The dance of the muon

Aida X. El-Khadra
University of Illinois

Jefferson Lab
Thomas Jefferson National Accelerator Facility

Theory Seminar
26 June 2023
Outline

- Introduction
- Fermilab and JPARC muon g-2 experiments
- SM contribution to g-2
- Muon g-2 Theory Initiative
  - dispersive/data driven methods
    - Hadronic Vacuum Polarization (HVP)
    - Hadronic Light-by-Light (HLLbL)
  - lattice QCD
    - lattice HVP
    - lattice HLLbL
- Connections
- Summary and Outlook


"Prospects for precise predictions of $\alpha_\mu$ in the SM": 2022 Snowmass Summer Study, arXiv:2203.15810
The Higgs boson (discovered in 2012) completes the very successful SM

- The muon ($\mu$) was the first unexpected discovery, prompting the question “Who ordered that?”, now phrased as: “Why three generations?”
- Many other questions:
  - Dark matter
  - Dark energy
  - How (a lot) more matter than antimatter?
  - Why $m_{\text{Higgs}} \simeq 125$ GeV?
  - …
- Answers to these questions will yield deeper insights and generically give rise to new particles and/or new forces.

If experimental measurements of the muon’s anomalous magnetic moment disagree with Standard Model theory, this could be evidence for new particles and/or forces.

➤ discovery potential of precision measurements
Introduction: magnetic moment

- All leptons (electron, muon, tau-lepton, neutrinos) carry spin (intrinsic angular momentum), with spin = 1/2.

Electric charge ($e$, $\mu$, $\tau$) + spin $\Rightarrow$ magnetic moment

$$\vec{\mu} = g \frac{e}{2m} \vec{S}$$

- A muon is a (tiny) magnetic dipole.

- In a magnetic field it can precess similar to a spinning top.
- This precession can be measured very precisely.
Anomalous magnetic moment

The magnetic moment of charged leptons \((e, \mu, \tau)\):

\[
\bar{\mu} = g \frac{e}{2m} S
\]

Dirac: 
\[g = 2\]

Quantum corrections change how a muon interacts with a magnetic field.

\[\alpha \equiv \frac{g - 2}{2} = 0.00116 \ldots\]

Julian Schwinger: [1948]

\[g = 2 \left( 1 + \frac{\alpha}{2\pi} \right) = 2(1 + 0.00116 \ldots) \quad \alpha \approx \frac{1}{137}\]

All known particles contribute ...

crucial test of QED and perturbation theory
Muon g-2: history of experiment vs theory

🌟 1959: start of first muon g-2 experiments at CERN

https://cds.cern.ch/record/41876

- Experimental measurements with increasing precision, down to ~10ppm
- Required more precise theoretical calculations
- Good agreement on final results

🌟 1984: start of Brookhaven experiment

By 2000: very precise measurements which disagree with theory, but not yet significant enough...

By 2000: very precise measurements which disagree with theory, but not yet significant enough...

[SciPost Physics Proceedings]

Summary and Conclusions

The measurement of the muon magnetic moment spans six decades, and the story is not over. With the recent significant improvements in the Standard Model value of the muon anomaly [55–57], evidence for a possible deviation between the experimental value and the Standard Model value continues to grow. The E821 result now differs by more than three and a half standard deviations from the Standard Model value. Fortunately there are two new experiments that should be able to clarify this discrepancy. The Fermilab experiment E989, which represents the next level of improvement in the series of “magic” storage ring experiments, is now collecting data with the goal of a fourfold improvement over BNL E821. A new experiment, E34 at J-PARC, discussed at this conference by Tsutomu Mibe, is developing a very different technique to measure the anomaly.

Acknowledgments

For many years I have been deeply involved in the muon (g-2) experiment E821 at Brookhaven, and in the new Fermilab experiment E989 that is now coming on line. I wish to thank all of my collaborators for their contributions, and for the many things that they taught me. I especially wish to recognize the very senior members, many of whom have died over the past decade or so. I wish to acknowledge the enormous amount of knowledge that I gained from Francis J.M. Farley (1920 - 2018), who played a leading role in all three CERN (g-2) experiments as well as being a collaborator on E821; Vernon W. Hughes (1920 - 2008), founder and Co-spokesperson for E821; Frank Krienen (1917-2008), who designed the inflector magnet and electric quadrupoles for the CERN-3 experiment, and made many contributions to E821; L. Roberts, arXiv:1811.06974, SciPost Phys. Proc.]

<table>
<thead>
<tr>
<th>YEAR</th>
<th>BNL</th>
<th>E821</th>
</tr>
</thead>
<tbody>
<tr>
<td>1984</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Fermilab experiment released the measurement result from their run 1 data on 7 April 2021. [B. Abi et al, Phys. Rev. Lett. 124, 141801 (2021)]

Runs 2 and 3 measurement: August 2023; Run 6 completed Spring 2023.
JPARC experiment

T. Mibe for E34 @ INT g-2 workshop

Muon g-2/EDM experiment at J-PARC

- 2018: Stage II approval by IPNS and IMSS directors.
- March 2019: Endorsed by KEK-SAC as a near-term priority
- 2020: Funding request
- 2024+: data taking runs

Features:
- Low emittance muon beam (1/1000)
- No strong focusing (1/1000) & good injection eff. (x10)
- Compact storage ring (1/20)
- Tracking detector with large acceptance
- Completely different from BNL/FNAL method
Muon g-2: SM contributions

\[ a_\mu = a_\mu (\text{QED}) + a_\mu (\text{EW}) + a_\mu (\text{hadronic}) \]

**QED**

\[ a_\mu (\text{QED}) = A_1 + A_2 \left( \frac{m_\mu}{m_e} \right) + A_2 \left( \frac{m_\mu}{m_e} \right) + A_3 \left( \frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau} \right) \]

\[ A_i = \sum_{n=0}^{\infty} \left( \frac{\alpha}{\pi} \right)^n A_i^{2n} \]

<table>
<thead>
<tr>
<th>( n )</th>
<th># of diagrams</th>
<th>Contribution x 10^{11}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>116140973.32</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>413 217.63</td>
</tr>
<tr>
<td>3</td>
<td>71</td>
<td>30141.90</td>
</tr>
<tr>
<td>4</td>
<td>891</td>
<td>381.00</td>
</tr>
<tr>
<td>5</td>
<td>12672</td>
<td>5.08</td>
</tr>
</tbody>
</table>

\[ a_\mu (\text{QED}) = 116 584 718.9 \times 10^{-11} \]

4.8\( \sigma \) difference for \( A_1 \) at 5th order in \( \alpha \). [Volkov, 2019]

\[ T. \text{Aoyama et al, arXiv:1205.5370, PRL; } \]
Muon $g-2$: SM contributions

\[ a_\mu = a_\mu(\text{QED}) + a_\mu(\text{EW}) + a_\mu(\text{hadronic}) \]

Electroweak

(contributions from W,Z,H bosons)

1-loop

\[ \begin{aligned}
    \mu &\rightarrow \mu \\
    W &\rightarrow \mu
\end{aligned} \]

2-loop

\[ \begin{aligned}
    \mu &\rightarrow \mu \\
    W &\rightarrow \mu
\end{aligned} \]

Compared to QED, \( \frac{m_\mu^2}{M_W^2} \approx 10^{-6} \)

\[ a_\mu(\text{EW}) = 153.6 \ (1.0) \times 10^{-11} \]

Muon g-2: SM contributions

The hadronic contributions are written as:

\[ a_\mu = a_\mu(QED) + a_\mu(EW) + a_\mu(\text{hadronic}) \]

The hadronic contributions are written as:

\[ a_\ell(\text{hadronic}) = a_\ell^{\text{HVP, LO}} + a_\ell^{\text{HVP, NLO}} + a_\ell^{\text{HVP, NNLO}} + \ldots \]

\[ + a_\ell^{\text{HLbL}} + a_\ell^{\text{HLbL, NLO}} + \ldots \]

\[ \sim 10^{-7} \]
**HVP: higher order (NLO, NNLO)**

\[ a_{\mu}^{\text{HVP,NLO}} = -9.83(7) \times 10^{-10} \]  
[based on KNT 2019]

\[ a_{\mu}^{\text{HVP,NNLO}} = 1.24(1) \times 10^{-10} \]  

**space-like NLO and NNLO HVP kernels for LQCD evaluations and MUonE**  

**mixed leptonic, hadronic (double bubble) contributions** to \( a_{\mu} \) are \( < 10^{-11} \)  
[Hoferichter + Teubner, arXiv:2112.06929]
Muon $g-2$: SM contributions

$$a_\mu = a_\mu(QED) + a_\mu(EW) + a_\mu(\text{hadronic})$$

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Error</th>
<th>Value</th>
<th>Error (%)</th>
<th>Precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED</td>
<td>0.001 ppm</td>
<td>$116,584,718.9 \times 10^{-11}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HVP</td>
<td>0.34 ppm</td>
<td>$6,845 (40) \times 10^{-11}$</td>
<td>0.6%</td>
<td></td>
</tr>
<tr>
<td>HLVbL</td>
<td>0.15 ppm</td>
<td>$92 (18) \times 10^{-11}$</td>
<td>20%</td>
<td></td>
</tr>
</tbody>
</table>
Hadronic Corrections

Two different, independent strategies:

For HVP: use dispersion relations to rewrite integral in terms of hadronic cross section:

$$Im \left[ \text{hadrons} \right] \sim |\text{hadrons}|^2$$

Many experiments (over 20+ years) have measured the $e^+e^-$ cross sections for the different channels over the needed energy range with increasing precision.

For HLbL: new dispersive approach

Direct calculation using Euclidean Lattice QCD

Approximations:
- discrete space-time (spacing $a$)
- finite spatial volume ($L$), and time extent ($T$)

Integrals are evaluated numerically using Monte Carlo methods.

$ab\text{-initio}$ method to quantify QCD effects
- already used for simple hadronic quantities with high precision
- requires large-scale computational resources
- allows for entirely SM theory based evaluations
Steering Committee

- Gilberto Colangelo (Bern)
- Michel Davier (Orsay) co-chair
- Aida El-Khadra (UIUC & Fermilab) chair
- Martin Hoferichter (Bern)
- Christoph Lehner (Regensburg University) co-chair
- Laurent Lellouch (Marseille)
- Tsutomu Mibe (KEK) J-PARC Muon g-2/EDM experiment
- Lee Roberts (Boston) Fermilab Muon g-2 experiment
- Thomas Teubner (Liverpool)
- Hartmut Wittig (Mainz)

Maximize the impact of the Fermilab and J-PARC experiments
- quantify and reduce the theoretical uncertainties on the hadronic corrections
- summarize the theory status and assess reliability of uncertainty estimates
- organize workshops to bring the different communities together:
  - First plenary workshop @ Fermilab: 3-6 June 2017
  - HVP workshop @ KEK: 12-14 February 2018
  - HLbL workshop @ U Connecticut: 12-14 March 2018
  - Second plenary workshop @ HIM (Mainz): 18-22 June 2018
  - Third plenary workshop @ INT (Seattle): 9-13 September 2019
  - Lattice HVP at high precision workshop (virtual): 16-20 November 2020
  - Fourth plenary workshop @ KEK (virtual): 28 June - 02 July 2021
  - Fifth plenary workshop @ Higgs Centre (Edinburgh): 5-9 September 2022
  - Sixth plenary workshop @ University of Bern: 4-8 September 2023
  - Seventh plenary workshop @ KEK (Japan): June 2024

https://muon-gm2-theory.illinois.edu
Near-term Timeline

FNAL E989

Run 4

Run 5

Run 6

J-PARC E34

Muon g-2 TI WP published

Run 1 result announced

WP update

Theory Initiative:
- ongoing activities: develop method average for Lattice HVP
- CMD-3 seminar (virtual): 27 March 2023 at 8:00am US CDT
- WP update with all available results ~ late 2023

!* TI workshops:
  - Jun 2021 @ KEK (virtual)
  - Sep 2022 @ Higgscentre
  - Sep 2023 @ Bern
  - Summer 2024 @ KEK
Hadronic Corrections: Comparisons

$\mu$

HVP from:
- LM20
- BMW20
- ETM18/19
- Mainz/CLS19
- FHM19
- PACS19
- RBC/UKQCD18
- BMW17
- RBC/UKQCD data/lattice
- BDJ19
- J17
- DHMZ19
- KNT19
- WP20

BNL+FNAL

$a_{\mu}^{SM} < a_{\mu}^{HVP} + [a_{\mu}^{QED} + a_{\mu}^{Weak} + a_{\mu}^{HLbL}]$

Glasgow consensus (09)
- N/JN09
- J17

Mainz21 (+ charm-loop)

RBC/UKQCD19 (+ charm-loop)

WP20 data-driven dispersive

WP20

Lattice QCD + QED

dispersive/data driven

models of QCD +EFT, large $N_c$

BNL+FNAL data-driven

RBC/UKQCD19 not used in WP20

Mainz21 (+ charm-loop)

WP20 data-driven dispersive

WP20

models of QCD +EFT, large $N_c$
Outline

1. Introduction
2. Fermilab and JPARC muon g-2 experiments
3. SM contribution to g-2
4. Muon g-2 Theory Initiative
   - dispersive/data driven methods
     - Hadronic Vacuum Polariztion (HVP)
     - Hadronic Light-by-Light (HLbL)
5. lattice QCD
     - lattice HVP
     - lattice HLbL
6. Connections
7. Summary and Outlook
Hadronic vacuum polarization

\[ \Pi_{\mu\nu} = \int d^4x e^{iqx} \langle j_\mu(x)j_\nu(0) \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) \quad \hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) \]

Leading order HVP correction:

\[ a_{\mu,\text{HVP,LO}} = \left( \frac{\alpha}{\pi} \right)^2 \int dq^2 \omega(q^2) \hat{\Pi}(q^2) \]

• Use optical theorem and dispersion relation to rewrite the integral in terms of the hadronic \( e^+e^- \) cross section:

\[ a_{\mu,\text{HVP,LO}} = \frac{m^2_\mu}{12\pi^3} \int ds \frac{\bar{K}(s)}{s} \sigma_{\text{exp}}(s) \]

Dominant contributions from low energies; \( \pi^+\pi^- \) channel: 73% of total

• Use direct integration method, summing up cross sections for all possible hadronic channels up to \( \sim 2 \) GeV

---

**e^+e^- facilities involved in HVP measurement**

- BNL-821
- FNAL E989
- J-PARC g-2/EDM E-34
- BaBar
- SND
- CMD-3

HVP measurements

KLOE

BES-III

BELLE-II

KEDR


- Total hadronic cross section $\sigma_{\text{had}}$ from > 100 data sets in more than 35 channels summed up to ~ 2 GeV
- $\sqrt{s} > 2$ GeV: inclusive data + pQCD + narrow resonances
- $\sigma_{\text{had}}$ defined to include real & virtual photons
- Direct integration method: no need to specify resonances ($\rho, \ldots$)
- Two independent compilations (DHMZ, KNT)

Tensions between BaBar and KLOE data sets:

- Cross checks using analyticity and unitarity relating pion form factor to $\pi\pi$ scattering
- Combinations of data sets affected by tensions between measurements and treatment of correlations:
  - Conservative merging procedure
In 2020 WP:

Conservative merging procedure to obtain a realistic assessment of the underlying uncertainties:

- account for tensions between data sets
- account for differences in methodologies for compilation of experimental inputs
- include correlations between systematic errors
- cross checks from unitarity & analyticity constraints

[Colangelo et al, 2018; Anantharayan et al, 2018; Davier et al, 2019; Hoferichter et al, 2019]

- Full NLO radiative corrections  [Campanario et al, 2019]

\[ a^\text{HVP,LO}_\mu = 693.1 (2.8)_{\exp} (0.7)_{\text{DV}+\text{PQCD}} (2.8)_{\text{BaBar}+\text{KLOE}} \times 10^{-10} \]

\[ = 693.1 (4.0) \times 10^{-10} \]


A new puzzle!

- discrepancies between experiments now \( \gtrsim (3 - 5) \sigma \) need to be understood/resolved

(virtual) scientific seminar + discussion panel on CMD-3 measurement

March 27 (8:00 –11:00 am US CDT)

Discussions are continuing!

- 6th Muon g-2 Theory Initiative workshop (4-8 Sep 2023, Bern)
HVP: data-driven

In 2020 WP:

Conservative merging procedure to obtain a realistic assessment of the underlying uncertainties:

- account for tensions between data sets
- account for differences in methodologies for compilation of experimental inputs
- include correlations between systematic errors
- cross checks from unitarity & analyticity constraints

[Colangelo et al, 2018; Anantharayan et al, 2018; Davier et al, 2019; Hoferichter et al, 2019]

- Full NLO radiative corrections  [Campanario et al, 2019]

\[ a_{\mu}^{\text{HVP,LO}} = 693.1 \times 10^{-10} \]

\[ = 693.1 \times 10^{-10} \]

Ongoing work on experimental inputs:

- **BaBar**: new analysis of large data set in \( \pi\pi \) channel, also \( \pi\pi\pi \), other channels
- **KLOE**: new analysis of large data in \( \pi\pi \) channel, other channels
- **SND**: new results for \( \pi\pi \) channel, other channels in progress
- **BESIII**: new results in 2021 for \( \pi\pi \) channel, continued analysis also for \( \pi\pi\pi \), other channels

Better statistics than BaBar or KLOE; similar or better systematics for low-energy cross sections

- Most collaborations proceeding with blind analyses

---

A. El-Khadra

JLAB Theory seminar, 26 June 2023
In 2020 WP:

Conservative merging procedure to obtain a realistic assessment of the underlying uncertainties:

- account for tensions between data sets
- account for differences in methodologies for compilation of experimental inputs
- include correlations between systematic errors
- cross checks from unitarity & analyticity constraints

[Colangelo et al, 2018; Anantharayan et al, 2018; Davier et al, 2019; Hoferichter et al, 2019]

- Full NLO radiative corrections [Campanario et al, 2019]

\[ a_\mu^{HVP,LO} = 693.1 (2.8)_{\text{exp}} (0.7)_{\text{DV+pQCD}} (2.8)_{\text{BaBar–KLOE}} \times 10^{-10} \]

\[ = 693.1 (4.0) \times 10^{-10} \]


Ongoing work on theoretical aspects:

- Developing NNLO Monte Carlo generators (STRONG 2020 workshop https://agenda.infn.it/event/28089/) [Campanario et al, 2019 appendix]
- radiative corrections using FsQED (scalar QED + pion form factor)
- charge asymmetry (CMD-3 measurement) vs radiative corrections [Ignatov + Lee, arXiv:2204.12235]
- development of new dispersive treatment of radiative corrections in \( \pi \pi \) channel [Colangelo et al, arXiv:2207.03495]
- new focus on structure-dependent NLO effects: source of difference between ISR and direct scan measurements? [Strong 2020 workshop]

- including \( \tau \) decay data: requires nonperturbative evaluation of IB correction [M. Bruno et al, arXiv:1811.00508]

If the differences between experiments are resolved: data-driven evaluations of HVP with \( \sim 0.3\% \) feasible by \(~2025\)

A. El-Khadra

JLAB Theory seminar, 26 June 2023

23
Dispersive approach:
[Colangelo at al, 2014; Pauk & Vanderhaegen 2014; …]
❖ model independent
❖ significantly more complicated than for HVP
❖ provides a framework for data-driven evaluations
❖ can also use lattice results as inputs

Dominant contributions (∼ 75% of total):
❖ Well quantified with ∼ 6% uncertainty
❖ η, η' pole contributions: Canterbury approximants only
❖ Ongoing work: consolidation of η, η' pole contributions using disp. relations and LQCD

Subleading contributions (∼ 25% of total):
❖ Not yet well known
❖ dominant contribution to total uncertainty
❖ Ongoing work:
  - Implementation of short-distance constraints (now at 2-loop)
  - DR implementation for axial vector contributions
  - new \( q_4 = 0 \) DR program for higher spin intermediate states
  [Luedtke @ Higgscentre workshop with Procura and Stoffer, in progress]
  - Mainz and BESIII ramping up \( \gamma(*)\gamma* \) programs
  [A. Denig and C. Redmer @ Higgscentre workshop]

Dispersive, data-driven evaluation of HLbL with ≤ 10% total uncertainty feasible by ~2025.
A. El-Khadra

Comparison:

<table>
<thead>
<tr>
<th>Contribution</th>
<th>PdRV(09) [471]</th>
<th>N/JN(09) [472, 573]</th>
<th>J(17) [27]</th>
<th>Our estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0, \eta, \eta'$-poles</td>
<td>114(13)</td>
<td>99(16)</td>
<td>95.45(12.40)</td>
<td>93.8(4.0)</td>
</tr>
<tr>
<td>$\pi, K$-loops/boxes</td>
<td>−19(19)</td>
<td>−19(13)</td>
<td>−20(5)</td>
<td>−16.4(2)</td>
</tr>
<tr>
<td>S-wave $\pi\pi$ rescattering</td>
<td>−7(7)</td>
<td>−7(2)</td>
<td>−5.98(1.20)</td>
<td>−8(1)</td>
</tr>
<tr>
<td>subtotal</td>
<td>88(24)</td>
<td>73(21)</td>
<td>69.5(13.4)</td>
<td>69.4(4.1)</td>
</tr>
<tr>
<td>scalars</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>1(3)</td>
</tr>
<tr>
<td>tensors</td>
<td>−</td>
<td>−</td>
<td>1.1(1)</td>
<td>−</td>
</tr>
<tr>
<td>axial vectors</td>
<td>15(10)</td>
<td>22(5)</td>
<td>7.55(2.71)</td>
<td>6(6)</td>
</tr>
<tr>
<td>$u, d, s$-loops / short-distance</td>
<td>−</td>
<td>21(3)</td>
<td>20(4)</td>
<td>15(10)</td>
</tr>
<tr>
<td>$c$-loop</td>
<td>2.3</td>
<td>−</td>
<td>2.3(2)</td>
<td>3(1)</td>
</tr>
<tr>
<td>total</td>
<td>105(26)</td>
<td>116(39)</td>
<td>100.4(28.2)</td>
<td>92(19)</td>
</tr>
</tbody>
</table>

NLO HLbL contribution:

$$a_\mu^{\text{HLbL,NLO}} = 2(1) \times 10^{-11}$$
Introduction
Fermilab and JPARC muon g-2 experiments
SM contribution to g-2
Muon g-2 Theory Initiative
dispersive/data driven methods
  • Hadronic Vacuum Polariztion (HVP)
  • Hadronic Light-by-Light (HLbL)
lattice QCD
  • lattice HVP
  • lattice HLbL
Connections
Summary and Outlook
Lattice QCD Introduction

\[ \mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (\mathcal{D} + m_f) \psi_f + \frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} \]

- discrete Euclidean space-time (spacing \(a\))
- derivatives \(\rightarrow\) difference operators, etc...
- finite spatial volume (\(L\))
- finite time extent (\(T\))

Integrals are evaluated numerically using monte carlo methods.

adjustable parameters
- lattice spacing: \(a \rightarrow 0\)
- finite volume, time: \(L \rightarrow \infty, T > L\)
- quark masses (\(m_f\)):
  tune using hadron masses extrapolations/interpolations

\(M_{H,\text{lat}} = M_{H,\text{exp}}\)
\(m_f \rightarrow m_{f,\text{phys}}\)

MILC \(n_f = 2 + 1 + 1\)
Lattice QCD Introduction

- typical momentum scale of quarks gluons inside hadrons: $\sim \Lambda_{QCD}$
- make $a$ small to separate the scales: $\Lambda_{QCD} \ll 1/a$
- Symanzik EFT: $\langle \mathcal{O} \rangle^{\text{lat}} = \langle \mathcal{O} \rangle^{\text{cont}} + O(a\Lambda)^n$, $n \geq 2$
  - provides functional form for extrapolation (depends on the details of the lattice action)
  - can be used to build improved lattice actions
  - can be used to anticipate the size of discretization effects

Discretization effects — continuum extrapolation
...of lattice spacing, chiral, heavy quark, and finite volume effects is based on Effective Field Theory (EFT) descriptions of QCD $\Rightarrow$ ab initio

- finite $a$: Symanzik EFT
- light quark masses: Chiral Perturbation Theory
- heavy quarks: HQET
- finite $L$: finite volume EFT

In practice:

stability and control over systematic errors depends on the lattice action(s) employed, underlying simulation parameters, available computational resources, analysis choices, ...
Lattice QCD calculations of simple quantities (with at most one stable meson in initial/final state) that quantitatively account for all systematic effects (discretization, finite volume, renormalization,…) in some cases with

- sub percent precision.
- total errors that are commensurate (or smaller) than corresponding experimental uncertainties.

Progress due to a virtuous cycle of theoretical developments, improved algorithms/methods and increases in computational resources (“Moore’s law”)

Scope of LQCD calculations is increasing due to continual development of new methods:

- nucleon matrix elements
- nonleptonic kaon decays ($K \to \pi \pi, \epsilon’,…$)
- resonances, scattering ($\pi \pi \to \rho, …$)
- long-distance effects ($\Delta M_K, …$)
- QED corrections
- radiative decay rates
- structure: PDFs, GPDs, TMDs, …
- inclusive decay rates ($B \to X_c \ell \nu, …$)
- …
Lattice QCD Introduction


<table>
<thead>
<tr>
<th>Quantity</th>
<th>CKM element</th>
<th>2013 expt. error</th>
<th>2007 forecast lattice error</th>
<th>2013 forecast lattice error</th>
<th>2021 FLAG Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_K/f_\pi$</td>
<td>$</td>
<td>V_{us}</td>
<td>$</td>
<td>0.2%</td>
<td>0.5%</td>
</tr>
<tr>
<td>$f_K^+/f_\pi^+(0)$</td>
<td>$</td>
<td>V_{us}</td>
<td>$</td>
<td>0.2%</td>
<td>–</td>
</tr>
<tr>
<td>$f_D$</td>
<td>$</td>
<td>V_{cd}</td>
<td>$</td>
<td>4.3%</td>
<td>5%</td>
</tr>
<tr>
<td>$f_{Ds}$</td>
<td>$</td>
<td>V_{cs}</td>
<td>$</td>
<td>2.1%</td>
<td>5%</td>
</tr>
<tr>
<td>$D \rightarrow \pi \ell \nu$</td>
<td>$</td>
<td>V_{cd}</td>
<td>$</td>
<td>2.6%</td>
<td>–</td>
</tr>
<tr>
<td>$D \rightarrow K \ell \nu$</td>
<td>$</td>
<td>V_{cs}</td>
<td>$</td>
<td>1.1%</td>
<td>–</td>
</tr>
<tr>
<td>$B \rightarrow D^* \ell \nu$</td>
<td>$</td>
<td>V_{cb}</td>
<td>$</td>
<td>1.3%</td>
<td>–</td>
</tr>
<tr>
<td>$B \rightarrow \pi \ell \nu$</td>
<td>$</td>
<td>V_{ub}</td>
<td>$</td>
<td>4.1%</td>
<td>–</td>
</tr>
<tr>
<td>$f_B$</td>
<td>$</td>
<td>V_{ub}</td>
<td>$</td>
<td>9%</td>
<td>–</td>
</tr>
<tr>
<td>$\xi$</td>
<td>$</td>
<td>V_{ts}/V_{td}</td>
<td>$</td>
<td>0.4%</td>
<td>2–4%</td>
</tr>
<tr>
<td>$\Delta m_s$</td>
<td>$</td>
<td>V_{ts}V_{td}</td>
<td>^2$</td>
<td>0.24%</td>
<td>7–12%</td>
</tr>
<tr>
<td>$B_K$</td>
<td>Im($V_{td}^2$)</td>
<td>0.5%</td>
<td>3.5–6%</td>
<td>1.3%</td>
<td>&lt; 1%</td>
</tr>
</tbody>
</table>

QED threshold: QED corrections important/dominant source of theory error in SM predictions

\[1.3\%\] (0.6) ~1.5% [fro] ~3% 0.7 % (0.6) 1.3 % 4.5 % 1.3 %
Lattice HVP: Introduction

\[ \hat{\Pi}(q^2) \]

Leading order HVP correction:

\[ a_{\mu}^{\text{HVP,LO}} = \left( \frac{\alpha}{\pi} \right)^2 \int dq^2 \omega(q^2) \hat{\Pi}(q^2) \]

- Calculate \( a_{\mu}^{\text{HVP,LO}} \) in Lattice QCD

Compute correlation function:

\[ C(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x,t) j_i(0,0) \rangle \]

and

\[ \hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt C(t) \left[ t^2 - \frac{4}{Q^2} \sin^2 \left( \frac{Qt}{2} \right) \right] \]

Obtain \( a_{\mu}^{\text{HVP,LO}} \) from an integral over Euclidean time:

\[ a_{\mu}^{\text{HVP,LO}} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dt \tilde{w}(t) C(t) \]


Lattice HVP: Introduction

Calculate $a_{\mu}^{\text{HVP}}$ in Lattice QCD:

$$a_{\mu}^{\text{HVP,LO}} = \sum_f a_{\mu,f}^{\text{HVP,LO}} + a_{\mu,\text{disc}}^{\text{HVP,LO}}$$

- Separate into connected for each quark flavor + disconnected contributions
  (gluon and sea-quark background not shown in diagrams)

  Note: almost always $m_u = m_d$

  $$f= ud, s, c, b$$

  $$\sum_f$$

  +

  $$a_{\mu,\text{disc}}^{\text{HVP,LO}} \sim 2\% \text{ of total}$$

  Isospinbreaking (QED + $m_u \neq m_d$) corrections:

  $$\delta a_{\mu}^{\text{HVP,LO}} \sim 1\% \text{ of total}$$

  $$a_{\mu}^{\text{HVP,LO}} = a_{\mu}^{\text{HVP,LO}}(ud) + a_{\mu}^{\text{HVP,LO}}(s) + a_{\mu}^{\text{HVP,LO}}(c) + a_{\mu,\text{disc}}^{\text{HVP,LO}} + \delta a_{\mu}^{\text{HVP,LO}}$$

A. El-Khadra

JLAB Theory seminar, 26 June 2023
Long-distance tail

\[ G(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x, t) j_i(0, 0) \rangle \]

- Use noise reduction methods (AMA, LMA, ...): Aubin et al, RBC/UKQCD, BMWc, Mainz, ...
- Spectral reconstruction (RBC/UKQCD, Mainz):
  - obtain low-lying finite-volume spectrum \((E_n, a_n)\) in dedicated study using additional operators that couple to two-pion states
  - use to reconstruct \(G(t > t_c)\)
  - can be used to improve bounding method:
    \[ G(t) \rightarrow G(t) - \sum_{n=0}^{N} A_n^2 e^{-E_n t} \]

**Shaun Lahert**
(Fermilab-HPQCD-MILC)
@ Lattice 2021

- First calculation with staggered multi-pion operators
Lattice HVP: subleading corrections

H. Wittig @ Lattice HVP workshop

- Charm, strange contributions already well determined.
- Mild tensions for light contribution

Ongoing efforts by FNAL-HPQCD-MILC RBC/UKQCD, Mainz

Consistent results with increasing precision

V. Gülbers @ Lattice HVP workshop

Overview of published results - contributions to $\delta a_\mu^B \times 10^{10}$

- BMW 20
- Anbin et al. 19
- Mainz/CLS 19
- FHM 19
- PACS 19
- ETM 19
- RBC/UKQCD 18
- BMW 17

Some tensions between lattice results for individual contributions.
Large cancellations between individual contributions:

$\delta a_\mu^B \lesssim 1\%$
In 2020 WP:

- Lattice HVP average at 2.6% total uncertainty: 
  \[ a_{\mu}^{\text{HVP, LO}} = 711.6 (18.4) \times 10^{10} \]
- BMW 20 (published in 2021) first LQCD calculation with sub-percent (0.8%) error in tension with data-driven HVP (2.1\sigma)
- Further tensions for intermediate window

-3.7\sigma tension with data-driven evaluation
-2.2\sigma tension with RBC/UKQCD18

HVP: lattice

Use windows in Euclidean time to consider the different time regions separately. [T. Blum et al, arXiv:1801.07224, 2018 PRL]

Short Distance (SD) \( t : 0 \to t_0 \)
Intermediate (W) \( t : t_0 \to t_1 \)
Long Distance (LD) \( t : t_1 \to \infty \)

\[ t_0 = 0.4 \text{ fm}, t_1 = 1.0 \text{ fm} \]

- disentangle systematics/statistics from long distance/FV and discretization effects
- intermediate window: easy to compute in lattice QCD & using disperse approach:
- Internal cross check: compute each window separately (in continuum, infinite volume limits, ...) and combine:

\[ a_\mu = a_\mu^{\text{SD}} + a_\mu^{\text{W}} + a_\mu^{\text{LD}} \]
In 2020 WP:
- Lattice HVP average at 2.6% total uncertainty: 
  \[ a_{\mu}^{\text{HVP,LO}} = 711.6(18.4) \times 10^{10} \]
- BMW 20 (published April 2021) first LQCD calculation with sub-percent (0.8%) error in tension with data-driven HVP (2.1σ)
- Further tensions for intermediate window: 2.6% \( a_{\mu} \) HVP
- COLO, \( \mu = 711.6(18.4) \times 10^{10} \)
- Further tensions for intermediate window:

New results in 2022 for intermediate window, \( a_{\mu}^W \) from six different lattice groups:
- Most recently announced unblinded results by RBC/UKQCD and Fermilab/HPQCD/MILC
- Lattice-only comparison of light-quark connected contribution to intermediate window:

- \[ \text{Fermilab/HPQCD/MILC} \]
- \[ \text{RBC/UKQCD} \]
- \[ \text{ETMC} \]
- \[ \text{Mainz/CLS} \]
- \[ \text{Aubin et al.} \]
- \[ \chiQCD \text{ OV/HISQ} \]
- \[ \chiQCD \text{ OV/DWF} \]
- \[ \text{BMW} \]
- \[ \text{Lehner & Meyer} \]
- \[ \text{Aubin et al.} \]
- \[ \text{RBC/UKQCD} \]

R-ratio data compiled by M. Hoferichter

- \[ \text{RBC/UKQCD 2022} \]
- \[ \text{ETMC 2022} \]
- \[ \text{Mainz 2022} \]
- \[ \text{ETMC 2021} \]
- \[ \text{BMW 2020} \]
- \[ \text{RBC/UKQCD 2018} \]

\[ a_{\mu}^{\text{HVP, win}} \times 10^{10} \]

Gaurav Ray, U. Plymouth
Michael Lynch, UIUC
Yin Lin, MIT
Shaun Lahert, UIUC
Curtis Peterson, CUBoulder

Noise reduction via low-mode improvement

QED corrections
Lattice-spacing determination from baryon mass/units

Strong-isospin breaking

Intermediate window

Isospin-symmetric value

Two-pion contribution

Strong-isospin breaking

Ongoing projects

- \[ 200 \]
- \[ 202 \]
- \[ 204 \]
- \[ 206 \]
- \[ 208 \]
- \[ 210 \]
- \[ 212 \]

Fermilab/HPQCD/MILC: arXiv:2301.08274
ETM: arXiv:2206.15084
Mainz: arXiv:2206.06582
χQCD: arXiv:2204.01280
χQCD OV/DWF 22
χQCD OV/HISQ 22
RBC/UKQCD 18: arXiv:1801.07224
In 2020 WP:
- Lattice HVP average at 2.6% total uncertainty:
  \[ a^\text{HVP,LO}_\mu = 711.6 (18.4) \times 10^{10} \]
- BMW 20 (published April 2021)
  first LQCD calculation with sub-percent (0.8%) error
  in tension with data-driven HVP (2.1σ)
- Further tensions for intermediate window:

Ongoing work:
Evaluations of short-distance windows [ETMC, RBC/UKQCD]
Proposals for computing more windows:
- Use linear combinations of finer windows to locate the tension (if it persists) in \( \sqrt{s} \) [Colangelo et al, arXiv:12963]
- Use larger windows, excluding the long-distance region \( t \gtrsim 2 \text{ fm} \) to maximize the significance of any tension [Davies at at, arXiv:2207.04765]
For total HVP:
- independent lattice results at sub-percent precision: coming soon!
- Including \( \pi\pi \) states for refined long-distance computation
  (Mainz, RBC/UKQCD, FNAL/MILC)
- include smaller lattice spacings to test continuum extrapolations (needs adequate computational resources)

If results are consistent, Lattice HVP (average) with ~ 0.5% errors feasible by 2025

Note: int window \( \sim 1/3 \) of \( a^\text{HVP,LO}_\mu \)
Still need independent LQCD calculations of all windows and sub-leading contributions.
Lattice QCD+QED: Two independent and complete direct calculations of $a_{\mu}^{HLbL}$

- **RBC/UKQCD**
  - QCD + QED$_L$ (finite volume)

- **Mainz group**
  - [E. Chao et al, arXiv:2104.02632]
  - QCD + QED (infinite volume & continuum)

DWF ensembles at/near phys mass, $a \approx 0.08 - 0.2$ fm, $L \sim 4.5 - 9.3$ fm

- Cross checks between RBC/UKQCD & Mainz approaches in White Paper at unphysical pion mass
- Both groups are continuing to improve their calculations, adding more statistics, lattice spacings, physical mass ensemble (Mainz)
- update from RBC/UKQCD [T. Blum @ Higgscentre workshop] preliminary results from QCD + QED (inf.)

Lattice HLbL results with 10% total uncertainty feasible by ~2025
Connections

\[ \sigma(e^+e^- \to \text{hadrons}) \iff a_{\mu}^{HVP} \iff \Delta \alpha_{\text{had}}(M_Z^2) \]

\( a_{\mu}^{HVP} \) also depends on the hadronic vacuum polarization function, and can be written as an integral over \( \sigma(e^+e^- \to \text{hadrons}) \), but weighted towards higher energies.

A shift in \( a_{\mu}^{HVP} \) also changes \( \Delta \alpha_{\text{had}}(M_Z^2) \): EW fits [Passera, et al, 2008, Crivellin et al 2020, Keshavarsi et al 2020, Malaeescu & Scott 2020]

If the shift in \( a_{\mu}^{HVP} \) is in the low-energy region (\( \lesssim 1 \text{ GeV} \)), the impact on \( \Delta \alpha_{\text{had}}(M_Z^2) \) and EW fits is small.

A shift in \( a_{\mu}^{HVP} \) from low (\( \lesssim 2 \text{ GeV} \)) energies
\[ \Rightarrow \sigma(e^+e^- \to \pi\pi) \]
must satisfy unitarity & analyticity constraints
\[ \Rightarrow F_\pi(s) \]
can be tested with lattice calculations [Colangelo, Hoferichter, Stoffer, arXiv:2010.07943]

\[ \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) \times 10^4 \]

<table>
<thead>
<tr>
<th>( e^+e^- )</th>
<th>KNT, DHMZ</th>
<th>EW fit HEPFit</th>
<th>EW fit GFitter</th>
<th>guess based on BMWc</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) \times 10^4 )</td>
<td>276.1(1.1)</td>
<td>270.2(3.0)</td>
<td>271.6(3.9)</td>
<td>277.8(1.3)</td>
</tr>
<tr>
<td>difference to ( e^+e^- )</td>
<td>(-1.8\sigma)</td>
<td>(-1.1\sigma)</td>
<td>(+1.0\sigma)</td>
<td></td>
</tr>
</tbody>
</table>

- **Time-like formulation:**
  \[ \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = \frac{\alpha M_Z^2}{3\pi} P \int_0^\infty ds \frac{R_{\text{had}}(s)}{s(M_Z^2 - s)} \]

- **Space-like formulation:**
  \[ \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = \frac{\alpha}{\pi} \hat{n}(\xi M_Z^2) + \frac{\alpha}{\pi} (\hat{n}(M_Z^2) - \hat{n}(\xi M_Z^2)) \]

- **Global EW fit**
  - Difference between HEPFit and GFitter
  - Implementation mainly treatment of \( M_W \)
  - Pull goes into opposite direction
Connections

\[ \sigma(e^+e^- \to \text{hadrons}) \Leftrightarrow a_{\mu}^{\text{HVP}} \Leftrightarrow \Delta \alpha_{\text{had}}(M_Z^2) \]

\( \Delta \alpha_{\text{had}}(M_Z^2) \) also depends on the hadronic vacuum polarization function, and can be written as an integral over \( \sigma(e^+e^- \to \text{hadrons}) \), but weighted towards higher energies.

A shift in \( a_{\mu}^{\text{HVP}} \) also changes \( \Delta \alpha_{\text{had}}(M_Z^2) \): ➤ EW fits
If the shift in \( a_{\mu}^{\text{HVP}} \) is in the low-energy region ( \( \lesssim 1 \text{ GeV} \)), the impact on \( \Delta \alpha_{\text{had}}(M_Z^2) \) and EW fits is small.

A shift in \( a_{\mu}^{\text{HVP}} \) from low ( \( \lesssim 2 \text{ GeV} \)) energies ➤ \( \sigma(e^+e^- \to \pi\pi) \)
must satisfy unitarity & analyticity constraints ➤ \( F_\pi(s) \)
can be tested with lattice calculations

Modifying \( a_{\mu}^{\pi\pi} \lesssim 1 \text{ GeV} \)

• “low-energy” scenario: local changes in cross section of \( \sim 8\% \text{ around } \rho \)
• “high-energy” scenario: impact on pion charge radius and space-like VFF ⇒ chance for independent lattice-QCD checks

● requires factor \( \sim 3 \)
improvement over \( \chi \text{QCD result:} \)
\( \langle r_\pi^2 \rangle = 0.433(9)(13) \text{ fm}^2 \)
Summary

🌟 consistent results from independent, precise LQCD calculations for light-quark connected contribution to intermediate window \( a_{\mu}^{W} \) (~ 1/3 of \( a_{\mu}^{\text{HVP, LO}} \)) ➔ 3 – 4 \( \sigma \) tension with data-driven results?

🌟 still need independent LQCD results for long-distance contribution, total HVP: coming soon

► develop method average for lattice HVP results, assess tensions (if any) with data-driven average

🌟 Programs and plans in place to improve by 2025:

 tableView

<table>
<thead>
<tr>
<th>Program</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>data-driven HVP</td>
<td>if differences are resolved/understood, ~ 0.3 %</td>
</tr>
<tr>
<td></td>
<td>new measurements from BaBar, KLOE, Belle II, …. will shed light on current discrepancies (blind analyses are paramount!)</td>
</tr>
<tr>
<td>lattice HVP</td>
<td>if no tensions between independent lattice results, ~ 0.5 %</td>
</tr>
<tr>
<td>dispersive HLbL and lattice HLbL</td>
<td>no puzzles, steady progress, ~ 10 %</td>
</tr>
</tbody>
</table>

🌟 IF tensions/differences between data-driven HVP and lattice HVP are resolved, SM prediction will likely match precision goal of the Fermilab experiment.

🌟 IF NOT, will need detailed comparisons, explore connections between HVP, \( \sigma(e^+e^-) \), \( \Delta \alpha \), global EW fits.

► continued coordination by Theory Initiative: workshops, WPs, …
Beyond the SM possibilities

\( a_\mu \) is loop-induced, conserves CP & flavor, flips chirality.

The difference between Exp-SM is large:
\[
\Delta a_\mu = 251(59) \times 10^{-11} > a_\mu(\text{EW})
\]

Can be accommodated by many BSM theories (800+ papers)

D. Stöckinger @ g-2 Days (http://pheno.csic.es/g-2Days21/)

SUSY: MSSM, MRSSM
- MSugra...many other generic scenarios
- Bino-dark matter+some coannihil.+mass splittings
- Wino-LSP+specific mass patterns

Two-Higgs doublet model
- Type I, II, Y, Type X(lepton-specific), flavour-aligned

Lepto-quarks, vector-like leptons
- scenarios with muon-specific couplings to \( \mu_L \) and \( \mu_R \)

Simple models (one or two new fields)
- Mostly excluded
- light N.P. (ALPs, Dark Photon, Light \( L_\mu - L_\tau \))

Generically expect:
\[
a_\mu^{\text{NP}} \sim a_\mu^{\text{EW}} \times \frac{M_W^2}{\Lambda^2} \times \text{couplings}
\]
Beyond the SM possibilities

\( a_\mu \) is loop-induced, conserves CP & flavor, flips chirality.

The difference between Exp-SM is large:

\[
\Delta a_\mu = 251(59) \times 10^{-11} > a_\mu(\text{EW})
\]

Generically expect:

\[
a_\mu^{\text{NP}} \sim a_\mu^{\text{EW}} \times \frac{M_W^2}{\Lambda^2} \times \text{couplings}
\]

Can be accommodated by many BSM theories (800+ papers)

Can new physics hide in the low-energy \( \sigma(e^+e^- \rightarrow \pi\pi) \) cross section?

\[ \Rightarrow \text{No} \quad [\text{Luzio, et al, arXiv:2112.08312}] \]

New boson at \( \sim 1\text{GeV} \) decays into \( \mu^+\mu^- , e^+e^- \), affects \( \sigma(e^+e^- \rightarrow \pi\pi) \) indirectly

[\text{L. Darmé et al, arXiv:2112.09139}] \[ \Rightarrow \text{Neutral, long-lived hadrons, heretofore undetected?} \quad [\text{Farrar, arXiv:2206.13460}] \]

\( Z' \) at \(< 1\text{ GeV} \), coupling to 1st gen matter particles [\text{Coyle, Wagner, arXiv:2305.02354}]

A. El-Khadra

JLAB Theory seminar, 26 June 2023
Outlook

🌟 Experimental program beyond 2025:
- **J-PARC**: Muon g-2/EDM
- **CERN**: MUonE
- **Fermilab**: future muon campus experiments?
- **Belle II, BESIII, Novosibirsk, ...**
- **Chiral Belle (?)**

🌟 Data-driven/dispersive program beyond 2025:
- Development of NNLO MC generators for HLbL, improved experimental/lattice inputs together with further development of dispersive approach
- **MUonE** will provide a space-like determination of HVP

🌟 Lattice QCD beyond 2025:
- Access to future computational resources (coming Exascale) will enable improvements of all errors (statistical and systematic)
- Concurrent development of better methods and algorithms (gauge-field sampling, noise reduction) will accelerate progress
- **Beyond g-2**: a rich program relevant for all areas of HEP
Lepton moments summary

<table>
<thead>
<tr>
<th>Harvard’08</th>
<th>FNAL+BNL average</th>
<th>Sensitivity to heavy new physics:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cs</td>
<td>2.4σ</td>
<td>$a_{\ell}^{SM} - a_{\ell}^{Exp}$</td>
</tr>
<tr>
<td></td>
<td>1.6σ</td>
<td>$a_{\ell}^{NP} \sim \frac{m_{\ell}^2}{\Lambda^2}$</td>
</tr>
<tr>
<td>Rb</td>
<td>4.2σ</td>
<td>$(m_{\mu}/m_e)^2 \sim 4 \times 10^4$</td>
</tr>
<tr>
<td>WP SM</td>
<td>-5.2 x 10^5</td>
<td>[X. Fan et al (Gabrielse group), arXiv:2209.13084]</td>
</tr>
</tbody>
</table>

$10^{14} \times a_e$  $10^{11} \times a_\mu$  $10^7 \times a_\tau$

Cs: $\alpha$ from Berkeley group [Parker et al, Science 360, 6385 (2018)]
Rb: $\alpha$ from Paris group [Morel et al, Nature 588, 61–65(2020)]

Prospects for tau moment measurement:

Chiral Belle [arXiv:2205.12847]
- use polarized $e^-$ beam
- with $40ab^{-1}$ measurement of $a_\tau$ at $10^{-5}$ feasible
- with more statistics measurement at $10^{-6}$ possible
**Outlook**

\[ m_{ud}, m_u/m_d, m_s \quad \leftarrow \quad a_{\mu}^{\text{HVP LO}} \quad a_{\mu}^{\text{HLbL}} \]

\[ m_c, m_b \]

\[ \alpha_{\overline{MS}}(m_z) \]

\[ \langle \overline{B}_{0}\mid O_{i}^{\Delta B=2}\mid B_{0}\rangle \]

\[ \langle \overline{D}^{0}\mid O_{i}^{\Delta C=2}\mid D^{0}\rangle \quad \Lambda_{b} \rightarrow (p, \Lambda_c, \Lambda) \ell\nu \]

\[ \hat{B}_{K}, \ldots \]

\[ f_{B(s)}, \ldots \]

\[ g_{A}, g_{T}, g_{S} \]

\[ f^{K\rightarrow \pi}(q^{2}), f^{B\rightarrow D}(q^{2}), \ldots \]

\[ \langle \pi\pi(\ell=0)\mid H^{S=1}\mid K^{0}\rangle \]

\[ \langle \pi\pi(\ell=2)\mid H^{S=1}\mid K^{0}\rangle \]

\[ \Delta M_{K}, \epsilon_{K} \]

\[ B \rightarrow \pi\pi \ell\nu \ldots \]

\[ D_{s} \rightarrow \ell\nu\gamma \]

\[ K^{+} \rightarrow \ell^{+}\nu(\gamma) \]

\[ B \rightarrow X_{c}\ell\nu, \text{ other inclusive decay rates,} \]

\[ \pi \rightarrow \ell^{+}\ell^{-}, K \rightarrow \gamma\gamma \]

\[ K^{+} \rightarrow \pi^{+}\ell^{+}\ell^{-} \ldots \]

\[ m_{ud}, m_{u}/m_{d}, m_{s} \]

\[ \alpha_{\overline{MS}}(m_{z}) \]

\[ m_{c}, m_{b} \]

\[ f_{B(s)}, \ldots \]

\[ g_{A}, g_{T}, g_{S} \]

\[ f^{K\rightarrow \pi}(q^{2}), f^{B\rightarrow D}(q^{2}), \ldots \]

\[ \langle \pi\pi(\ell=0)\mid H^{S=1}\mid K^{0}\rangle \]

\[ \langle \pi\pi(\ell=2)\mid H^{S=1}\mid K^{0}\rangle \]

\[ \Delta M_{K}, \epsilon_{K} \]

\[ K^{+} \rightarrow \pi^{+}\nu\bar{\nu} \]

**Complexity**

- **LQCD flagship results**
- **First complete LQCD results, large(ish) errors**
- **First results, physical params, incomplete systematics**
- **new methods, pilot projects, unphysical kinematics**
- **new ideas, first studies**

**Outlook**

- **MEs for light nuclei**
- **PDFs, GPDs, TMDs**
- **\( \langle NN\mid O_{i}\mid NN\rangle \)**
- **\( D_{s} \rightarrow \ell\nu\gamma \)**
- **\( K^{+} \rightarrow \ell^{+}\nu(\gamma) \)**
- **\( B \rightarrow X_{c}\ell\nu, \text{ other inclusive decay rates,} \)**
- **\( \pi \rightarrow \ell^{+}\ell^{-}, K \rightarrow \gamma\gamma \)**
- **\( K^{+} \rightarrow \pi^{+}\ell^{+}\ell^{-} \ldots \)**
Thank you!
Appendix
### Updated WP Summary Table

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value ( \times 10^{11} )</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental average (E989+E821)</td>
<td>116592061(41)</td>
<td>Phys.Rev.Lett. 124, 141801</td>
</tr>
<tr>
<td>HVP LO (( e^+e^- ))</td>
<td>6931(40)</td>
<td>Refs. [2–7]</td>
</tr>
<tr>
<td>HVP NLO (( e^+e^- ))</td>
<td>–98.3(7)</td>
<td>Ref. [7]</td>
</tr>
<tr>
<td>HVP NNLO (( e^+e^- ))</td>
<td>12.4(1)</td>
<td>Ref. [8]</td>
</tr>
<tr>
<td>HVP LO (lattice, ( udsc ))</td>
<td>7116(184)</td>
<td>Refs. [9–17]</td>
</tr>
<tr>
<td>HLbL (phenomenology)</td>
<td>92(19)</td>
<td>Refs. [18–30]</td>
</tr>
<tr>
<td>HLbL NLO (phenomenology)</td>
<td>2(1)</td>
<td>Ref. [31]</td>
</tr>
<tr>
<td>HLbL (lattice, ( uds ))</td>
<td>79(35)</td>
<td>Ref. [32]</td>
</tr>
<tr>
<td>HLbL (phenomenology + lattice)</td>
<td>90(17)</td>
<td>Refs. [18–30, 32]</td>
</tr>
<tr>
<td>QED</td>
<td>116 584 718.931(104)</td>
<td>Refs. [33, 34]</td>
</tr>
<tr>
<td>Electroweak</td>
<td>153.6(1.0)</td>
<td>Refs. [35, 36]</td>
</tr>
<tr>
<td>HVP (( e^+e^- ), LO + NLO + NNLO)</td>
<td>6845(40)</td>
<td>Refs. [2–8]</td>
</tr>
<tr>
<td>HLbL (phenomenology + lattice + NLO)</td>
<td>92(18)</td>
<td>Refs. [18–32]</td>
</tr>
<tr>
<td>Total SM Value</td>
<td>116 591 810(43)</td>
<td>Refs. [2–8, 18–24, 31–36]</td>
</tr>
<tr>
<td>Difference: ( \Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}} )</td>
<td>251(59)</td>
<td></td>
</tr>
</tbody>
</table>

[website: https://muon-gm2-theory.illinois.edu](https://muon-gm2-theory.illinois.edu)
HVP: data-driven

Conservative merging procedure to obtain a realistic assessment of the underlying uncertainties:

[B. Malaescu @ INT g-2 workshop]

Detailed comparisons by-channel and energy range between direct integration results:

<table>
<thead>
<tr>
<th>Energy range</th>
<th>ACD18</th>
<th>CHS18</th>
<th>DHMZ19</th>
<th>DHMZ19'</th>
<th>KNT19</th>
</tr>
</thead>
<tbody>
<tr>
<td>≤ 0.6 GeV</td>
<td>110.1(9)</td>
<td>110.4(4)(5)</td>
<td>110.3(4)</td>
<td>108.7(9)</td>
<td></td>
</tr>
<tr>
<td>≤ 0.7 GeV</td>
<td>214.8(1.7)</td>
<td>214.7(0.8)(1.1)</td>
<td>214.8(8)</td>
<td>213.1(1.2)</td>
<td></td>
</tr>
<tr>
<td>≤ 0.8 GeV</td>
<td>413.2(2.3)</td>
<td>414.4(1.5)(2.3)</td>
<td>414.2(1.5)</td>
<td>412.0(1.7)</td>
<td></td>
</tr>
<tr>
<td>≤ 0.9 GeV</td>
<td>479.8(2.6)</td>
<td>481.9(1.8)(2.9)</td>
<td>481.4(1.8)</td>
<td>478.5(1.8)</td>
<td></td>
</tr>
<tr>
<td>≤ 1.0 GeV</td>
<td>495.0(2.6)</td>
<td>497.4(1.8)(3.1)</td>
<td>496.8(1.9)</td>
<td>493.8(1.9)</td>
<td></td>
</tr>
<tr>
<td>[0.6, 0.7] GeV</td>
<td>104.7(7)</td>
<td>104.2(5)(5)</td>
<td>104.5(5)</td>
<td>104.4(5)</td>
<td></td>
</tr>
<tr>
<td>[0.7, 0.8] GeV</td>
<td>198.3(9)</td>
<td>199.8(0.9)(1.2)</td>
<td>199.3(9)</td>
<td>198.9(7)</td>
<td></td>
</tr>
<tr>
<td>[0.8, 0.9] GeV</td>
<td>66.6(4)</td>
<td>67.5(4)(6)</td>
<td>67.2(4)</td>
<td>66.6(3)</td>
<td></td>
</tr>
<tr>
<td>[0.9, 1.0] GeV</td>
<td>15.3(1)</td>
<td>15.5(1)(2)</td>
<td>15.5(1)</td>
<td>15.3(1)</td>
<td></td>
</tr>
<tr>
<td>≤ 0.63 GeV</td>
<td>132.9(8)</td>
<td>132.8(1.1)</td>
<td>132.9(5)(6)</td>
<td>132.9(5)</td>
<td>131.2(1.0)</td>
</tr>
<tr>
<td>[0.6, 0.9] GeV</td>
<td>369.6(1.7)</td>
<td>371.5(1.5)(2.3)</td>
<td>371.0(1.6)</td>
<td>369.8(1.3)</td>
<td></td>
</tr>
<tr>
<td>[√0.1, √0.95] GeV</td>
<td>490.7(2.6)</td>
<td>493.1(1.8)(3.1)</td>
<td>492.5(1.9)</td>
<td>489.5(1.9)</td>
<td></td>
</tr>
</tbody>
</table>

Include constraints using unitarity & analyticity for ππ and πππ channels


\[ a_\mu^{\text{HVP,LO}} = 693.1 (2.8) \text{exp} (2.8) \text{sys} (0.7)_{\text{DV+pQCD}} \times 10^{-10} = 693.1 (4.0) \times 10^{-10} \]
Efforts on Radiative Corrections for low energy $e^+e^- \rightarrow$ hadrons

Workshop+Conference in Zurich 5-9 June 2023 (LOC: A. Signer, G. Stagnitto, Y. Ulrich)

Three-day in-person (Workstop) + a three half day hybrid conference

5 Working Groups:
- WP1: Leptonic processes at NNLO
- WP2: Form factor contributions at N3LO
- WP3: Processes with hadrons
- WP4: Parton showers
- WP5: Experimental input

Final goal: full NNLO MC. Aim to write a report by Autumn 2023
Alternative measurement of HVP for $a^\text{HVP}_\mu$: MUonE at CERN

- Space-like determination of $a^\text{HVP}_\mu$ at <0.5% through the scattering of 160 GeV muons on electron target
- Much progress in the last years, inc. detector optimization and development of $\mu - e$ (N)NLO MC
- Staged approach towards the full experiment: one station (2022), two stations (2023); possible 10 stations before LS3 (2026) (2% accuracy)
- Technical proposal towards full experiment in preparation
- Growing interest from both experimental and theory community

- C. M. Carloni Calame et al PLB 746 (2015) 325
### TABLE IV. Approximate error budgets for $a_{\mu}(\text{conn.})$ and $a_{\mu}^{W}(\text{conn.})$.

<table>
<thead>
<tr>
<th>Source</th>
<th>$\delta a_{\mu}^{W}$ (conn.) (%)</th>
<th>$\delta a_{\mu}^{W2}(\text{conn.})$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo statistics</td>
<td>0.19</td>
<td>2.44</td>
</tr>
<tr>
<td>Continuum extrapolation ($a \to 0$, $\Delta_{TB}$)</td>
<td>0.34</td>
<td>1.05</td>
</tr>
<tr>
<td>Finite-volume correction ($\Delta_{FV}$)</td>
<td>0.16</td>
<td>0.23</td>
</tr>
<tr>
<td>Pion-mass adjustment ($\Delta_{M}$)</td>
<td>0.06</td>
<td>0.96</td>
</tr>
<tr>
<td>Scale setting ($w_0$ (fm), $w_0/a$)</td>
<td>0.21</td>
<td>1.28</td>
</tr>
<tr>
<td>Current renormalization ($Z_{\mu}$)</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>Total</td>
<td>0.50%</td>
<td>3.18%</td>
</tr>
</tbody>
</table>