Quark orbital angular momentum in the proton
from Lattice QCD

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Acknowledgments:
Gauge ensembles provided by JLab/W&M Collaboration, and by RBC/UKQCD
Supported by U.S. DOE, Office of Science, Office of Nuclear Physics
Proton spin decompositions

\[ \frac{1}{2} = \frac{1}{2} \sum_q \Delta q + \sum_q L_q + J_g \]  
(Ji)

\[ \frac{1}{2} = \frac{1}{2} \sum_q \Delta q + \sum_q L_q + \Delta g + L_g \]  
(Jaffe-Manohar)

... and many more (in fact, we will see a continuous interpolation between the two ...)

There isn’t one unique way of separating quark and gluon orbital angular momentum – the different decompositions have different, legitimate meanings.
Quark orbital angular momentum

Interpreting terms in the energy-momentum tensor:

\[ L_q \sim -i\psi^\dagger (\vec{r} \times \vec{D}) z \psi \]

Can be obtained from \( L_q = J_q - S_q \), where \( S_q \) and \( J_q \) can be related to GPDs (Ji sum rule) – this is the traditional method in Lattice QCD.

\[ \mathcal{L}_q \sim -i\psi^\dagger (\vec{r} \times \vec{D}) z \psi \quad \text{in light cone gauge} \]

Needs a different method - GTMDs.
Quark OAM and parton distributions

Ji sum rule:

\[ L_3 = J_3 - S_3 = \frac{1}{2} \int dx \, x \, (H + E) - \frac{1}{2} \int dx \, H \]

Twist-2 GTMD \( F_{14} \):

\[ L_3 = - \int dx \int d^2 k_T \frac{k_T^2}{M^2} F_{14} \]

\( \rightarrow \) from Ji to Jaffe-Manohar OAM

Twist-3 GPD \( E_{2T} \):

\[ L_3 = (L_3 + 2S_3) - 2S_3 = - \int dx \, x \, E_{2T} - \int dx \, H \]

Note: A version of this approach was first noted by M. Polyakov, denoting the relevant twist-3 GPD variously as \( G_2 \) or \( G_3 \). Relation to OAM derived using OPE. Here, will review alternative approach based on GTMDs.

Note: All distribution functions above taken in the forward limit!
Equation of motion relation

\[-x E_{2T} - H = - \int d^2 k_T \frac{k_T^2}{M^2} F_{14} + M\quad \Rightarrow \quad L_3 = - \int dx \int d^2 k_T \frac{k_T^2}{M^2} F_{14} = - \int dx x E_{2T} - \int dx H\]

(forward limit)

How? Consider correlator (straight gauge link $U$)

$$W_{\Lambda' \Lambda}^{\Gamma} = \frac{1}{2} \int \frac{dz^- d^2 z_T}{(2\pi)^3} e^{ix P^+ z^- - i k_T \cdot z_T} \langle p', \Lambda' | \bar{\psi}(z/2) \Gamma U \psi(z/2) | p, \Lambda \rangle |_{z^+ = 0}$$

Use $\Gamma = i \sigma^i + \gamma^5$ and quark field equation of motion $\rightarrow (i \slashed{D} - m) \psi = 0$

to generate (note zero skewness throughout)

$$ik^+ e^{ij} W_{\Lambda' \Lambda}^{\gamma^j} + \frac{\Delta_i}{2} W_{\Lambda' \Lambda}^{\gamma^+ \gamma^5} - i e^{ij} k^j W_{\Lambda' \Lambda}^{\gamma^+} + M_{\Lambda' \Lambda}^i = 0$$

Form combination $\frac{\Delta_i}{\Delta_T^{\gamma}} (W_{+++}^{\Gamma} - W_{++}^{\Gamma})$, insert parametrizations of $W_{\Lambda' \Lambda}^{\Gamma}$ in terms of GTMDs

Integrate over $k_T$, identify GPDs
Lorentz invariance relation

\[ \frac{d}{dx} \int d^2k T \frac{k_T^2}{M^2} F_{14} = \bar{E}_2 T + H + E \]

How? Consider completely unintegrated correlator (straight gauge link \( U \))

\[ W_{\Lambda'\Lambda}^{\Gamma} = \frac{1}{2} \int \frac{d^4z}{(2\pi)^4} e^{ik \cdot z} \langle P + \Delta/2, \Lambda' | \bar{\psi} \left( -\frac{z}{2} \right) \Gamma U \psi \left( \frac{z}{2} \right) | P - \Delta/2, \Lambda \rangle \]

It can be parametrized in terms of Lorentz invariant amplitudes \( A(k^2, k \cdot P, k \cdot \Delta, \Delta^2, P \cdot \Delta) \).

Its \( k^- \) integral is the GTMD correlator,

\[ W_{\Lambda'\Lambda}^{\Gamma} = \int dk^- W_{\Lambda'\Lambda}^{\Gamma} \]

There are more GTMDs than \( A \)-amplitudes \( \longrightarrow \) relations between GTMDs induced.

Combine with EoM relation,

\[ x E_{2T} = -\bar{H} + \int d^2k T \frac{k_T^2}{M^2} F_{14} - \bar{M} \quad \Rightarrow \quad L_3 = -\int dx \int d^2k T \frac{k_T^2}{M^2} F_{14} = \frac{1}{2} \int dx \ x (H + E) - \frac{1}{2} \int dx \ \bar{H} \]

(forward limit)
Quark Orbital Angular Momentum from Wigner Distribution

\[ L_3^U = \int dx \int d^2 k_T \int d^2 r_T (r_T \times k_T)_3 \mathcal{W}^U(x, k_T, r_T) \]

\[ = - \int dx \int d^2 k_T \left[ \frac{k_T^2}{M^2} \right] F_{14}(x, k_T^2, k_T \cdot \Delta_T, \Delta_T^2) \bigg|_{\Delta_T = 0} \]

\[ = \epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle p', S | \overline{\psi}(-z/2) \gamma^+ U[-z/2, z/2] \psi(z/2) | p, S \rangle \bigg|_{z^+=z^-=0, \Delta_T=0, z_T\to0} \]

\( P, S \) in 3-direction, \( P \to \infty \)
\( p' = P + \Delta_T/2, \ p = P - \Delta_T/2 \)

Connection to GTMDs – A. Metz, M. Schlegel, C. Lorcé, B. Pasquini ...
Direct evaluation of quark orbital angular momentum

\[ \frac{L_U}{n} = \epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \left\langle p', S \middle| \overline{\psi}(-z/2)\gamma^+ U[-z/2, z/2] \psi(z/2) \middle| p, S \right\rangle \bigg|_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0} \]

Role of the gauge link \( U \):

Y. Hatta, M. Burkardt:

- Straight \( U[-z/2, z/2] \rightarrow \) Ji OAM
- Staple-shaped \( U[-z/2, z/2] \rightarrow \) Jaffe-Manohar OAM
- Difference is torque accumulated due to final state interaction
Direct evaluation of quark orbital angular momentum

\[
\frac{L_U^U}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}}}{\langle p', S | \overline{\psi}(-z/2)\gamma^+ U[-z/2, z/2] \psi(z/2) | p, S \rangle |_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0}}
\]

Role of the gauge link $U$:

Direction of staple taken off light cone (rapidity divergences)

Characterized by Collins-Soper parameter $\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$

Are interested in $\hat{\zeta} \to \infty$; synonymous with $P \to \infty$ in the frame of the lattice calculation $(v = e_3)$
**Ensemble details**

Clover ensemble provided by R. Edwards, B. Joó and K. Orginos (JLab / W&M Collaboration)

<table>
<thead>
<tr>
<th>$L^3 \times T$</th>
<th>$a$ (fm)</th>
<th>$M_\pi$ (MeV)</th>
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<th>#conf.</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$32^3 \times 96$</td>
<td>0.11403(77)</td>
<td>317(2)(2)</td>
<td>1.077(8)</td>
<td>967</td>
<td>23208</td>
</tr>
</tbody>
</table>
Direct evaluation of quark orbital angular momentum

\[ \frac{L^U_3}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}}}{\langle p', S | \bar{\psi}(-z/2)\gamma^+ U[-z/2, z/2] \psi(z/2) | p, S \rangle \mid z^+=z^-=0, \Delta_T=0, z_T \to 0} \]

\[ \langle p', S | \bar{\psi}(-z/2)\gamma^+ U[-z/2, z/2] \psi(z/2) | p, S \rangle \mid z^+=z^-=0, \Delta_T=0, z_T \to 0 \]

Parameters to consider: \( \Delta, z, \hat{\zeta}, \eta \)
Direct evaluation of quark orbital angular momentum

\[
\frac{L^U_3}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}}}{\epsilon_{ij}} \langle p', S | \bar{\psi}(-z/2)\gamma^+ U[-z/2, z/2] \psi(z/2) | p, S \rangle |_{\Delta_T=0, z_T\rightarrow 0} = 0.
\]

• Derivative w.r.t. \( \Delta_T \) taken using direct derivative method (Rome method)

• Derivative w.r.t. \( z_T \) taken as finite difference over lattice spacing \( a \)
  (cutoff on transverse momenta same as lattice cutoff)
Direct evaluation of quark orbital angular momentum

\[
\frac{L^U_3}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}}}{n} \langle p', S | \overline{\psi}(-z/2)\gamma^+ U[-z/2, z/2]\psi(z/2) | p, S \rangle |_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0} \\
\langle p', S | \overline{\psi}(-z/2)\gamma^+ U[-z/2, z/2]\psi(z/2) | p, S \rangle |_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0}
\]

Remaining parameters to consider: \( \hat{\zeta}, \eta \)
Ji quark orbital angular momentum: $\eta = 0$

$\hat{L}_3^{(\eta=0)} / m(\eta=0) = -0.3$

$u$–$d$ quarks

$m_\pi = 317$ MeV
From Ji to Jaffe-Manohar quark orbital angular momentum

\[ u-d \text{ quarks} \]
\[ m_\pi = 317 \text{ MeV} \]
\[ \hat{\zeta} = 0 \]
From Ji to Jaffe-Manohar quark orbital angular momentum

$\eta |v| / a$

$u-d$ quarks

$m_\pi = 317$ MeV

$\hat{\zeta} = 0.315$
From Ji to Jaffe-Manohar quark orbital angular momentum

- $m_\pi = 317$ MeV
- $\hat{\zeta} = 0.63$
Final state interaction torque – extrapolation in $\zeta$

$$\tau_3 = \left( \frac{L_3^{(\eta=\infty)}}{n^{(\eta=\infty)}} \right) - \left( \frac{L_3^{(\eta=0)}}{n^{(\eta=0)}} \right)$$

Integrated torque accumulated by struck quark leaving proton.
Flavor separation – from Ji to Jaffe-Manohar quark orbital angular momentum
Extraction of $\tilde{E}_{2T}$ moment from QCD matrix element

$L_3 + 2S_3 = -\int dx \, x \tilde{E}_{2T} = 2 \int dx \, x \int d^2 k_T \left( \frac{k_T \cdot \Delta_T}{\Delta_T^2} F_{27} + F_{28} \right) = -i P^+ \int dx \, x \int d^2 k_T \epsilon_{ij} \frac{\Delta^i_j}{\Delta_T^2} \left( W_{++}^{\gamma^j} - W_{--}^{\gamma^j} \right)$

(all in forward limit – note $W_{\Lambda' \Lambda}^{\gamma^j}$ is parametrized in terms of 8 twist-3 GTMDs $F_{21}, \ldots, F_{28}$).

$L_3 + 2S_3 = \epsilon_{ij} \frac{1}{2} \frac{\partial}{\partial (z \cdot P)} \frac{\partial}{\partial \Delta^i} \langle P + \Delta_T/2, + | \bar{\psi}(-z/2) \gamma^j U \psi(z/2) | P - \Delta_T/2, + \rangle \big|_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0}$

Renormalize using number of valence quarks $n$

$2 \, P^j \, n = \langle P, + | \bar{\psi}(-z/2) \gamma^j U \psi(z/2) | P, + \rangle \big|_{z^+ = z^- = 0, z_T \to 0}$

i.e., ultimately determine ratio $(L_3 + 2S_3)/n$
Setting up a lattice calculation

Lattice frame is boosted with respect to phenomenology frame; use invariants $z \cdot P, z^2$

Original frame: $z^+ = 0, \quad z \cdot P = z^- P^+, \quad z^2 = -z_T^2$

Lattice frame: $z_0 = 0, \quad z \cdot P = -z_3 P_3, \quad z^2 = -z_3^2 - z_T^2$

Use Rome direct derivative method to perform $\partial / \partial \Delta_i$

Perform $\partial / \partial z_3$ via finite difference at constant $z^2$
Setting up a lattice calculation

Lattice frame is boosted with respect to phenomenology frame; use invariants \( z \cdot P, z^2 \)

Original frame: \( z^+$ = 0, \quad z \cdot P = z^- P^+$, \quad \( z^2 = -z^2_T \)
Lattice frame: \( z_0 = 0, \quad z \cdot P = -z_3 P_3, \quad z^2 = -z_3^2 - z^2_T \)

Use Rome direct derivative method to perform \( \partial / \partial \Delta^i \)
Perform \( \partial / \partial z_3 \) via finite difference at constant \( z^2 \)
**Ensemble: same as before**

Clover fermion ensemble provided by R. Edwards, B. Joó and K. Orginos (JLab / W&M Collaboration)

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Dependence on $z^2$

$P_3=1$

$m_\pi = 317\ MeV$

$u\text{-}d\ quarks$
Dependence on $z^2$

$P_3=2$

$m_\pi = 317$ MeV

u–d quarks
result for $L_3$ and comparison to other approaches

- Combine with $2S_3$ previously obtained on same ensemble, same parameters (arXiv:1703.06703): $2S_3 = 1.18(2)$ (extrapolate $E_{2T}$ to $P = 0$ to match).

- Different systematics within the various approaches appear to remain smaller than statistical uncertainty.

- Larger uncertainty of twist-3 GPD approach: Difference of large numbers, incomplete cancellation of fluctuations due to use of rotational symmetry in normalization.

- It is feasible to extract OAM from twist-3 GPDs, but higher effort compared to other approaches for comparable accuracy.
Direct evaluation of quark spin-orbit correlations

\[ 2L_3 S_3 = \int dx \int d^2 k_T \int d^2 r_T (r_T \times k_T)_3 \Sigma \mathcal{W}_\Sigma^{\mathcal{U}}(x, k_T, r_T) \]

Wigner distribution

\[ \frac{2L_3 S_3}{n} = \frac{\epsilon_{ij} \frac{\partial}{\partial z_{T,i}} \frac{\partial}{\partial \Delta_{T,j}}}{\langle P + \Delta_T/2 \mid \bar{\psi}(-z/2)\gamma^+ \gamma^5 \mathcal{U}[-z/2, z/2] \psi(z/2) \mid P - \Delta_T/2 \rangle |_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0}} \]

\(\langle P + \Delta_T/2 \mid \bar{\psi}(-z/2)\gamma^+ \mathcal{U}[-z/2, z/2] \psi(z/2) \mid P - \Delta_T/2 \rangle |_{z^+ = z^- = 0, \Delta_T = 0, z_T \to 0}\)

\(P\) in 3-direction, \(P \to \infty\)

Renormalization: Form ratio with number of valence quarks \(n\) – note: are using Domain Wall Fermions!

Connection to GTMDs:

\[ 2L_3 S_3 = \int dx \int d^2 k_T \frac{k_T^2}{M^2} G_{11} \bigg|_{\Delta_T = 0} \]
**Ensemble details**

DWF ensemble provided by the RBC/UKQCD collaboration

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<td>$48^3 \times 96$</td>
<td>0.114</td>
<td>139</td>
<td>130</td>
<td>(33280 sloppy + 520 exact) AMA</td>
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For comparison, in a polarized proton (from C. Alexandrou et al., PRD 101 (2020) 094513; 2003.08486):

$\langle L^u \rangle = -0.22(3)$ , $\langle 2S^u \rangle = 0.86(2) \Rightarrow \langle L^u \rangle \langle 2S^u \rangle = -0.2$

$\langle L^d \rangle = 0.26(2)$ , $\langle 2S^d \rangle = -0.42(2) \Rightarrow \langle L^d \rangle \langle 2S^d \rangle = -0.1$

$\Rightarrow \langle L^u \rangle \langle 2S^u \rangle - \langle L^d \rangle \langle 2S^d \rangle = -0.1$
Lorentz invariance and equation of motion relations:

\[ 2L_3S_3 = \int dx \int d^2k_T \frac{k_T^2}{M^2} G_{11} = \frac{1}{2} \int dx \, x \bar{H} - \frac{1}{2} \int dx \, H + \frac{m}{2M} \int dx \, (2\bar{H}_T + E_T) \]

(in the forward limit). This is essentially the axial version of Ji’s sum rule.

M. Rodekamp, at the physical point: \( \int dx \, x \, \bar{H} = 0.21(2) \Rightarrow 2L_3S_3 = -0.395(10) \)

Sum rule satisfied.
Questions?