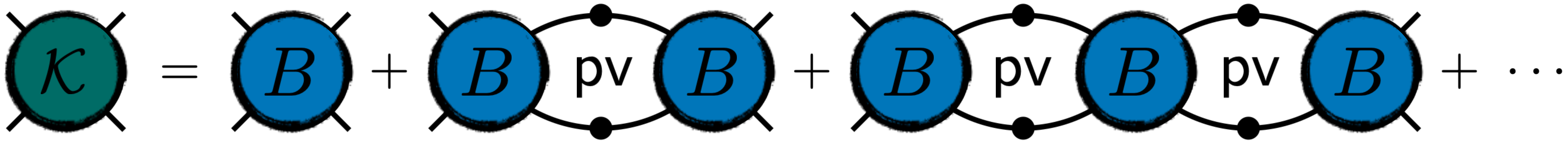
Structure of the scattering amplitude



Reorganise amplitude sum into series:



using ***K-matrix***:



$$i\mathcal{M}(s) = i\mathcal{K}(s) + i\mathcal{K}(s)\, i\rho(s)\, i\mathcal{K}(s) + \cdots$$

$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

Recovering the standard formalism

setting $g = 0$ → $\det_{\boldsymbol{k}_{\text{cm}}\ell m}[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}^{\text{os}}(P)\xi] = 0$ ⟷ $\det_{\ell m}[\xi S(P,L)\xi^\dagger + \overline{\mathcal{K}}^{\text{os}}(P)^{-1}] = 0$

$$\xi S(P,L)\xi^\dagger = F(P,L) + I(P)$$

simple algebraic relation to Lüscher F
$I(P)$ matrix of known geometric functions

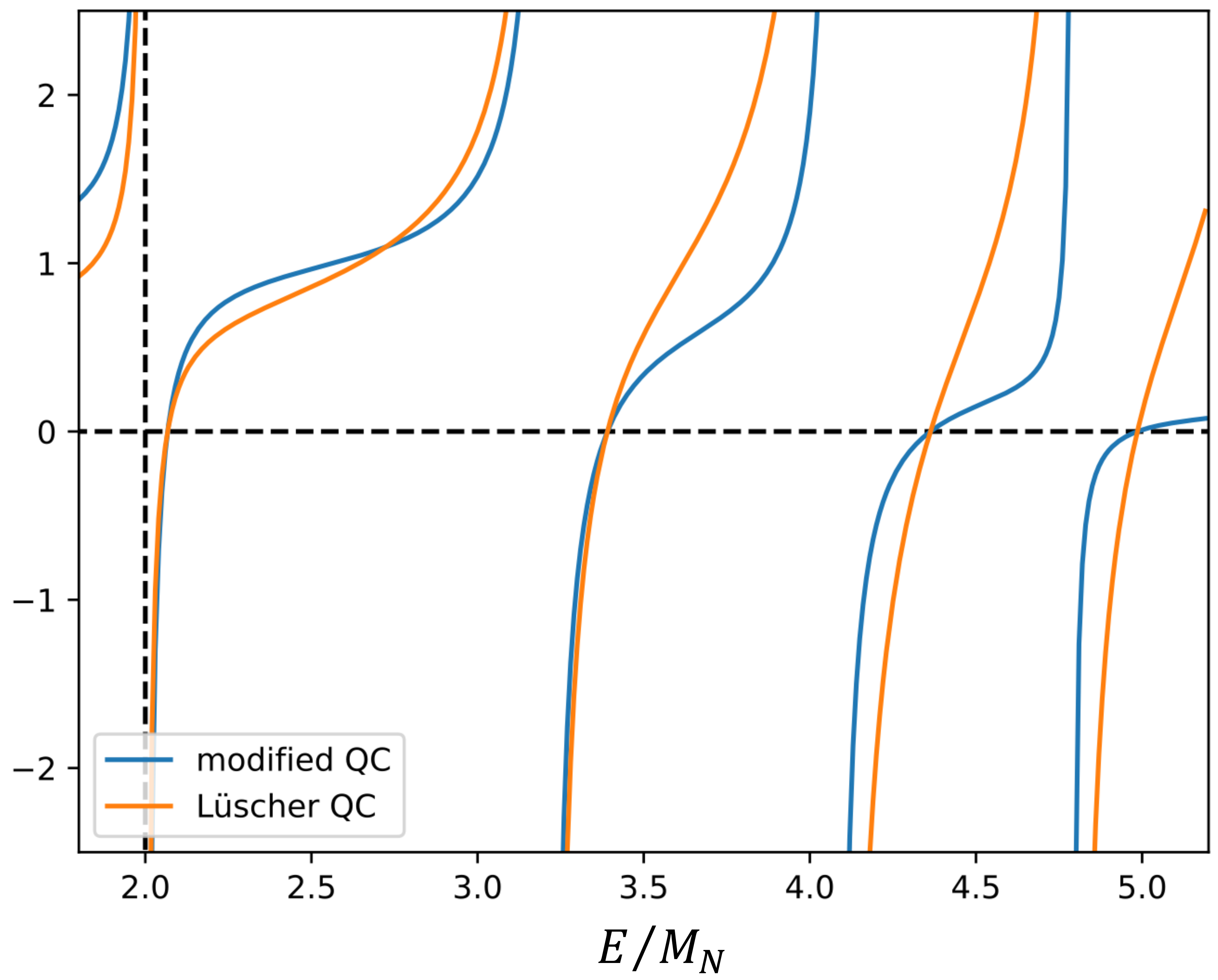
$$\overline{\mathcal{K}}^{\text{os}}(P)^{-1} = \mathcal{K}(P)^{-1} - I(P)$$

simplification of the integral equations in $g = 0$ case

→ $\det_{\ell m}[F(P,L) + \mathcal{K}(P)^{-1}] = 0$

we recover the standard condition

$a = 0.5/M_N\,,\ r = 0.1/M_N$



$$\mathcal{K}(s) = 16\pi\sqrt{s}\frac{1}{p\cot\delta}\,, \qquad p\cot\delta = -\frac{1}{a} + \frac{1}{2}rp^2 + \mathcal{O}(p^4)$$