The three-body problem

looking back at the last 9yrs

OLD DOMINION UNIVERSITY

Jefferson Lab

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hadspec

9+ years

- ☑ Summer of 2006: W&M / JLAb REU

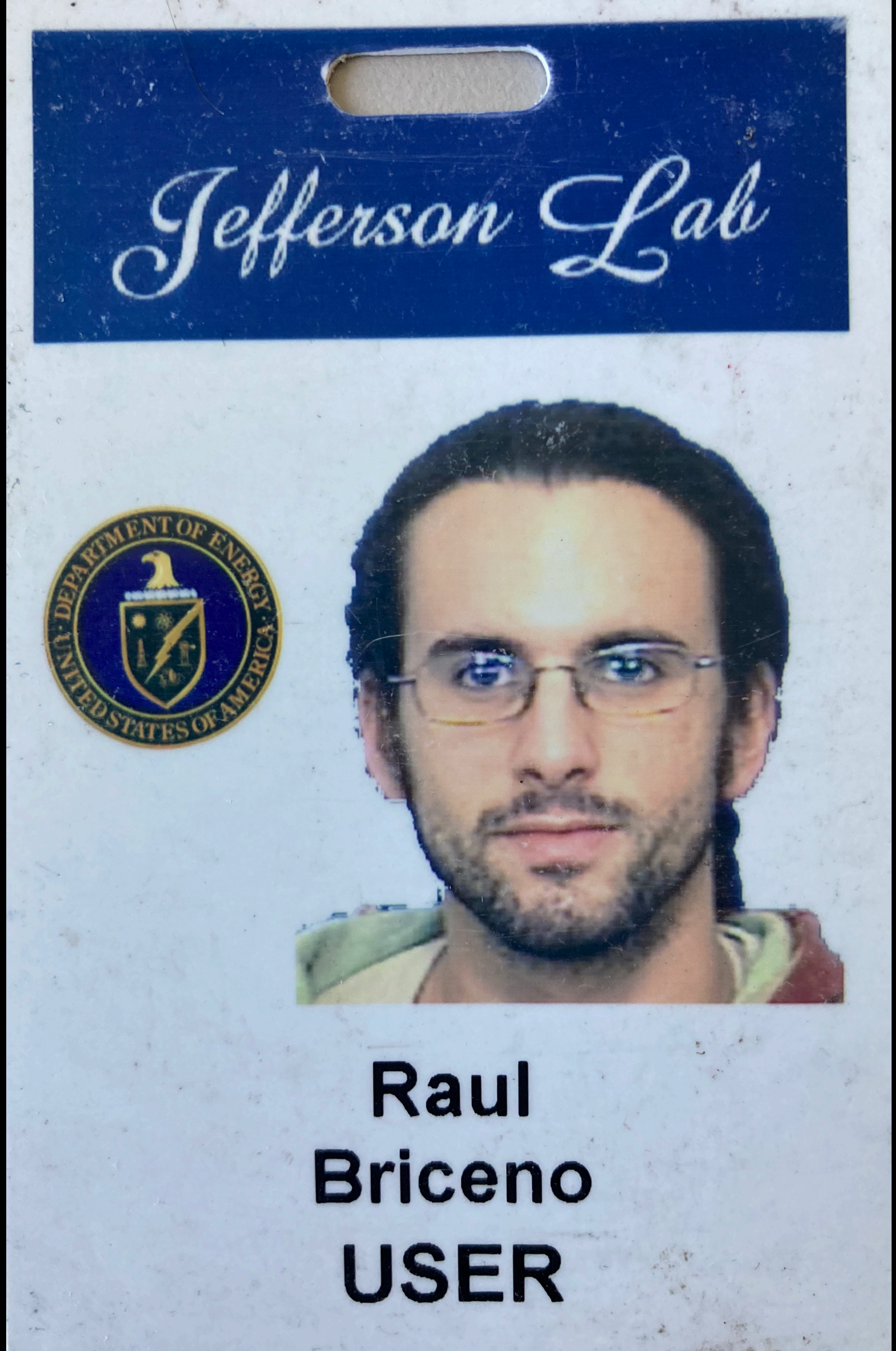
  realized that I was useless at the lab 😅

- ☑ 2012: first three-body paper with Zohreh Davoudi
- ☑ Fall of 2013: JLab postdoc

  "friendliest group!"

- ☑ Fall of 2015: ODU postdoc
- ☑ Fall of 2016: Isgur fellow
- ☑ Summer of 2017: Joint ODU/JLab staff

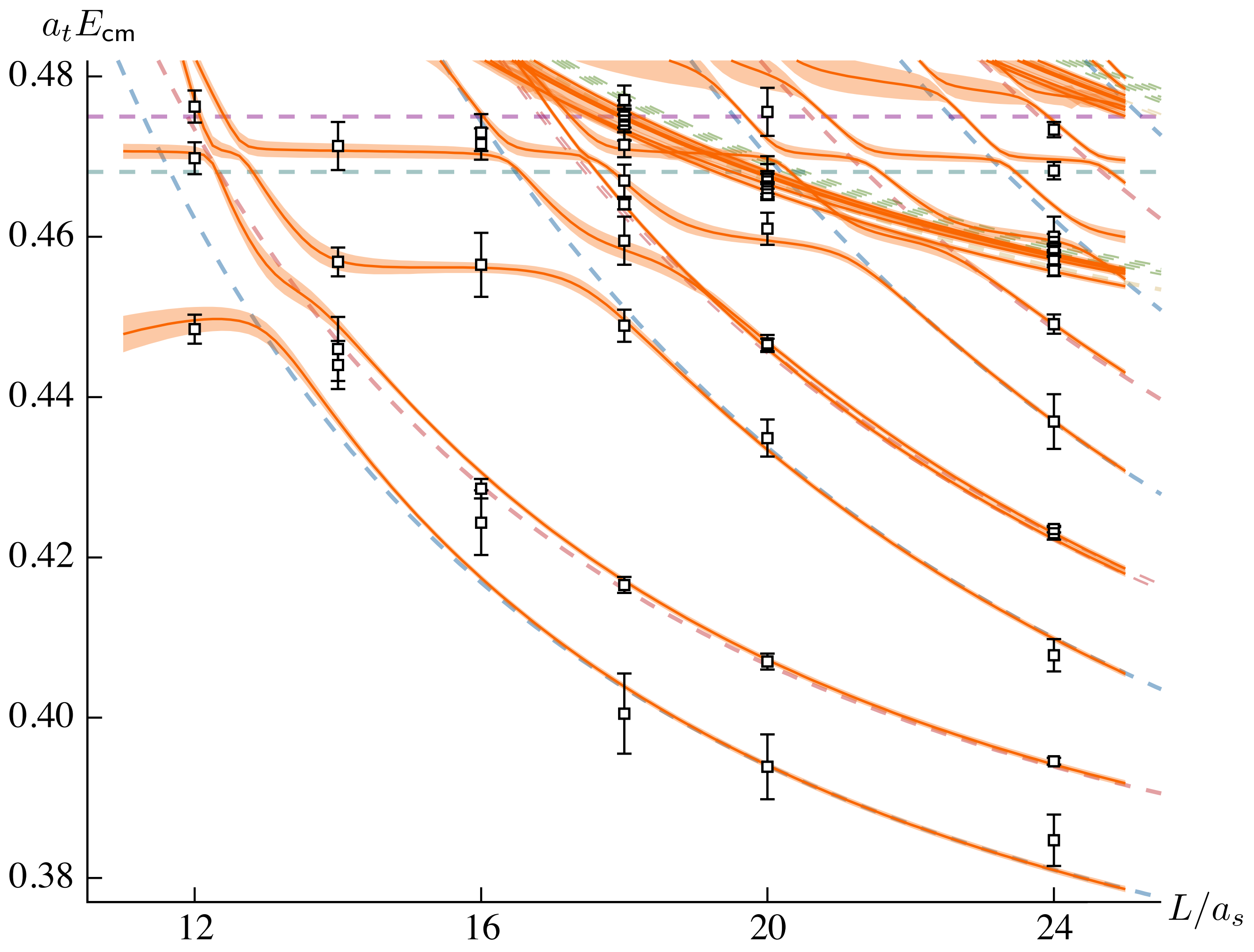
0 COVID cases reported! 😅



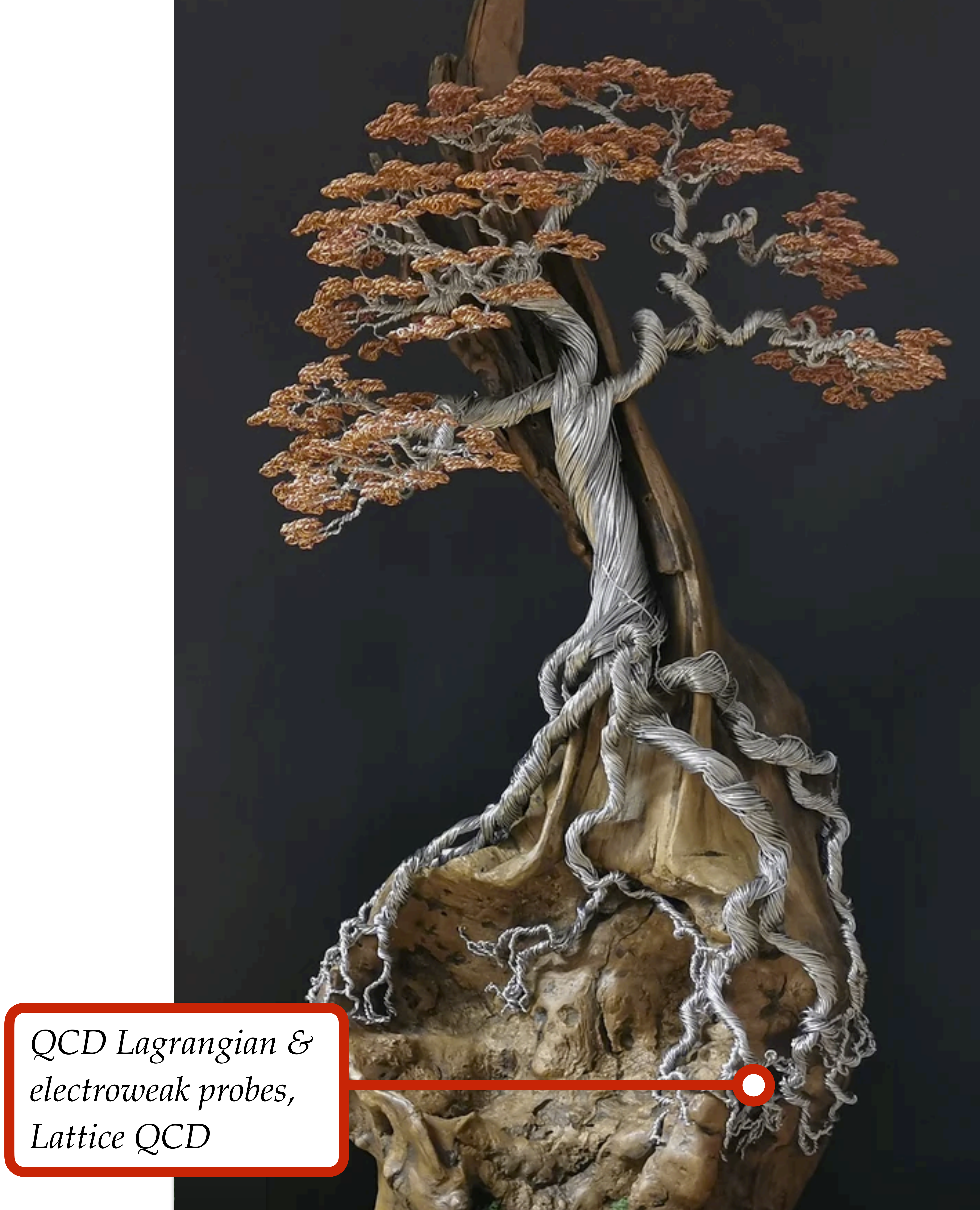
Modern-day spectroscopy rooted in QCD

$$\sum_f \overline{\psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \psi_f - \frac{1}{4}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$$

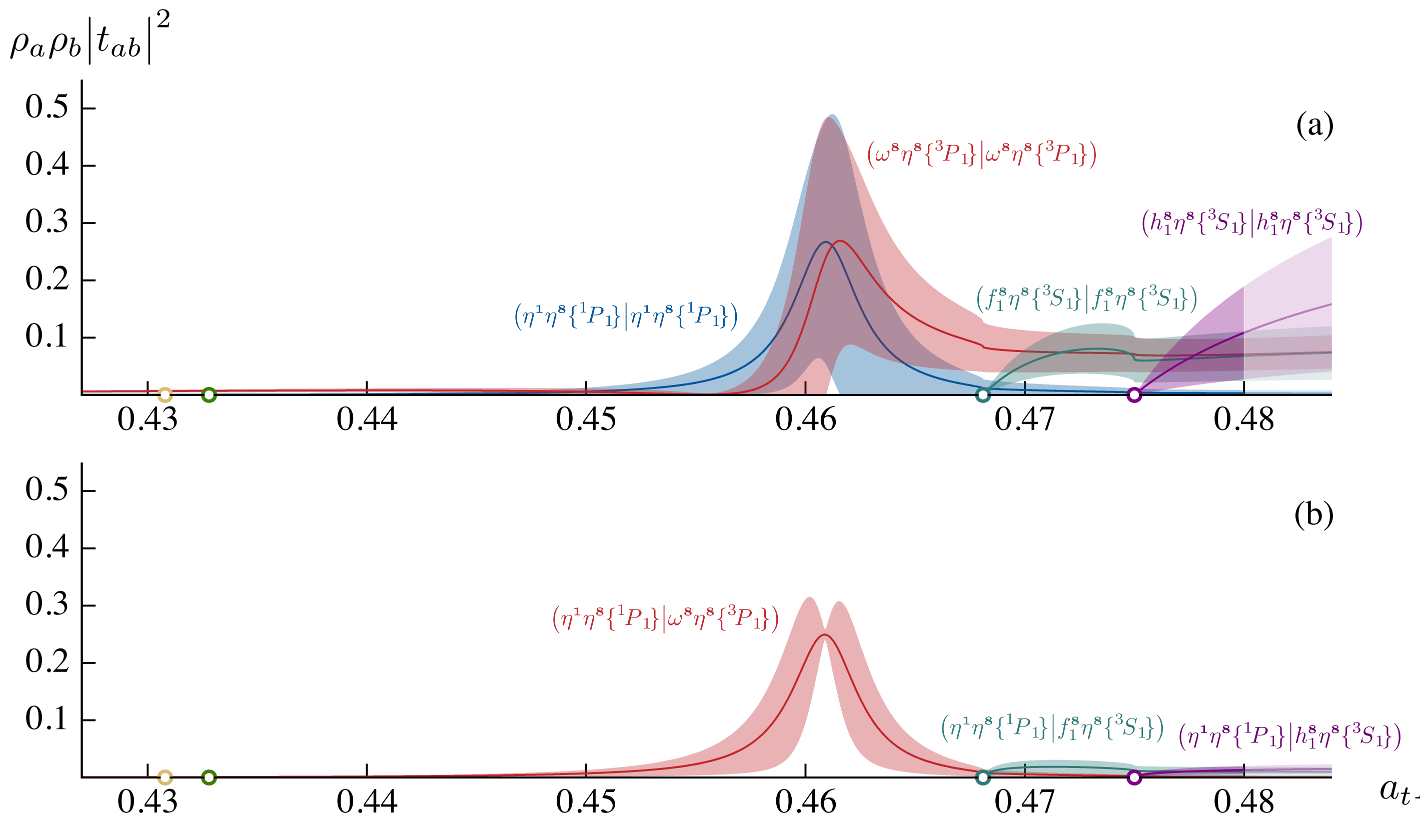
+ electroweak sector

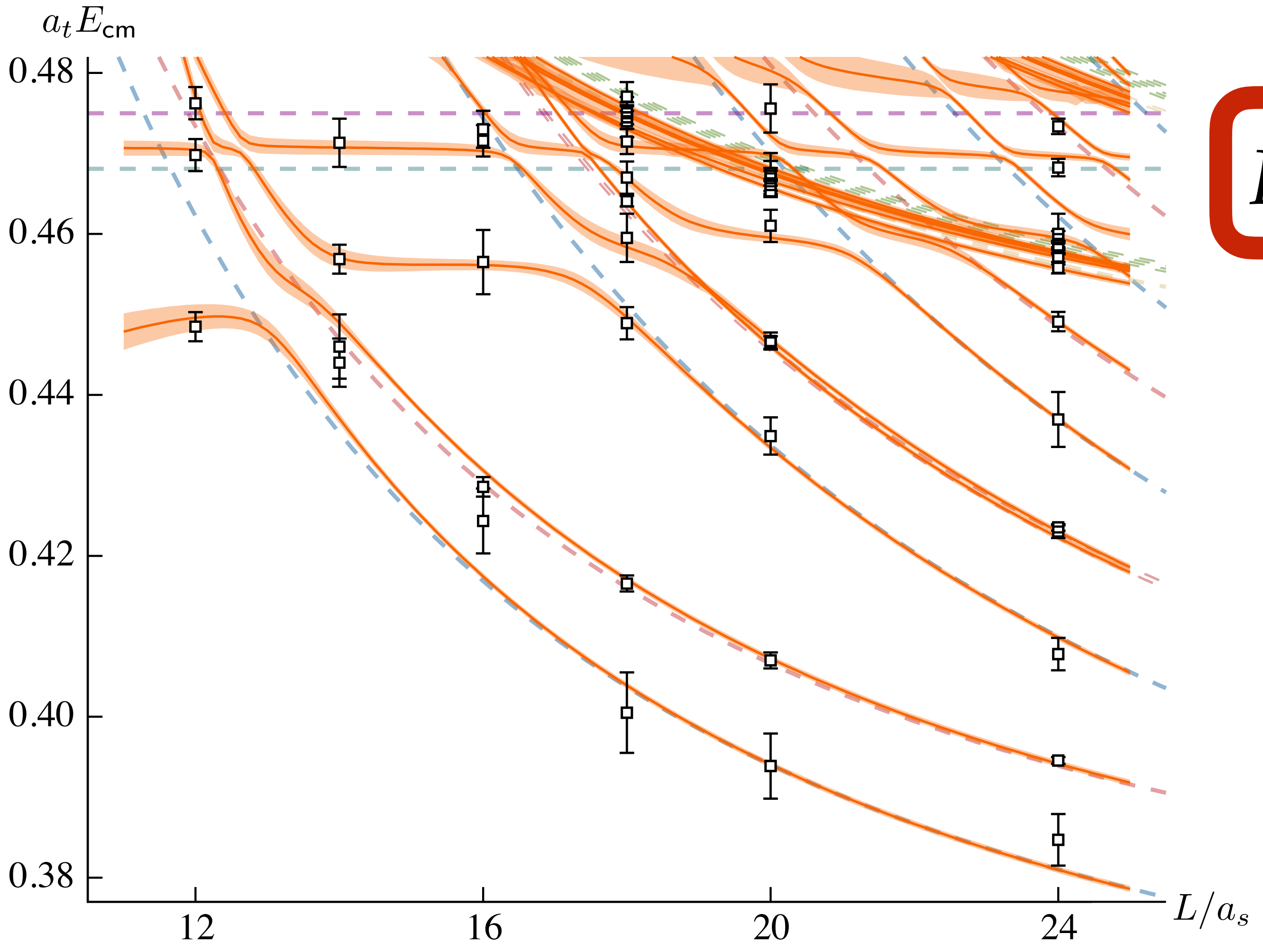


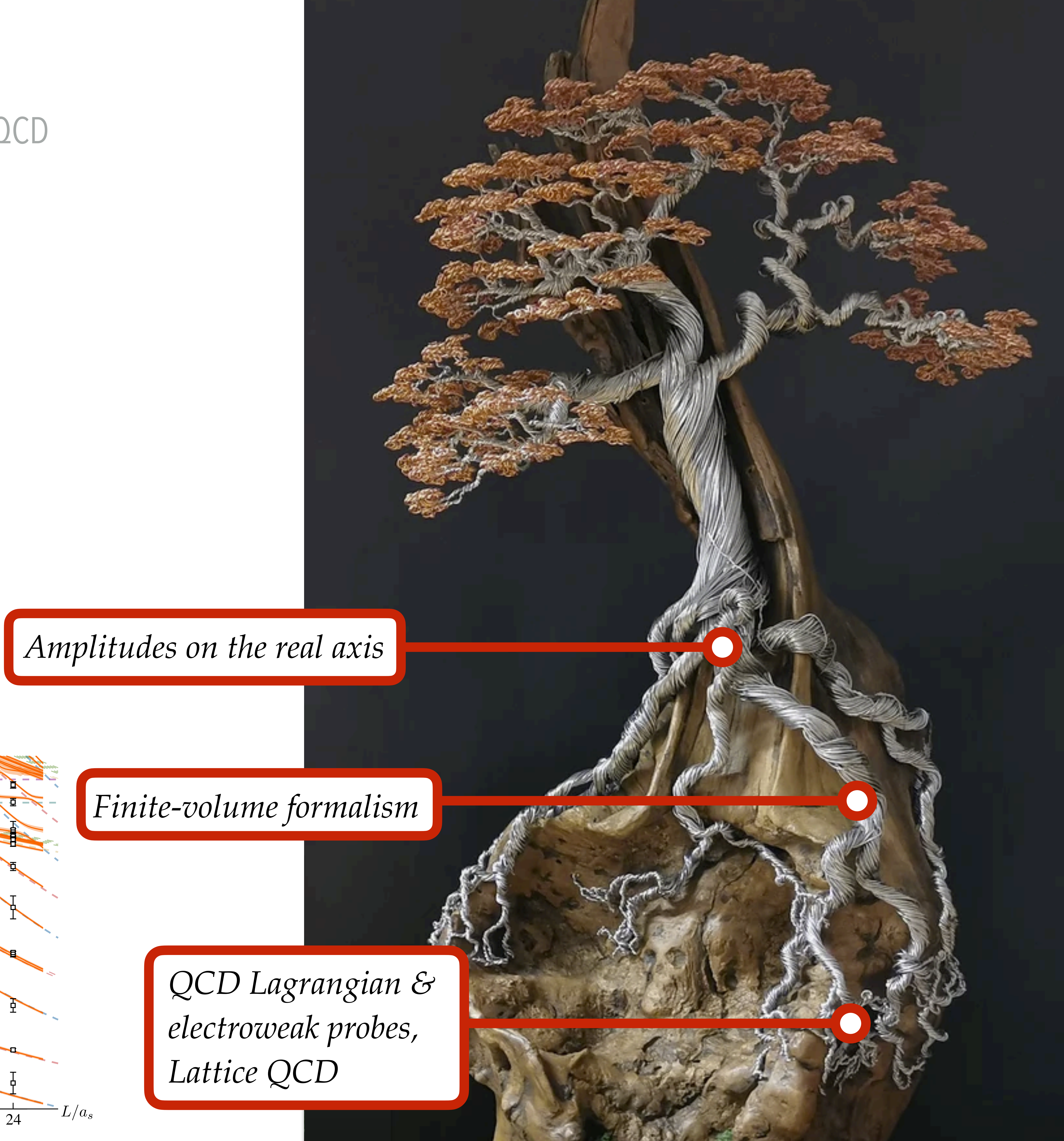
Woss, Dudek,
Edwards, Thomas, Wilson (2020)



Modern-day spectroscopy rooted in QCD

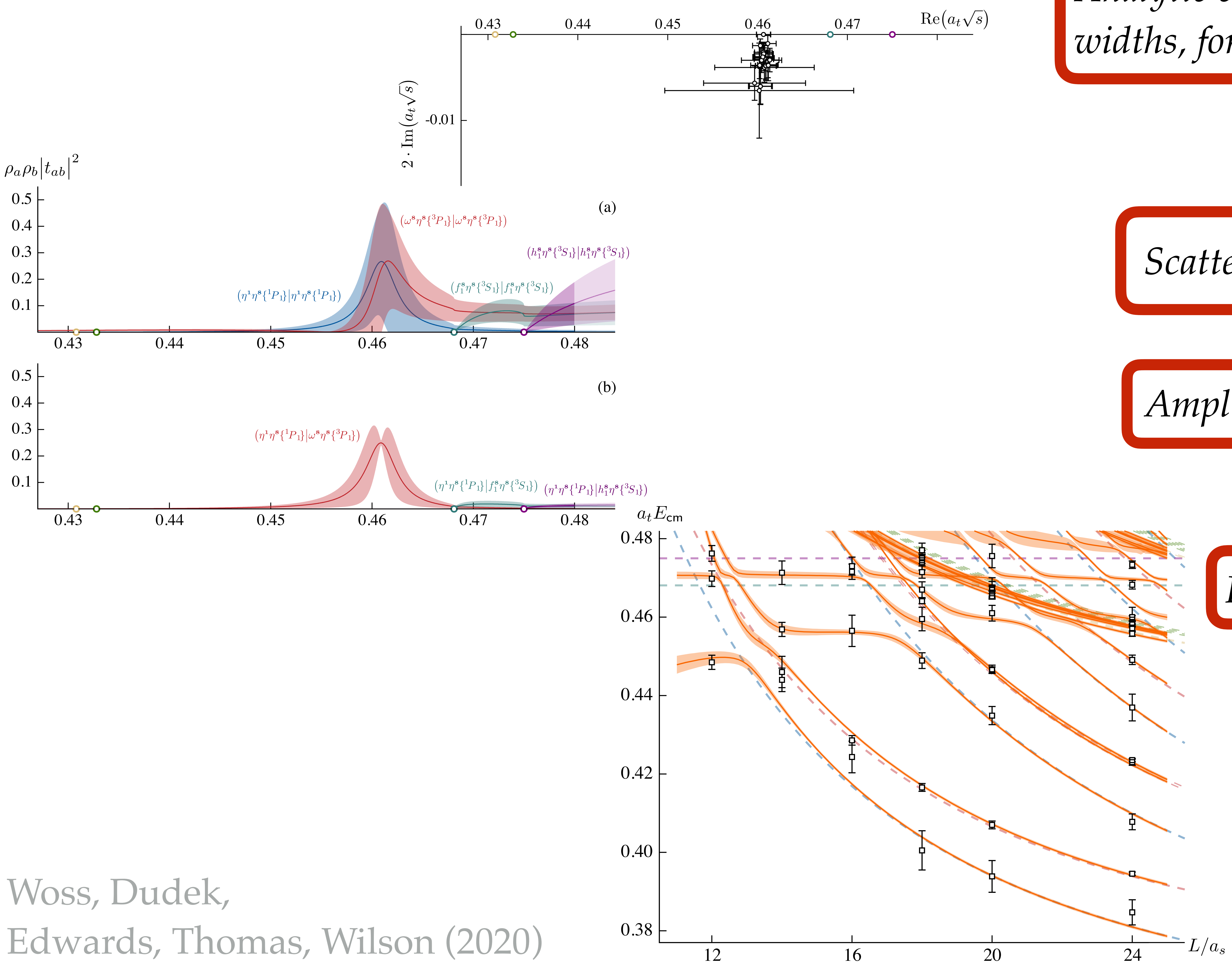




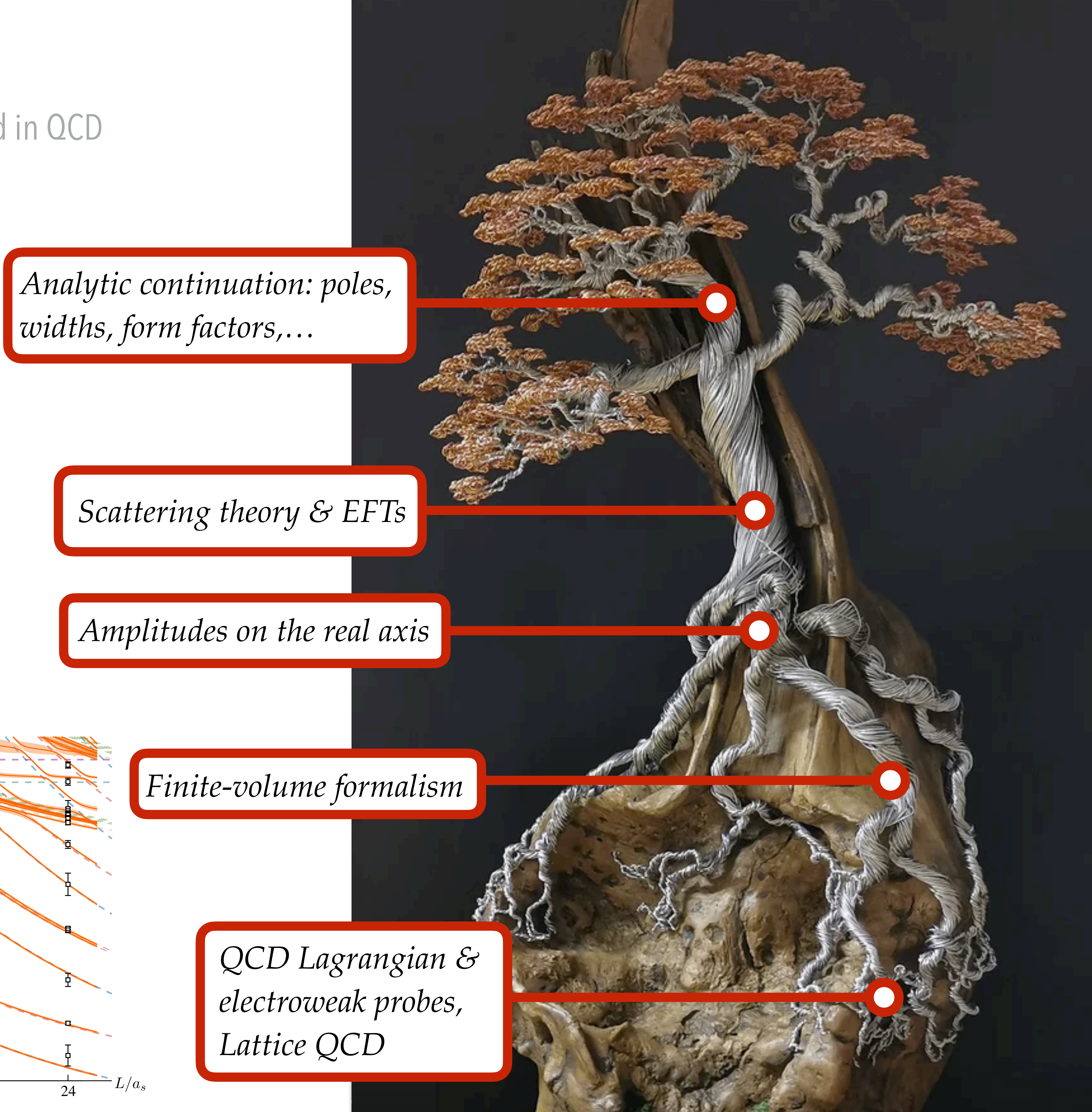


Woss, Dudek,
Edwards, Thomas, Wilson (2020)

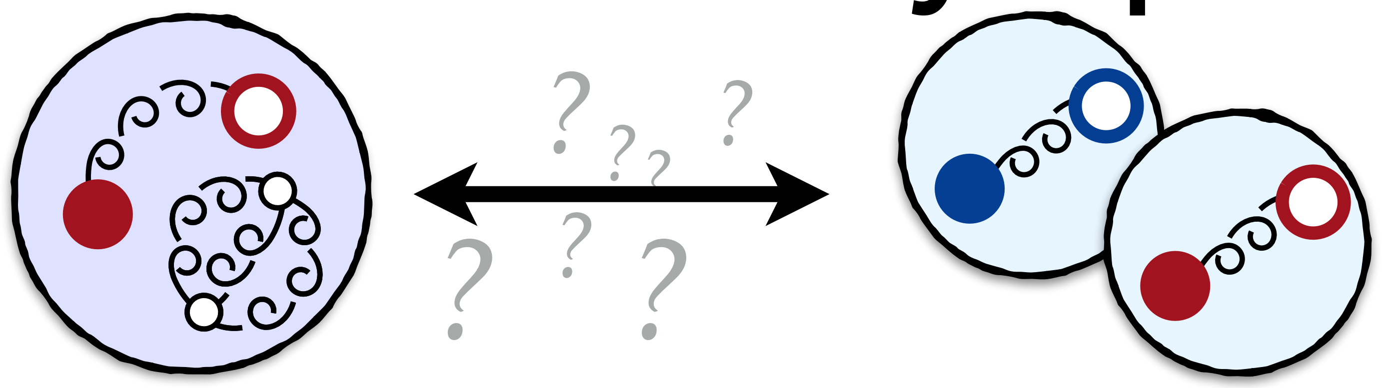
Modern-day spectroscopy rooted in QCD

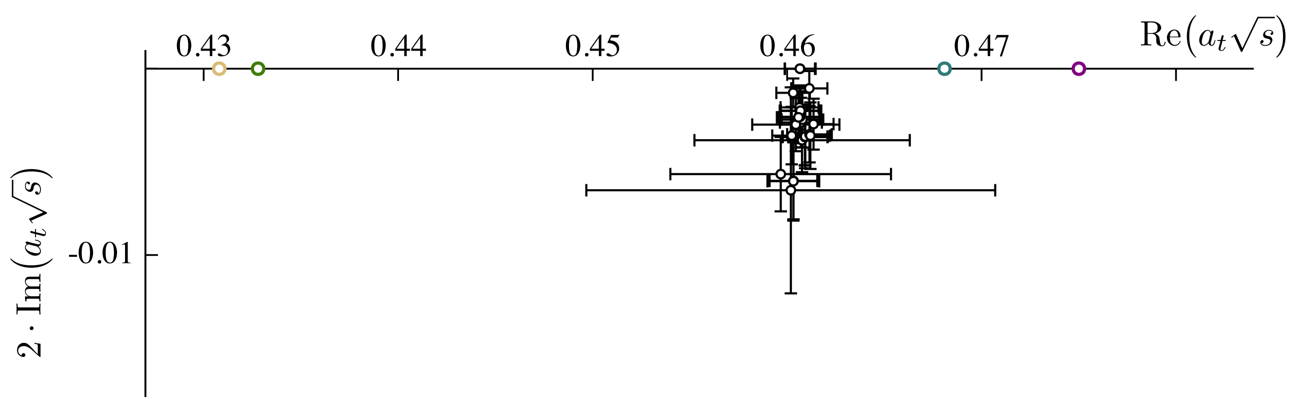


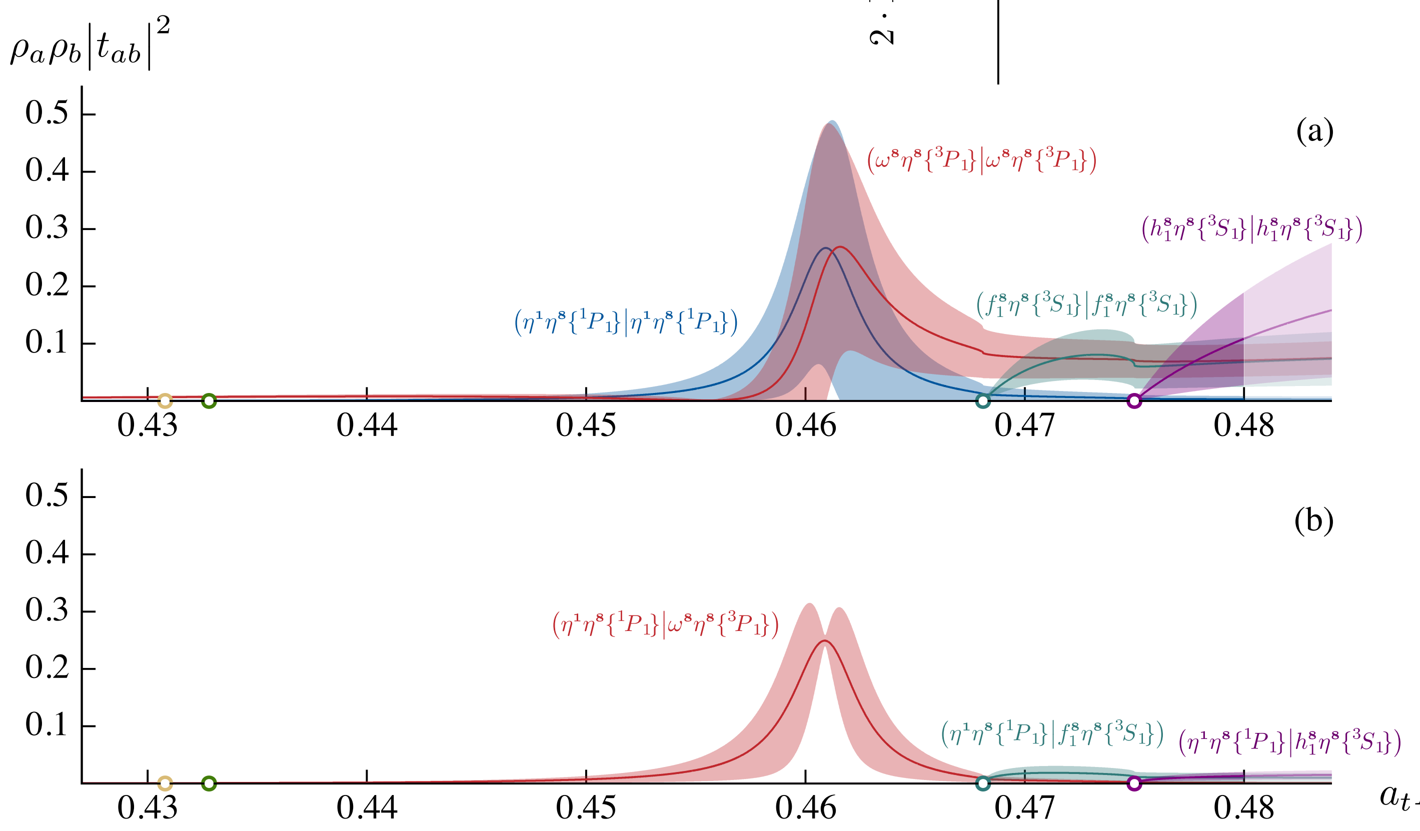
Woss, Dudek,
Edwards, Thomas, Wilson (2020)

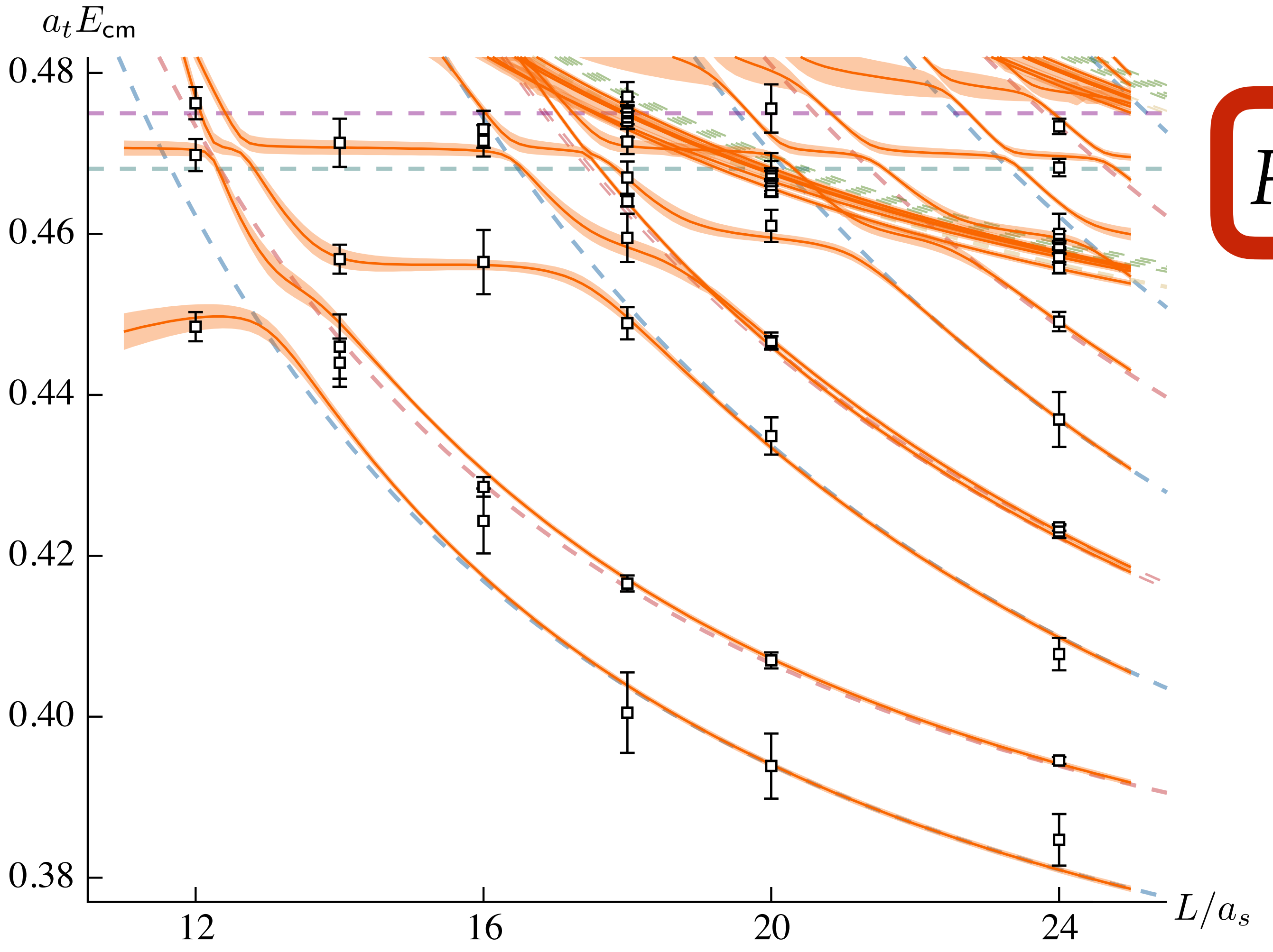


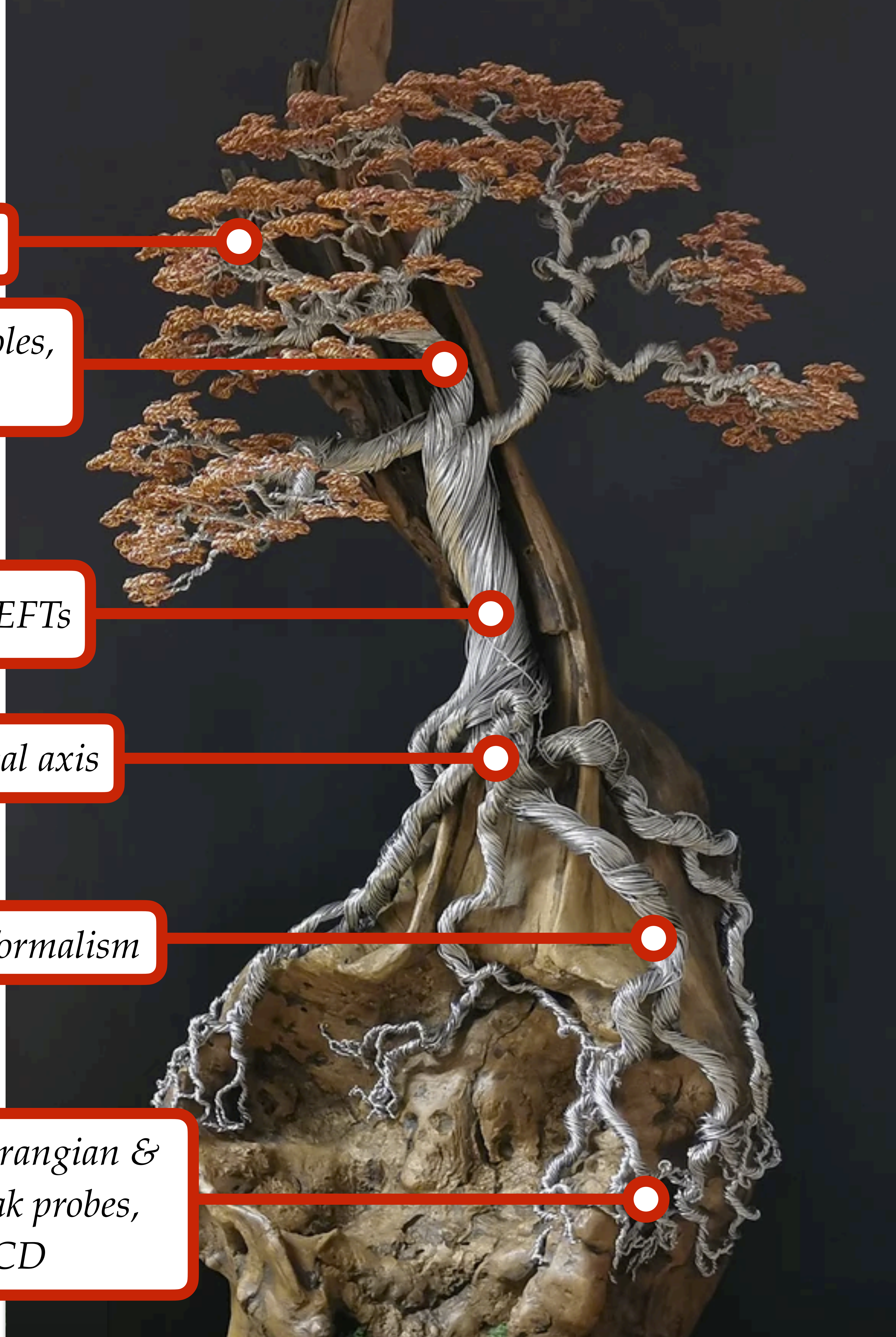
Modern-day spectroscopy rooted in QCD











Nature of states, patterns in the spectrum, …

Analytic continuation: poles, widths, form factors,…

Scattering theory & EFTs

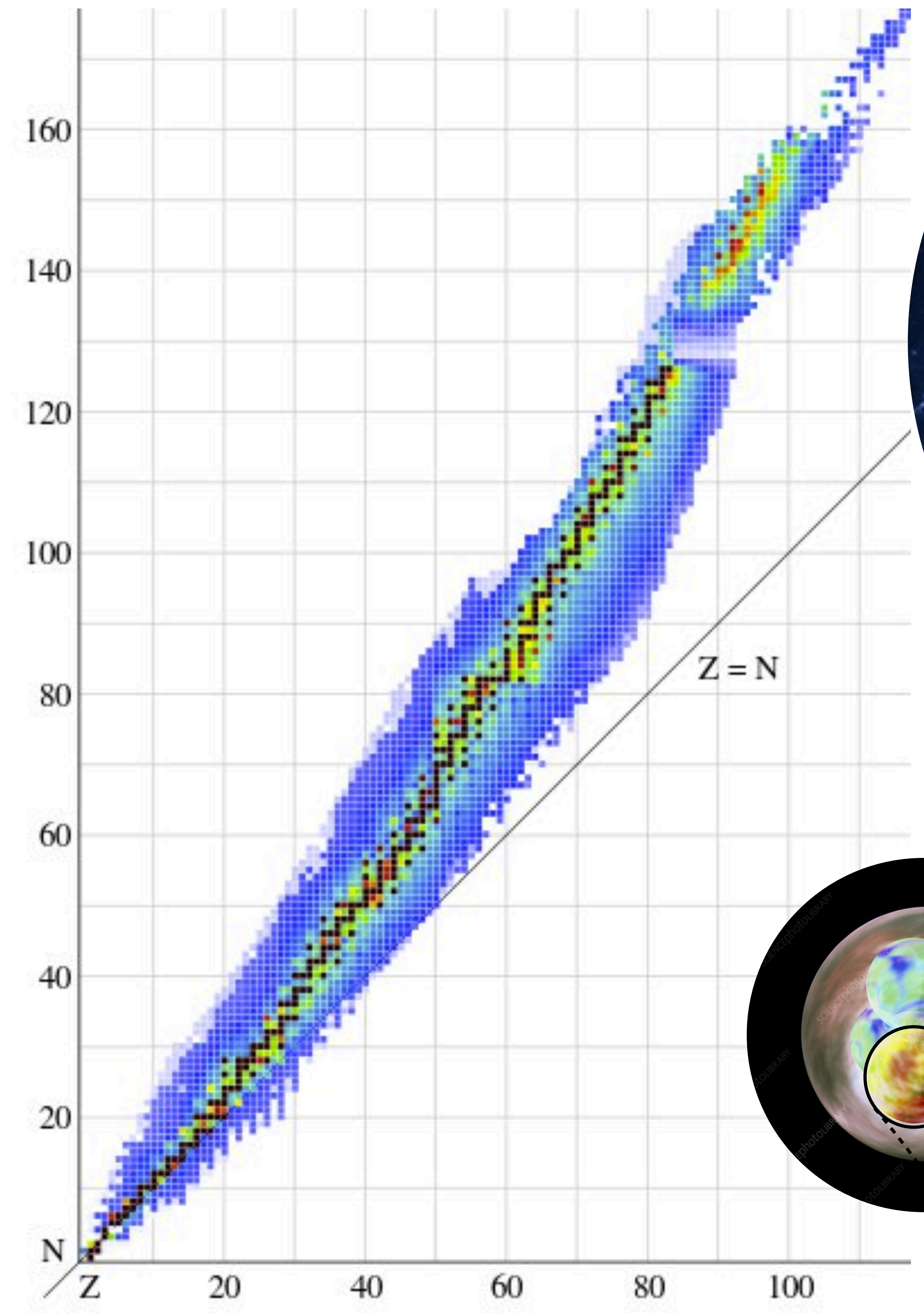
Amplitudes on the real axis

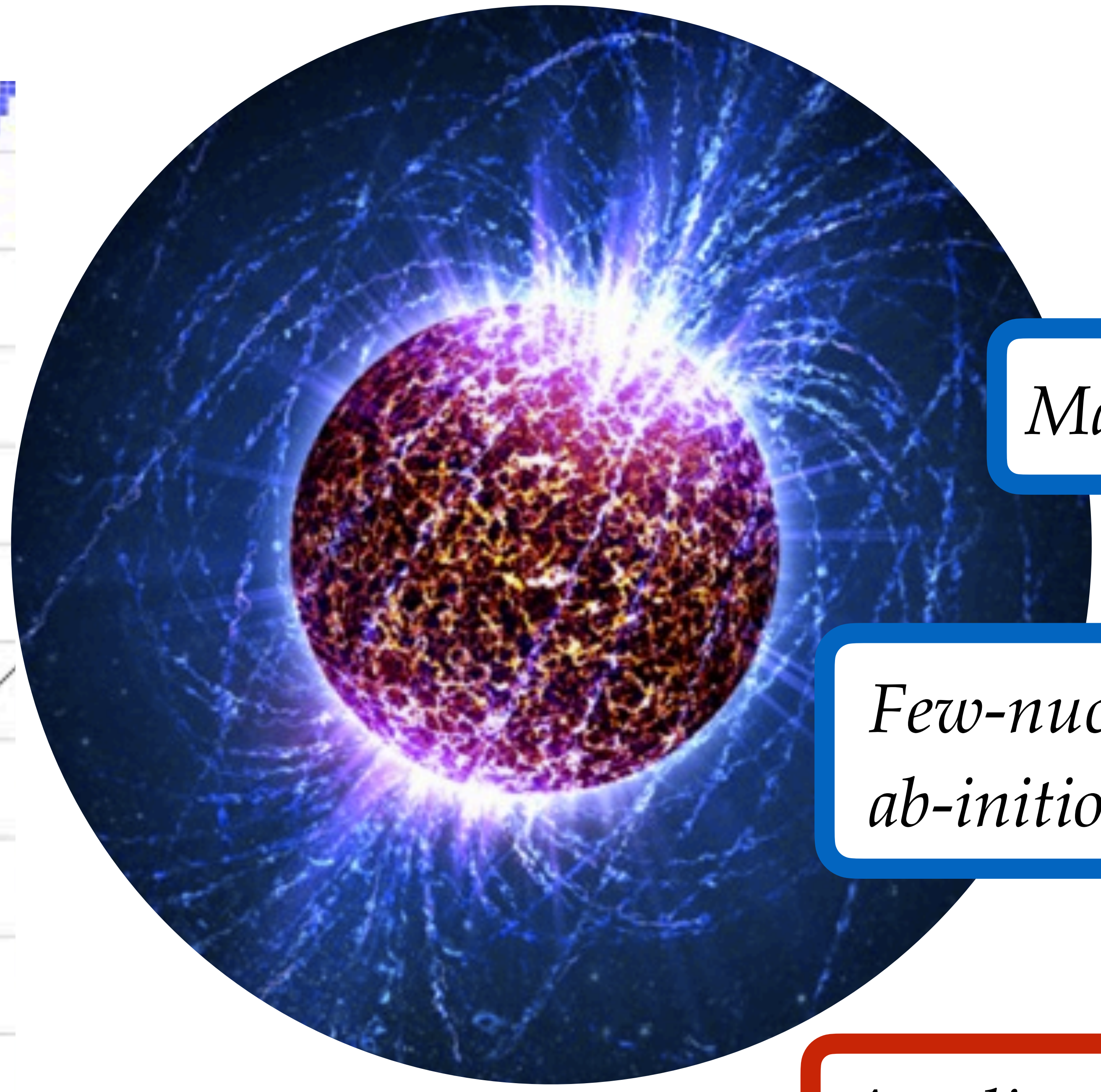
Finite-volume formalism

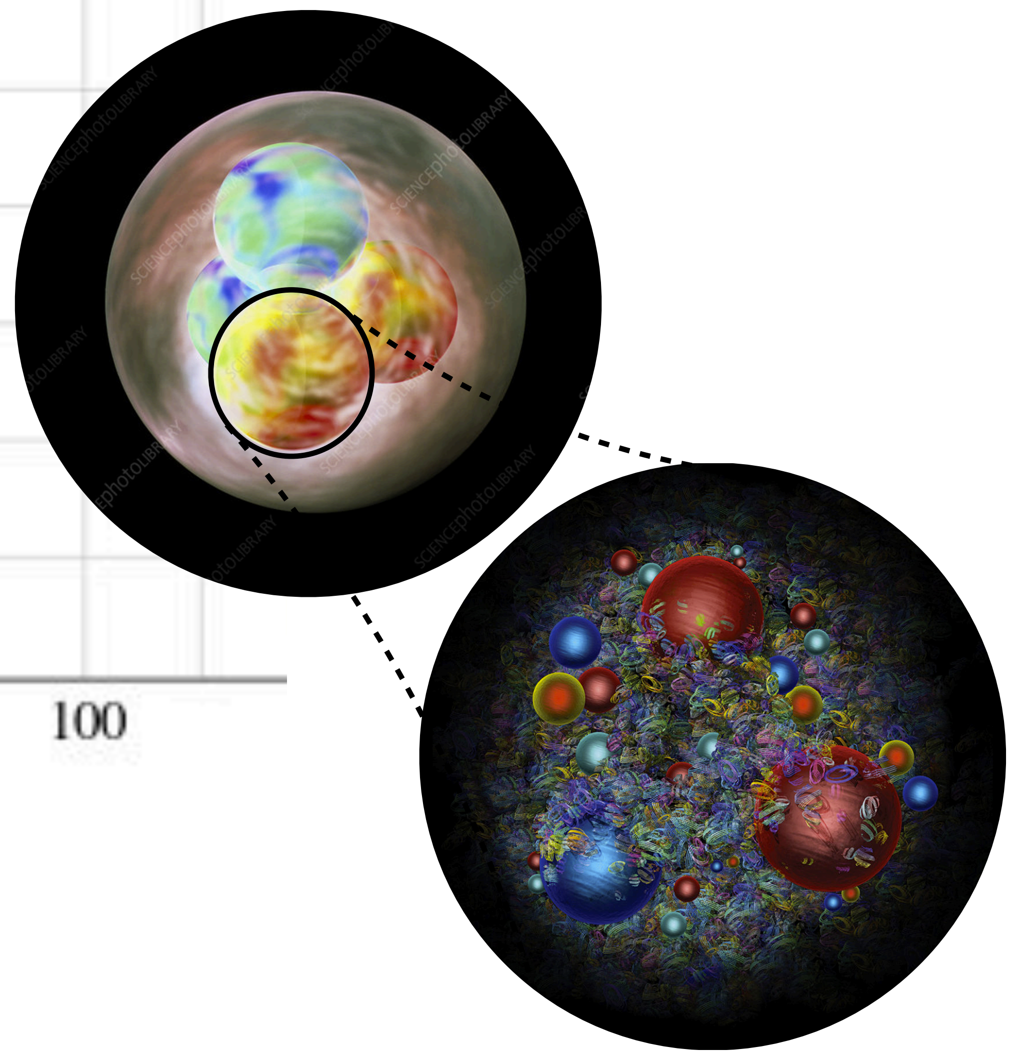
QCD Lagrangian & electroweak probes, Lattice QCD

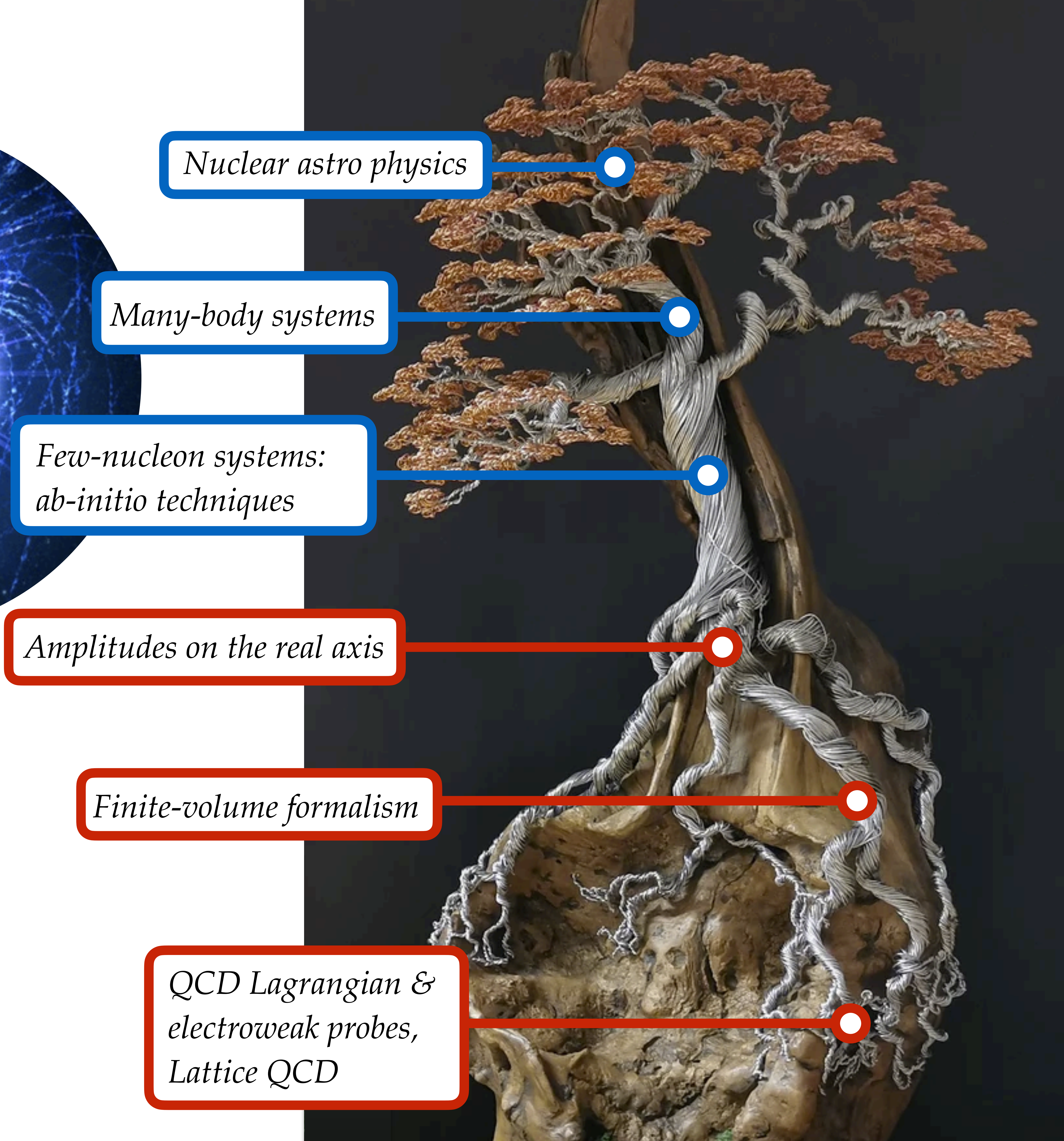
Woss, Dudek,
Edwards, Thomas, Wilson (2020)

Modern-day nuclear structure





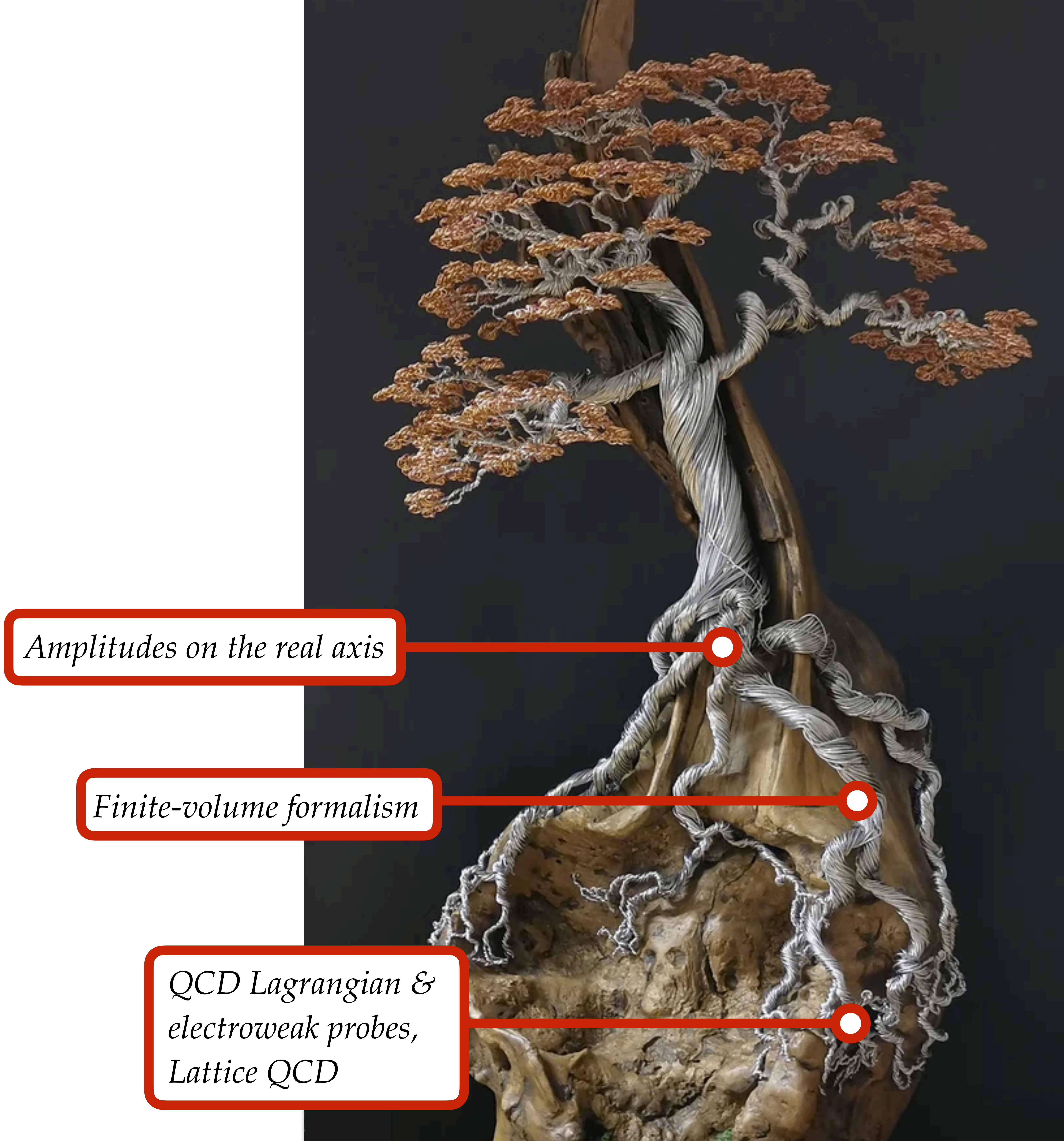




The common thread

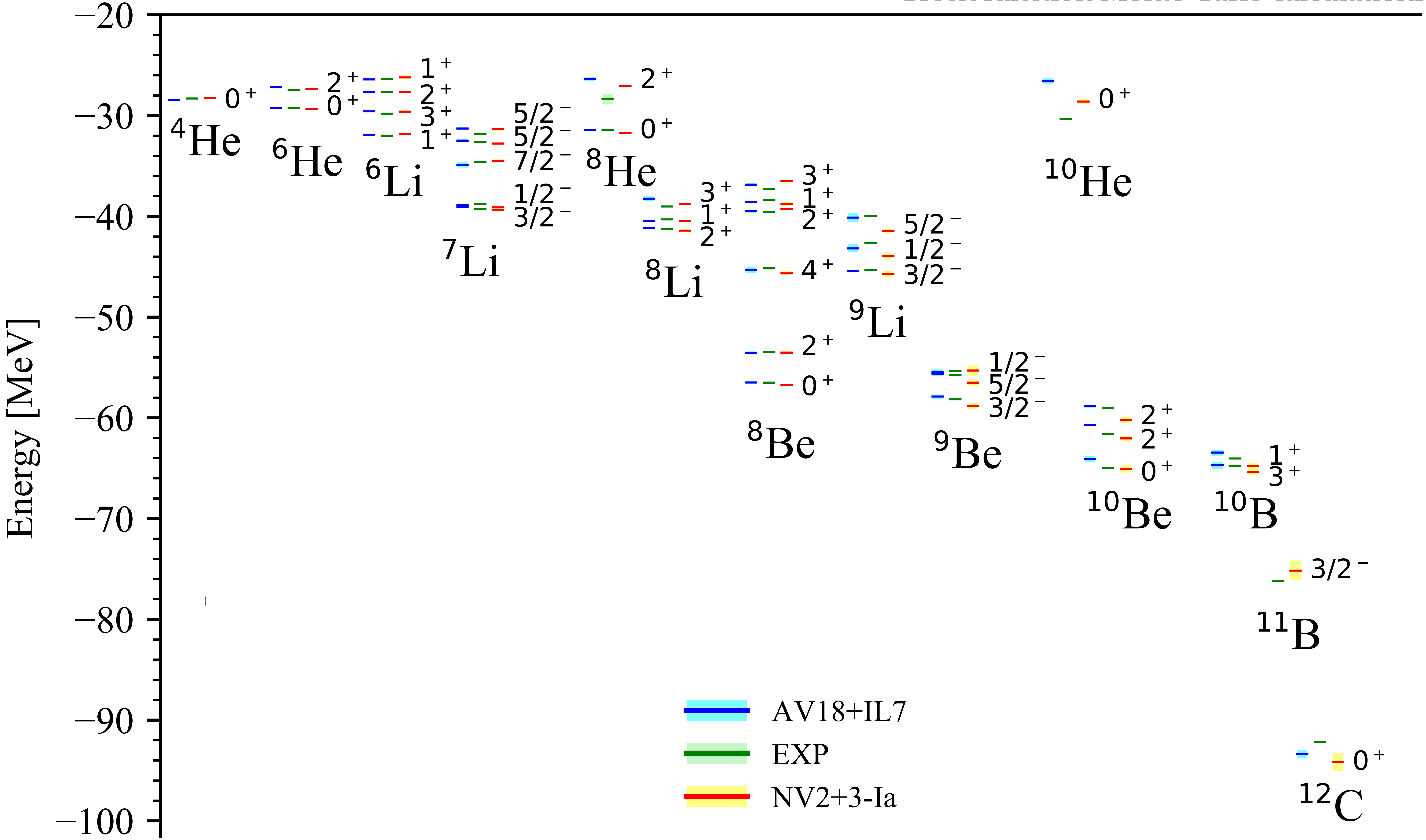
This is at the root of several areas of research:

- hadron spectroscopy,
- nuclear structure,
- hadron structure,
- electroweak precision decays,
- …



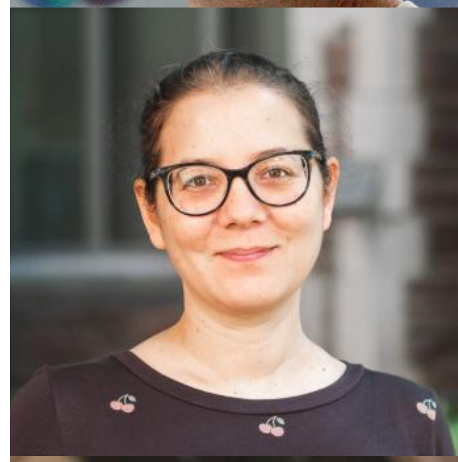
the three-body problem in nuclear physics

Green function Monte Carlo calculations



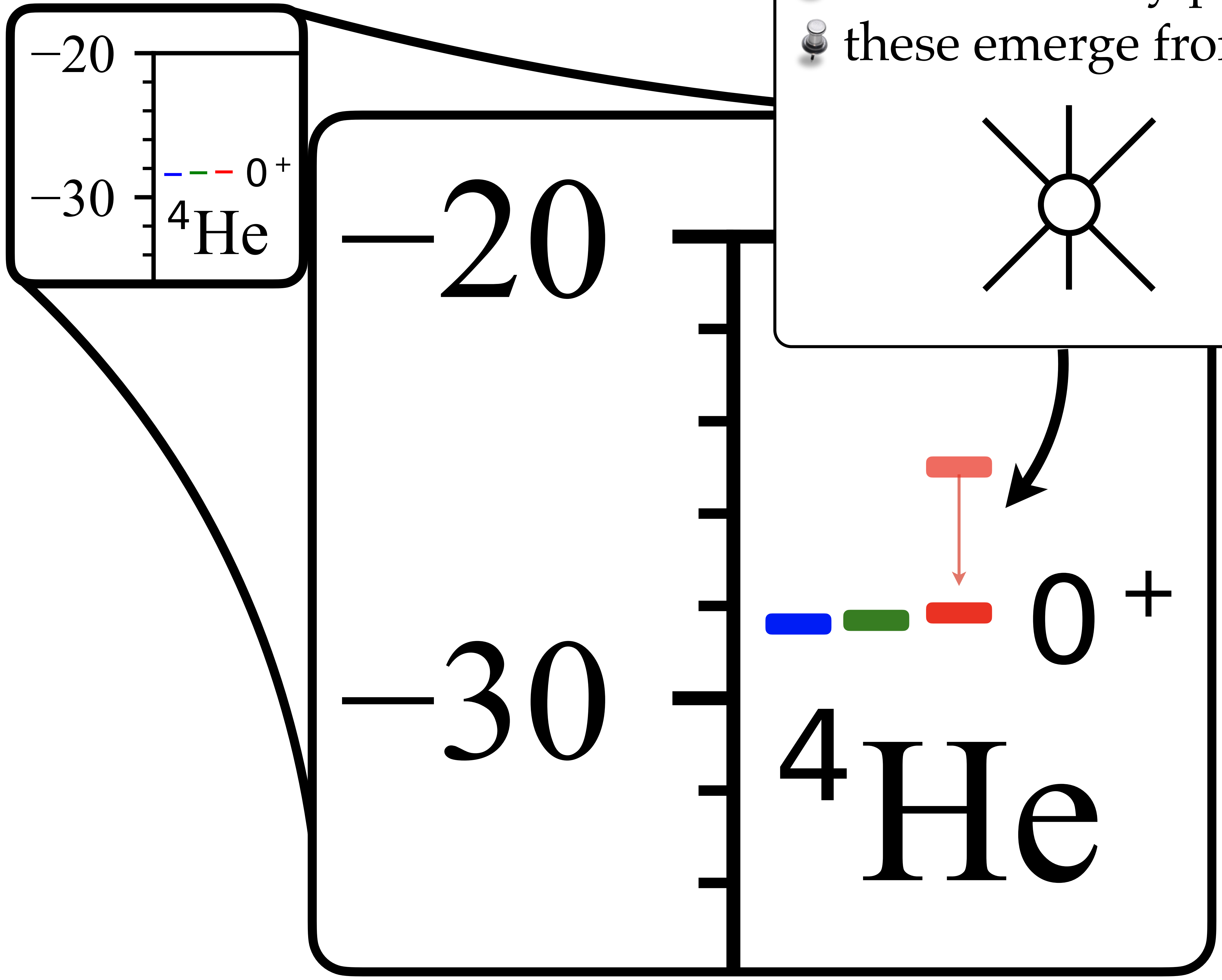
Piarulli, Baroni, Schiavilla et al. (2017)

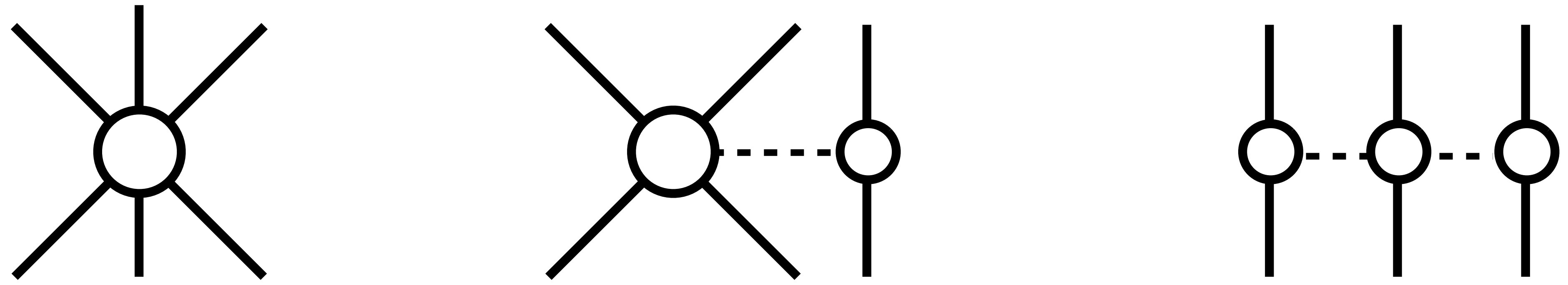
talento di ODU

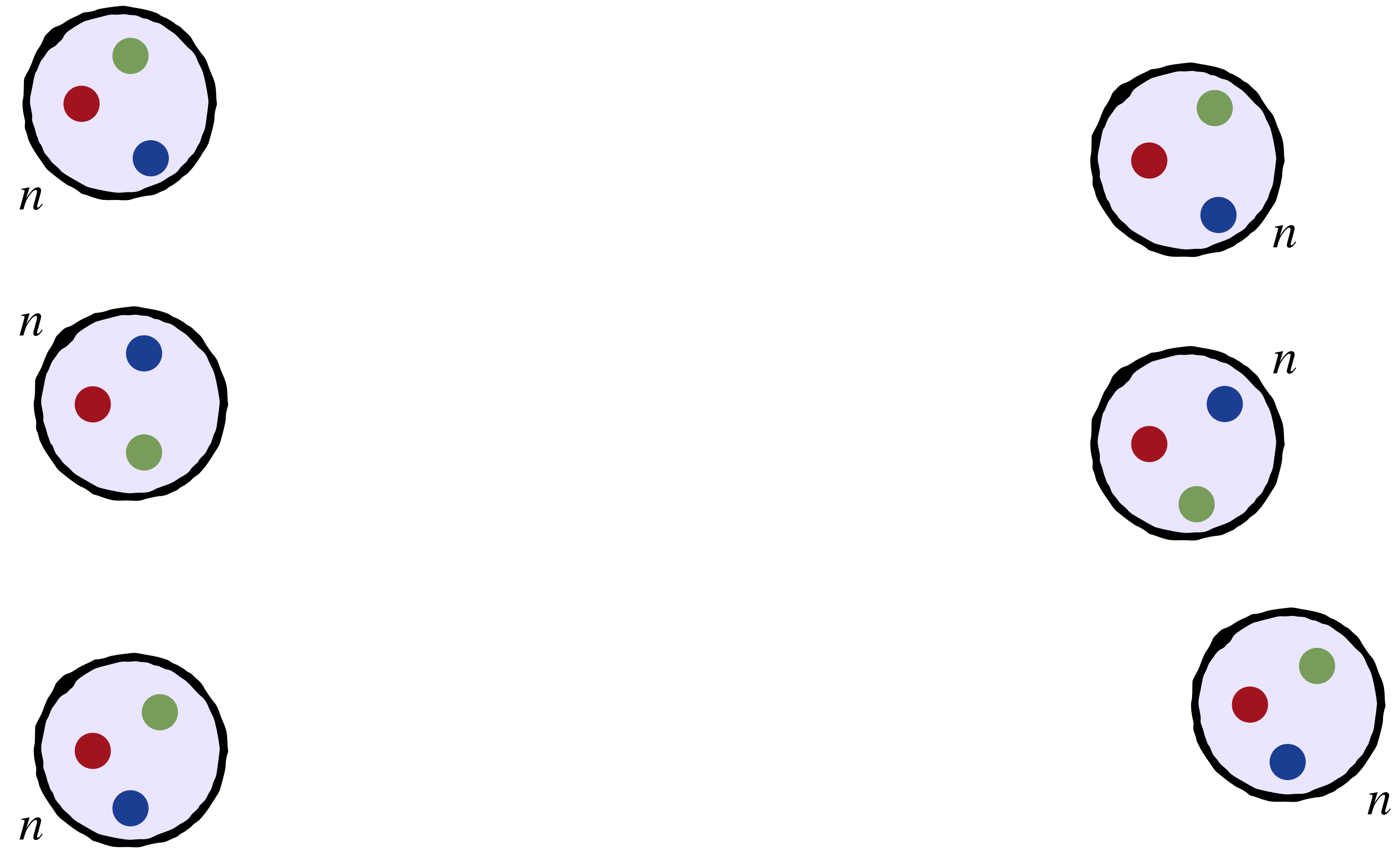


the three-body problem

- the three-body potentials shifts spectra by about 10%-20%
- these emerge from “local” & “non-local” interactions







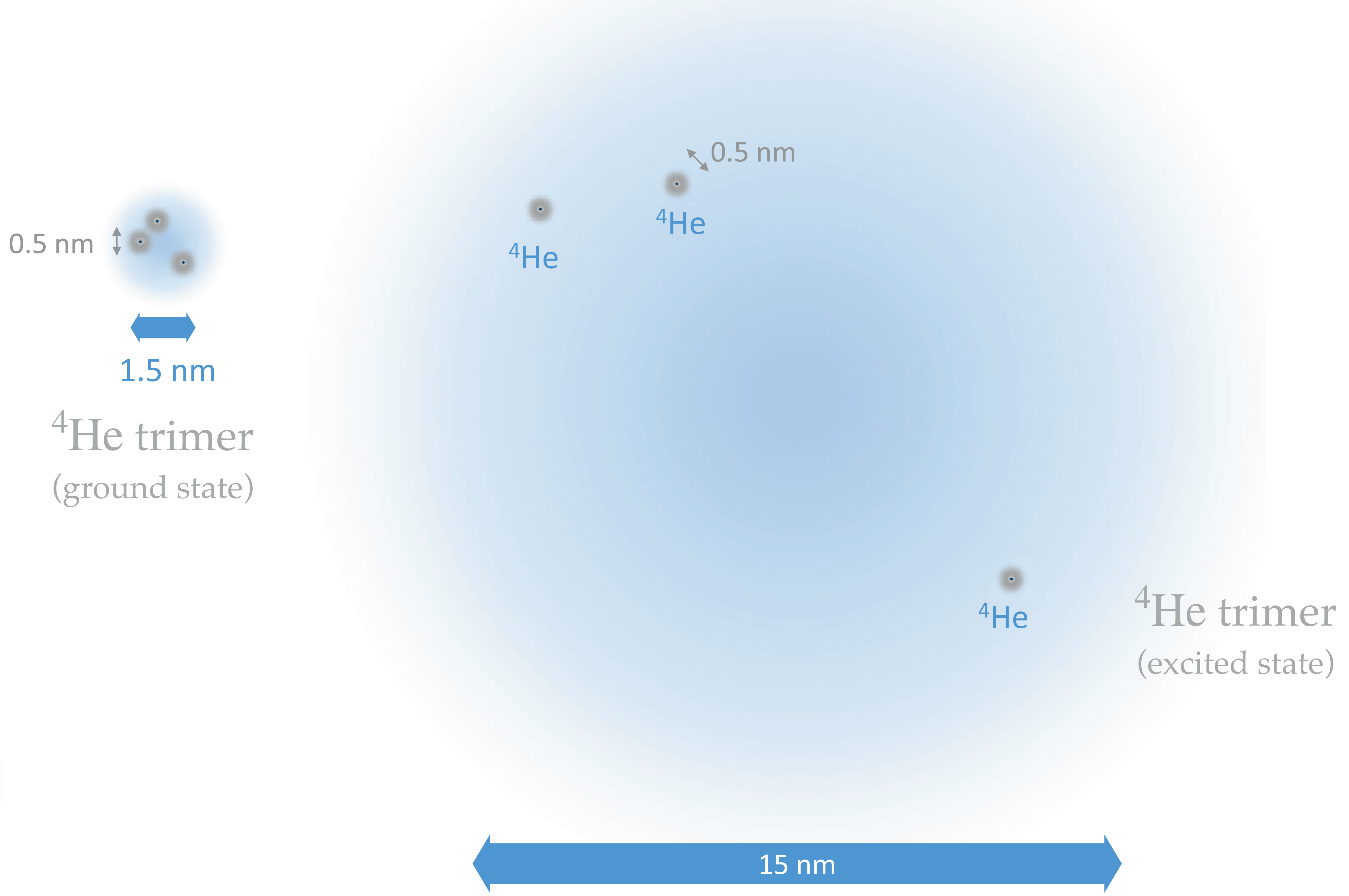
3n scattering

the three-body problem *Efimov physics*

Unitary limit: $$p \cot \delta = -\frac{1}{a} + \frac{rp^2}{2} + \cdots = 0$$

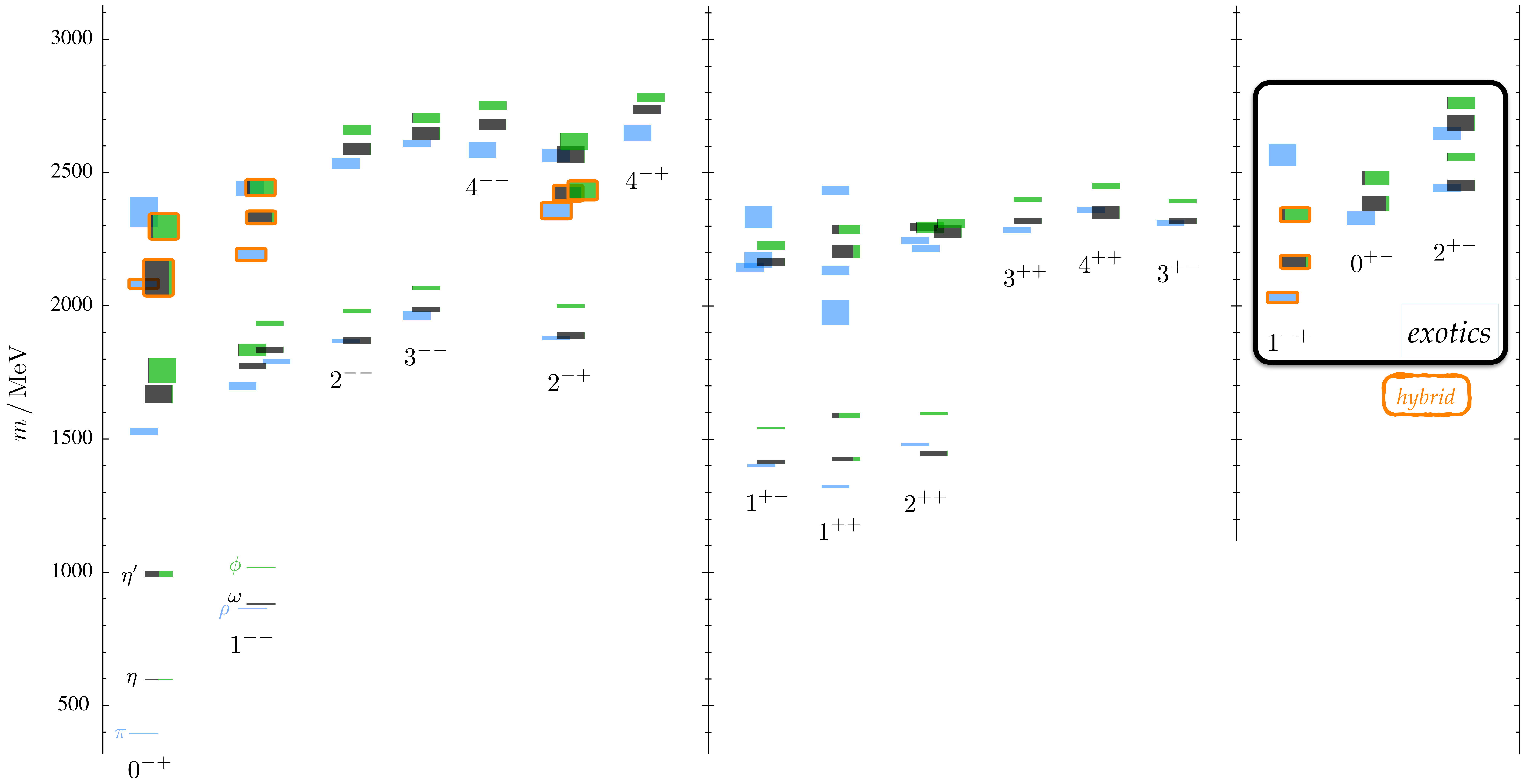
Pole in the two-body scattering amplitude at threshold: $$\mathcal{M} \sim \frac{1}{p \cot \delta - ip} = \frac{1}{ip}$$

Infinite tower of geometrically-separated three-body bound states: $E_{N+1} = E_N/\lambda$ where $\lambda = 22.69438$



Vitaly Efimov

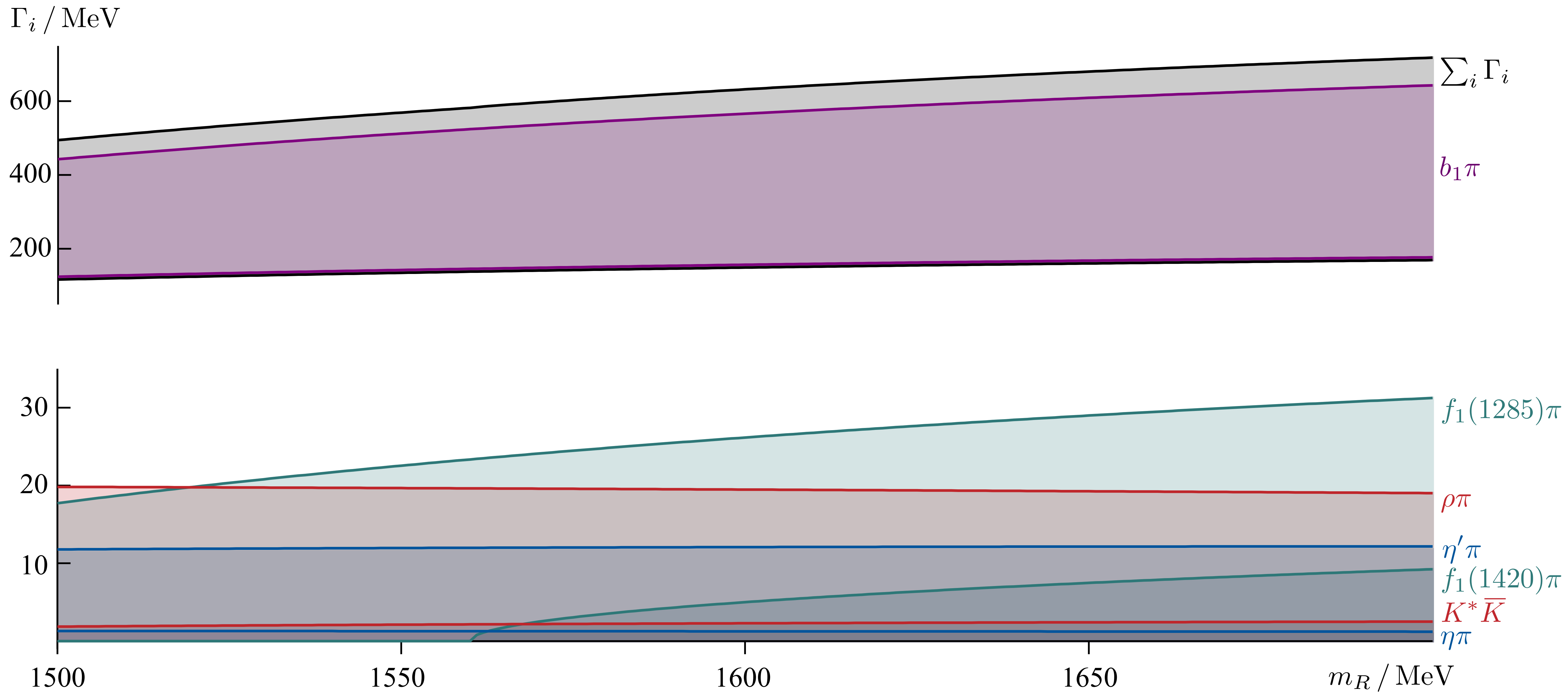
the three-body problem *QCD Spectroscopy*

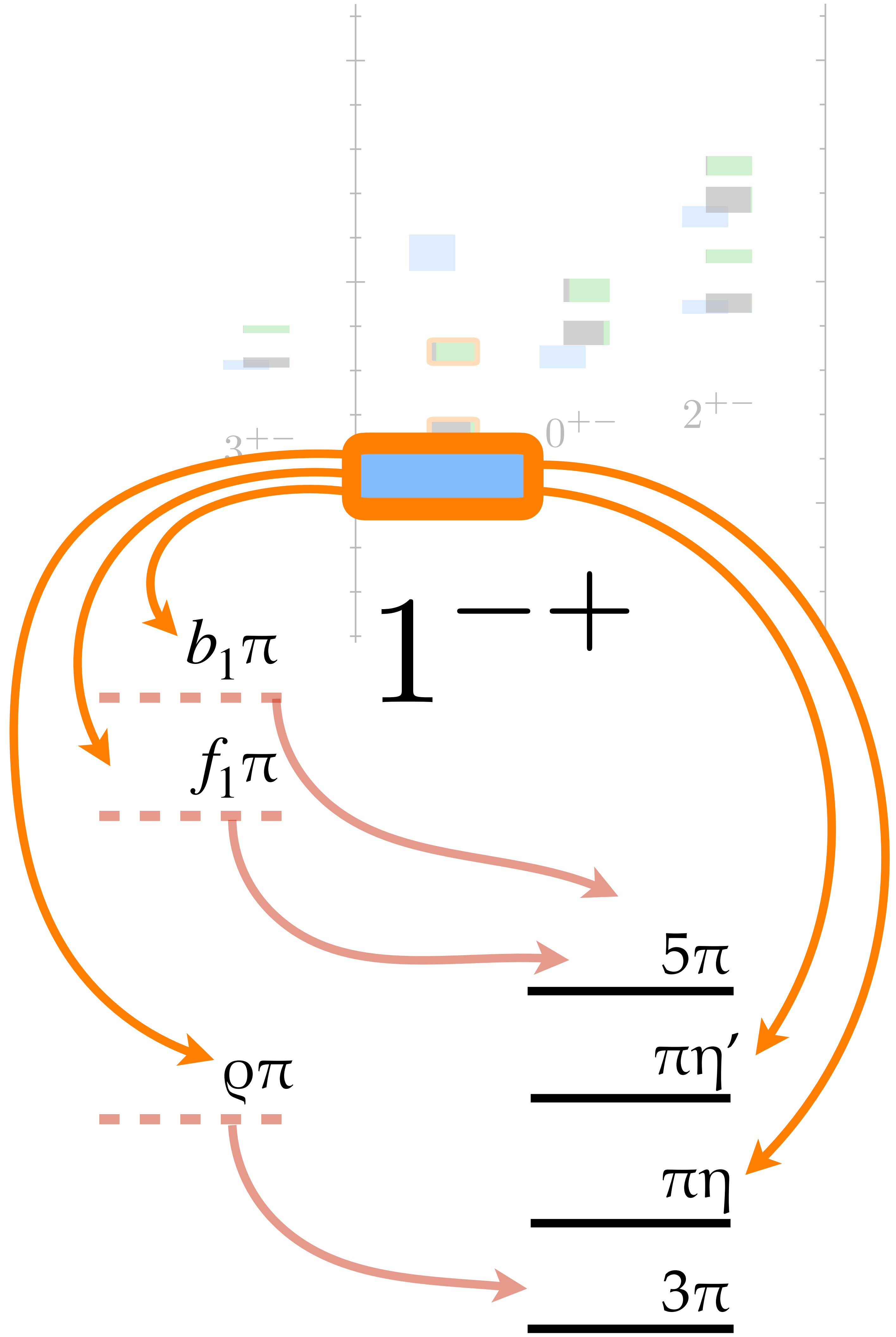


m_π=391 MeV

Dudek, Edwards, Guo, Thomas (2013) RB

the three-body problem QCD Spectroscopy





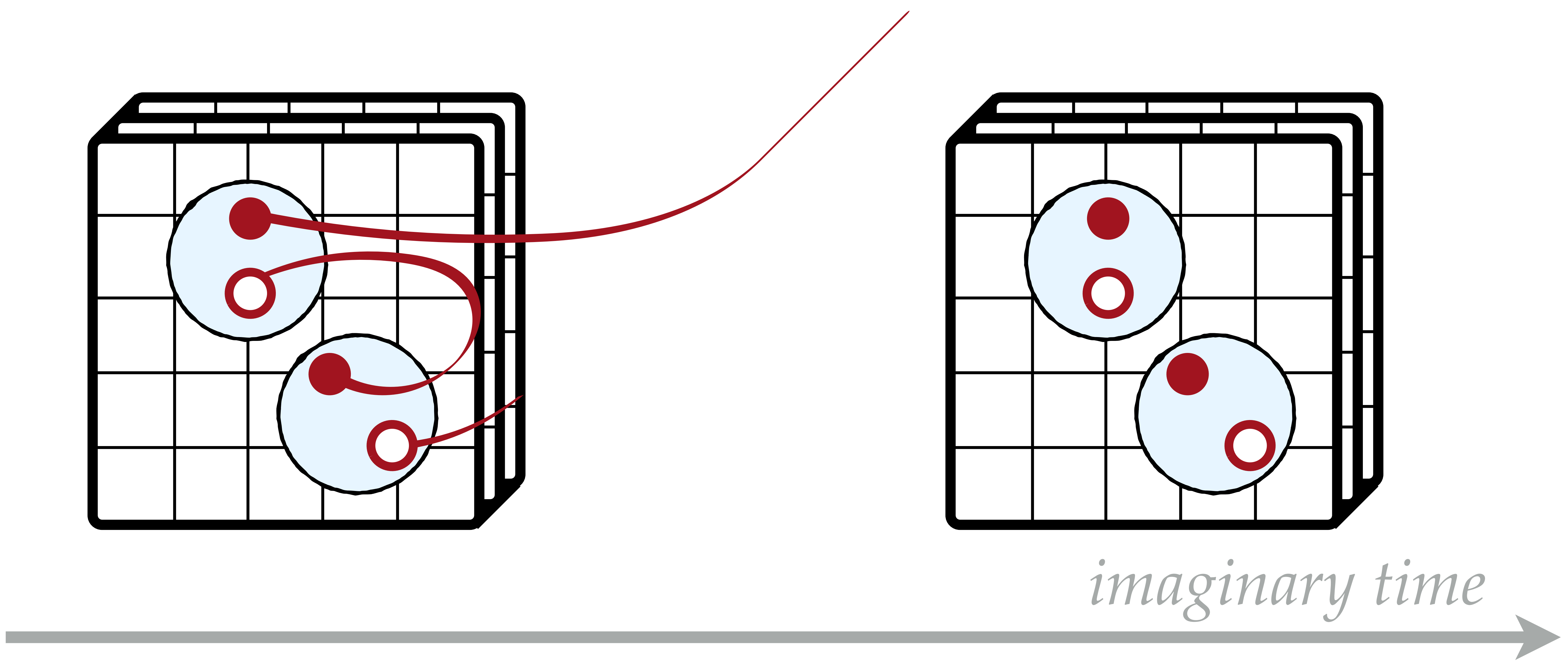
Woss, Dudek, Edwards, Thomas, Wilson (2020)

Dudek, Edwards, Guo, Thomas (2013) ~~RB~~

Lattice QCD in a nut shell

- lattice spacing: $a \sim 0.03 - 0.12$ fm
- finite volume: $L \sim 6 - 12$ fm
- quark masses
- Euclidean spacetime: $t_M \to -it_E$
- Importance sampling
- Correlation functions, …

$$C^{2pt.}(t_E) \equiv \langle 0|\mathcal{O}(t_E)\mathcal{O}^\dagger(0)|0\rangle = \sum_n c_n e^{-E_n t_E}$$



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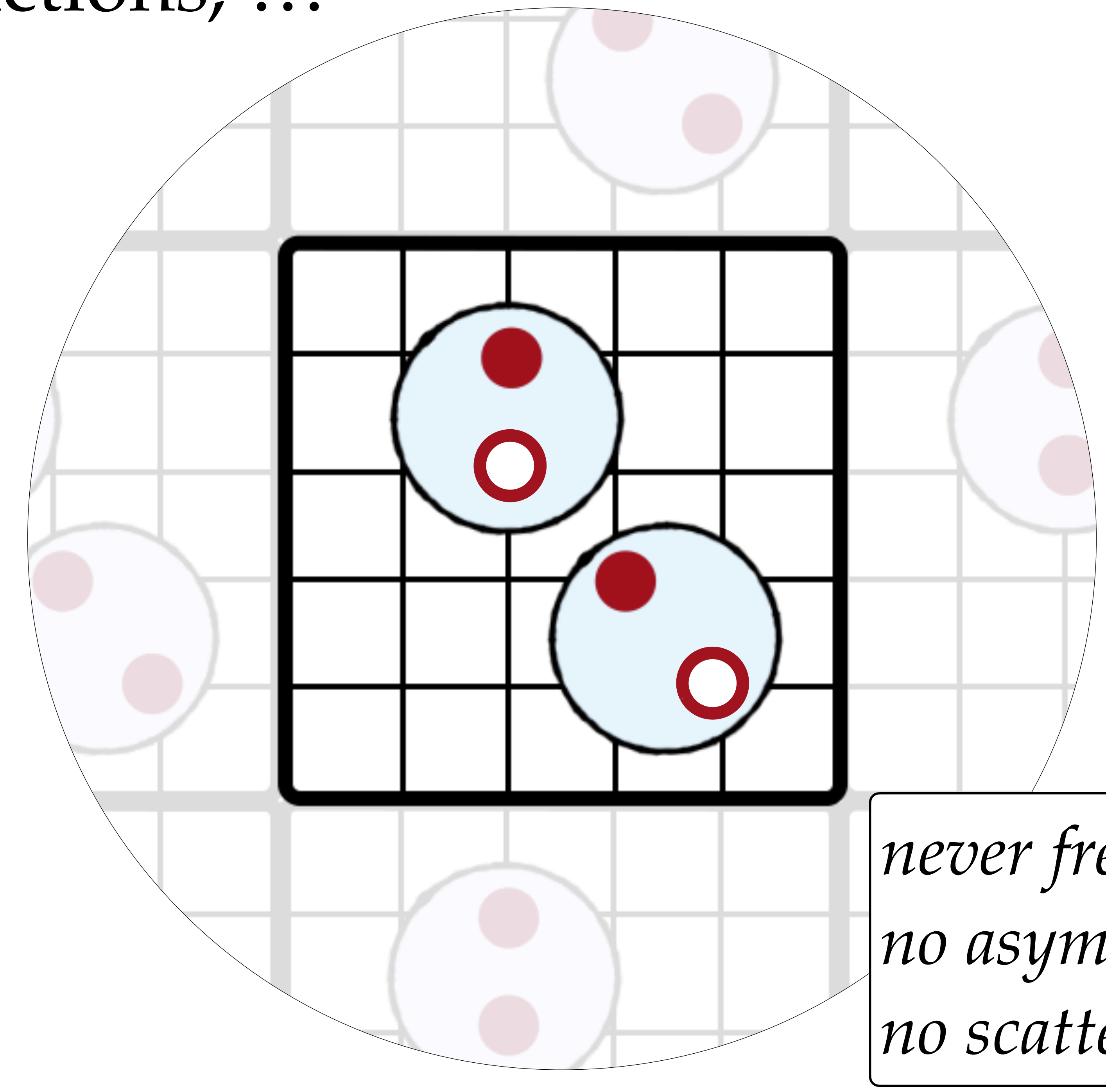
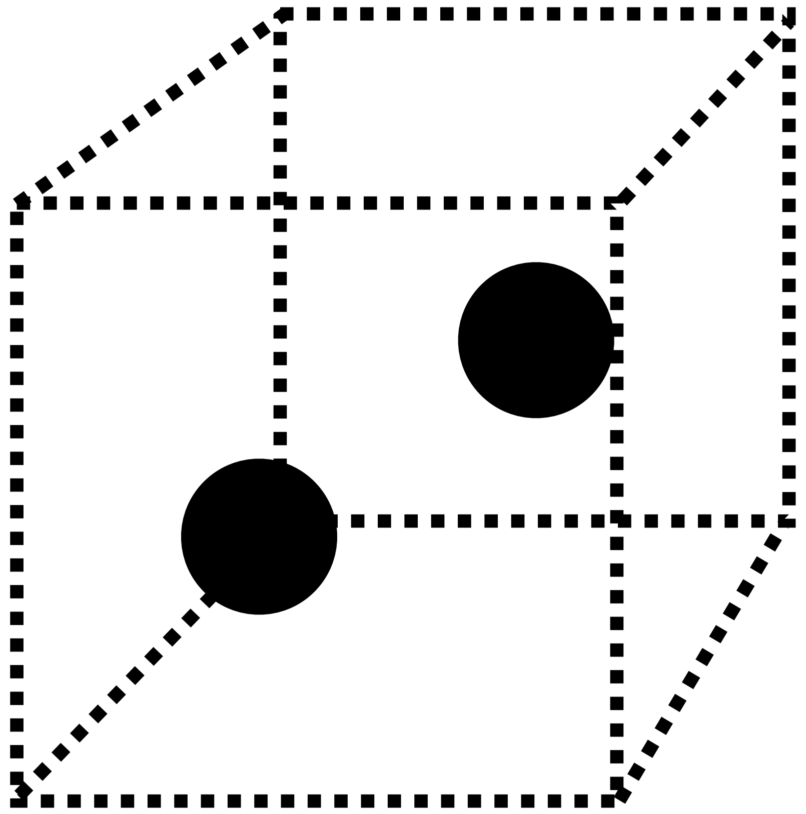


never free
no asymptotic states
no scattering

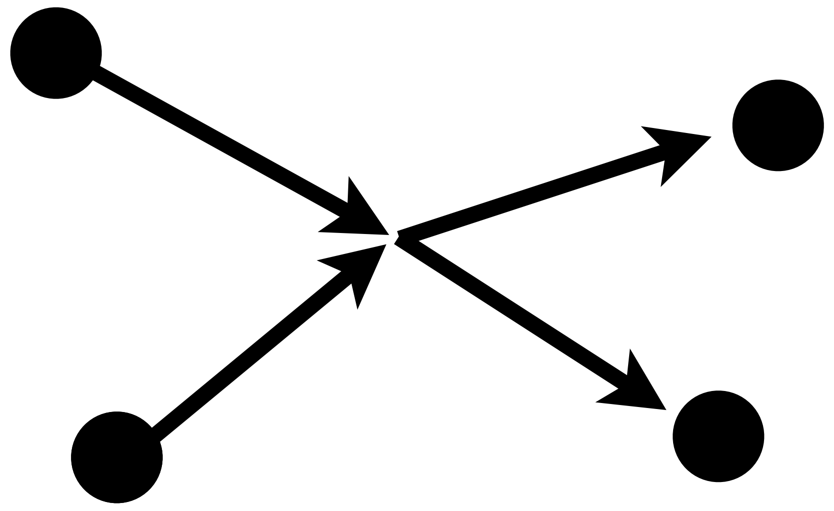
Two-hadron systems



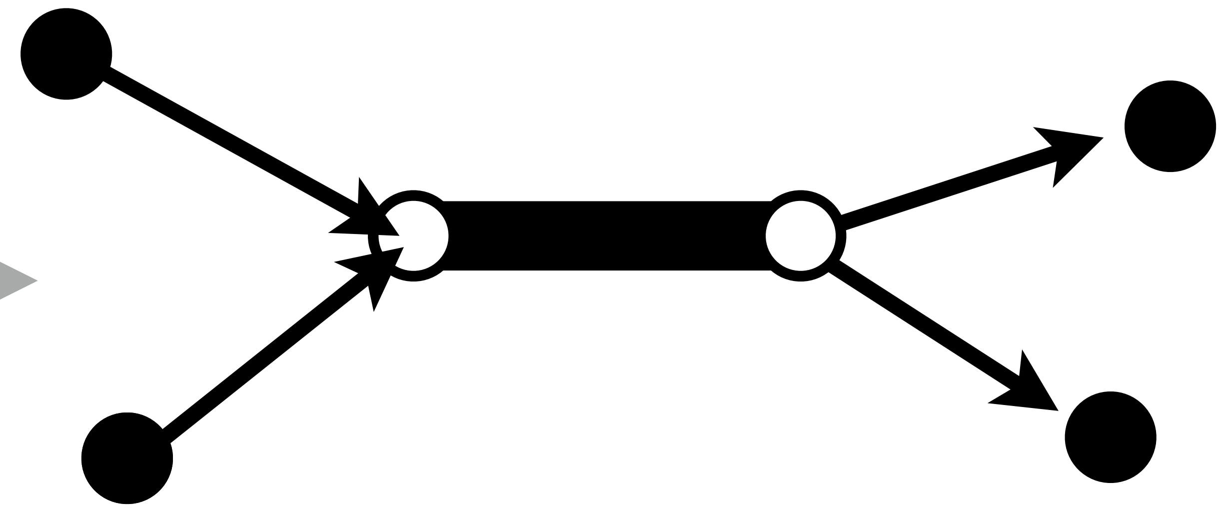
finite-volume spectroscopy

$$\det\left[F(E_L, L) + \mathcal{M}^{-1}(E_L)\right] = 0$$

Lüscher (1986, 1991)
Rummukainen & Gottlieb (1995)
Kim, Sachrajda, & Sharpe (2005)
Christ, Kim & Yamazaki (2005)
Feng, Li, & Liu (2004)
Hansen & Sharpe (2021)
RB & Davoudi (2012)
RB (2014)



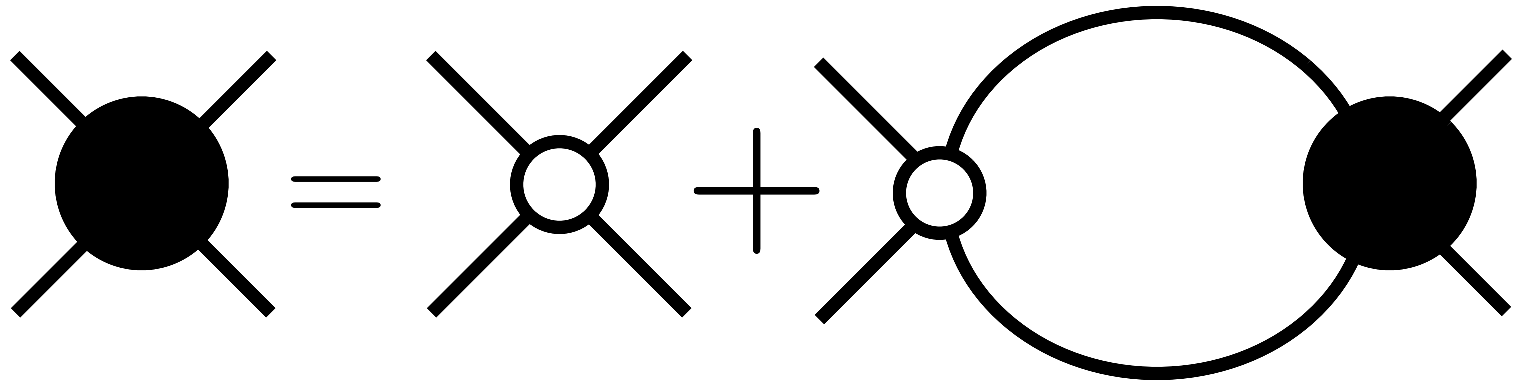
infinite-volume scattering amplitudes

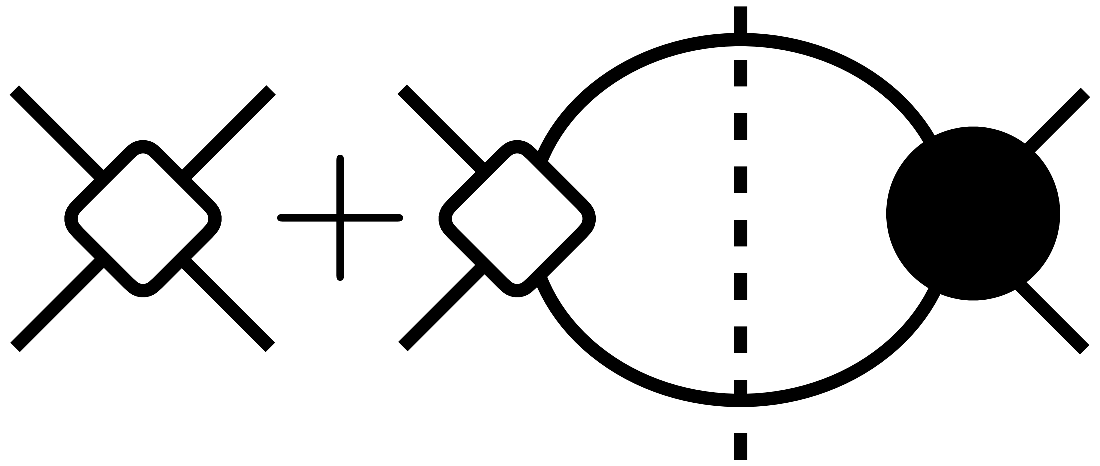


bound state and resonance poles

Two-hadron systems

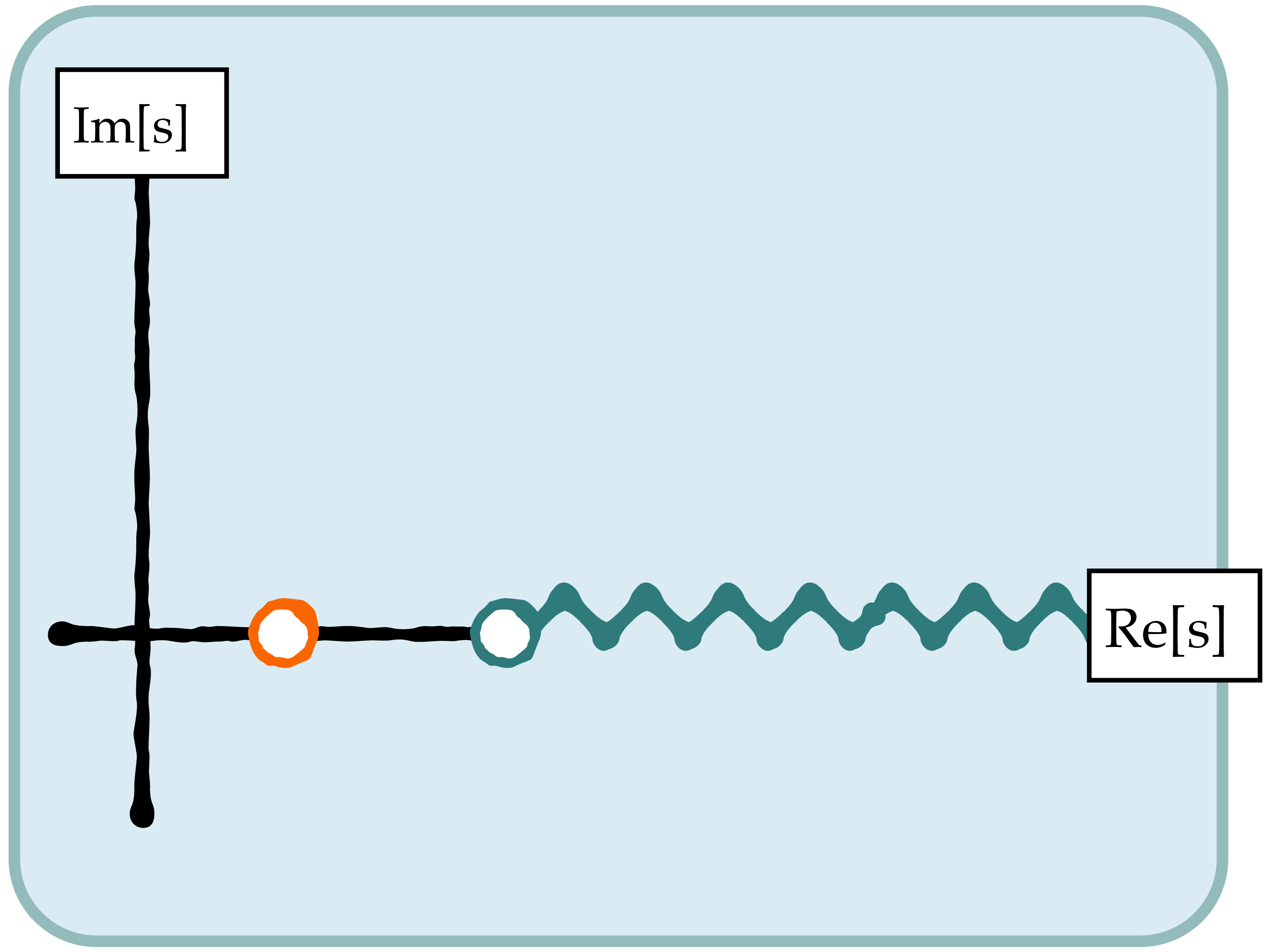
Infinite Minkowski spacetime

$i\mathcal{M} =$ 

$=$ 

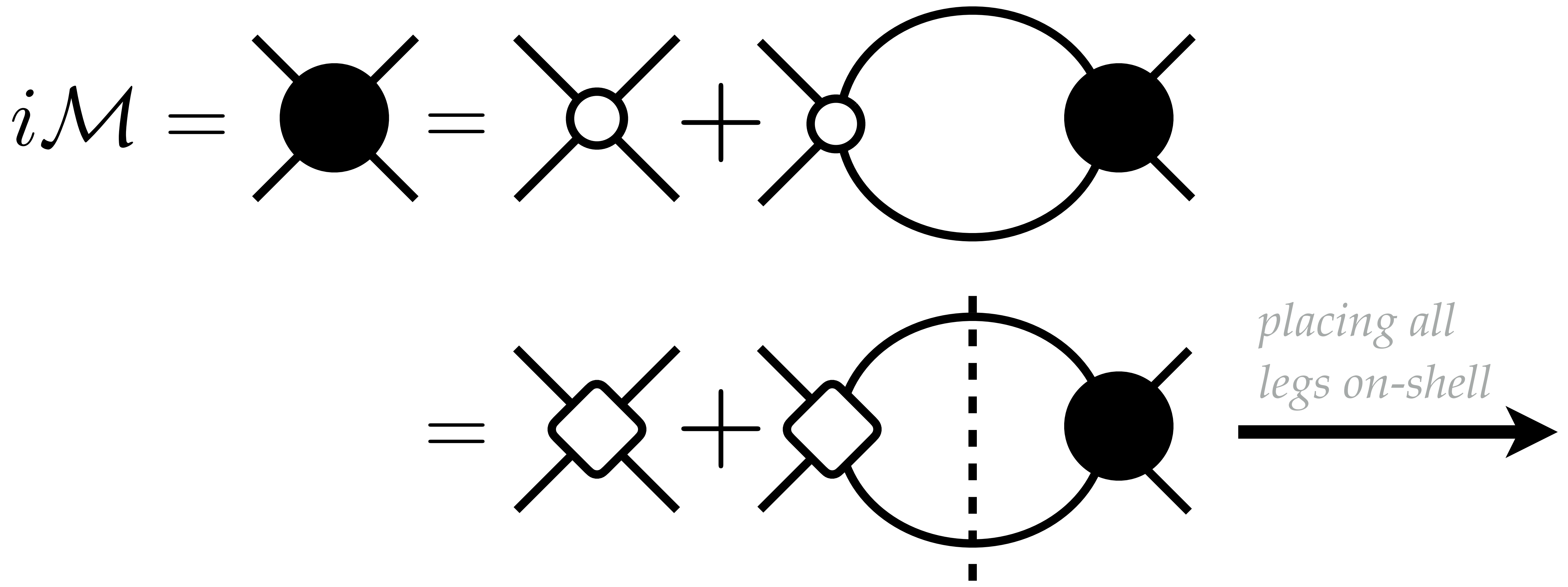
placing all legs on-shell $\longrightarrow \dfrac{i}{\mathcal{K}^{-1} - i\rho}$

$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}}$$

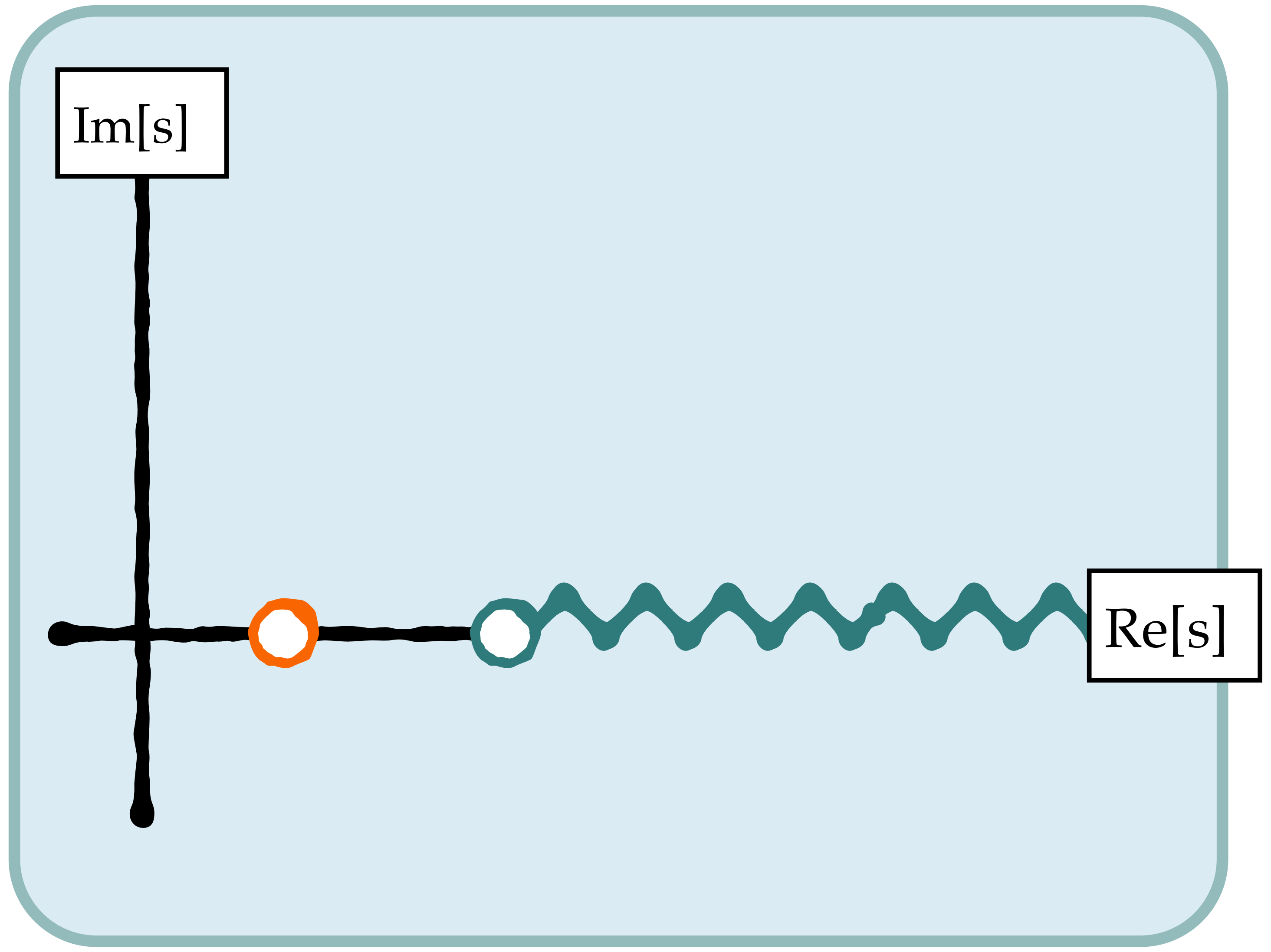


Two-hadron systems

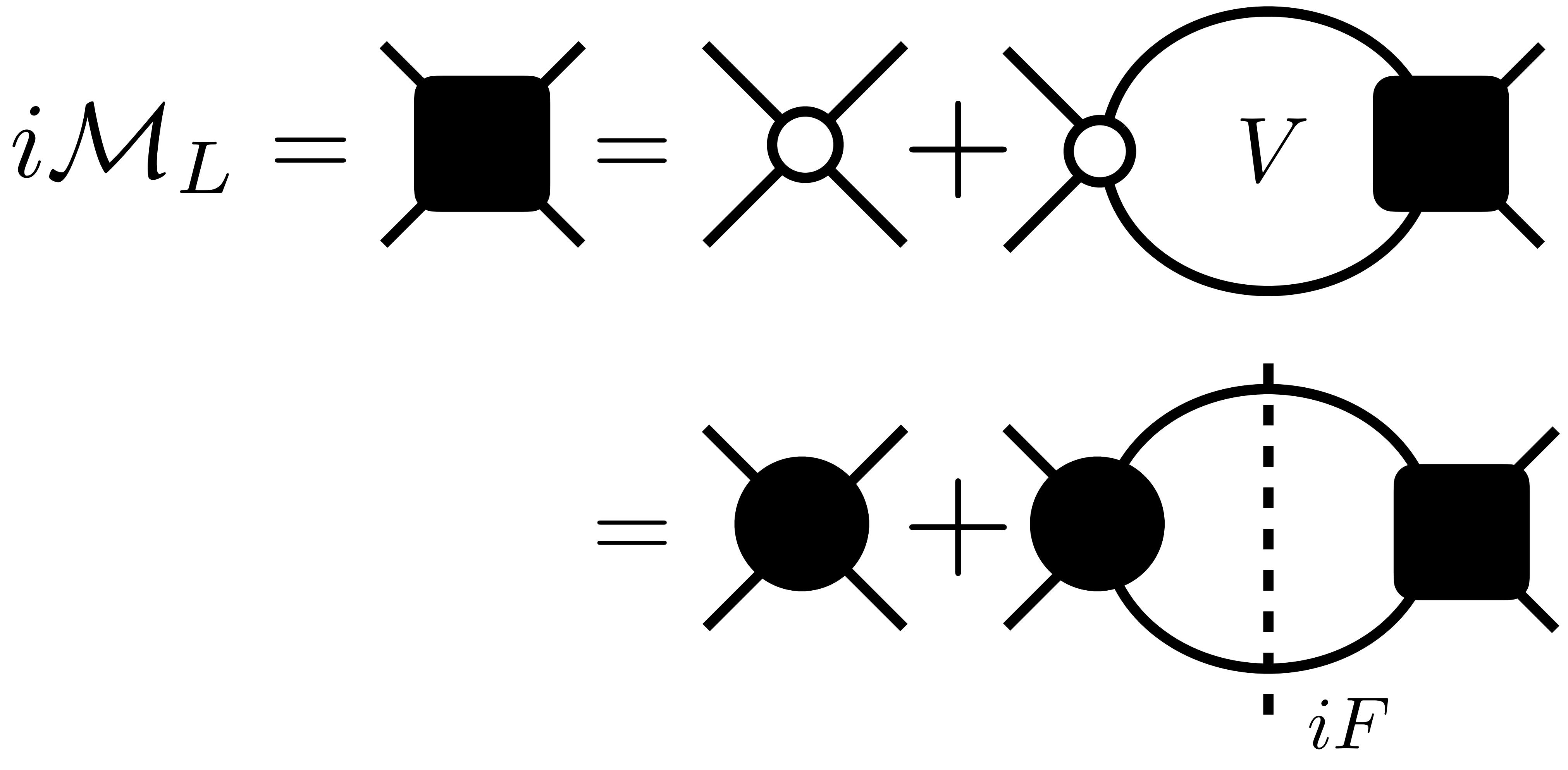
Infinite Minkowski spacetime

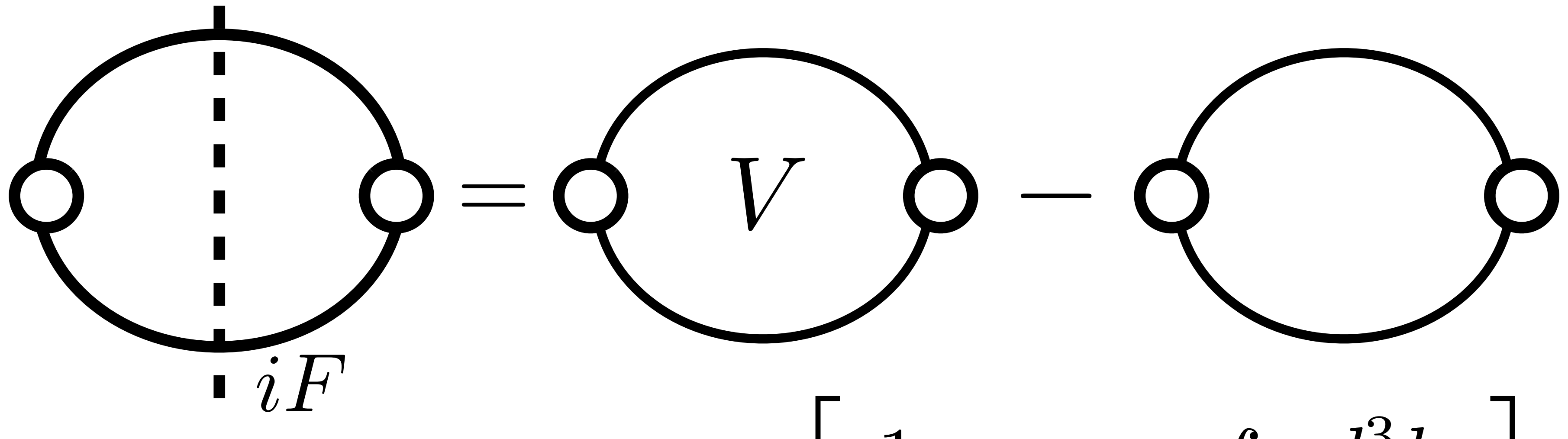


$$\frac{i}{\mathcal{K}^{-1} - i\rho}$$



Finite Minkowski spacetime



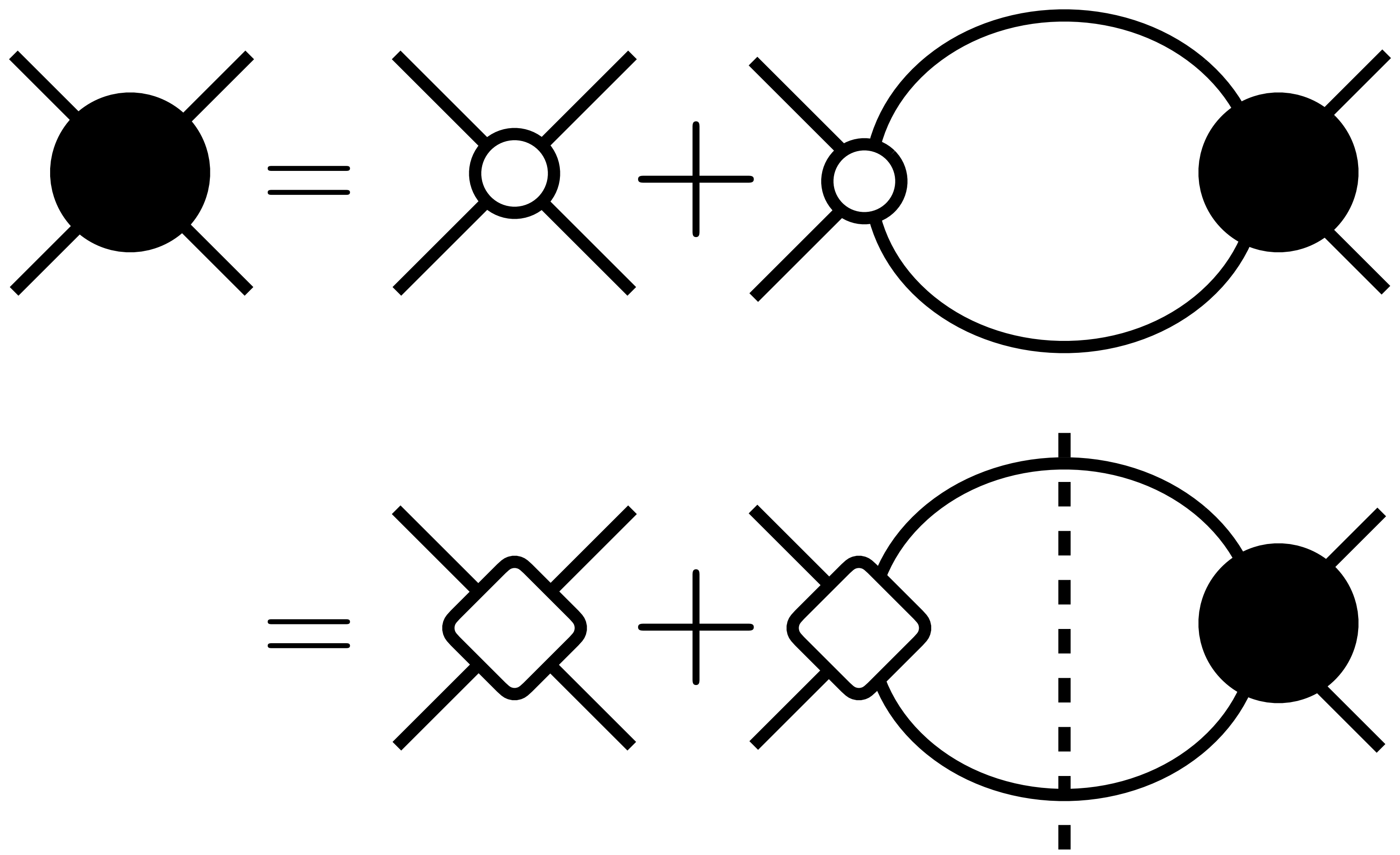


$$iF = \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{d^3k}{(2\pi)^3} \right] \frac{1}{(2\omega_k)^2} \frac{i}{E - 2\omega_k}$$

Two-hadron systems

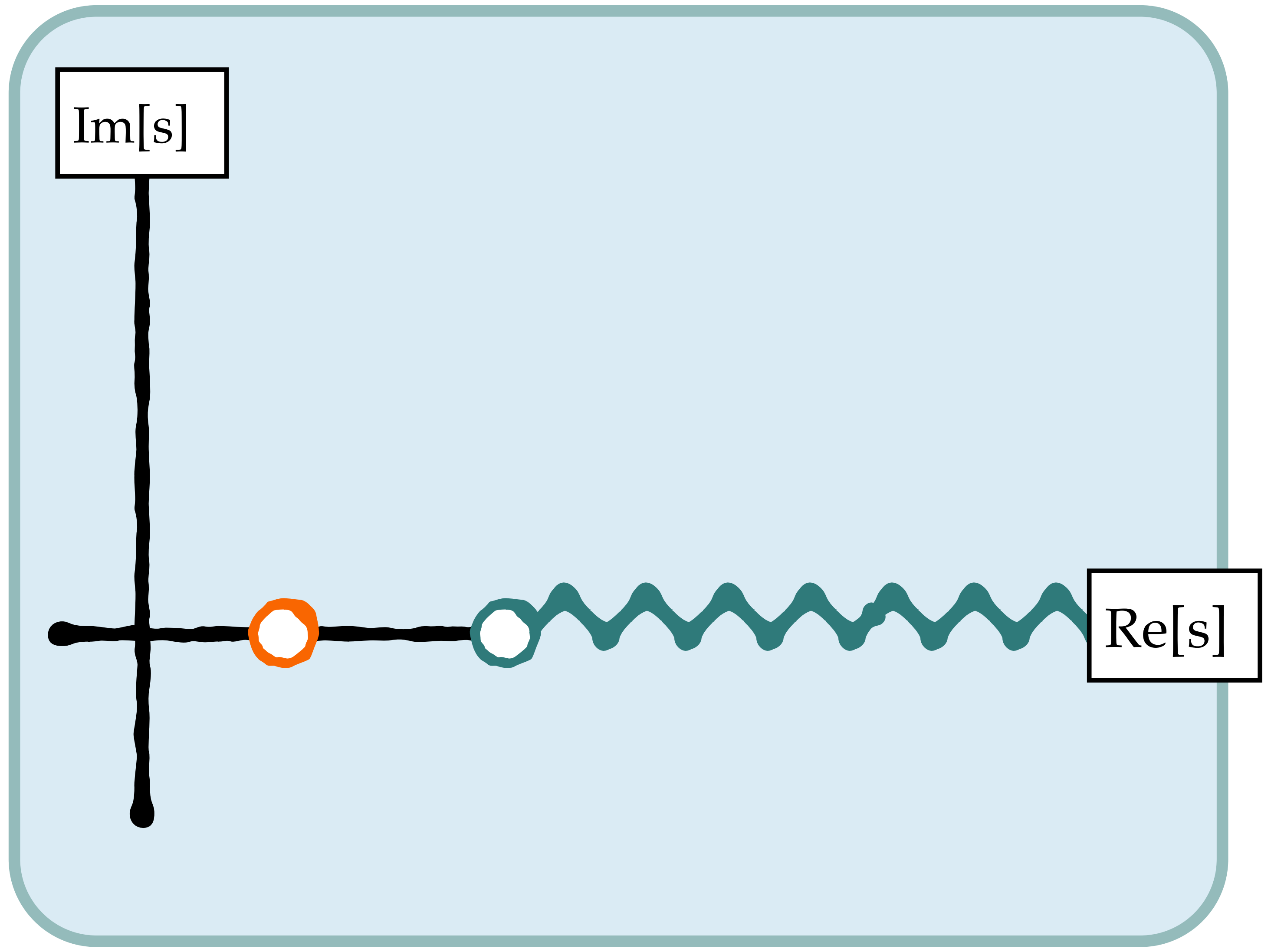
Infinite Minkowski spacetime

$i\mathcal{M} =$



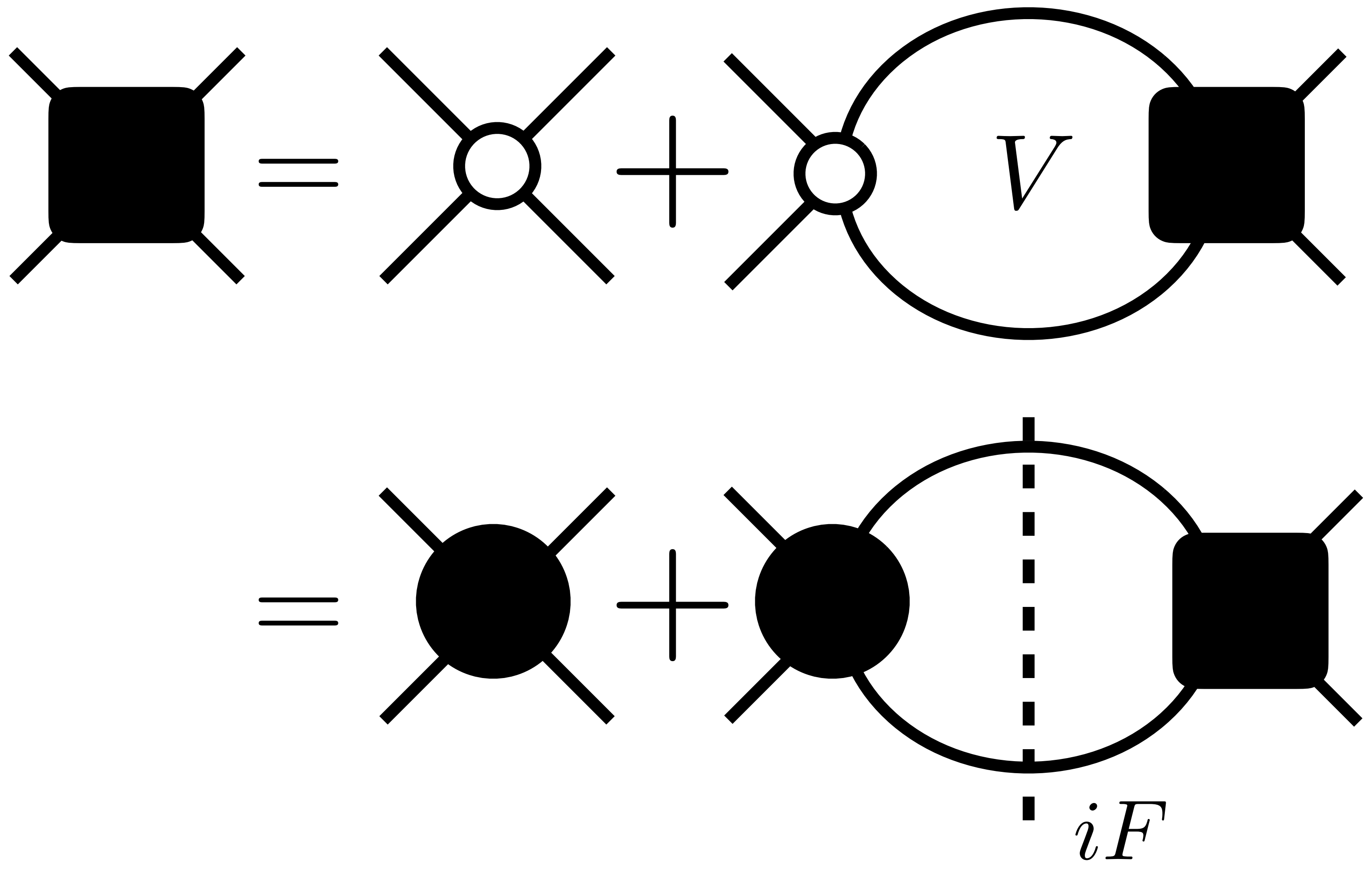
placing all legs on-shell →

$$\frac{i}{\mathcal{K}^{-1} - i\rho}$$

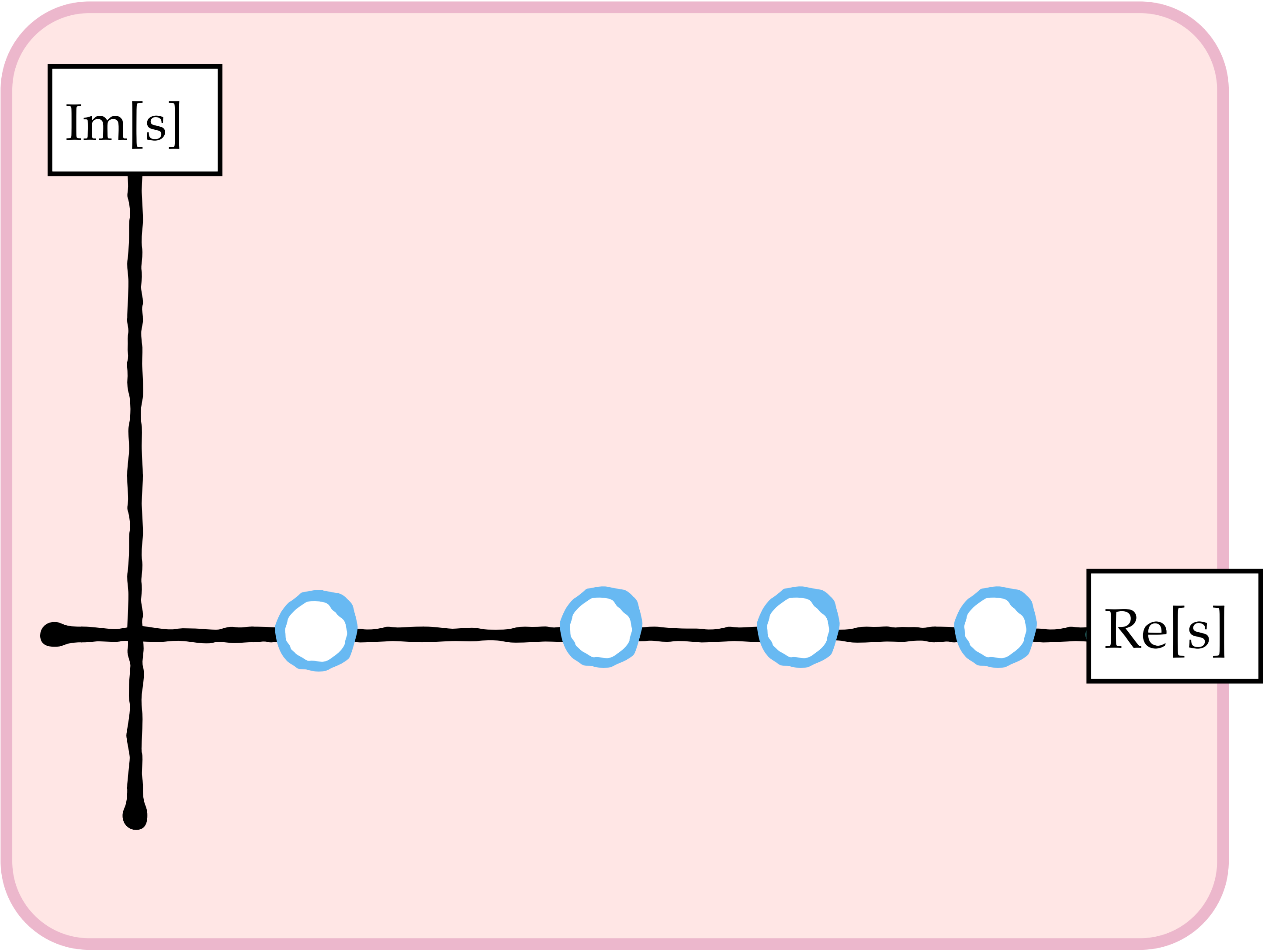


Finite Minkowski spacetime

$i\mathcal{M}_L =$

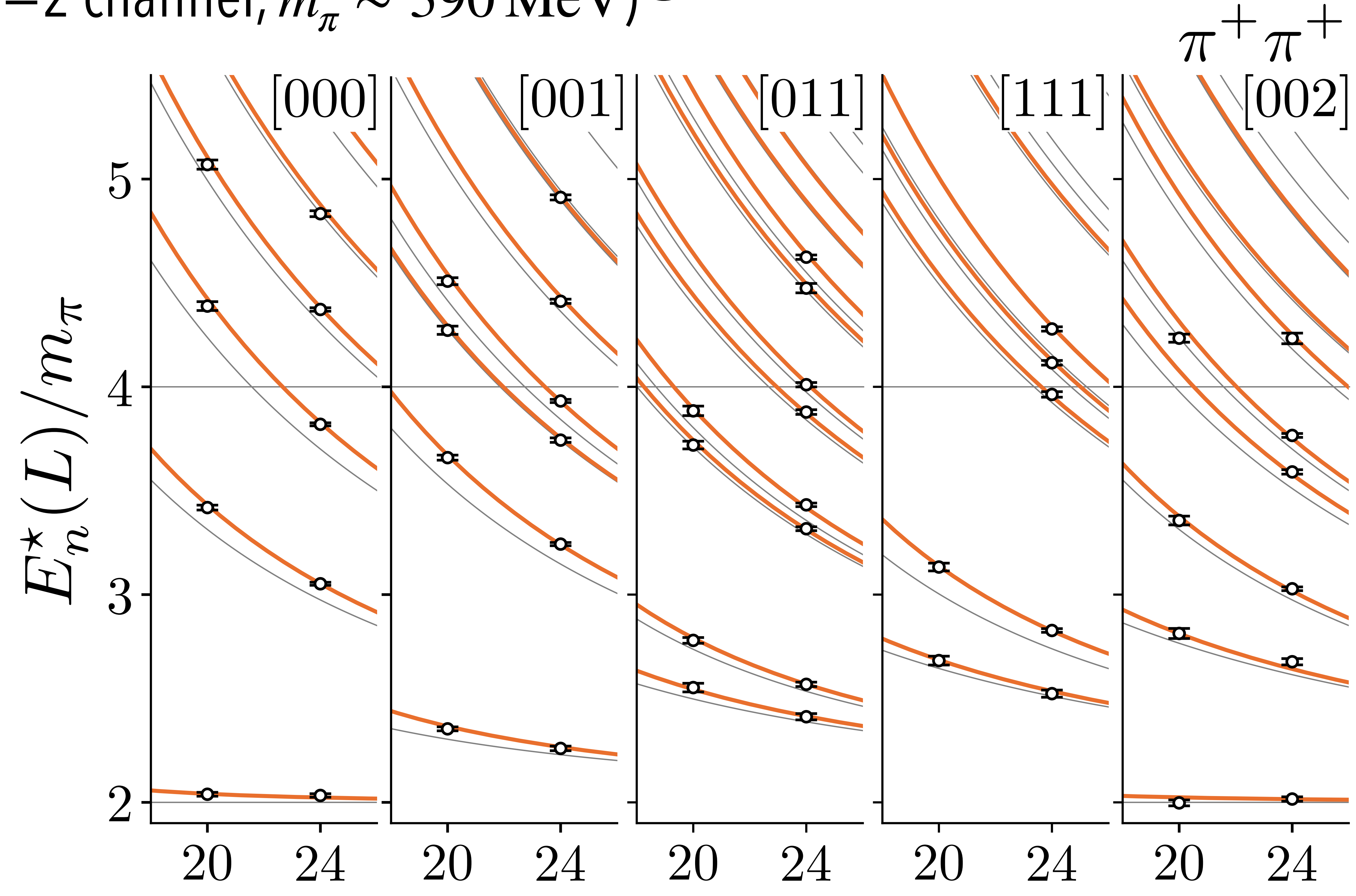


$$= \frac{1}{\mathcal{M}^{-1} + F}$$



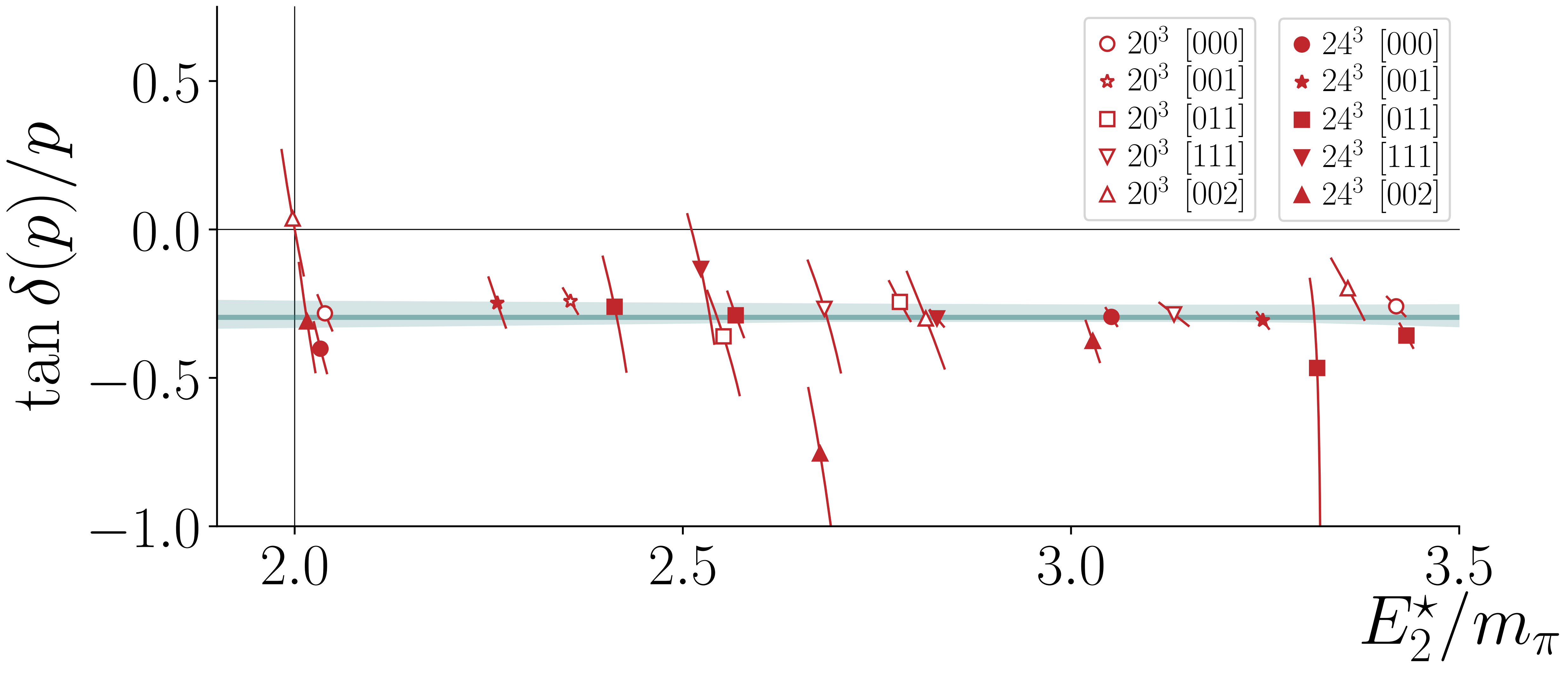
$\pi\pi$ scattering

(I=2 channel, $m_\pi \sim 390\,\mathrm{MeV}$)



$$\det[F^{-1}(P,L) + \mathcal{M}(P)] = 0$$

$$\mathcal{M} \sim \frac{1}{p\cot\delta - ip}$$



Hansen, RB, Edwards, Thomas, & Wilson (2020)

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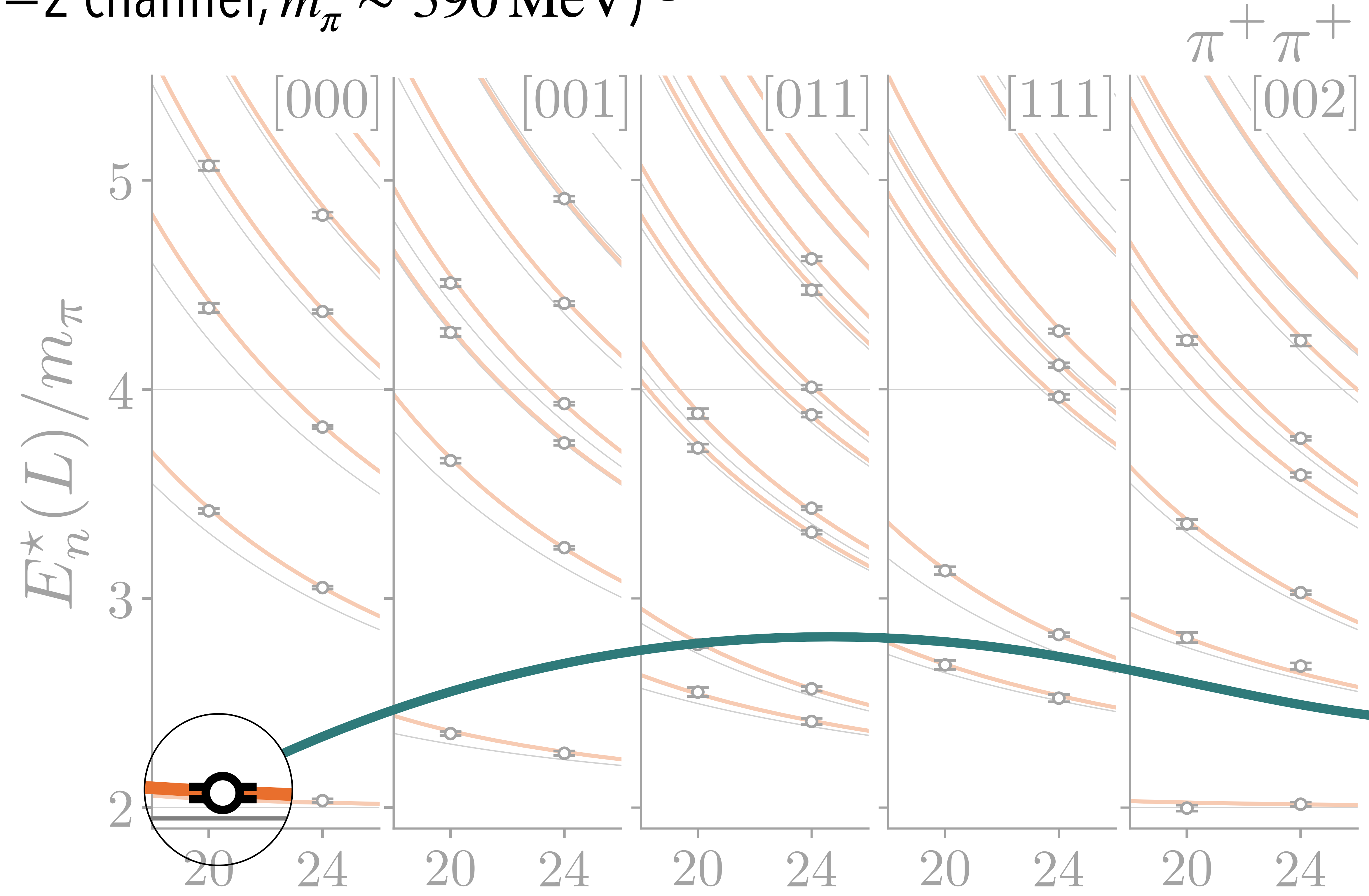

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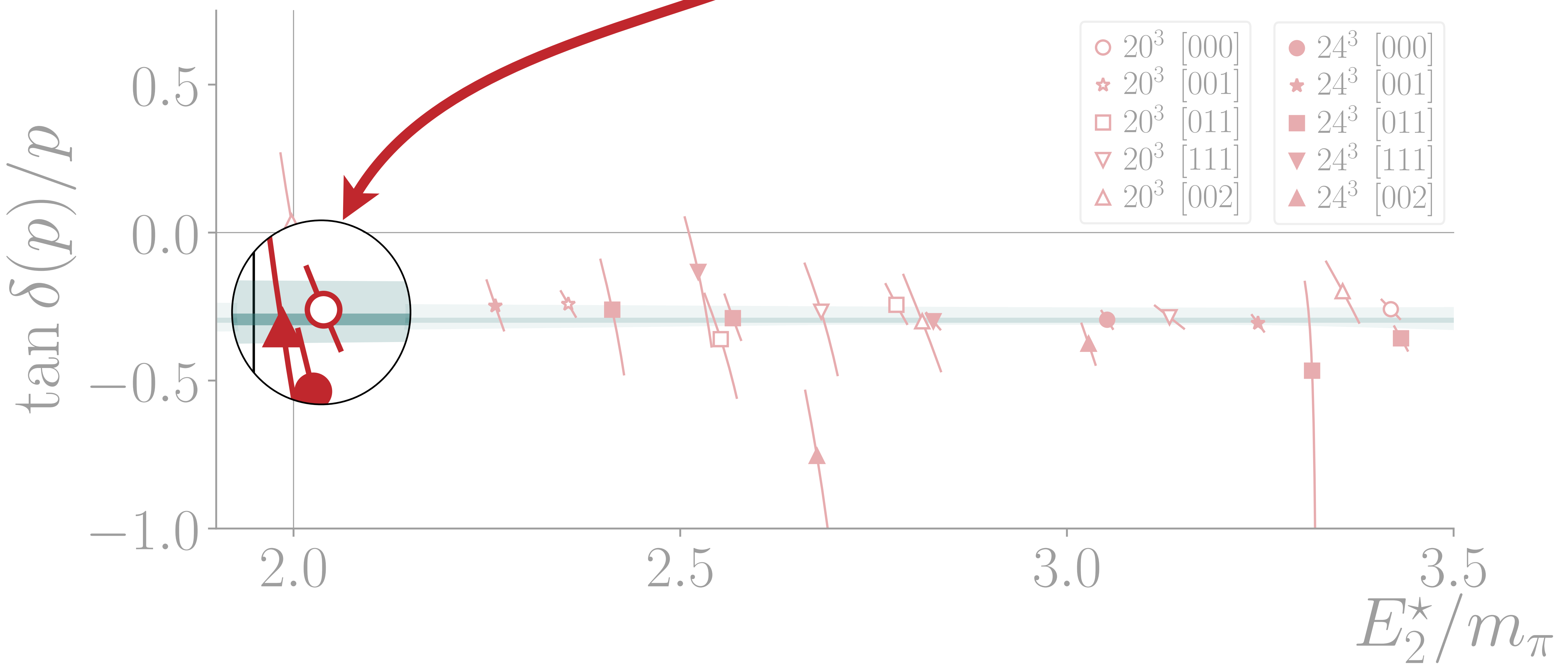
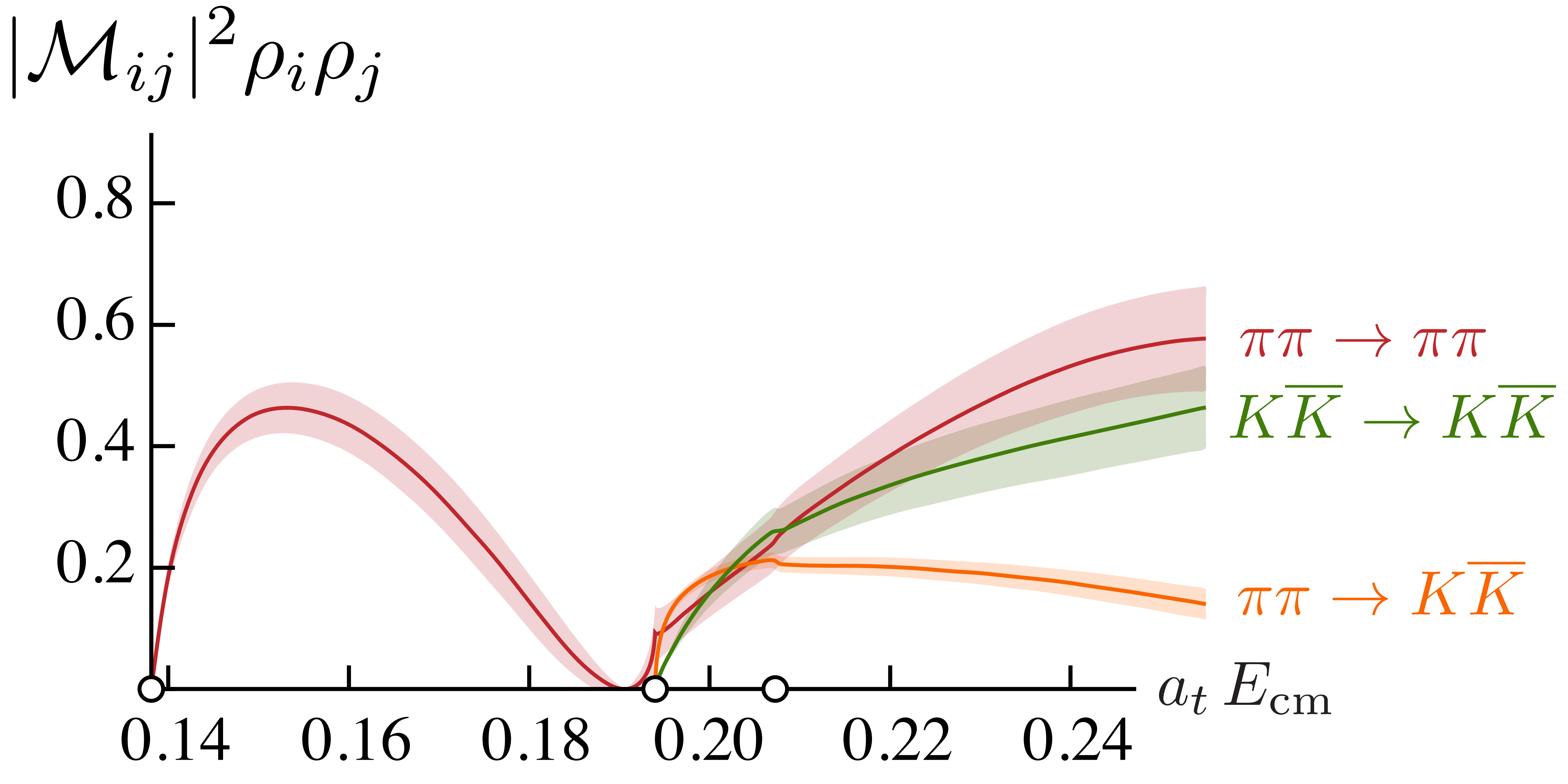
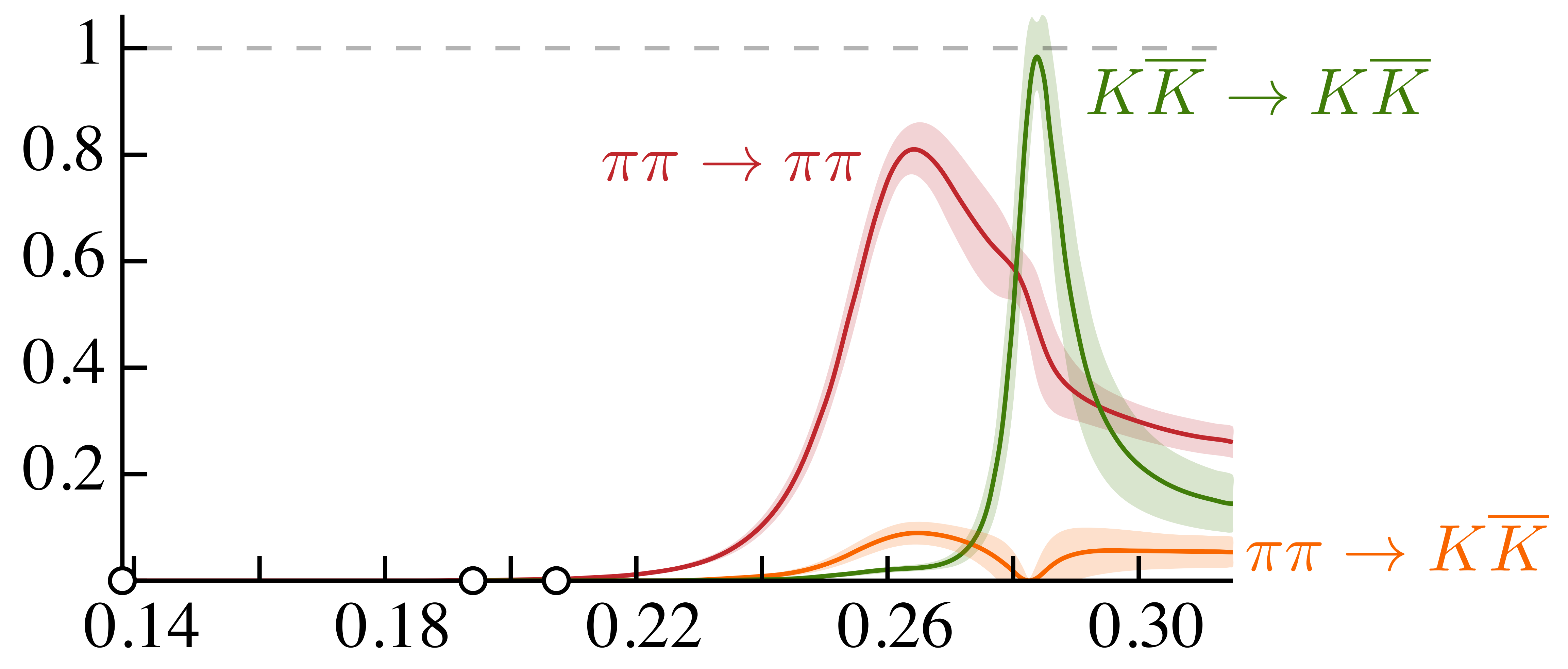
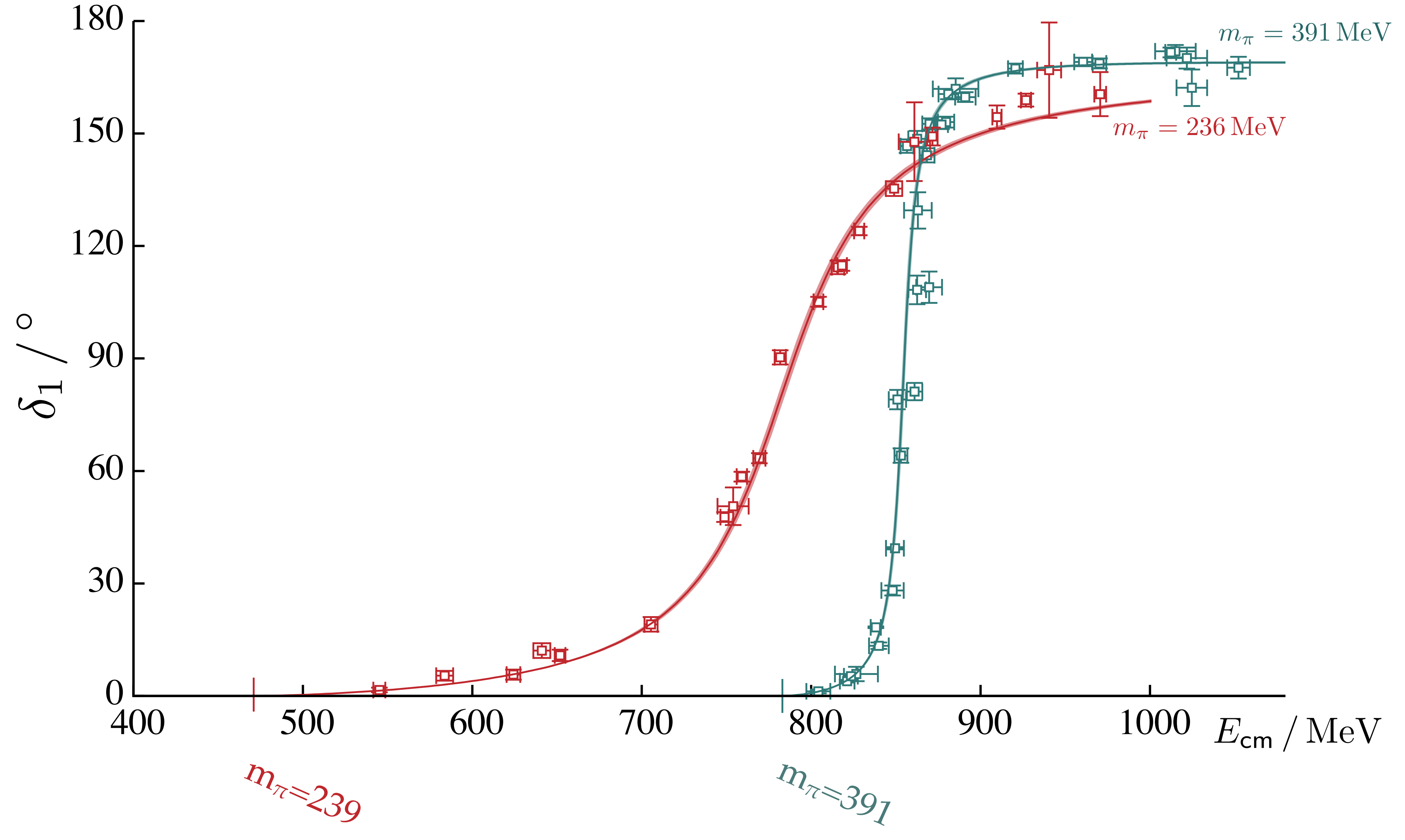


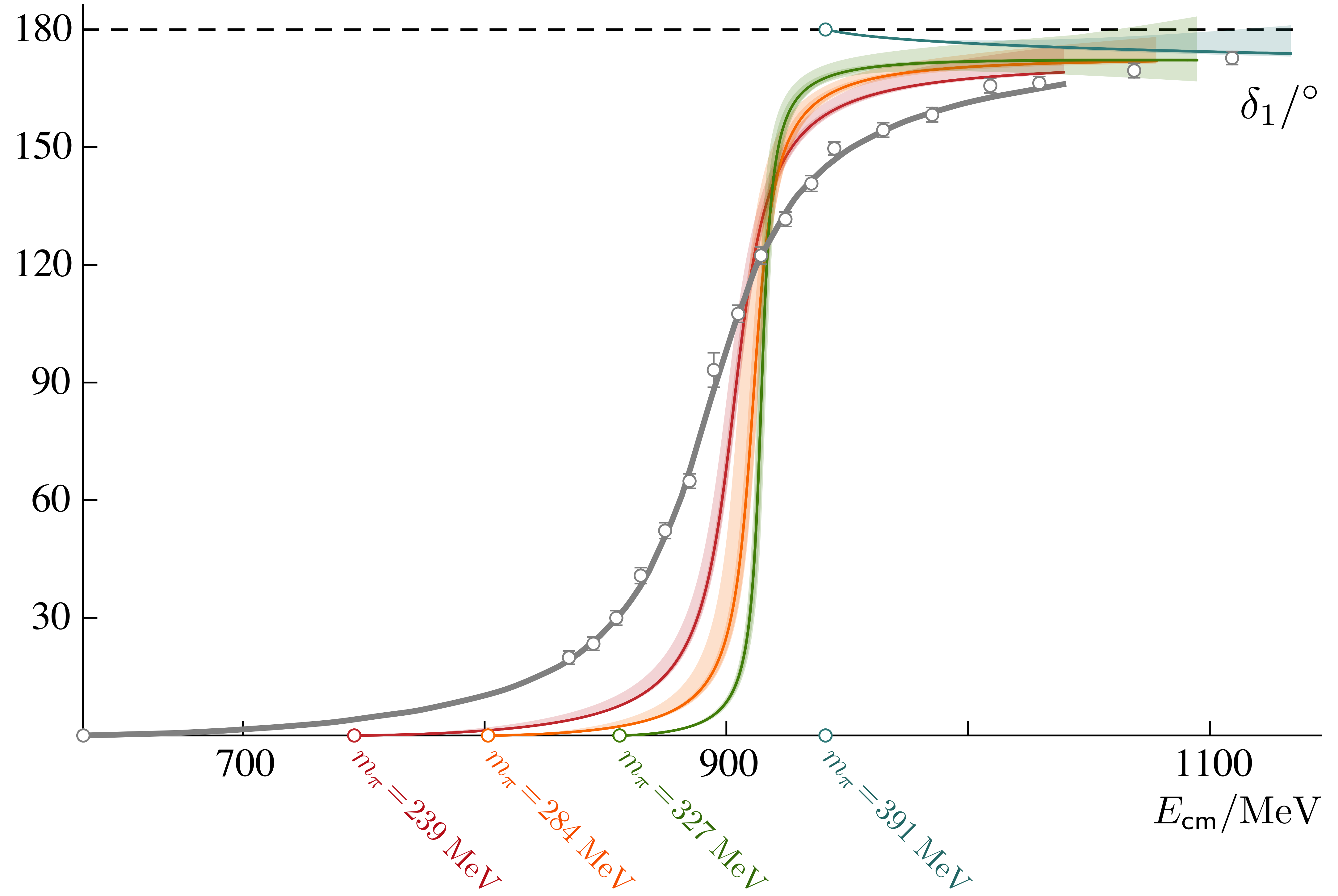
Hansen, RB, Edwards, Thomas, & Wilson (2020)

Many other calculations



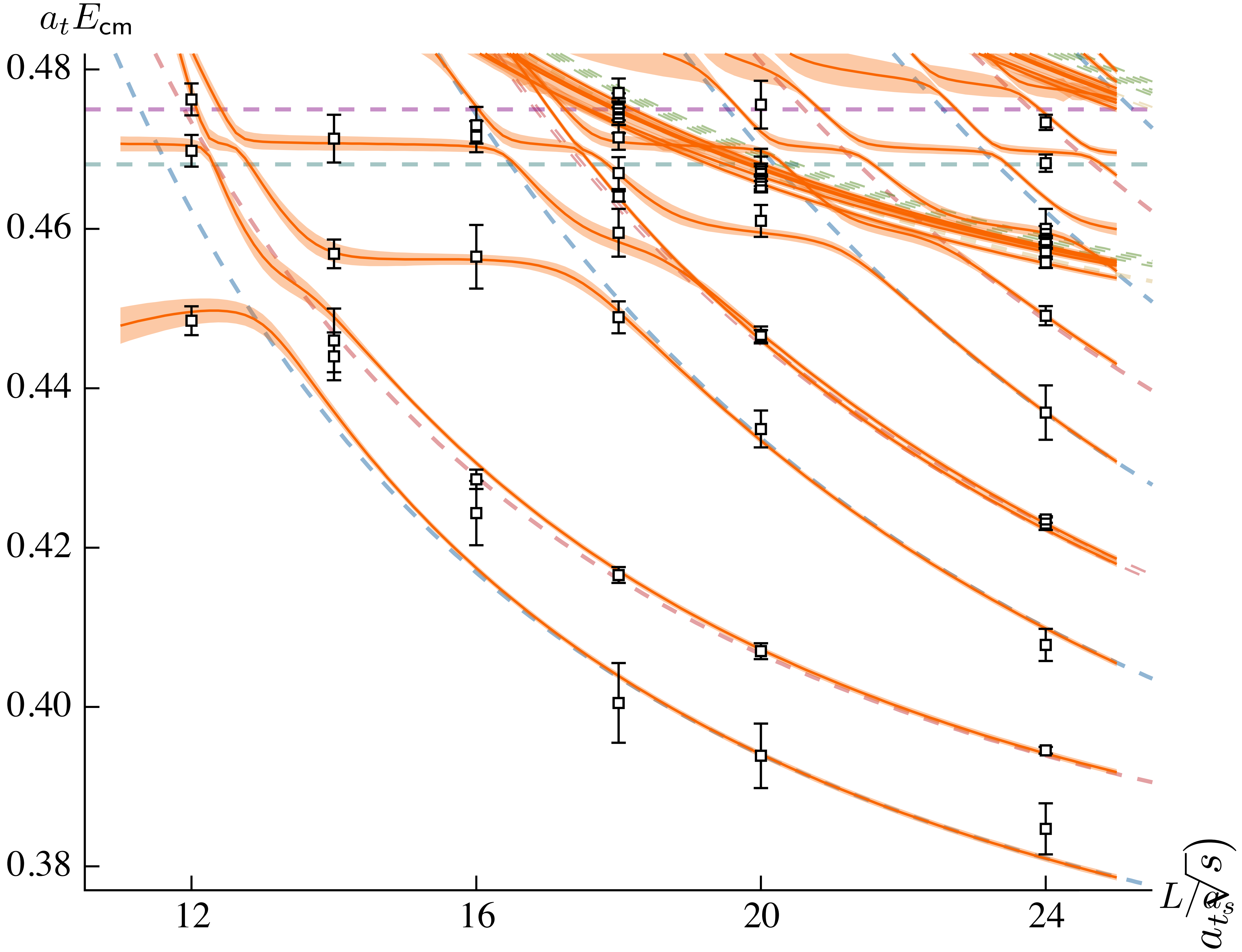


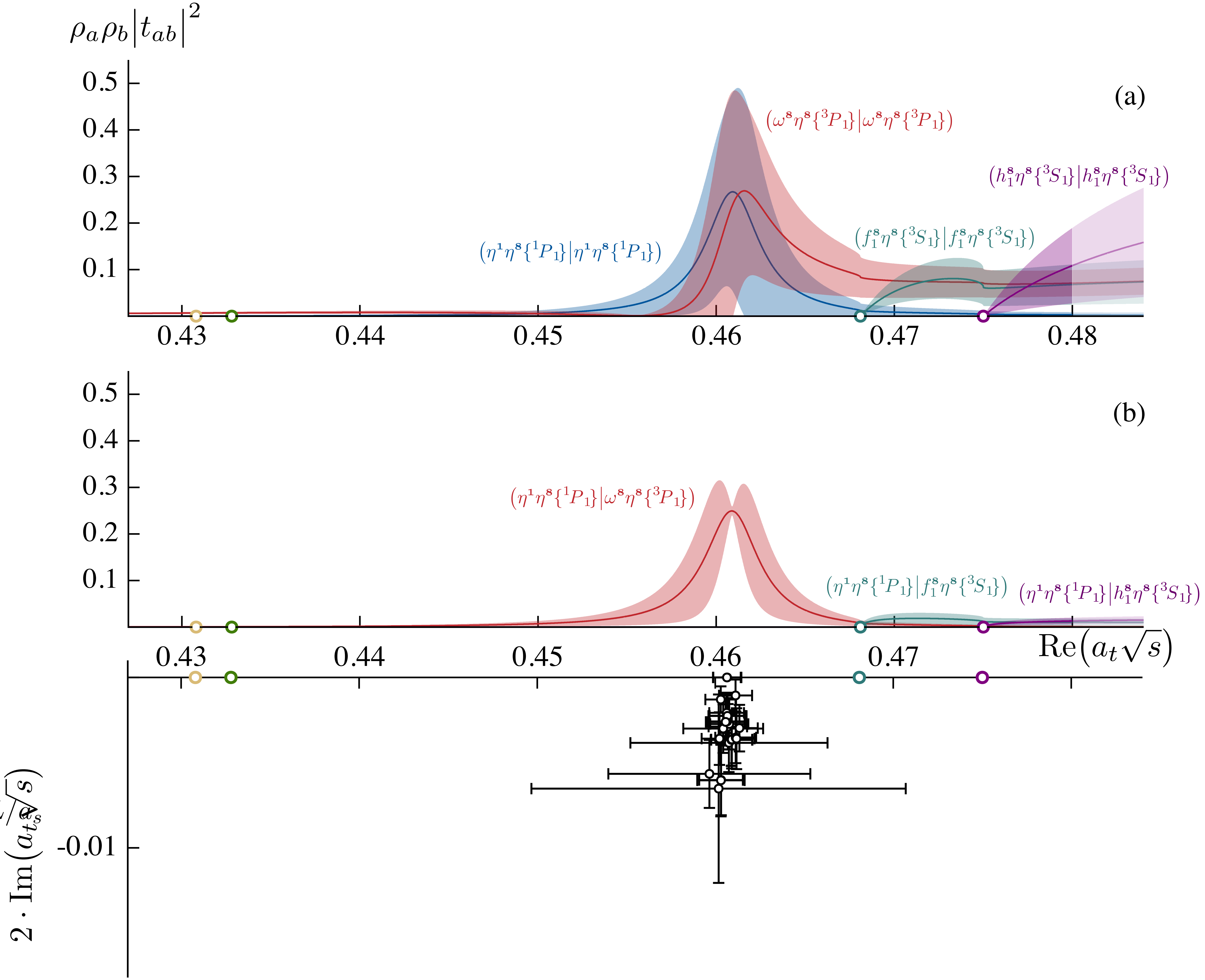




π_1 channel

($m_\pi \sim 700\,\text{MeV}$)



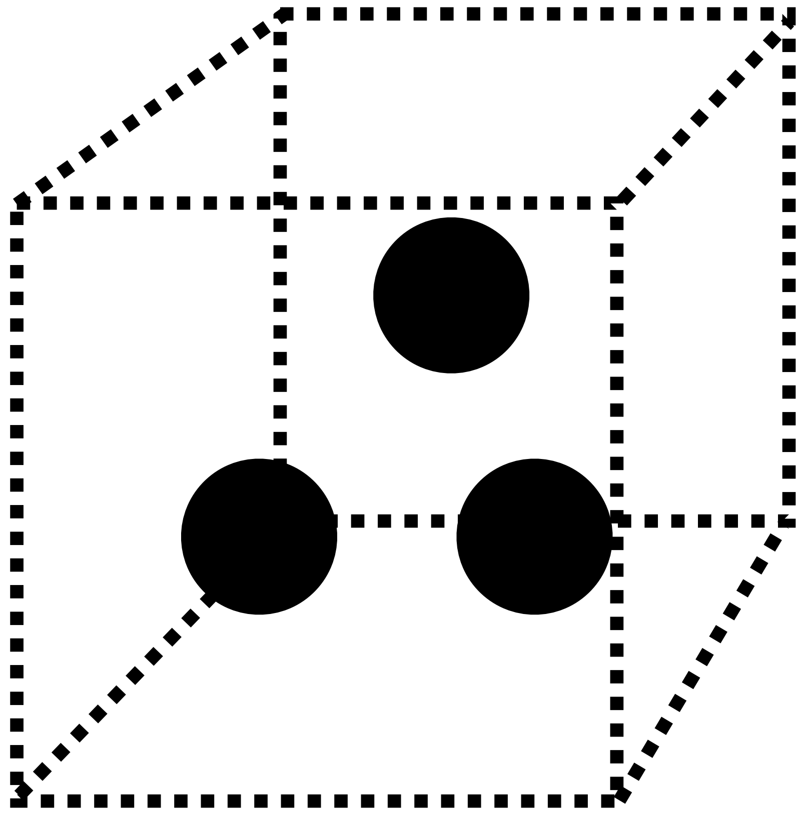


~~RB~~

Woss, Dudek, Edwards, Thomas, Wilson (2020)

$$b_1\pi \gg [f_1(1285)\pi \sim \rho\pi \sim \pi\eta \sim \pi\eta' \sim K^*K \sim f_1(1420)\pi]$$

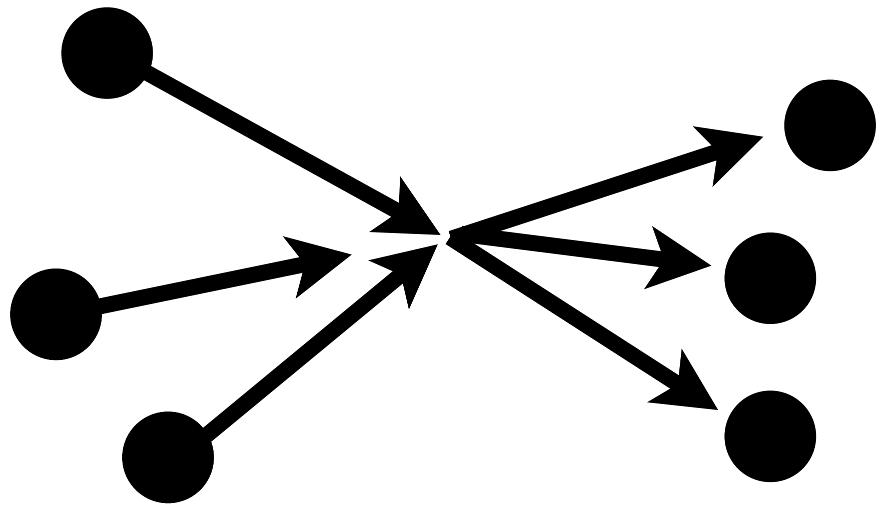
Three-hadron systems



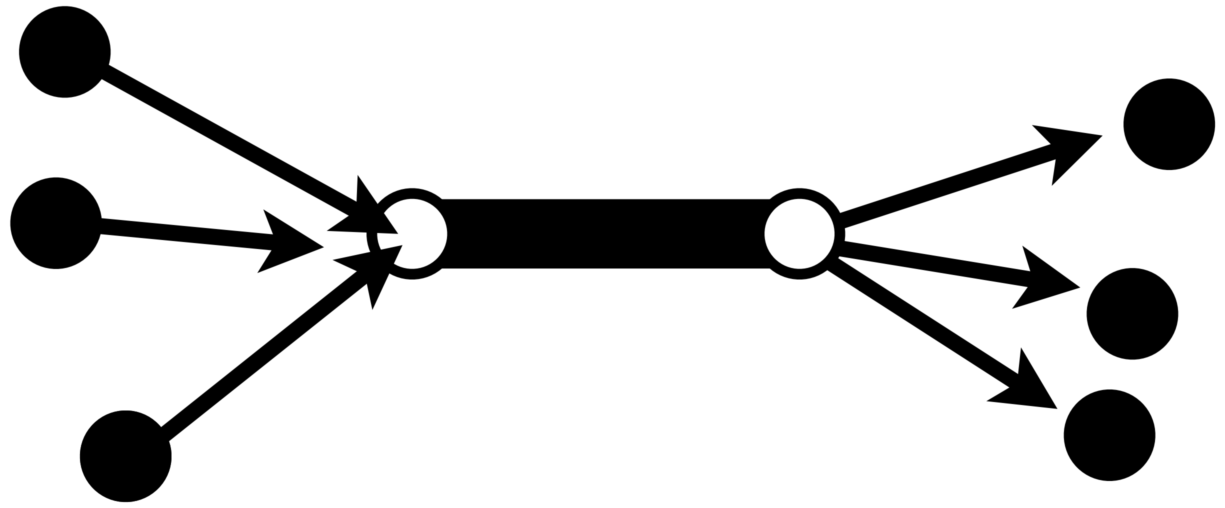
finite-volume
spectroscopy

$$\det\left[F_3(E_L,L)+\mathcal{K}^{-1}_{\mathrm{df},3}(E_L)\right]=0$$

Hansen & Sharpe ('14, '15)
Mai & Döring ('17)
RB, Hansen & Sharpe ('18)
Hansen, Romero-Lopez & Sharpe ('20)
Blanton & Sharpe ('20)
Jackura ('22)



infinite-volume
scattering amplitudes



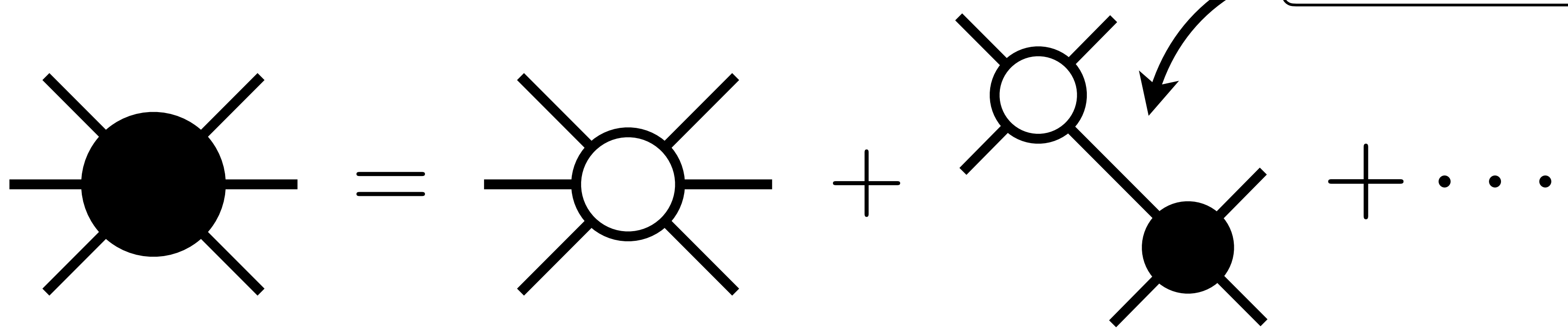
bound state and
resonance poles

Three-hadron systems

Infinite Minkowski spacetime

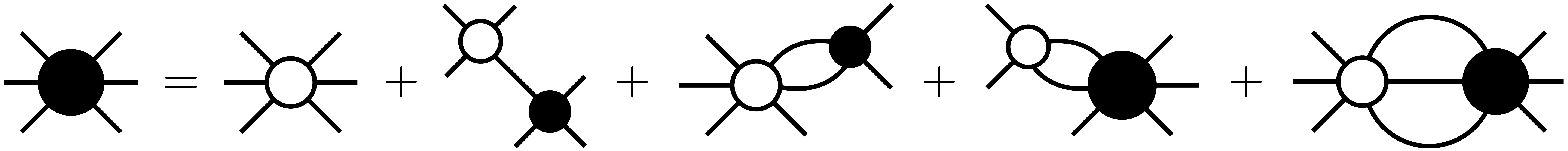
$$G \sim \frac{1}{(P-p-k)^2 - m^2}$$

$i\mathcal{M}_3 =$



Three-hadron systems

Infinite Minkowski spacetime

$i\mathcal{M}_3 =$ 

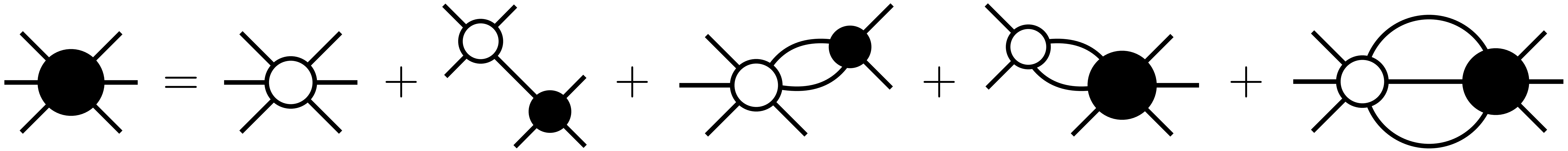
placing all legs on-shell → $i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_{3,\mathrm{df}}^{-1} + F_3^{\infty}}\mathcal{L}$

satisfies an integral equations

$$i\mathcal{D} = i\mathcal{M}_2 iG i\mathcal{M}_2 + \int i\mathcal{M}_2 iG\, i\mathcal{D}$$

Three-hadron systems

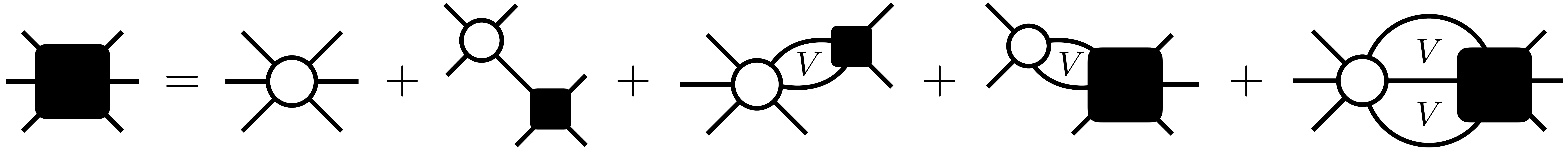
Infinite Minkowski spacetime

$i\mathcal{M}_3 =$ 

placing all legs on-shell

$$\longrightarrow \; i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_{3,\mathrm{df}}^{-1} + F_3^{\infty}}\mathcal{L}$$

Finite Minkowski spacetime

$i\mathcal{M}_{3,L} =$ 

$$\longrightarrow \; \frac{\#}{\mathcal{K}_{3,\mathrm{df}}^{-1} + F_3}$$

spectrum satisfies

$$\det\left[\mathcal{K}_{3,\mathrm{df}} + F_3^{-1}\right] = 0$$

Jackura [2208.10587, 2022]

Solving integral equations

- Introduce $\mathcal{D} = \mathcal{M}_2 d \mathcal{M}_2$
- Assuming $\mathscr{K}_{\text{df}} = 0$ and partial wave-projecting onto $\ell = 0$ sector,

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

- Soften or isolate possible singularities: non-zero epsilon, integrate out poles analytically, …
- Write as a matrix form:

$$\boldsymbol{d}(N, \epsilon) = -\boldsymbol{G}(\epsilon) - \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{d}(N, \epsilon)$$

- Recover the exact results when taking $\epsilon = \eta / N \to 0$.
- Including $\mathscr{K}_{\text{df}} \neq 0$ contributions is "easy"

Jackura, RB, Dawid, Islam, & McCarty (2020)

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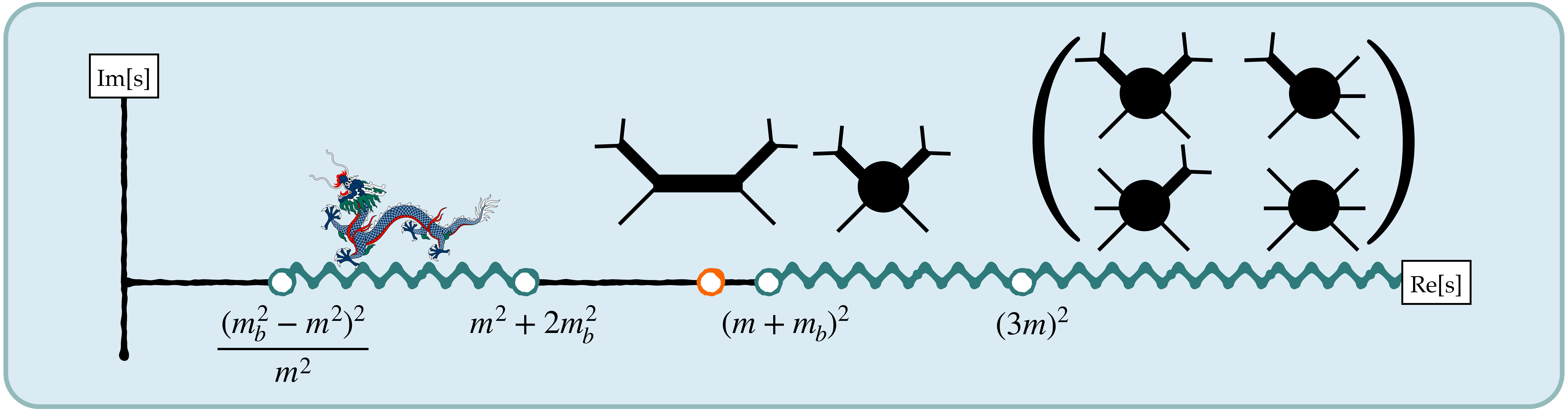
Consistency checks using toy models

Consider a toy theory with a two-body bound state

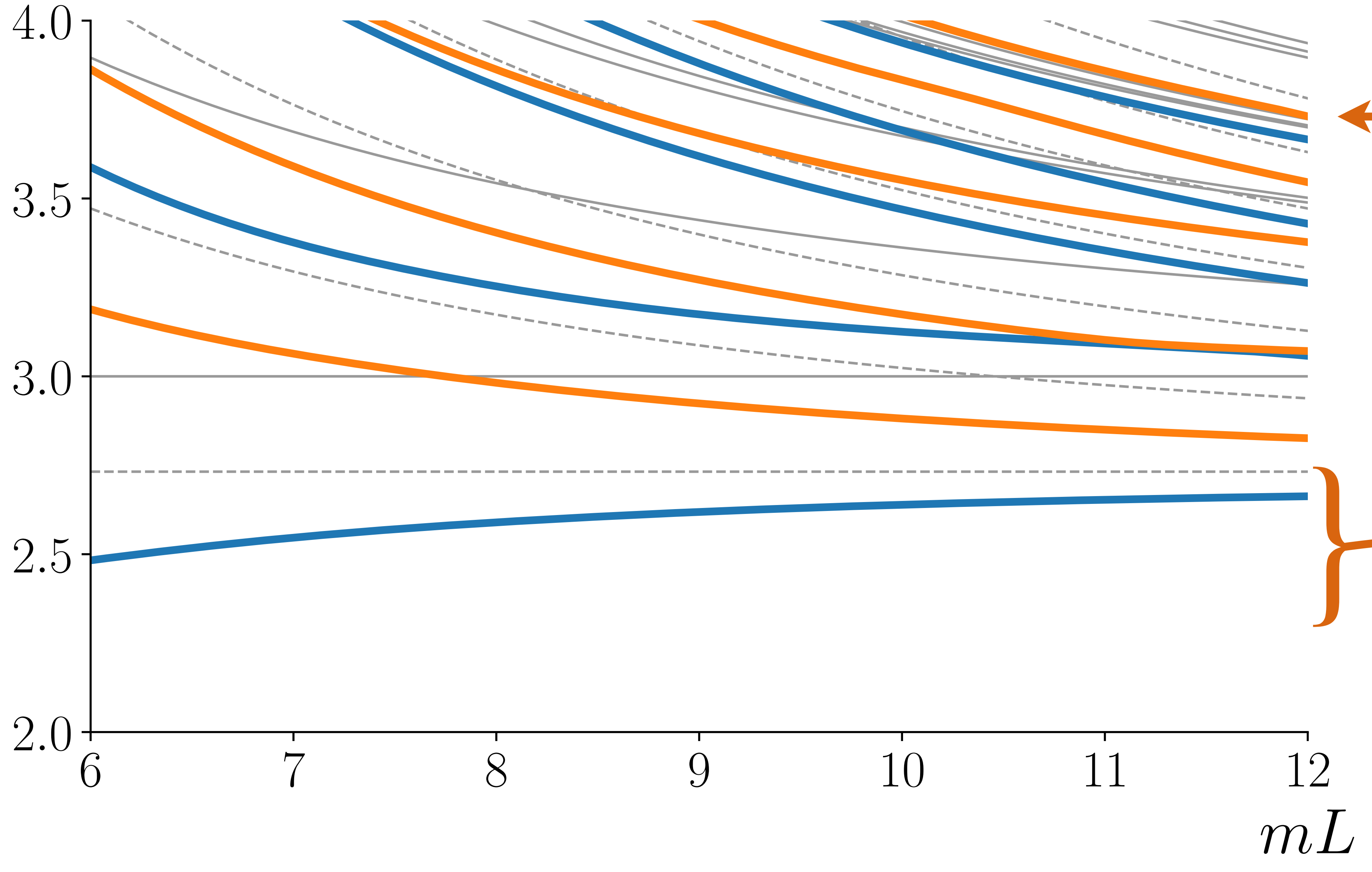
- Simple case $q \cot \delta = -\dfrac{1}{a}$
- if $a > 0$, there is a bound state with binding energy $m_b = 2\sqrt{m^2 - \dfrac{1}{a^2}}$



Consistency checks for toy model

Alternatively to solving the integral equations, one can instead:

- Fix $ma = 2$ and $\mathscr{K}_{\mathrm{df}} = 0$
- obtain E_L using 3-body quantization by Hansen & Sharpe (2014),
- obtain $q \cot \delta_{\varphi b}$ using the two-body Lüscher formalism.



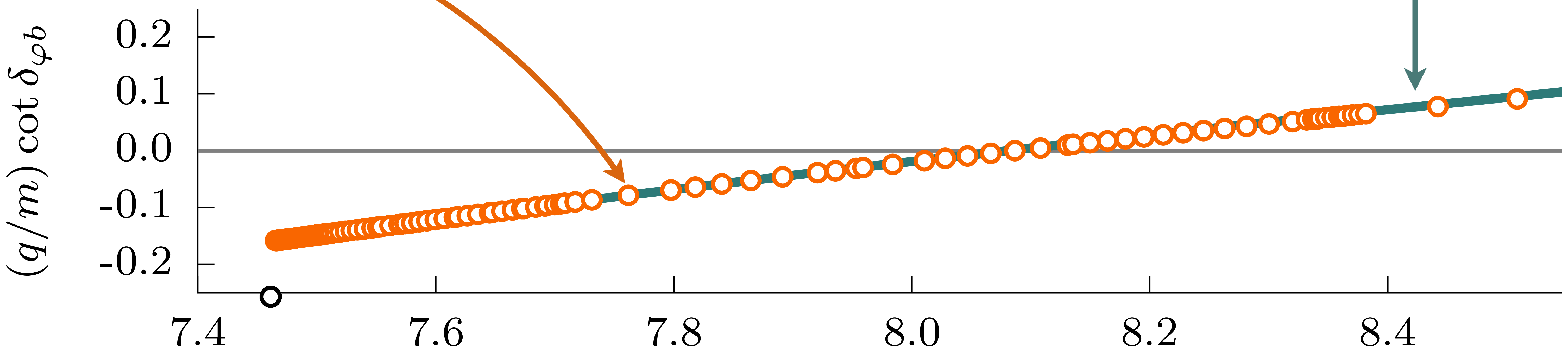
finite-volume physics

$$\det\left[F_3(E_L, L) + \mathcal{K}_{\mathrm{df},3}^{-1}(E_L)\right] = 0$$

integral equations

$$\mathcal{M}_{\varphi b} \propto d \propto \frac{1}{\cot \delta_{\varphi b} - i}$$

$$\det\left[F(E_L, L) + \mathcal{M}^{-1}(E_L)\right] = 0$$



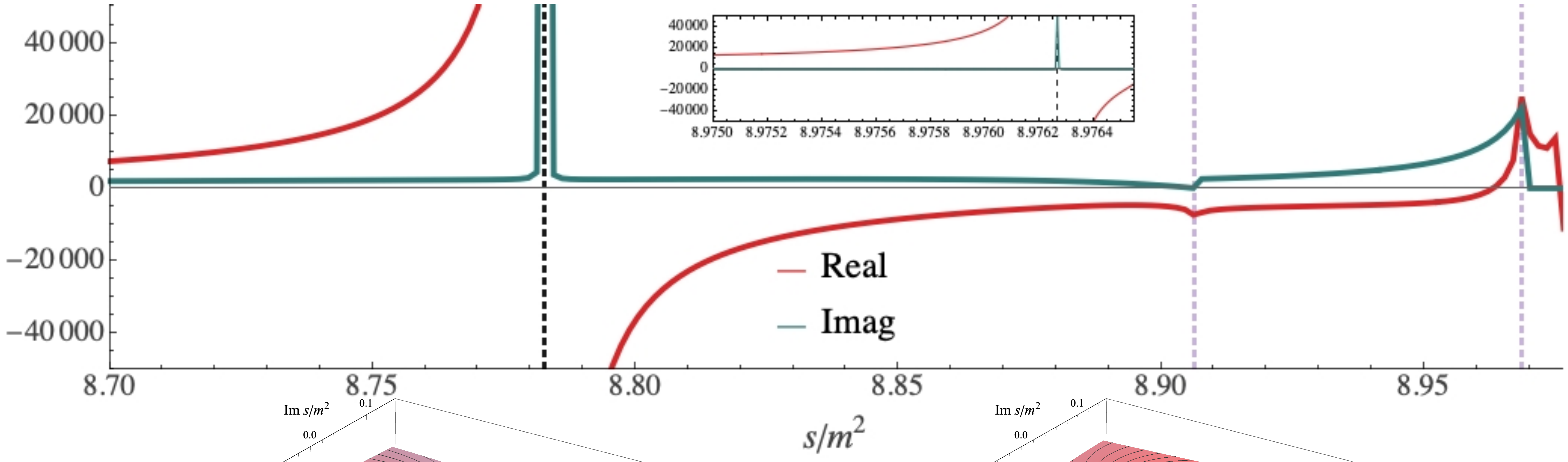
Jackura, RB, Dawid, Islam, & McCarty (2020)

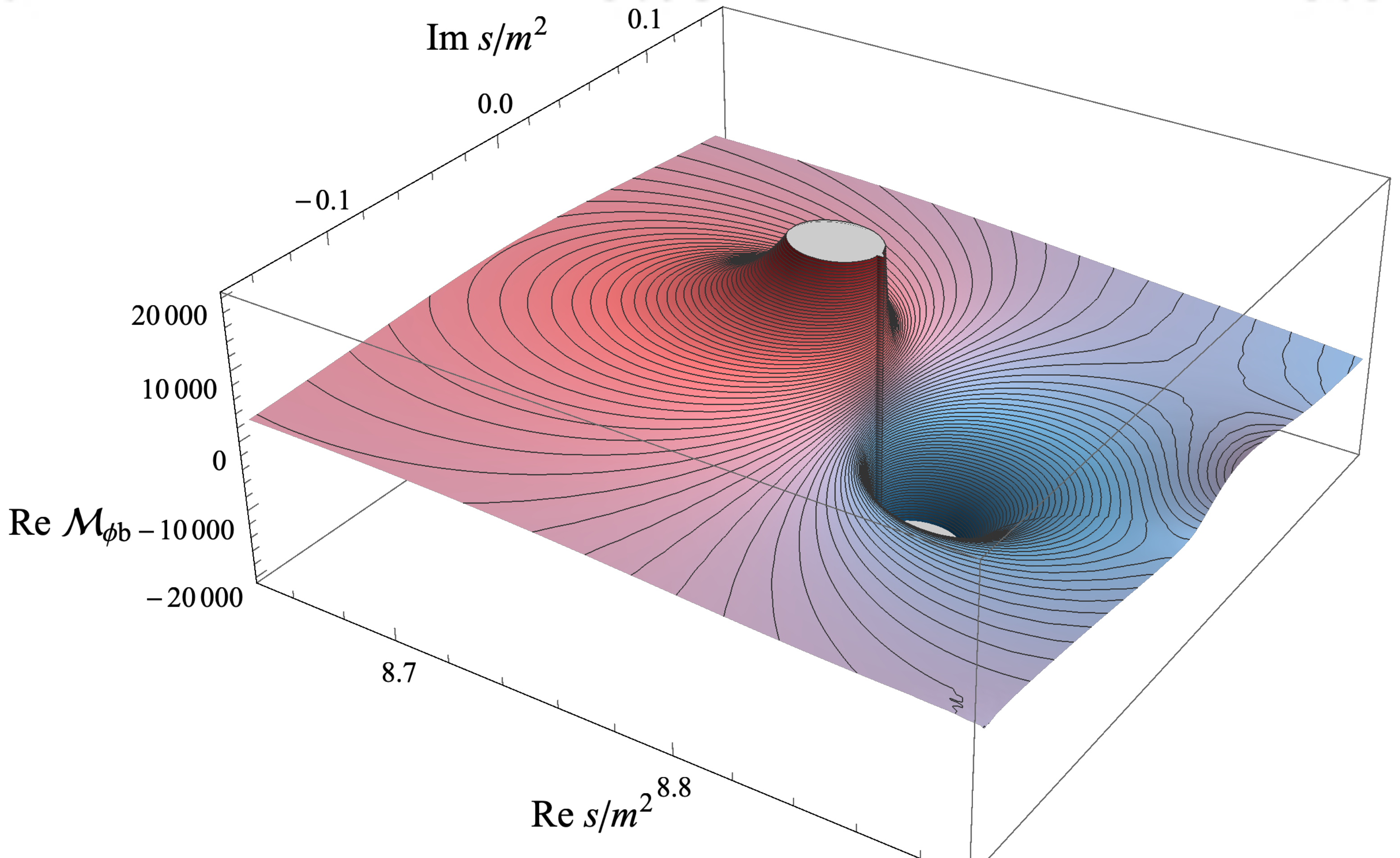
Romero-López, Sharpe, Blanton, RB, & Hansen (2019)

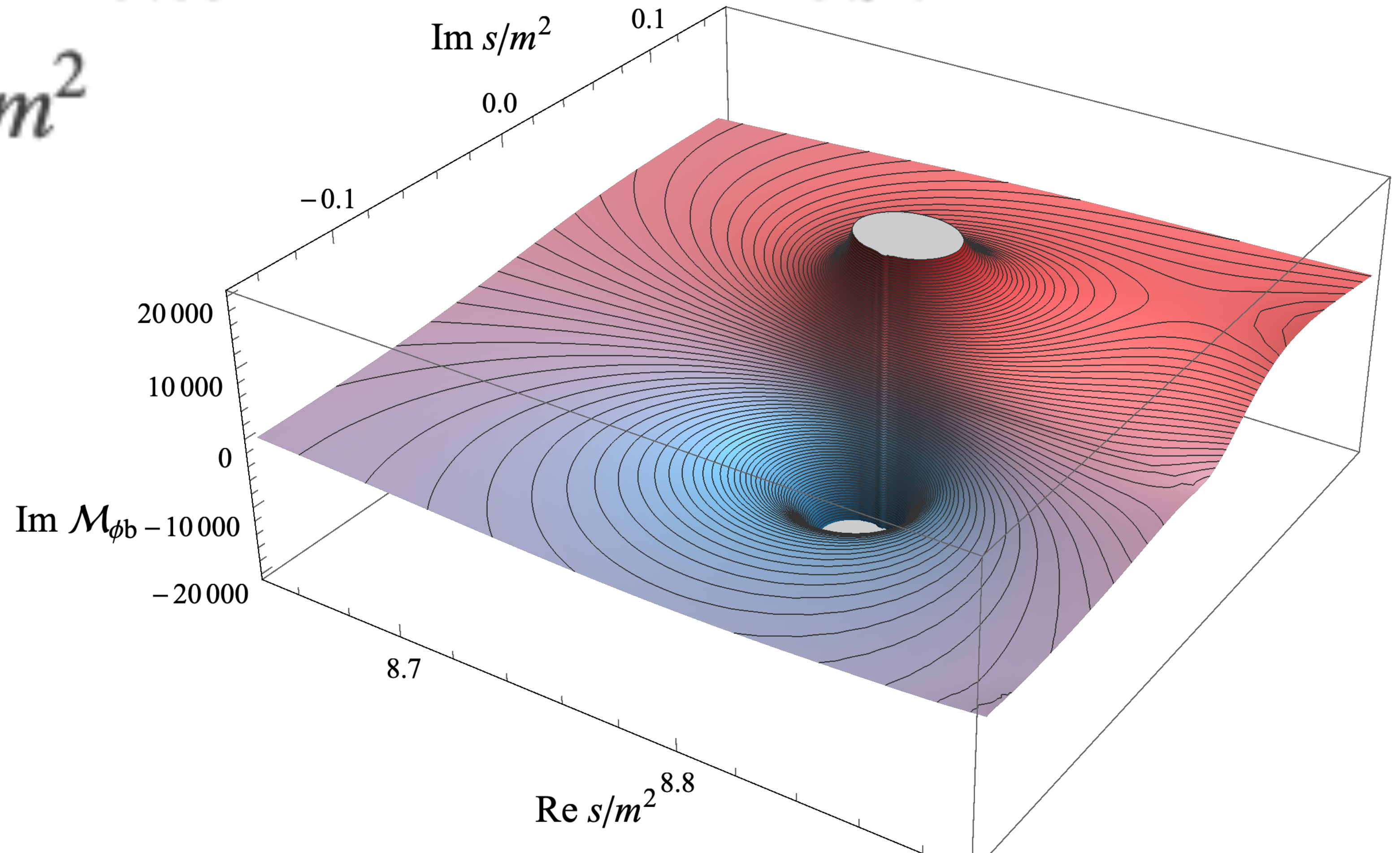
Trimers

Scan for trimer poles as a function of energy

- Fix $ma = 16$ and $\mathcal{K}_{\text{df}} = 0$





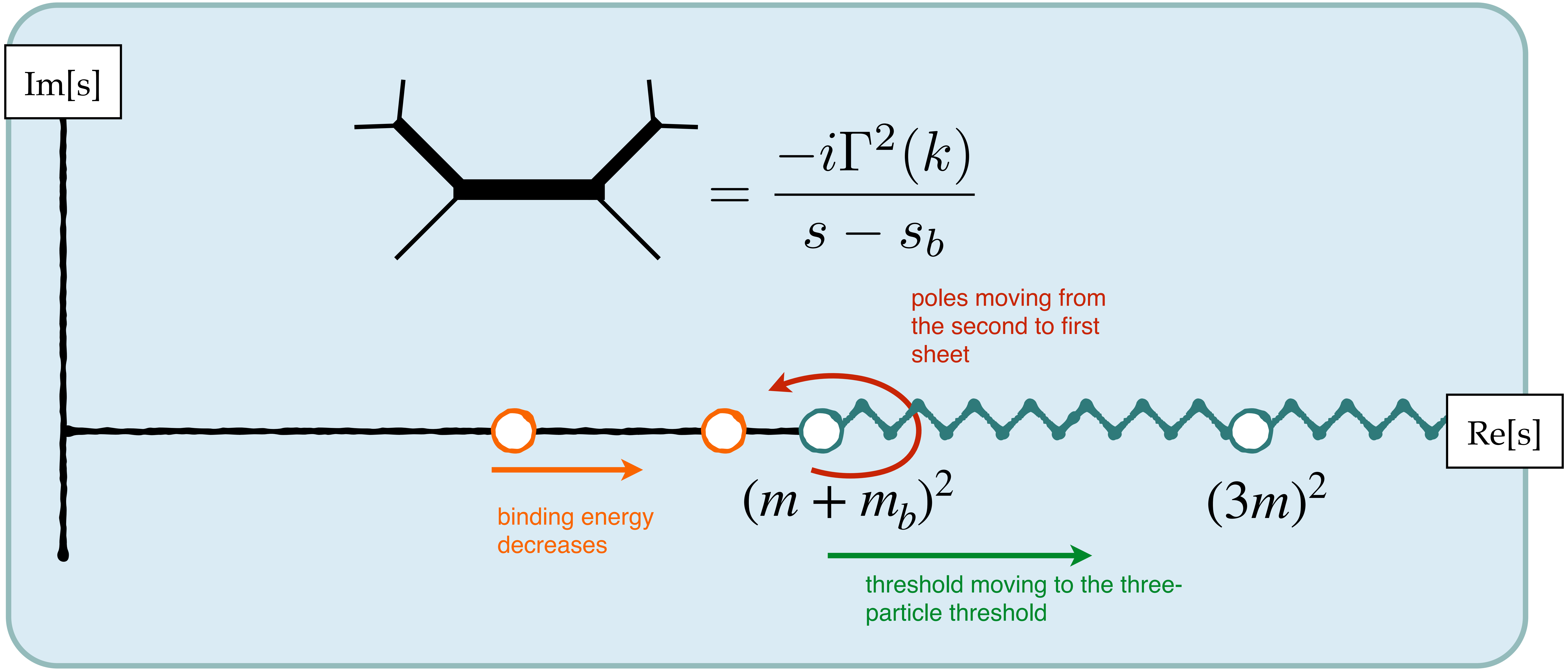


Recovering Efimov Physics

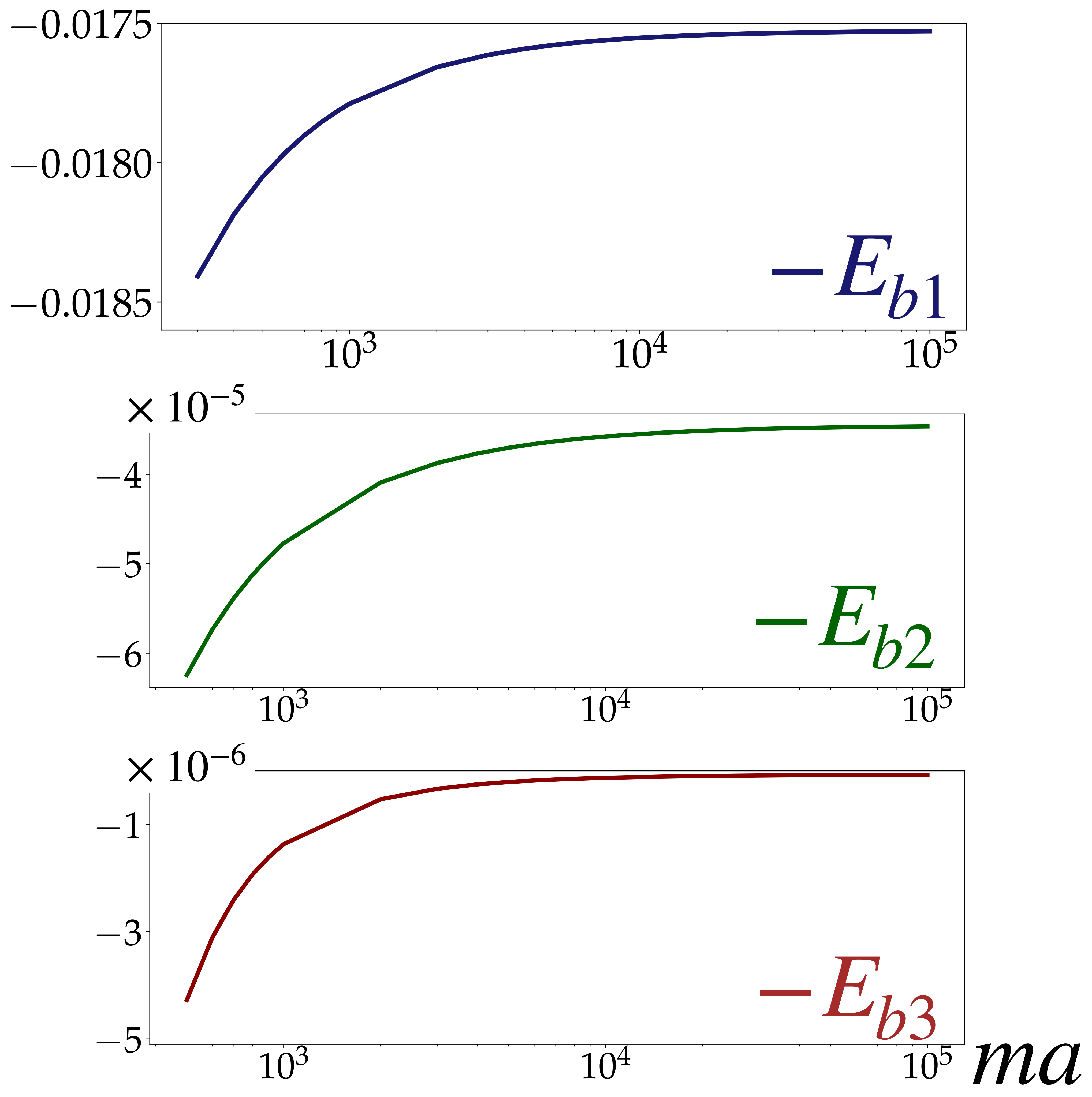
Remember $m_b = 2\sqrt{m^2 - \frac{1}{a^2}}$,

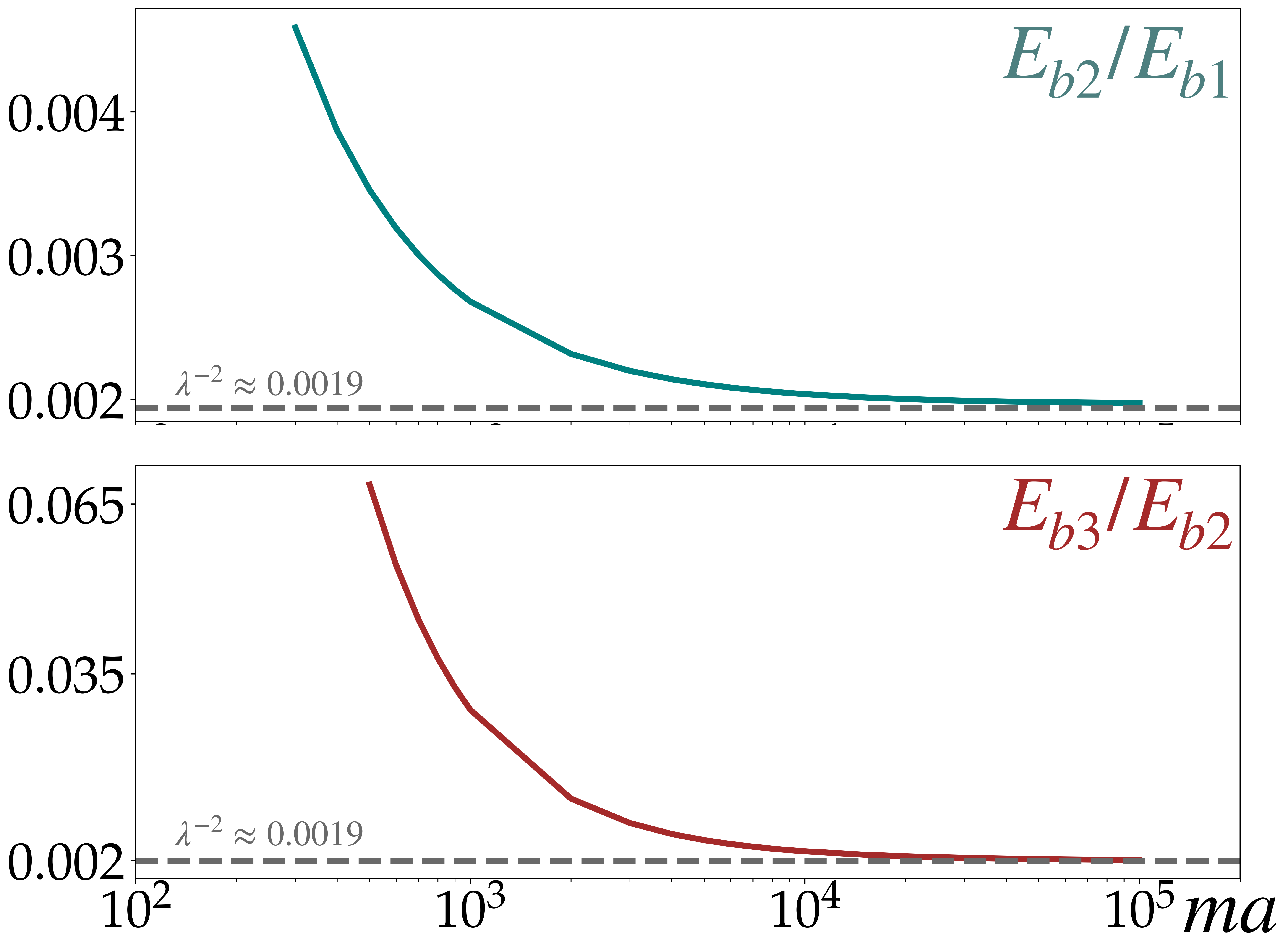
If $a \to \infty$, then $E_{N+1} = E_N/\lambda^2$ where $\lambda = 22.69438\ldots$

But as $a \to \infty$ life gets more complicated!

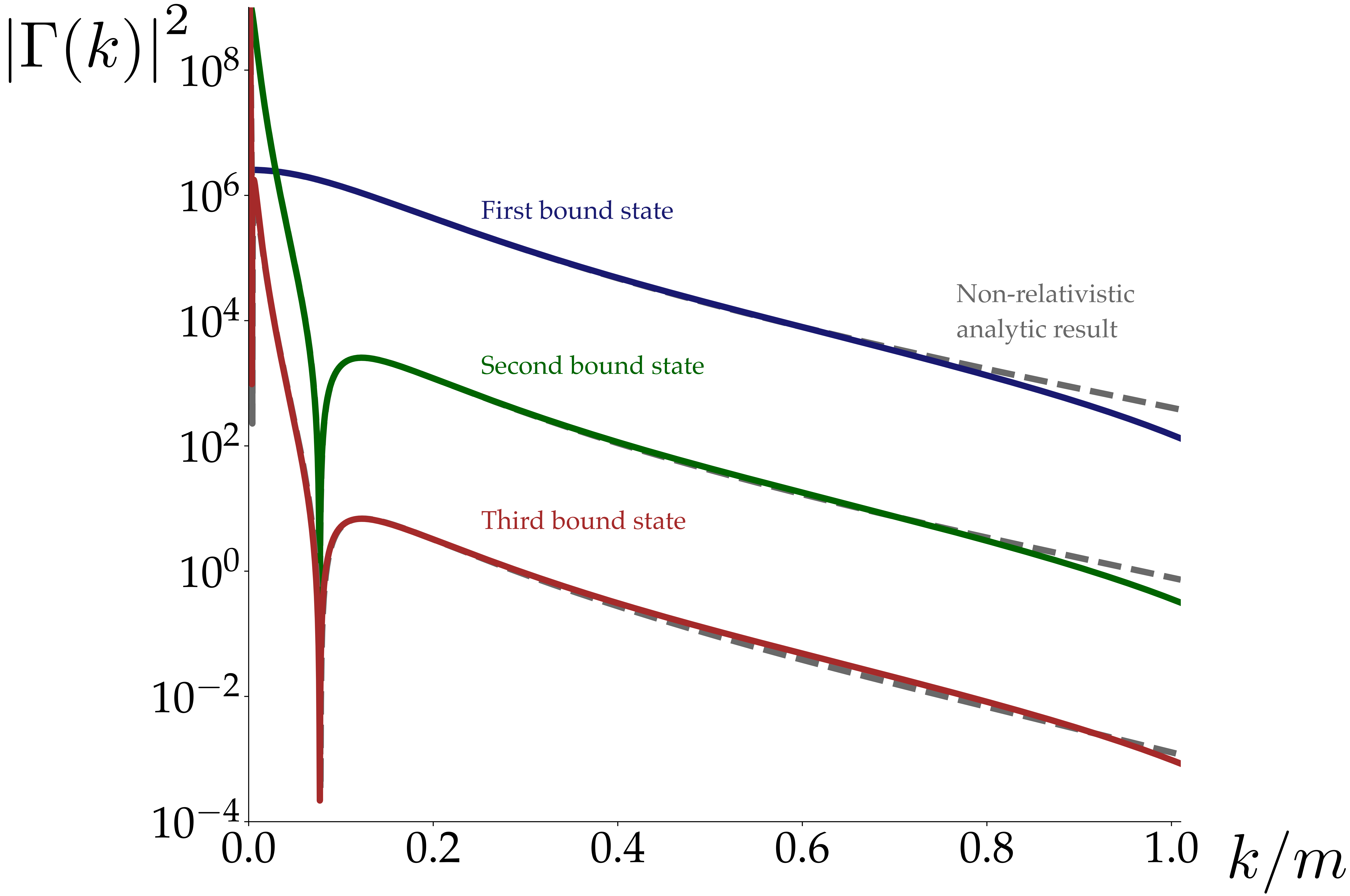


Recovering Efimov Physics *Binding energies*

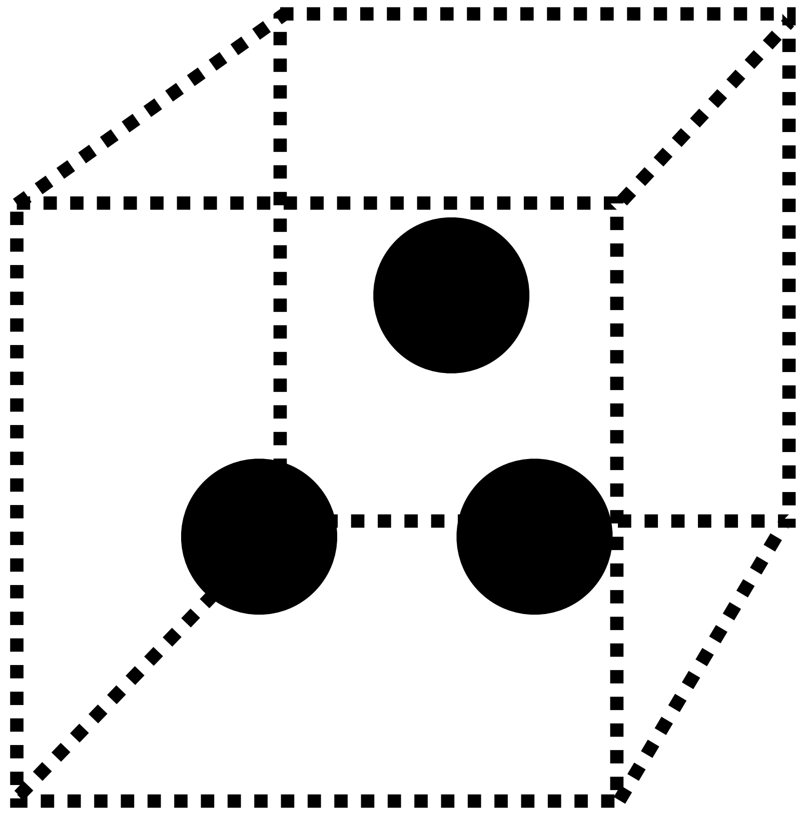




Recovering Efimov Physics *residues*



Back to lattice QCD



$$\det\left[F_3(E_L, L) + \mathcal{K}^{-1}_{\mathrm{df},3}(E_L)\right] = 0$$

Hansen & Sharpe ('14, '15)

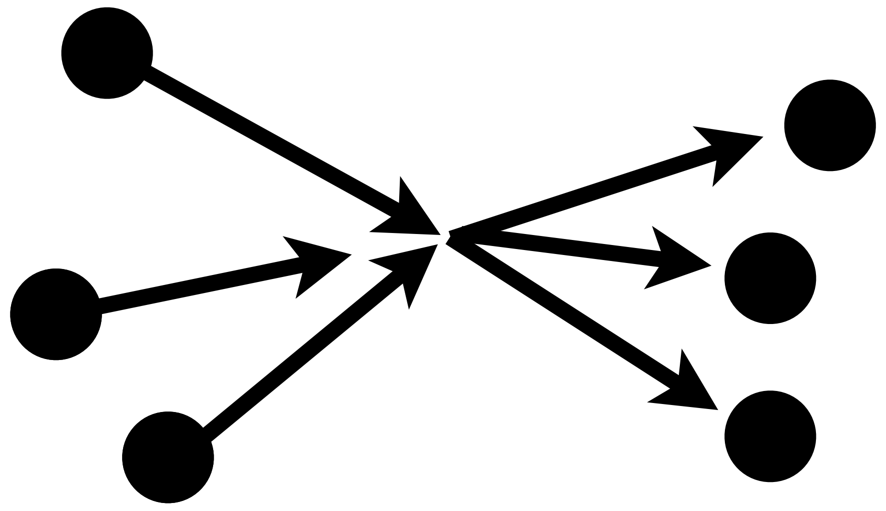
Mai & Döring ('17)

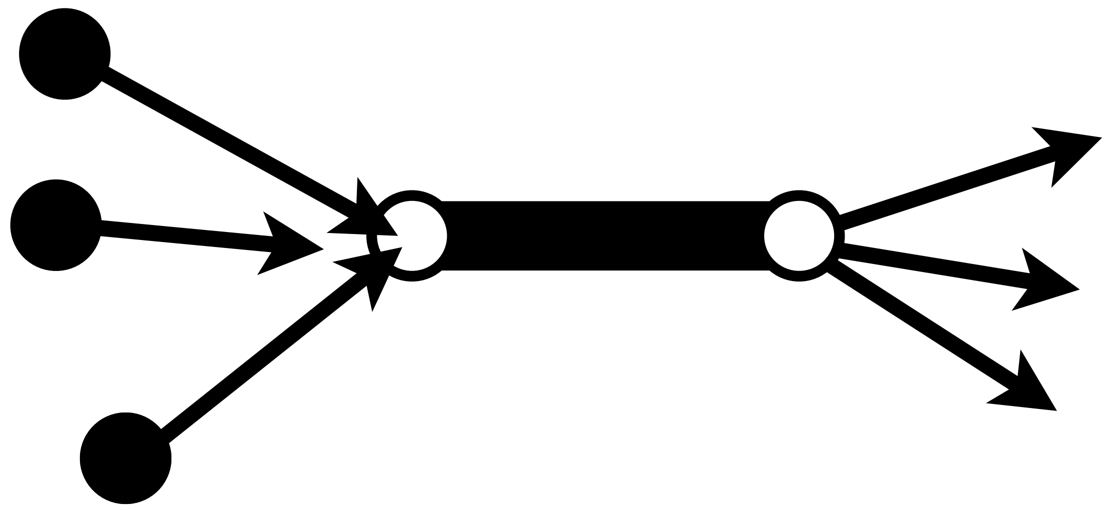
RB, Hansen & Sharpe ('18)

Hansen, Romero-Lopez & Sharpe ('20)

Blanton & Sharpe ('20)

Jackura ('22)





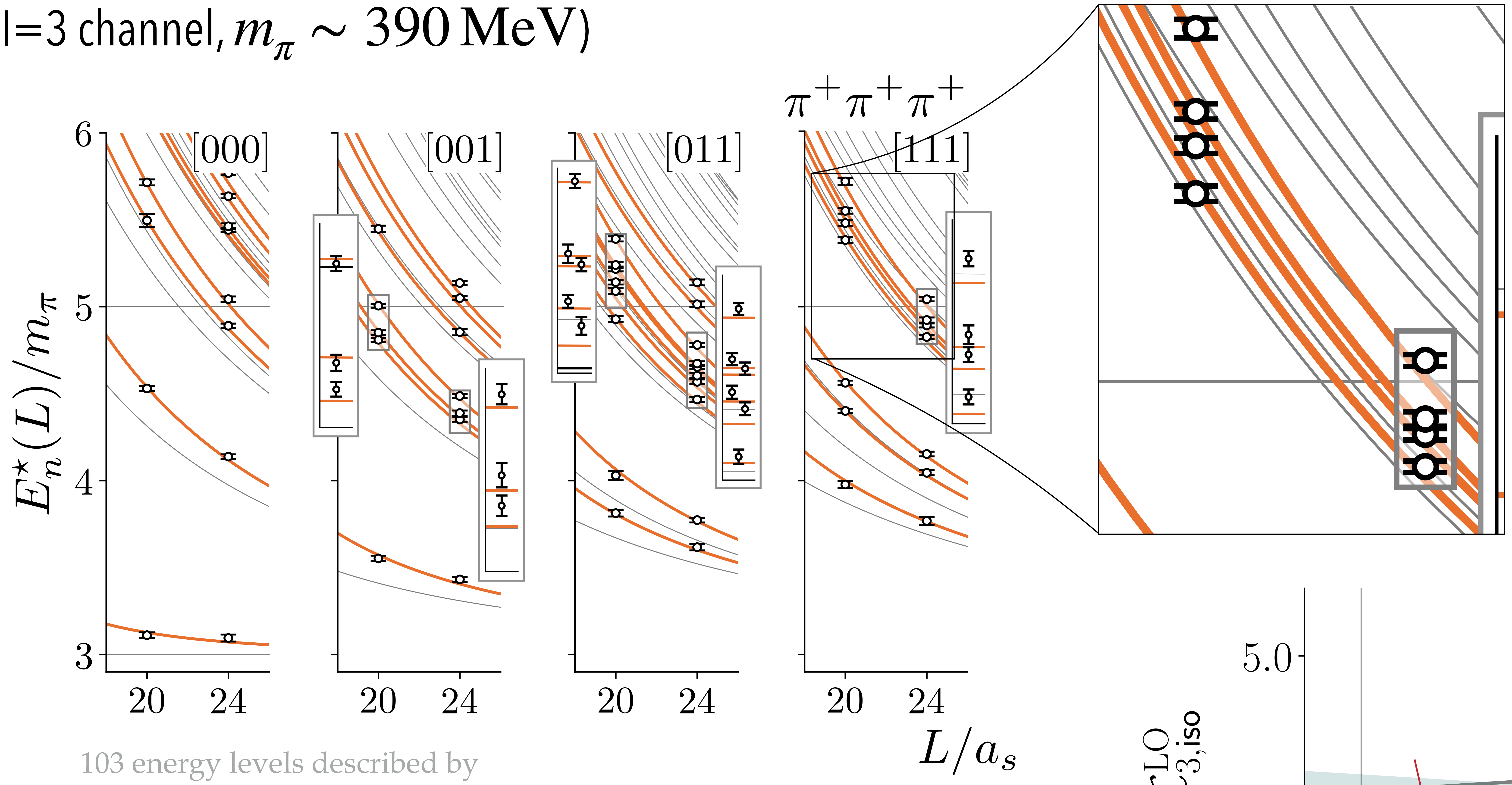
finite-volume
spectroscopy

infinite-volume
scattering amplitudes

bound state and
resonance poles

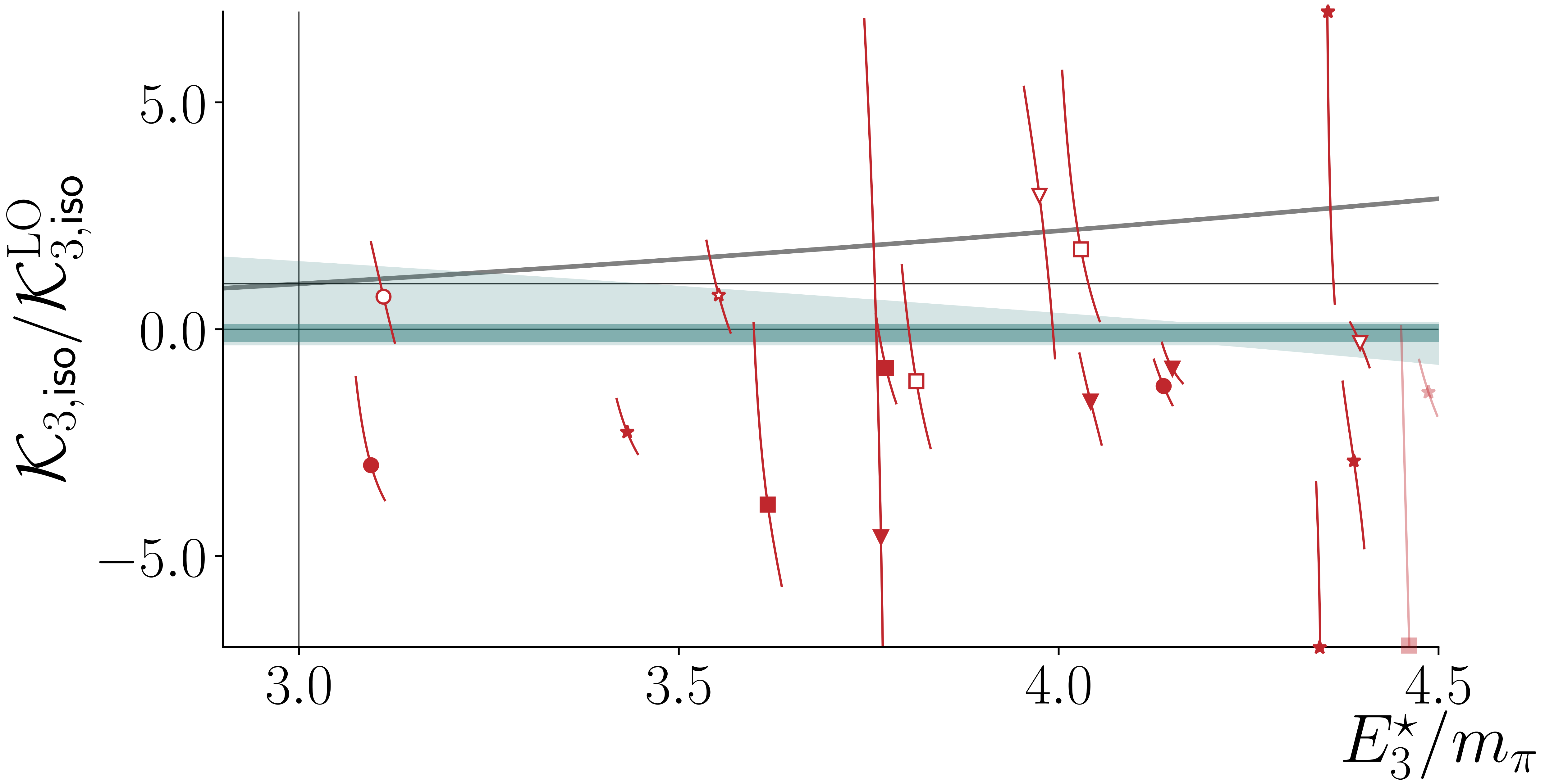
$\pi\pi\pi$

(I=3 channel, $m_\pi \sim 390\,\text{MeV}$)



$$F_{3,\text{iso}}^{-1}(P,L) + \mathcal{K}_{3,\text{iso}}(P^2) = 0$$

103 energy levels described by three numbers: m_π, $a_{\pi\pi}$, $\mathcal{K}_{3,\text{iso}}$



$m_\pi \sim 390\,\text{MeV}$

Hansen, RB, Edwards, Thomas, & Wilson (2020)

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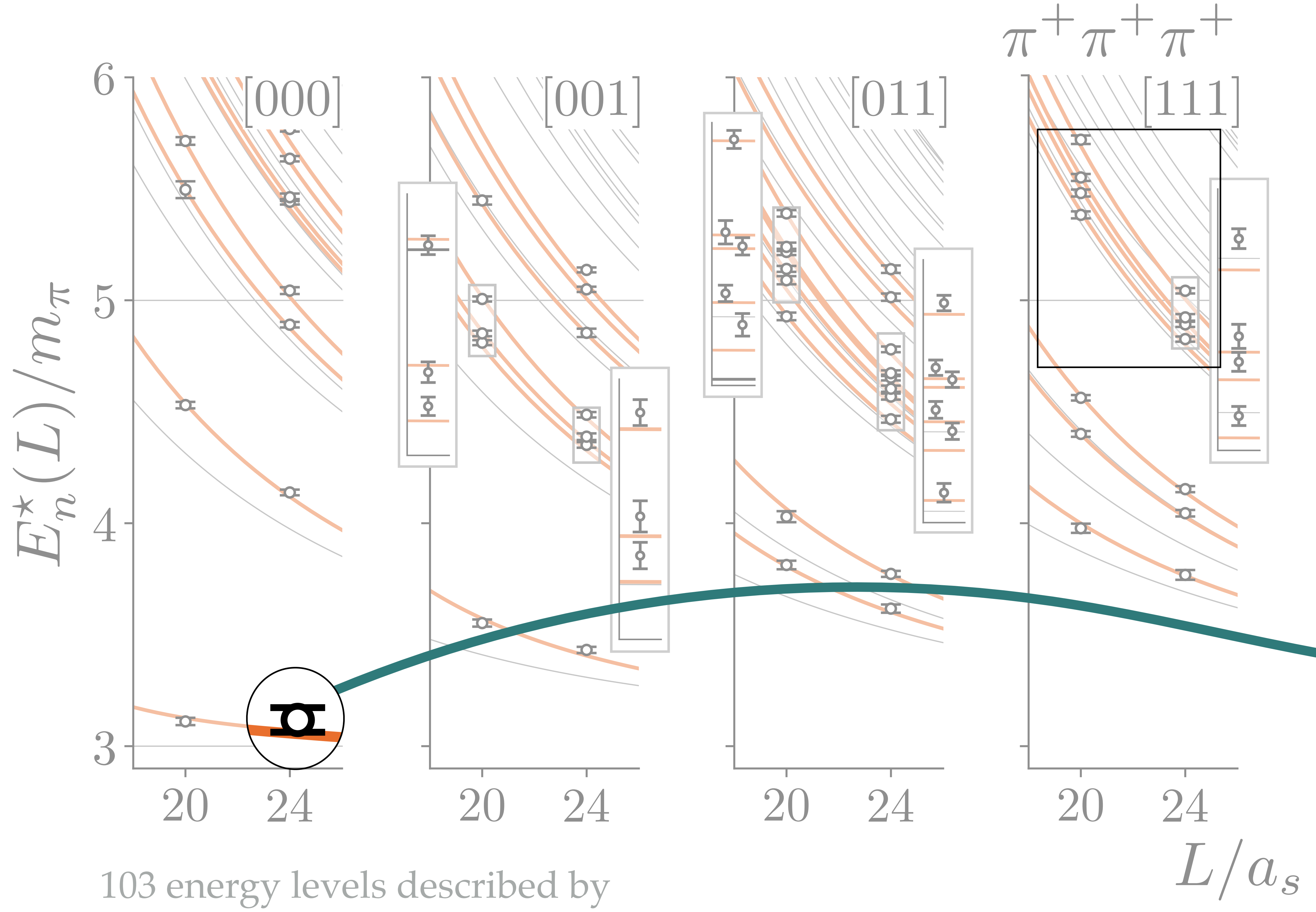

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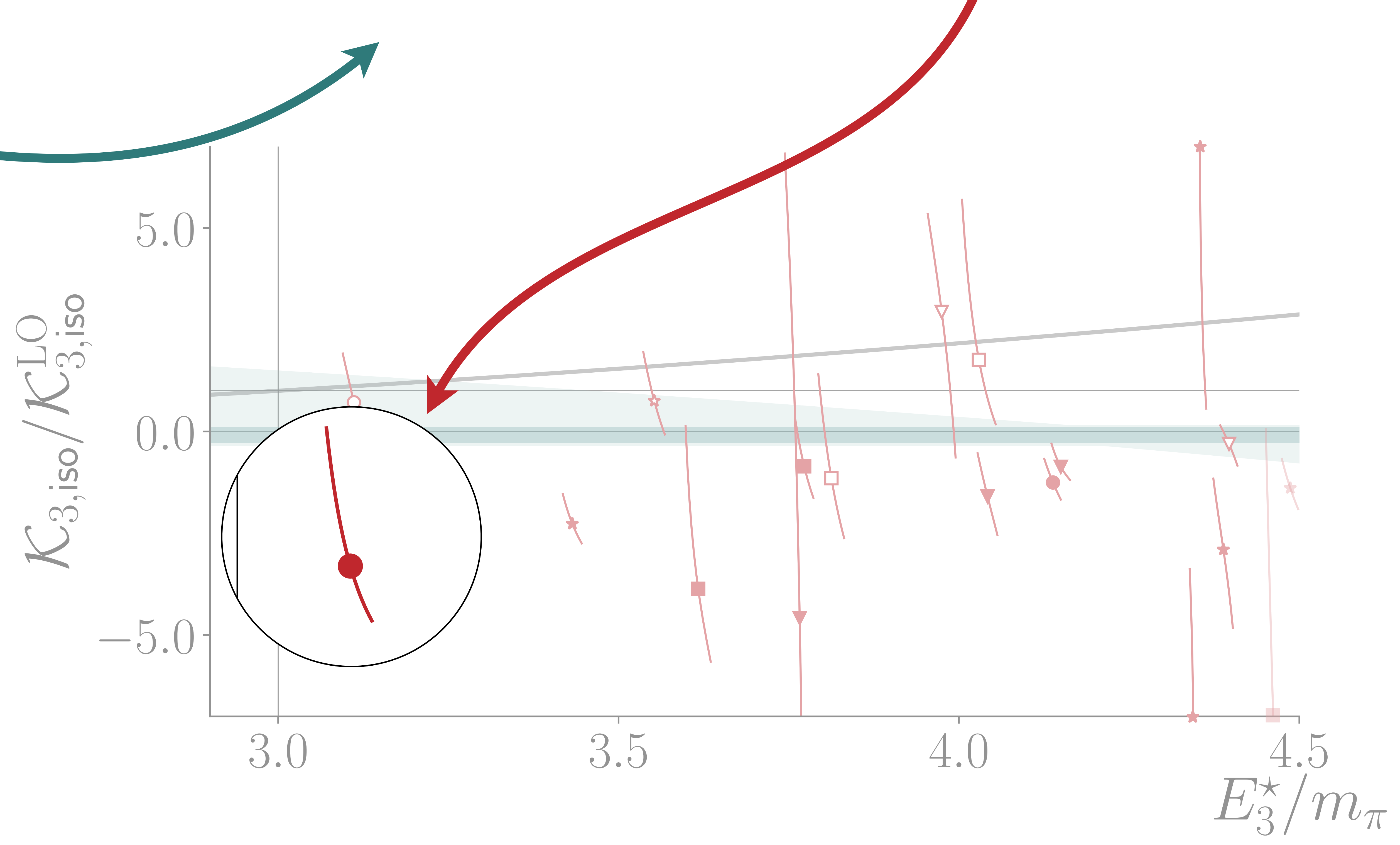

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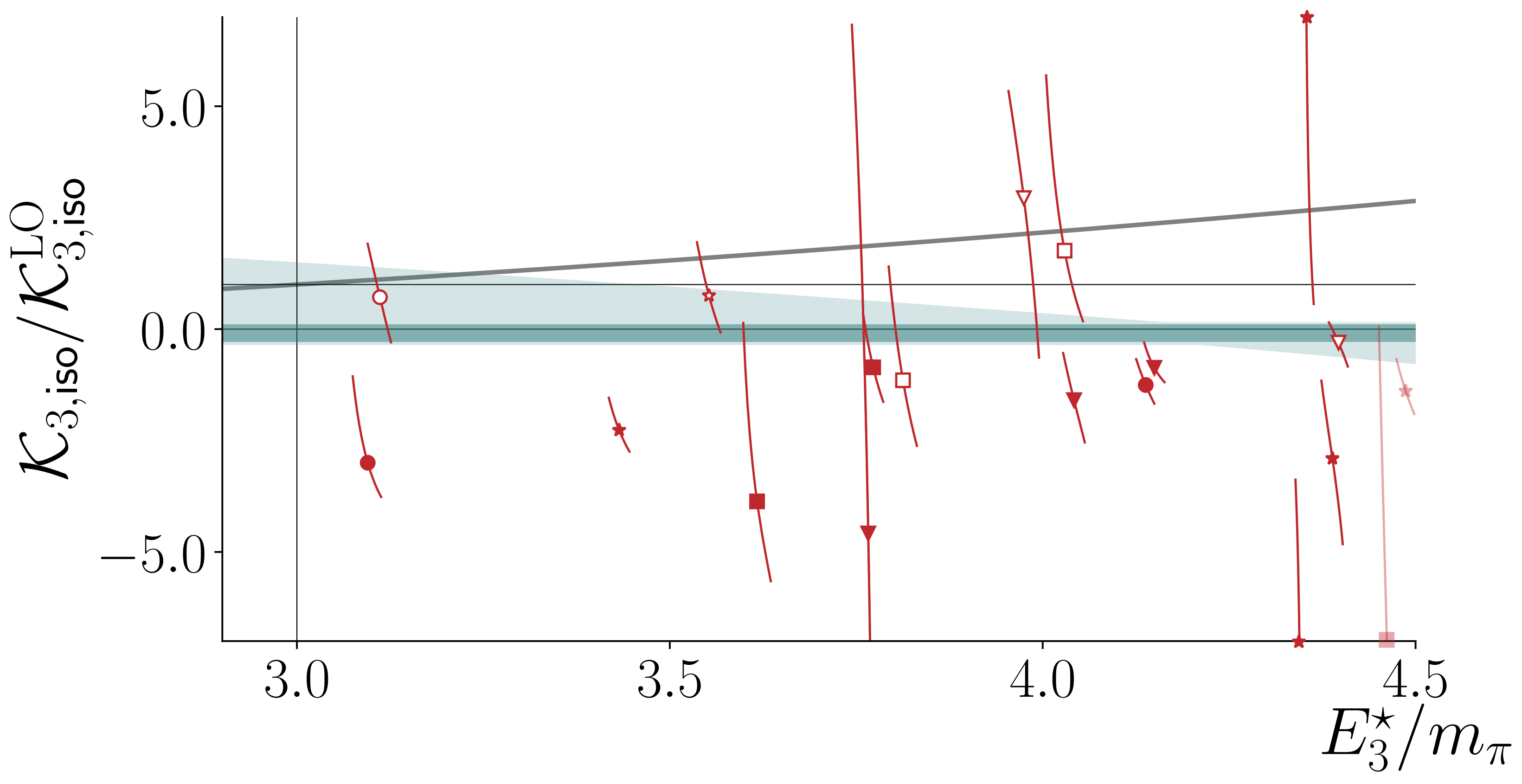
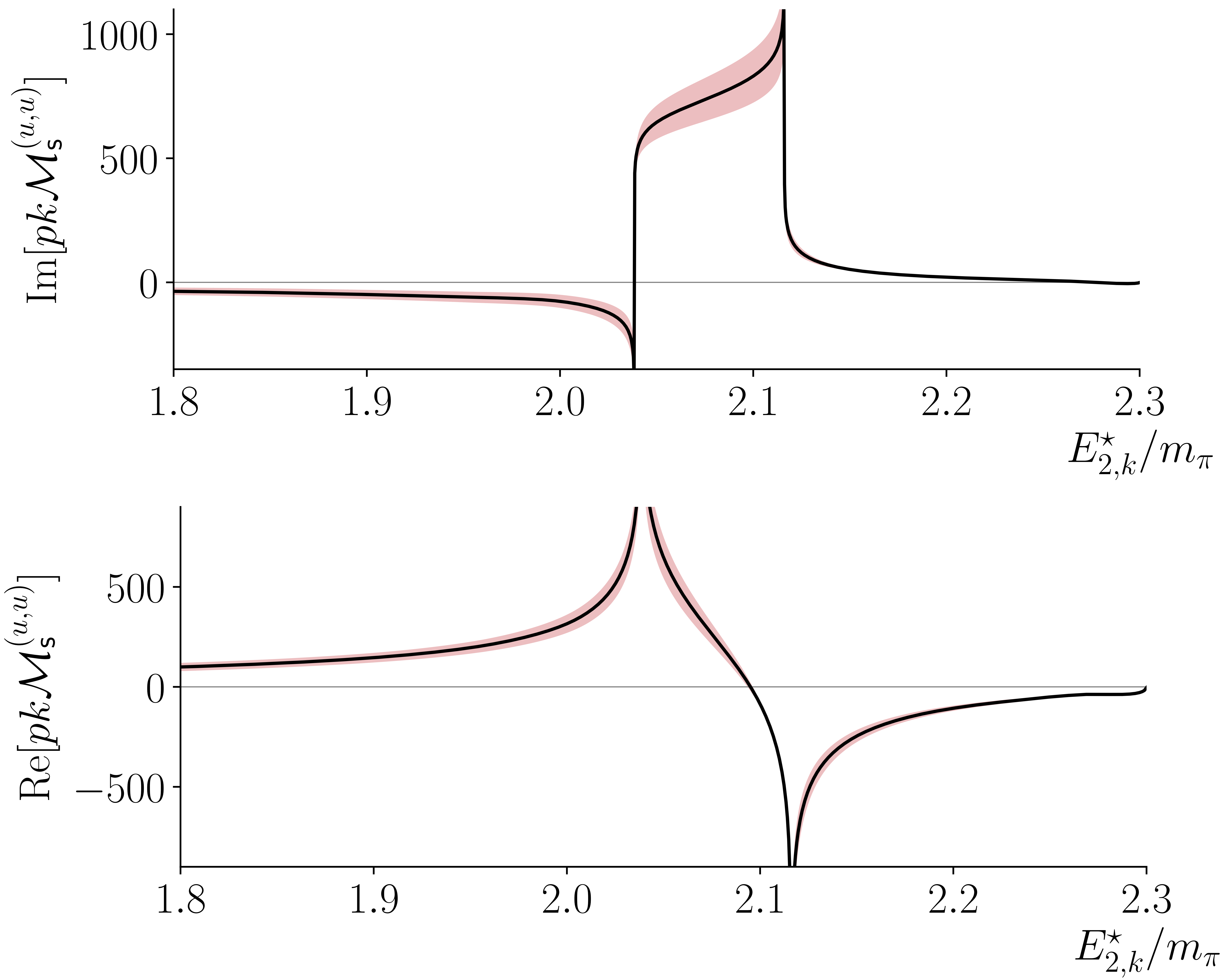
Hansen, RB, Edwards, Thomas, & Wilson (2020)

$\pi\pi\pi$ scattering
(I=3 channel, $m_\pi \sim 390\,\text{MeV}$)

first 3body scattering amplitude from the lattice QCD!



$$i\mathcal{D} + i\mathcal{L}\frac{1}{\mathcal{K}_{3,\text{df}}^{-1} + F_3^\infty}\mathcal{L}$$



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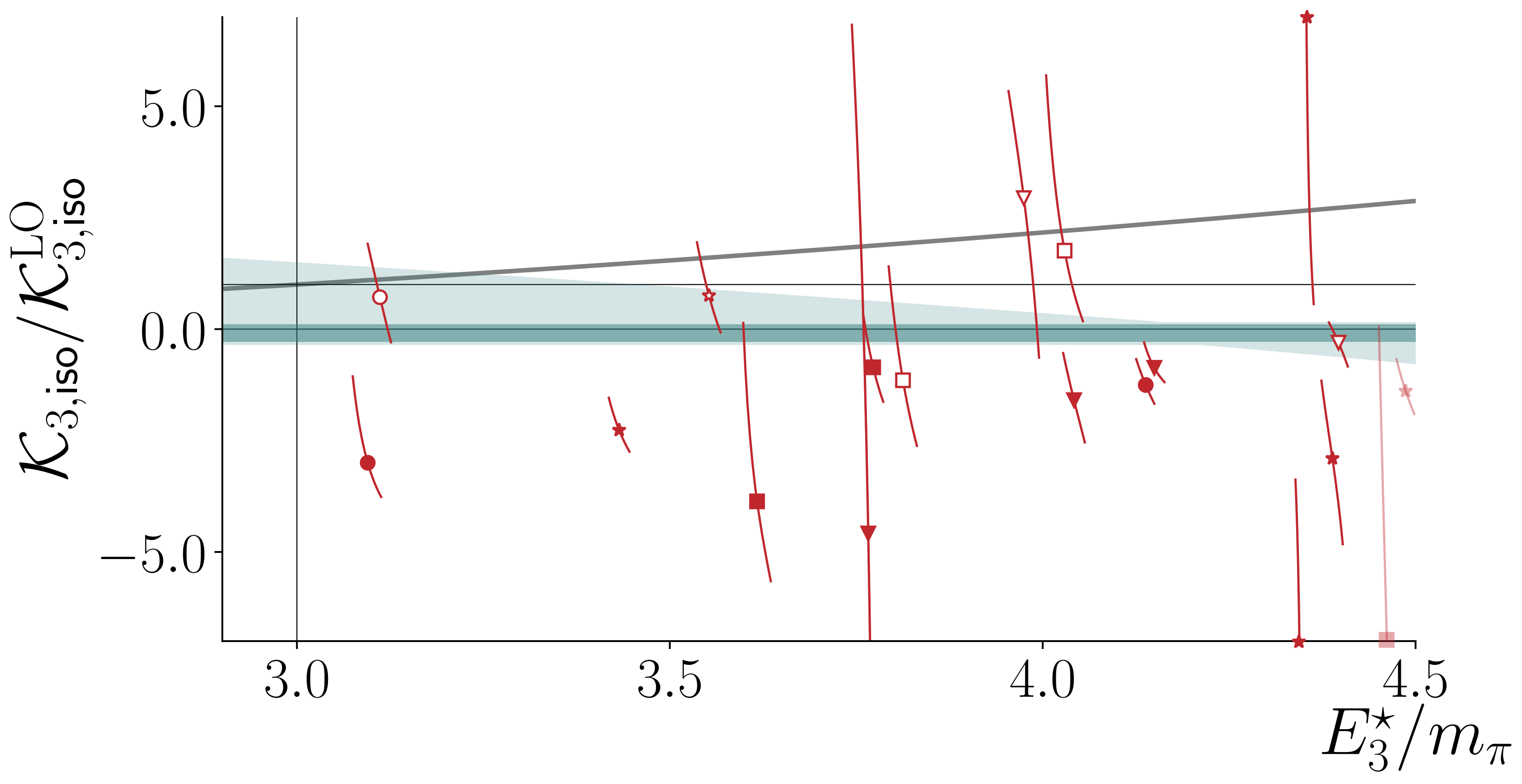

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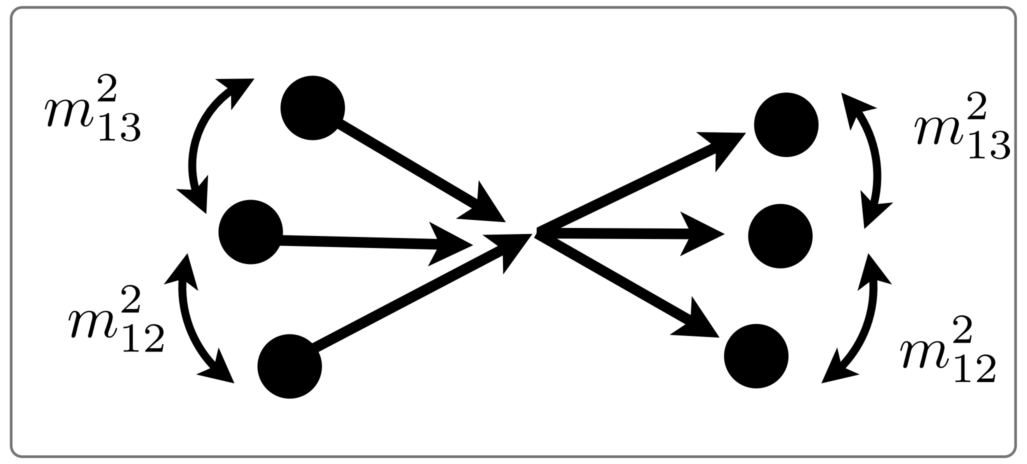

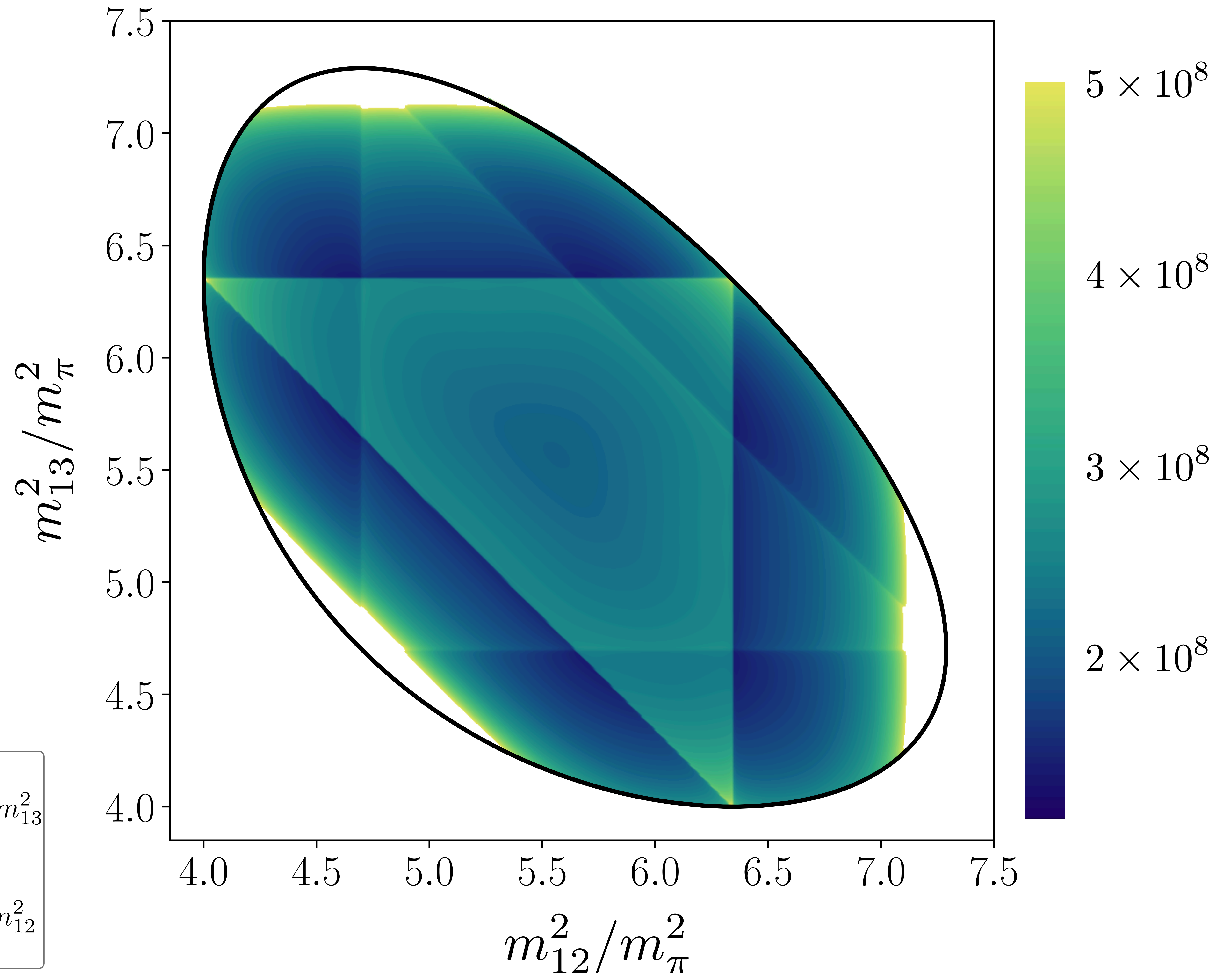

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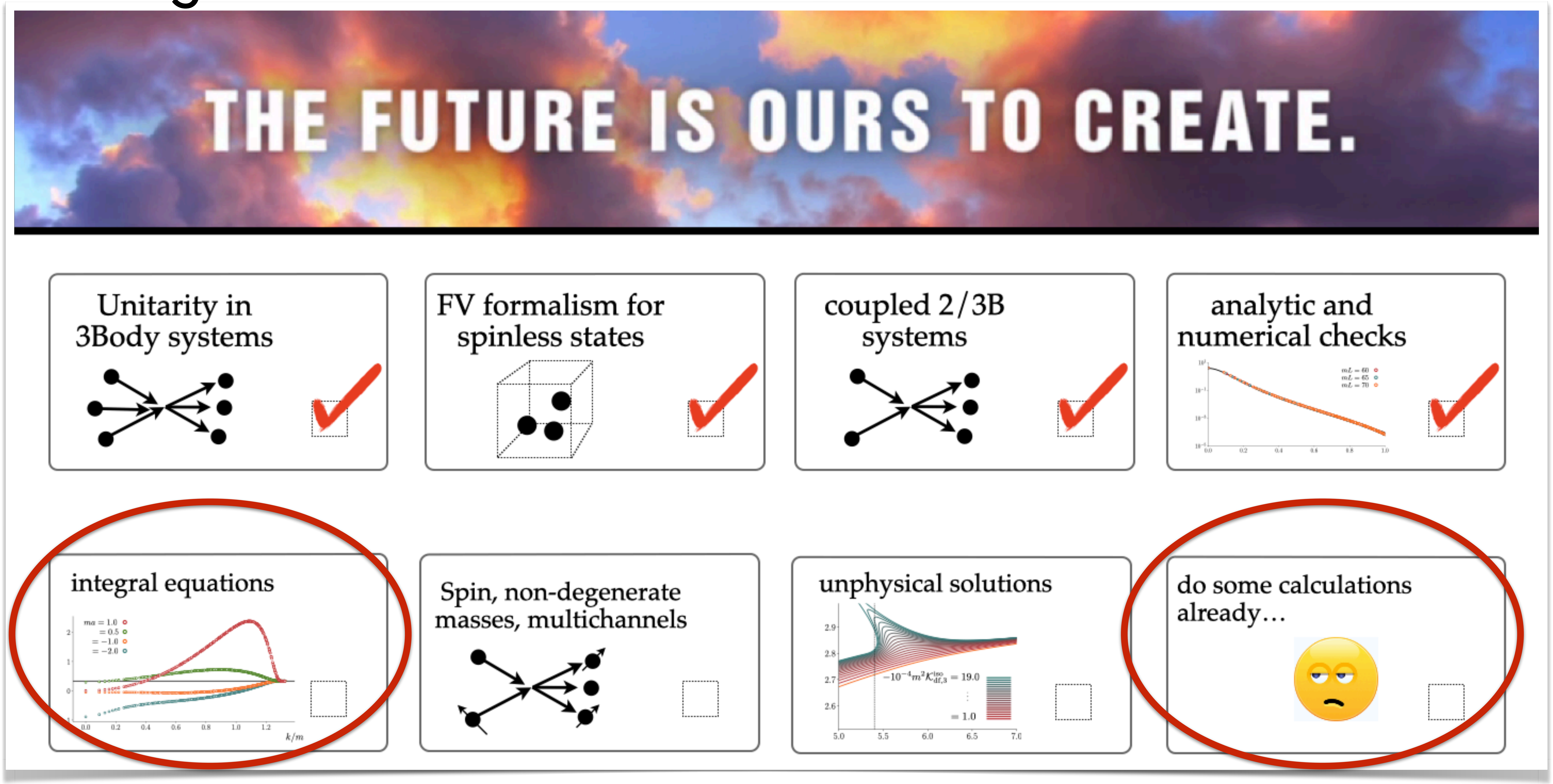
Looking back at 2019!

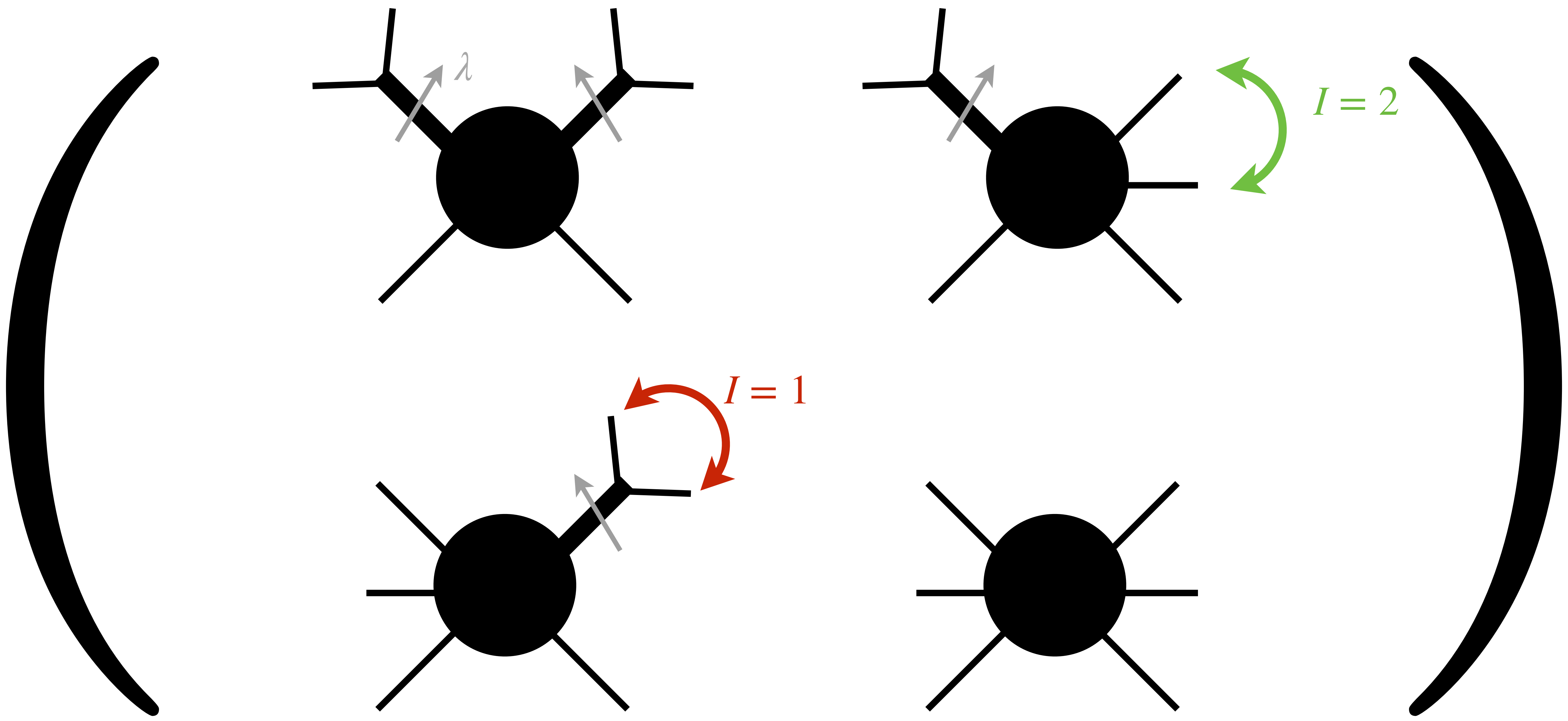


Looking into the future

- Lots of formal developments.
- Lattice QCD calculations are catching up.

Next obvious place to explore is the $I = 2$ channel, which is not so trivial.

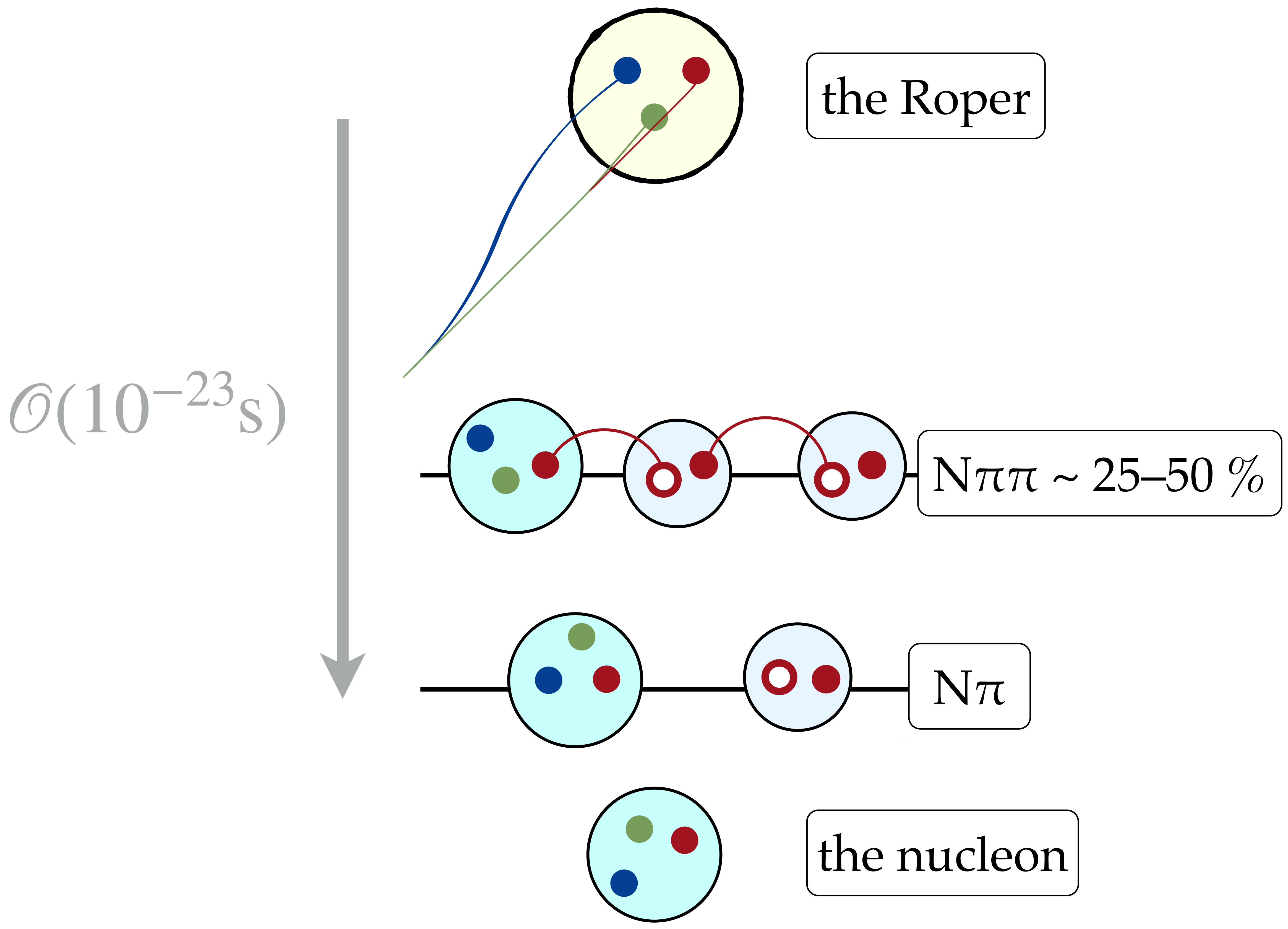
- Coupled channels $\left[\,\pi(\pi\pi)_{I=2},\;\pi(\pi\pi)_{I=1}\,\right]$,
- non-zero angular momentum,
- two-body resonances with non-zero helicity.



Looking into the future

Outstanding formal questions:

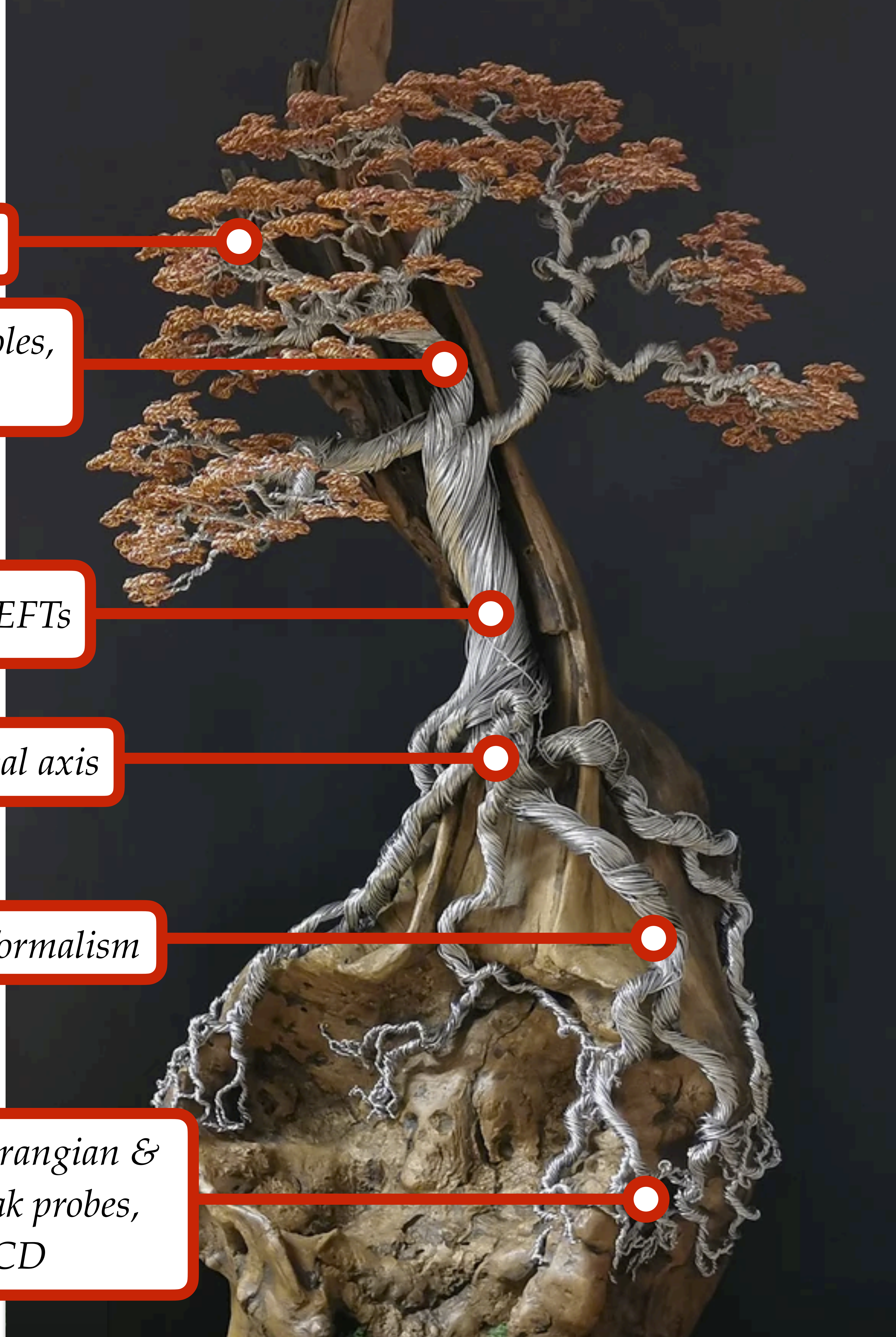
- [!] non-zero orbital angular momentum,
- [!] analytic continuation into the complex plane,
- [!] coupled 2-3 bodies,
- [!] non-identical particles,
- [!] electroweak production,
- [ ] electroweak probes,
- [!] long-range processes,
- [ ] non-zero intrinsic spin.



Modern-day few-body physics

Challenging but not impossible

- ☑ Enhancement of operator basis,
  - ☐ Tetraquarks, glueballs, pentraquarks, three-particles, ….
- ☑ Contraction costs,
  - ☐ Cost grows with the number of externals legs…
- ☑ Extensions of finite- and infinite-volume formalism,
  - ☐ Multi-channel 3-body, spin, etc.
  - ☐ Electroweak processes, …
- ☑ Scattering theory,
  - ☐ "Earnest amplitudes", analytic continuations,…



Lots of great memories!

"friendliest group!"