Lattice QCD calculation of Hyperon transition form factors using the Feynman-Hellmann method.

Mischa Batelaan
QCDSF-UKQCD-CSSM collaboration

The University of Adelaide

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• SM requires unitarity: \( |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \)

• \( |V_{us}| \) can be constrained by
  • (Semi-) leptonic Kaon decays
  • Semi-leptonic hyperon decays

• Transition matrix element for semi-leptonic hyperon decay:

\[
T = \frac{G_F}{\sqrt{2}} V_{us} \left[ \langle B' | \bar{u} \gamma_\mu \gamma^5 s | B \rangle - \langle B' | \bar{u} \gamma_\mu s | B \rangle \right] \bar{\nu}_\mu (1 - \gamma^5) \nu_l
\]

• Lattice QCD determinations can improve on phenomenological form factor values
How Does Lattice QCD Work?

• Discretize space-time
  • Quarks on the lattice sites
  • Gauge fields on the links

• Use Monte Carlo sampling to generate gauge fields

• Calculate expectation value by averaging over Gauge configurations

\[
\langle \Omega | \mathcal{O} | \Omega \rangle = \frac{1}{Z} \int D\bar{\psi} \, D\psi \, DA \, \mathcal{O} \, e^{iS}
\]

\[
\text{Discretise} \rightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{W}\{\mathcal{O}\} \left( U_i \right)
\]

• Then account for the systematics (finite volume, lattice spacing, larger-than-physical pion mass)
What can we calculate with Lattice QCD?

Nucleon Mass

- Creation operator which couples to nucleons:
  - Will couple to any state with the same quantum numbers

- The spectral decomposition of the correlation function includes a tower of states with increasing energies

- Effective energy of the correlator asymptotes towards the ground state energy

\[
C^{2pt}(x, 0) = \langle T \{ \chi_N(x, t) \bar{\chi}_N(0) \} \rangle
\]

\[
C^{2pt}(t) = \sum_{i=0} A_i e^{-E_i(\vec{p})t}
\]

\[
E_{\text{eff}} = \ln \left( \frac{C(t)}{C(t + 1)} \right)
\]

\[
\xrightarrow{t \gg 0} E_0
\]
What can we calculate with Lattice QCD?

Nucleon Mass

- Calculate correlation function on each gauge configuration
- Use bootstrap resampling to get uncertainties

- Fit the correlator using exponential function ansatz
  - Signal to Noise ratio decreases at large time
  - Ground state dominates signal only at large times
  - Signal strength decreases at large momenta

\[
C^{2pt}(t) = \sum_{i=0} A_i e^{-E_i(\vec{p})t}
\]
What can we calculate with Lattice QCD?

How can we deal with these issues:

- Include excited states in the ansatz
  - Can fit from earlier time
- Use multiple operators to improve overlap with the ground state
  - Signal improves
  - Easier to filter out excited states
- Weighted averaging over multiple fit ranges
  - Reduces effects of researcher's fit window choice

\[
C_0^{2pt}(t) = A_0 e^{-E_0 t}
\]

\[
C_1^{2pt}(t) = A_0 e^{-E_0 t} + A_1 e^{-E_1 t}
\]

Baryon mass, \((\kappa_1, \kappa_2) = (0.121040, 0.121040), 32x64\)
What can we calculate with Lattice QCD?

**Baryon Matrix Elements**

- Include a current insertion operator between creation and annihilation

\[ C^{3pt}(t) = \sum_{B, B'} e^{-E_B'(t-\tau)} e^{-E_B \tau} \langle \Omega | \chi(0) | B' \rangle \langle B' | \mathcal{O} | B \rangle \langle B | \bar{\chi}(0) | \Omega \rangle \]

- Two towers of exponentially decaying excited states
- Form factors are contained in this matrix element

\[ C^{3pt}(t) = \left\langle \chi_N(x_2, t) \hat{O}(x_1, \tau) \bar{\chi}_N(0, 0) \right\rangle \]

- Annihilate nucleon at the sink
- Insert current
- Create nucleon at the source
Three-point functions

On the lattice:
• Use a sequential source
• Fix the sink time and momentum
Three-point functions

On the lattice:
• Use a sequential source
• Fix the sink time and momentum
• Invert from the sequential source at the sink to the operator insertion
Three-point functions

On the lattice:
- Use a sequential source
- Fix the sink time and momentum
- Invert from the sequential source at the sink to the operator insertion
- Insert the current operator and connect it with the source
Three-point functions

• Advantages:
  • Allows for operator choice after all the inversions
  • Free choice of operator, momentum transfer

• Disadvantages:
  • It has two time windows for which excited states need to be controlled
  • Requires separate inversions for
    • Every sink momentum
    • Every source-sink time separation
    • Every polarisation
Feynman-Hellmann method

1. Modify the QCD action with an operator

\[ \mathcal{L} \rightarrow \mathcal{L} + \lambda \mathcal{O} \]

2. Calculate the energy spectrum with the modified action

3. Relate the change in energy to the matrix element:

\[ \frac{\partial E_X}{\partial \lambda} \bigg|_{\lambda=0} \propto \langle X | \mathcal{O} | X \rangle \]

Connected contributions => Invert the new fermion matrix
Disconnected contributions => generate new gauge configurations
1. Modify the QCD action with an operator including momentum

\[ \mathcal{L} \rightarrow \mathcal{L} + \lambda \left( e^{i \vec{q} \cdot \vec{x}} + e^{-i \vec{q} \cdot \vec{x}} \right) \mathcal{O} \]

2. Calculate energy spectrum

3. Relate change in energy to the matrix element

\[ \left. \frac{\partial E_X(p')}{\partial \lambda} \right|_{\lambda=0} \propto \langle X(p') | \mathcal{O} | X(p) \rangle \]

This requires Breit frame kinematics: \( E_X(p') = E_X(p) \)
Non-forward Feynman-Hellmann method results

Matrix elements up to high momentum are accessible:

![Graphs showing matrix elements vs. Q^2](image-url)

hep-lat [2202.01366]
Transition Form Factors

\[
\langle B' | \mathcal{V}_\mu | B \rangle = \gamma_\mu f_{1}^{BB'}(Q^2) + \sigma_{\mu \nu} q_\nu \left( \frac{f_{2}^{BB'}(Q^2)}{M_B + M_{B'}} - i q_\mu \right) \left( \frac{f_{3}^{BB'}(Q^2)}{M_B + M_{B'}} \right) = 0 \text{ if } B=B'
\]

\(f_1(Q^2=0)\) can be used to constrain \(V_{us}\) element of the CKM matrix

Breit frame condition cannot be satisfied as easily anymore

\(\Rightarrow\) Consider quasi-degenerate energy states

\[E_{B_r}(\vec{p}_r) = \bar{E} + \epsilon_r, \quad r = 1, \ldots, d_S\]

\(d_s\) states which are quasi-degenerate in energy

quasi-degenerate states must be well separated from any other states
Consider a two-point function with a Hamiltonian which includes a perturbing operator:

$$C_{\lambda B'B}(t; \vec{p}, \vec{q}) = \lambda \langle 0| \hat{B}'(0; \vec{p}') \hat{S}_\lambda(\vec{q})^t \hat{B}(0, \vec{0}) |0\rangle_\lambda$$

Transfer matrix with perturbed Hamiltonian:

$$\hat{S}_\lambda(\vec{q}) = e^{-\hat{H}_\lambda(\vec{q})}$$

$$\hat{H}_\lambda(\vec{q}) = \hat{H}_0 - \lambda \hat{O}(\vec{q})$$

Insert two complete sets-of-states into the two-point function:

$$C_{\lambda B'B}(t; \vec{p}, \vec{q}) = \sum_{X(\vec{p}_X)} \sum_{Y(\vec{p}_Y)} \lambda \langle 0| \hat{B}'(\vec{p}')|X(\vec{p}_X)\rangle \langle X(\vec{p}_X)| \hat{S}_\lambda(\vec{q})^t|Y(\vec{p}_Y)\rangle \langle Y(\vec{p}_Y)| \hat{B}(\vec{0}) |0\rangle_\lambda$$

We want this
Expand the transfer matrix for small values of $\lambda$:

$$e^{-(\hat{H}_0 - \lambda \hat{O})t} = e^{-\hat{H}_0 t} + \lambda \int_0^t dt' e^{-\hat{H}_0 (t-t')} \hat{O} e^{-\hat{H}_0 t'} + \lambda^2 \text{(compton terms)}$$

Consider the separate pieces:

$$\langle B_r | e^{-(\hat{H}_0 - \lambda \hat{O})t} | B_s \rangle = e^{-\bar{E}t} \left( \delta_{rs} + tD_{rs} + O(2) \right)$$

$$\langle B_r | e^{-(\hat{H}_0 - \lambda \hat{O})t} | Y \rangle = e^{-\bar{E}t} \left( \lambda \frac{\langle B_r | \hat{O} | Y \rangle}{E_Y - E_{B_r}} + O(2) \right) + \text{more damped}$$

$D_{rs}$ is defined as:

$$D_{rs}(\lambda, \epsilon) = -\epsilon e \delta_{rs} + \lambda \langle B_r (\vec{p}_r) | \hat{O} (\vec{q}) | B_s (\vec{p}_s) \rangle$$
We can diagonalise the matrix $D$, such that:

$$D_{rs} = \sum_{i=1}^{d_S} \mu^{(i)} e_r^{(i)} e_s^{(i)\ast}$$

This allows us to write the two-point function as:

$$C_{\lambda B'B}^{\lambda B} (t; \vec{p}, \vec{q}) = \sum_{i=1}^{d_S} A^{(i)}_{\lambda B'B} (\vec{p}, \vec{q}) e^{-E^{(i)}_\lambda (\vec{p}, \vec{q}) t}$$

With the energies being determined by the eigenvalues:

$$E^{(i)}_{\lambda} (\vec{p}, \vec{q}) = \bar{E}(\vec{p}, \vec{q}) - \mu^{(i)} (\epsilon, \lambda; \vec{p}, \vec{q}) , \quad i = 1, \ldots, d_S$$

The problem is now a Generalized Eigenvalue Problem in $B_r$, $B_s$ space to find $E^{(i)}_{\lambda} (\vec{p}, \vec{q})$
\[ \Sigma^- \rightarrow n \text{ Transition} \]

- Consider the action:
  \[ S = S_g + \int_x \frac{1}{(\bar{u}, \bar{s})} \begin{pmatrix} D_u & -\lambda \mathcal{T}' \\ -\lambda \mathcal{T} & D_s \end{pmatrix} \begin{pmatrix} u \\ s \end{pmatrix} + \int_x \bar{d} D_d d \]
  where \( \mathcal{T}(x, y; \bar{q}) = \gamma e^{i \bar{q} \cdot \bar{x}} \delta_{x,y} \)

- Construct a matrix of correlation functions:
  \[ C_{\lambda B'B} = \begin{pmatrix} C_{\lambda \Sigma \Sigma} & C_{\lambda \Sigma N} \\ C_{\lambda N \Sigma} & C_{\lambda NN} \end{pmatrix}_{B'B} \]

- After solving the GEVP, the energies are related to the matrix element
  \[ \Delta E_{\lambda \Sigma N} = \sqrt{(E_N(\bar{q}) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(\bar{q}) | \bar{u} \gamma_4 s | \Sigma(\bar{0}) \rangle \right|^2} \]
Each correlator is built from a Green's function:

\[
\begin{pmatrix}
G_{uu} & G_{us} \\
G_{su} & G_{ss}
\end{pmatrix}
= \begin{pmatrix}
(M^{-1})_{uu} & (M^{-1})_{us} \\
(M^{-1})_{su} & (M^{-1})_{ss}
\end{pmatrix}
\]

Where:

\[
G^{(uu)} = (1 - \lambda^2 D_u^{-1} T D_s^{-1} \gamma_5 T^\dagger \gamma_5)^{-1} D_u^{-1}
\]

\[
G^{(ss)} = (1 - \lambda^2 D_s^{-1} \gamma_5 T^\dagger \gamma_5 D_u^{-1} T)^{-1} D_s^{-1}
\]

\[
G^{(us)} = \lambda D_u^{-1} T G^{(ss)}
\]

\[
G^{(su)} = \lambda D_s^{-1} \gamma_5 T^\dagger \gamma_5 G^{(uu)}
\]

Problem: inversion within another inversion
Iterative Green's functions

- We can expand the Green's functions for small lambda.

- Will give the exact result as $n$ goes to infinity.

- We will consider up to order $O(\lambda^4)$.

- This allows changing the value of lambda after the inversions!

\[
\begin{pmatrix}
G_{uu} & G_{us} \\
G_{su} & G_{ss}
\end{pmatrix}
\]

\[
G^{(uu)}_{2n+2} = D_u^{-1} + \lambda^2 D_u^{-1} \mathcal{T} D_s^{-1} \gamma_5 \mathcal{T}^{\dagger} \gamma_5 G^{(uu)}_{2n},
\]

\[
G^{(ss)}_{2n+2} = D_s^{-1} + \lambda^2 D_s^{-1} \gamma_5 \mathcal{T}^{\dagger} \gamma_5 D_u^{-1} \mathcal{T} G^{(ss)}_{2n},
\]

\[
G^{(us)}_{2n+1} = \lambda D_u^{-1} \mathcal{T} G^{(ss)}_{2n}
\]

\[
G^{(su)}_{2n+1} = \lambda D_s^{-1} \gamma_5 \mathcal{T}^{\dagger} \gamma_5 G^{(uu)}_{2n}
\]
For example, take $\mathcal{O}(\lambda^3)$:

\[
G_3^{(us)} = \lambda D_u^{-1} \mathcal{T} D_s^{-1} + \lambda^3 D_u^{-1} \mathcal{T} D_s^{-1} \gamma_5 \mathcal{T}^\dagger \gamma_5 D_u^{-1} \mathcal{T} D_s^{-1}
\]

\[
G_3^{(us)} = \sum_{t_1} \lambda \begin{array}{c}
\text{s} \\
\text{d}
\end{array} \quad G_3^{(us)} = \sum_{t_1, t_2, t_3} \lambda^3 \begin{array}{c}
\text{s} \\
\text{d}
\end{array}
\]
Kinematics

• Choose the Sigma to be at rest and change the momentum of the nucleon

• Choose the operator to be the vector current $\gamma_4$

• For hyperons with quasi-degenerate energies the shift due to a perturbation in the action lambda is

$$\Delta E_{\lambda \Sigma N} = \sqrt{(E_N(q) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(q) | \bar{u} \gamma_4 s | \Sigma(0) \rangle \right|^2}$$

• When the energy of the nucleon equals the mass of the Sigma, this will be linear in lambda.
  • How far from the degenerate energy point can we make this work?
    • At $Q^2=0$?
• Momentum on the lattice is quantised
  • how do we get to the energy-degenerate point?

• Twisted boundary conditions add a complex phase to the boundary conditions
  • Gives lattice correlators any momentum

\[ q(\vec{x} + N_s \vec{e}_i, t) = e^{i\theta_i} q(\vec{x}, t) \]

• \( Q^2 \) points of interest:
  • Degenerate energy \( E_N(\vec{q}) = M_\Sigma \)
  • \( Q^2 = 0 \)

\[ Q^2 = -(M_\Sigma - E_N(\vec{q}))^2 + \vec{q}^2 \]
Lattice details

- $32^3\times64$ lattice size
- Lattice spacing $a=0.074\text{fm}$
- $N_f = 2 + 1$, $O(a)$-improved clover Wilson fermions
- Up and down quark are degenerate
- $O(500)$ configurations used for each choice of momentum

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<th>$\bar{q}^2$</th>
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<td>0.469(13)</td>
<td>-0.0037(78)</td>
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</tbody>
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Generalized EigenValue Problem (GEVP)

- Diagonalise the matrix
  \[
  C_{\lambda B'B} = \begin{pmatrix}
  C_{\lambda\Sigma\Sigma} & C_{\lambda\Sigma N} \\
  C_{\lambda N\Sigma} & C_{\lambda N N}
  \end{pmatrix}_{B'B}
  \]
  - Gives two eigenvectors and eigenvalues
  - Eigenvalues related to the energy

- Use the eigenvectors to project out two correlation functions:
  \[
  C_{\lambda}^{(i)}(t) = v^{(i)\dagger} C_{\lambda}(t) u^{(i)} , \quad i = \pm
  \]

- Take the ratio of the two correlators and fit to the energy shift \(\Delta E\).
  \[
  R_{\lambda}(t; \bar{0}, \bar{q}) = \frac{C_{\lambda}^{(-)}(t; \bar{0}, \bar{q})}{C_{\lambda}^{(+)}(t; \bar{0}, \bar{q})} \sim t \to \infty e^{-\Delta E_{\lambda}(\bar{0}, \bar{q}) t}
  \]
GEVP Stability

Stable under GEVP parameters?
• Do the GEVP for many values of $t_0$ and $\Delta t$
• Calculate the value of $\Delta E(c^+,c^-)$ from the eigenvalues
• Compare with $\Delta E$ from the fit to the ratio of correlators
• For each $\Delta t$ we show results from $t_0=1-8$

GEVP depends on two parameters ($t_0$ & $\Delta t$):

$$C^{-1}_\lambda(t_0)C_\lambda(t_0 + \Delta t)e^{(i)} = c^{(i)}e^{(i)}$$

Ground state saturated:

Result from GEVP are stable in range $\Delta t \geq 4$ and $t_0 \geq 6$
ΔE as a function of \( \lambda \) when \( E_N(\vec{q}) = M_\Sigma \)

- Iterative Method: higher orders in lambda increase the range over which our approximation holds
- We want to fit in the region where the dependence is linear
- Choose the region where the two highest order results agree
  - \( O(\lambda^3) \) and \( O(\lambda^4) \) agree up to \( \lambda = 0.05 \)

\[
\Delta E_{\lambda N} = \sqrt{(E_N(\vec{q}) - M_\Sigma)^2 + 4\lambda^2 \left| \langle N(\vec{q}) | \bar{u} \gamma_4 s | \Sigma(\vec{0}) \rangle \right|^2}
\]

\[
R_\lambda(t; \vec{0}, \vec{q}) = \frac{C^{(-)}_\lambda(t; \vec{0}, \vec{q})}{C^{(+)}_\lambda(t; \vec{0}, \vec{q})} \quad t \gg 0 \quad e^{-\Delta E_\lambda(\vec{0}, \vec{q}) t}
\]
Fitting the slope of $\Delta E$

- Consider the ratio at two values of $\lambda$ close to each other.

\[ \frac{R_{\lambda+\delta\lambda}(t)}{R_\lambda(t)} \sim 0, \quad t \gg 0 \quad e^{- (\Delta E_{\lambda+\delta\lambda} - \Delta E_\lambda) t} \]

- This cancels out more correlations
- Fit to get the slope in $\lambda$
- Relate the slope to the matrix element

\[ \Delta E_\lambda = \sqrt{(E_N(q') - M_S)^2 + 4\lambda^2 \langle ME \rangle_{lat}^2} \]
Does the method work at $Q^2=0$?

The expansion in $\lambda$ holds for a smaller range: $O(\lambda^3)$ and $O(\lambda^4)$ diverge around $\lambda = 0.03$

$O_{\lambda^3}$ and $O_{\lambda^4}$ diverge around $\lambda = 0.03$

<= Fitting to the slope at small $\lambda$ still produces stable results

$$\Delta E_{\lambda N} = \sqrt{(E_N(q) - M_S)^2 + 4\lambda^2 \left| \langle N(q) | \bar{u} \gamma_4 s | \Sigma(0) \rangle \right|^2}$$
How does this compare to three-point function results?

3-point function

• Same # of gauge configurations
• Both $\Sigma^- \rightarrow n$ and opposite 3-point functions used.
• 3 source-sink separations:
  • $t=10,13,16$ (0.74, 0.96, 1.18 fm)
• Fit ansatz includes all three $t_{sep}$ and the first excited state

![Ratio of 3pt and 2pt functions at $Q^2=0$](image)

The dependence on $t_{sep}$ has not been eliminated:

$\Rightarrow$ Has the ground state been saturated?
How does this compare to three-point function results?

Matrix element as a function of $Q^2$:

3-point function results at non-zero $Q^2$ only used one transition direction
How does this compare to three-point function results?

Compare two $Q^2$ points more closely:
Avoided level crossing

$\lambda = 0$

$\lambda = 0.029$
Eigenvectors

- The eigenvectors of the GEVP show the mixing between the states as a function of $q^2$

$$\bar{e'}(\pm) = \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix}$$

![Graph showing $|e_1^{(\pm)}|^2$ and $|e_2^{(\pm)}|^2$ as functions of $\tilde{q}^2$]
Conclusion

• The Feynman-Hellman method can be used to calculate hyperon transition form factors
• Only requires one Euclidean time parameter to be optimised to extract the ground state.
• Using multiple different operators will allow for the extraction of separate form factors
• Method should be tested on lattices with larger splittings between the light and strange quarks