Maximum entropy freeze-out of *fluctuations* in heavy-ion collisions

Based on PRL,130 (2023) 16
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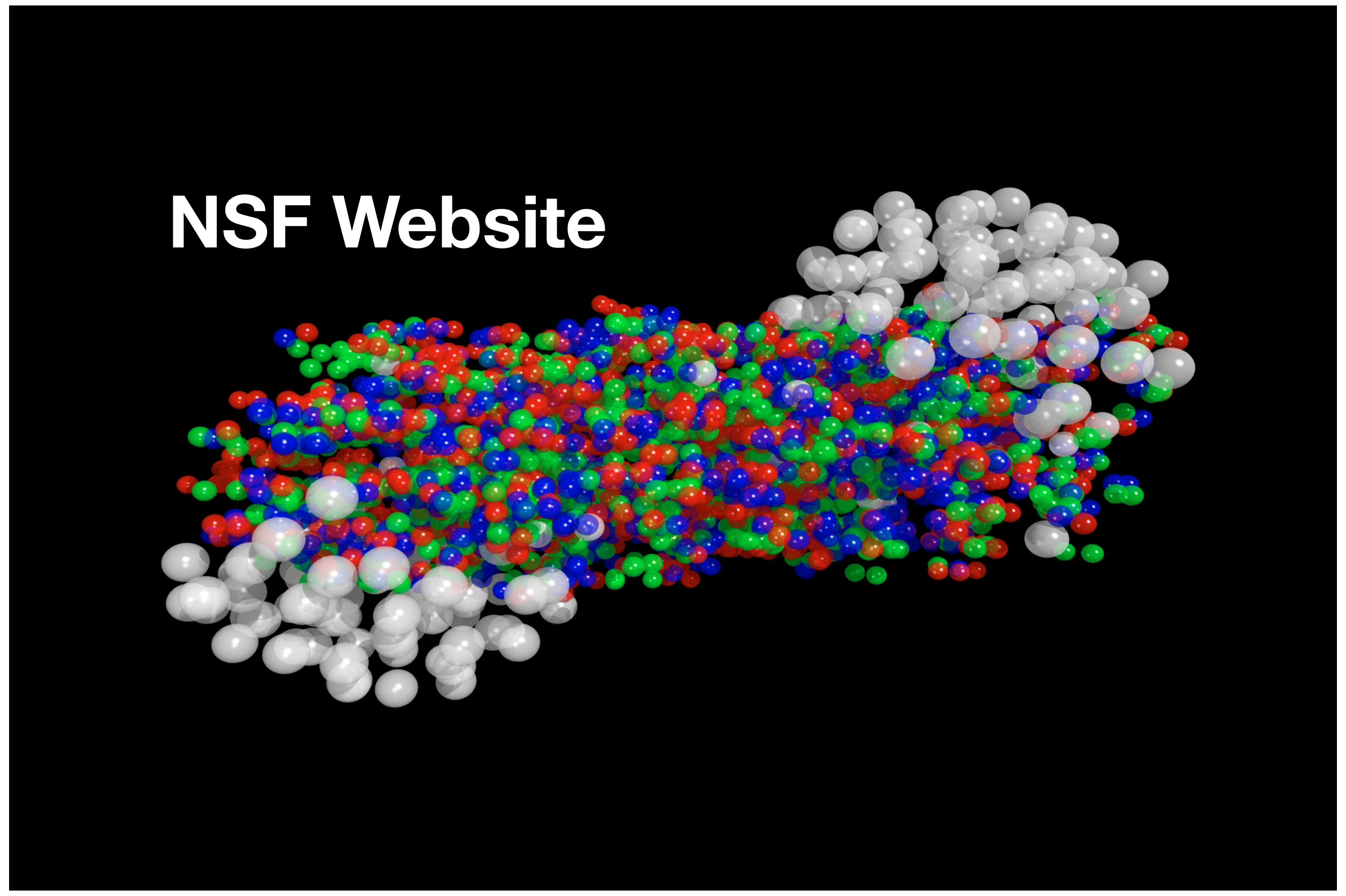
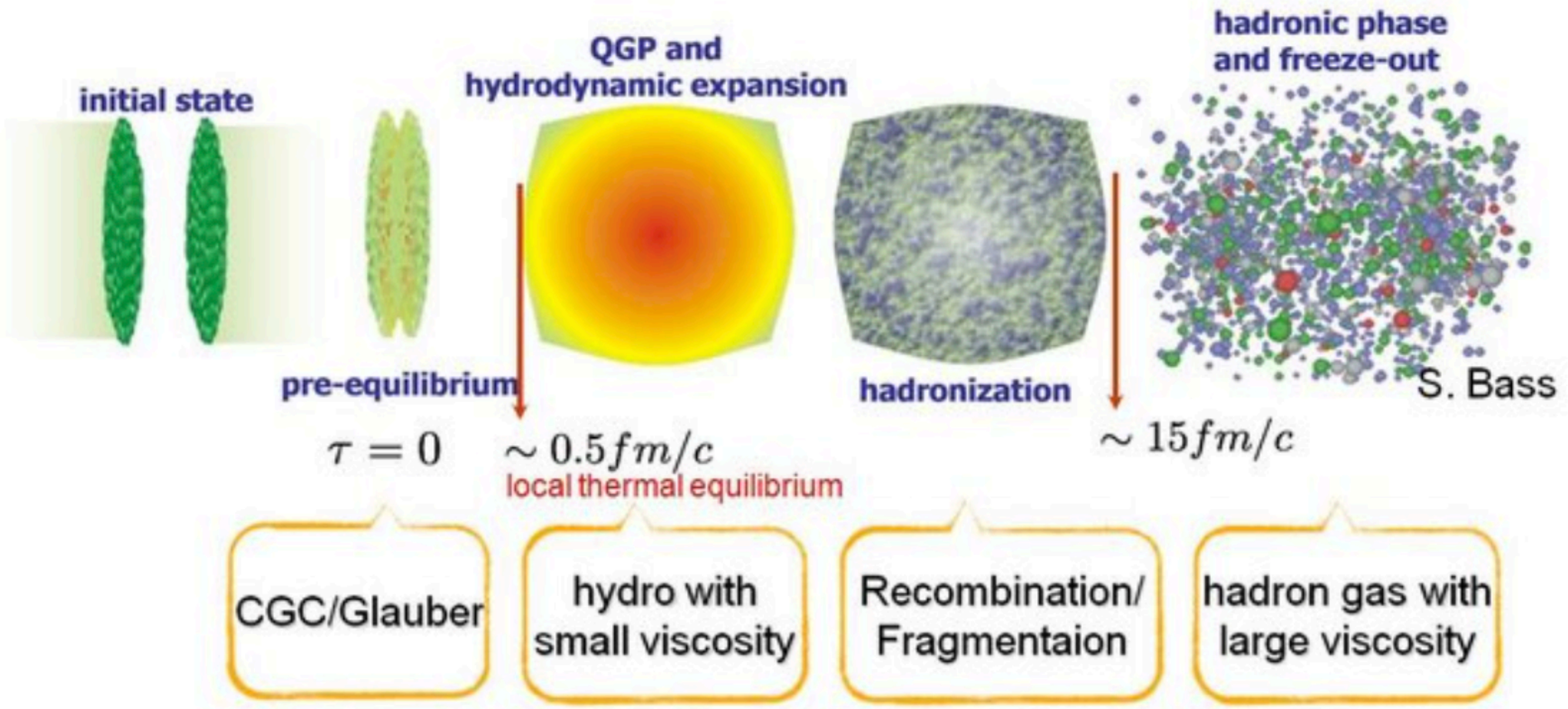
QCD thermodynamics via heavy-ion collision experiments





- Equation of State of QCD, transport

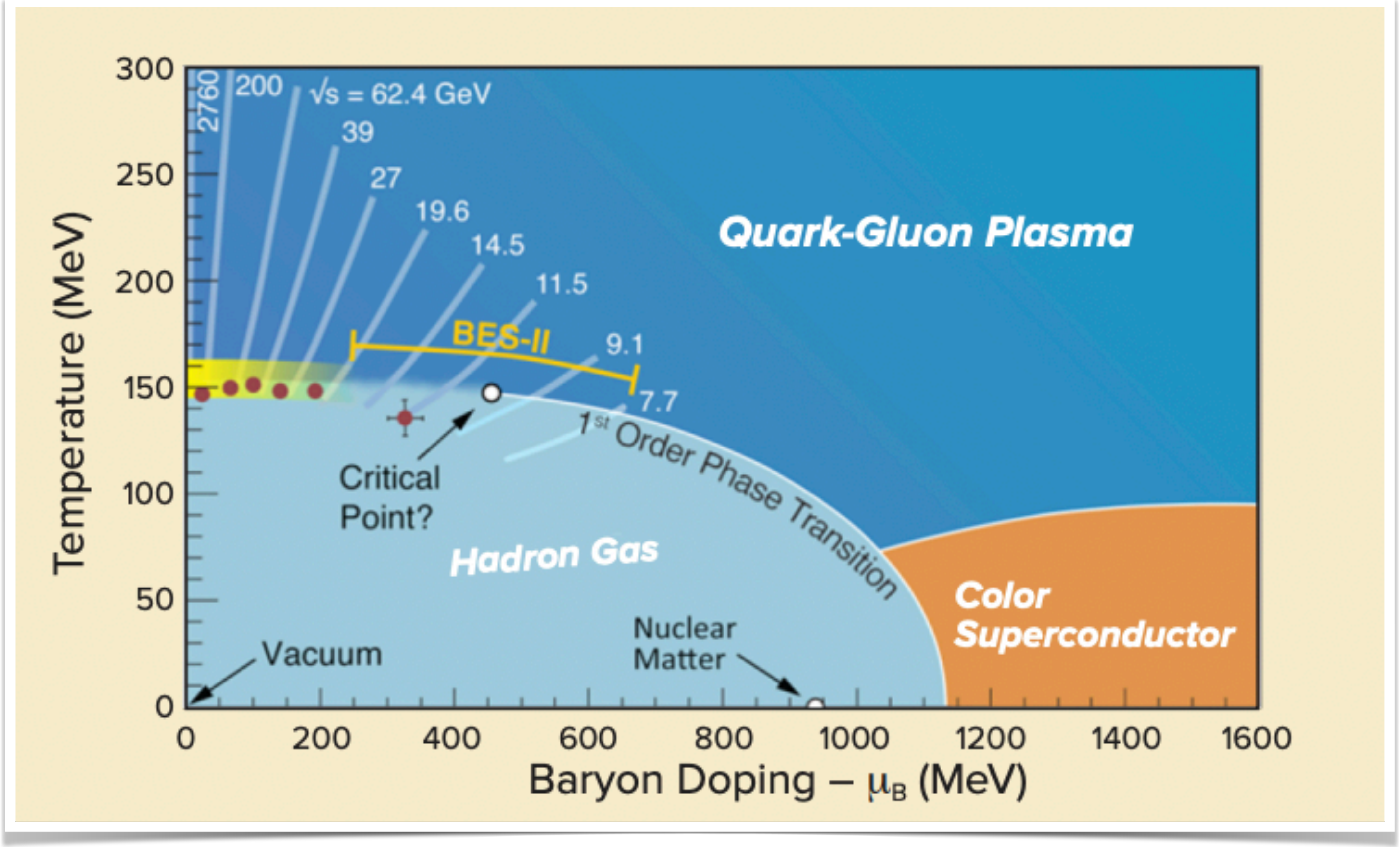
Not calculable from first-principles due to sign problem

- QCD Phase Diagram
    - Key goal; locate the **critical point** in T-μ plane

Macroscopic/ thermodynamic/emergent properties of QCD

- Anomalies in QCD - Chiral Magnetic Effect, Chiral Vortical Effect,..

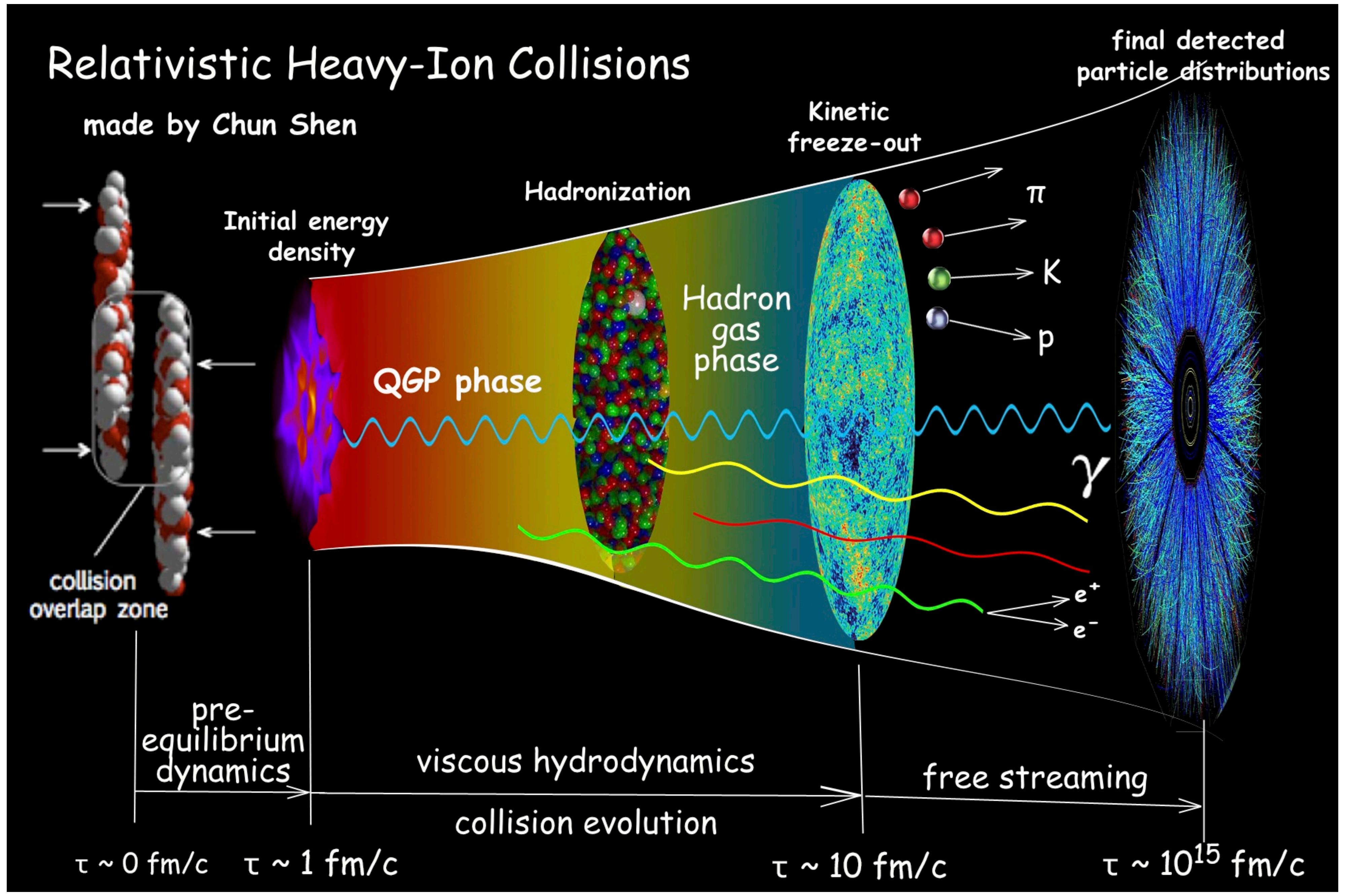
QCD Phase Diagram

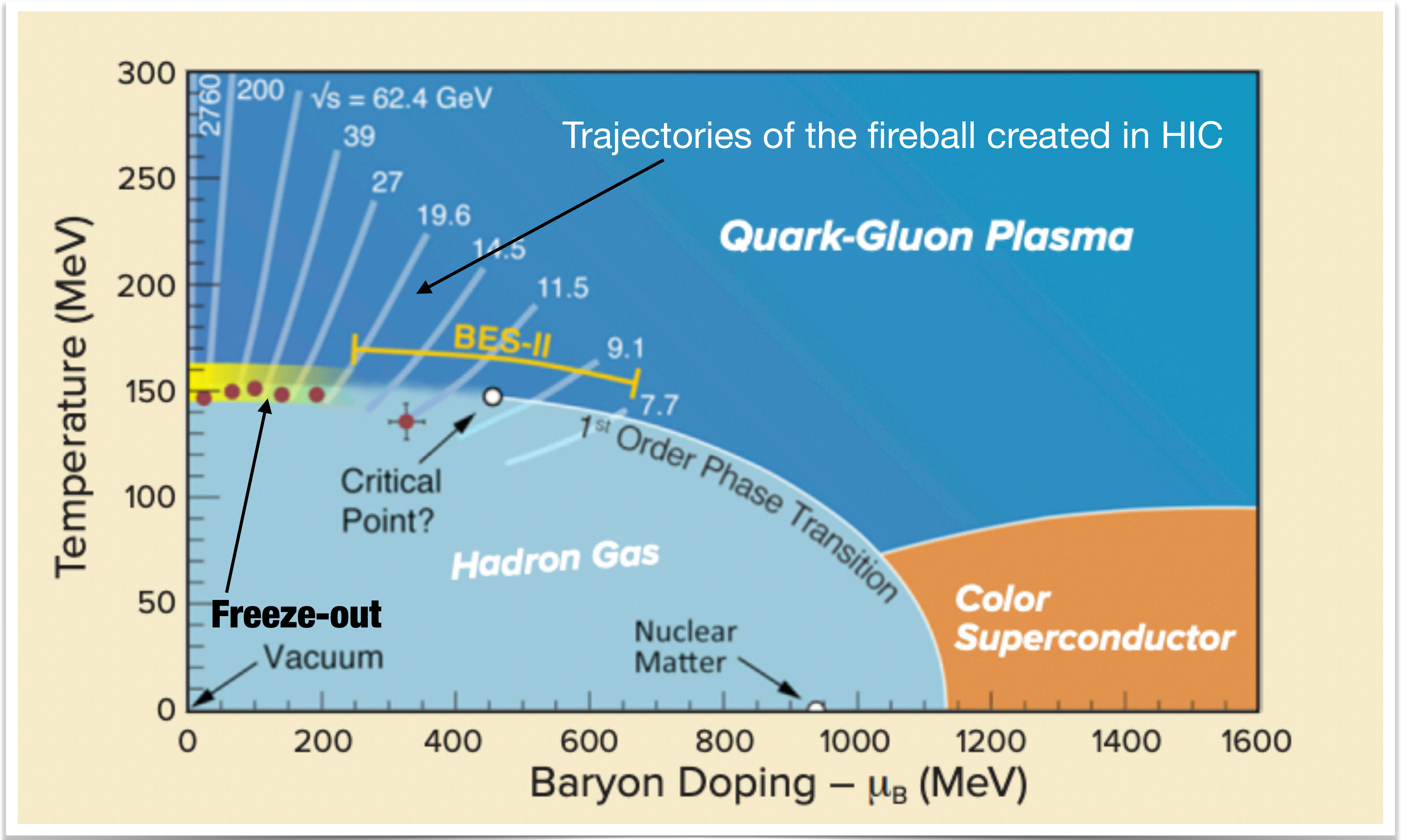


- Map of singularities of the Equation of State , $P(\mu, T)$
- At $\mu_B = 0$: It is a cross-over. — Lattice QCD
- At T= 0 : Models predict a first order phase transition

Heavy-ion collisions at varying center of mass energies can scan the phase diagram ,eg. Beam Energy Scan Program at RHIC

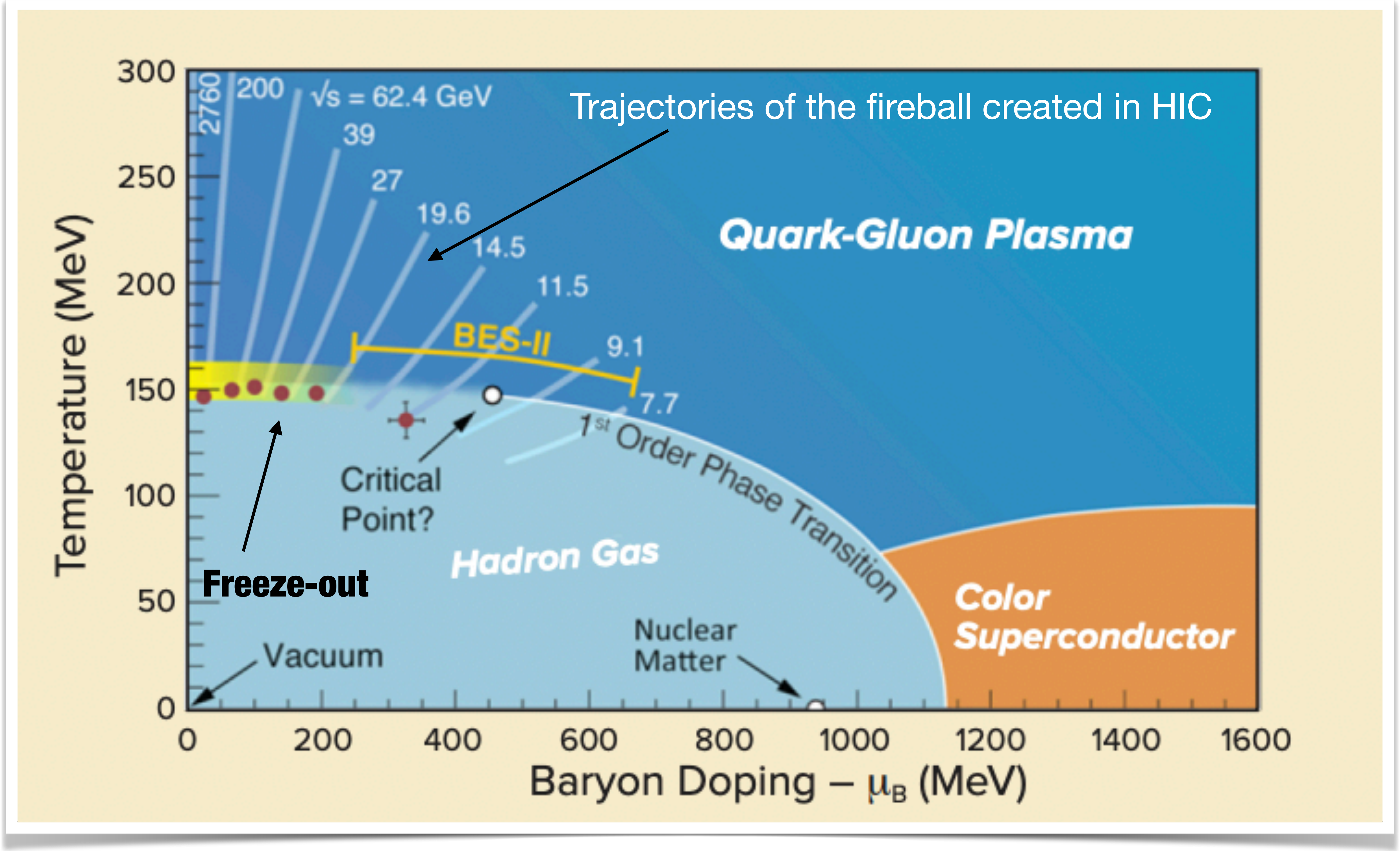
“Standard Model” of Heavy-Ion Collisions





- QGP thermalizes in a relatively short amount of time ~ 1fm/c
- Freeze-out at about 10 fm/c
- The event by event distribution of the particle multiplicities are fixed at freeze-out
- Freeze-out points move to lower values of baryon chemical potential as center of mass energy is increased

Signatures of the critical point



Fluctuations are enhanced near CP

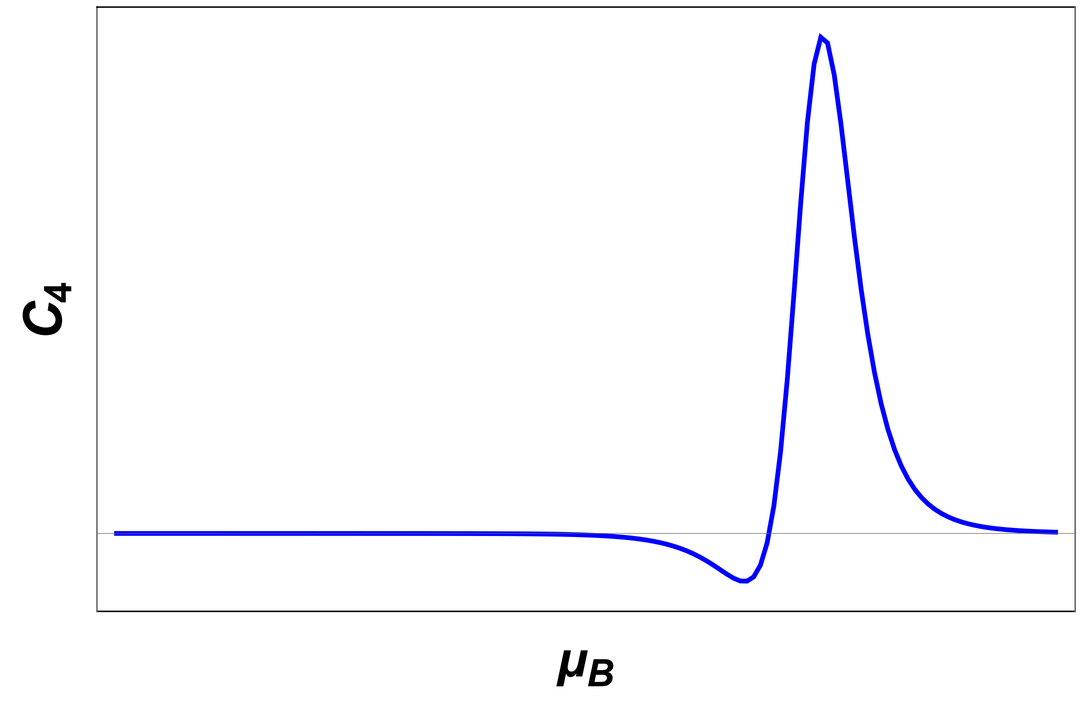
Diverges at CP

$$C_k \equiv \langle \delta N_B^k \rangle_c \overset{\text{in eq.}}{=} V T^{k-1} \frac{\partial^k P}{\partial \mu^k}$$

Connected event by event averages of baryon multiplicity fluctuations, also called cumulants

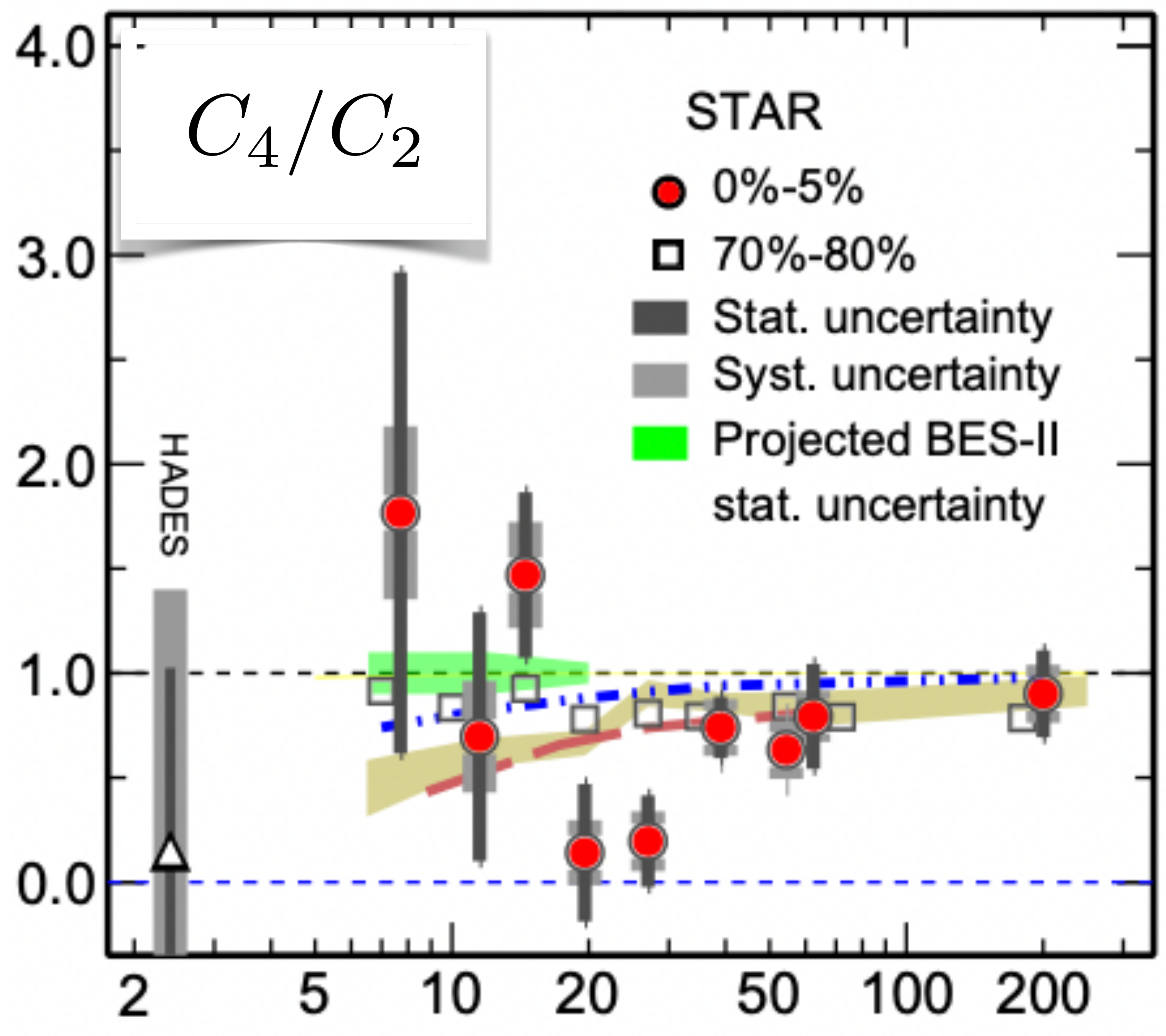
$$\sim \xi^{5(k-\eta)/2-3}$$

Correlation length



- Non-monotonic dependence of cumulants of baryon multiplicities as a function of collision energy is an indication of the presence of a critical point
- Higher cumulants of baryon multiplicities show more prominent enhancements Stephanov, 09

Relevant Observables



$$C_k = \langle (N_p - \langle N_p \rangle)^k \rangle_c$$

STAR Collaboration, 21

Beam Energy Scan- I results;

BES-II results with better precision

anticipated in near future

(For net proton multiplicity)

In a Boltzmann gas,

$$C_4/C_2 = 1$$

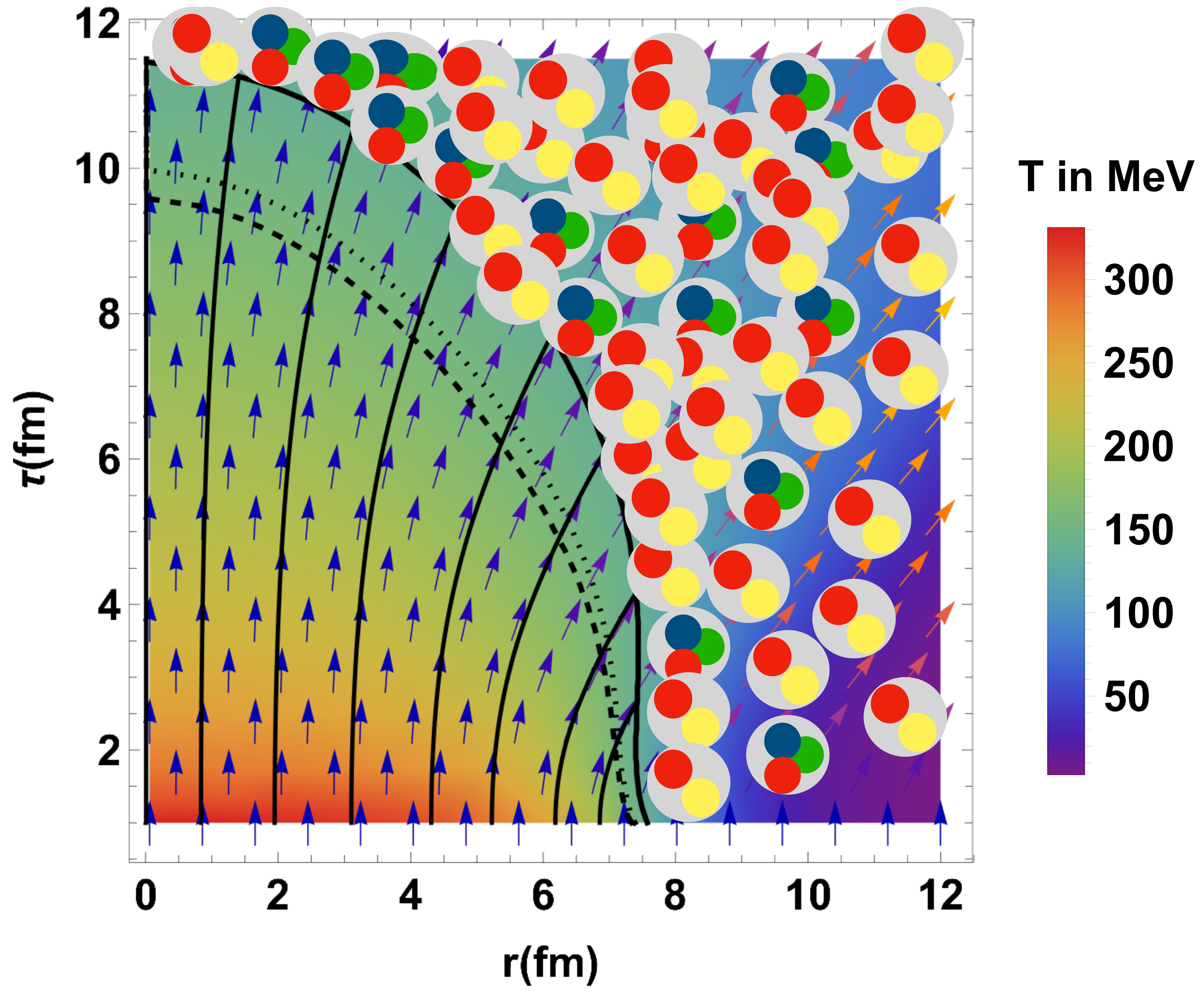
Particle Distribution Functions get frozen at Freeze-out

- Indications of a non-monotonic behavior? — signaling a Critical Point?

Require theoretical models for systematic exp-theory comparison

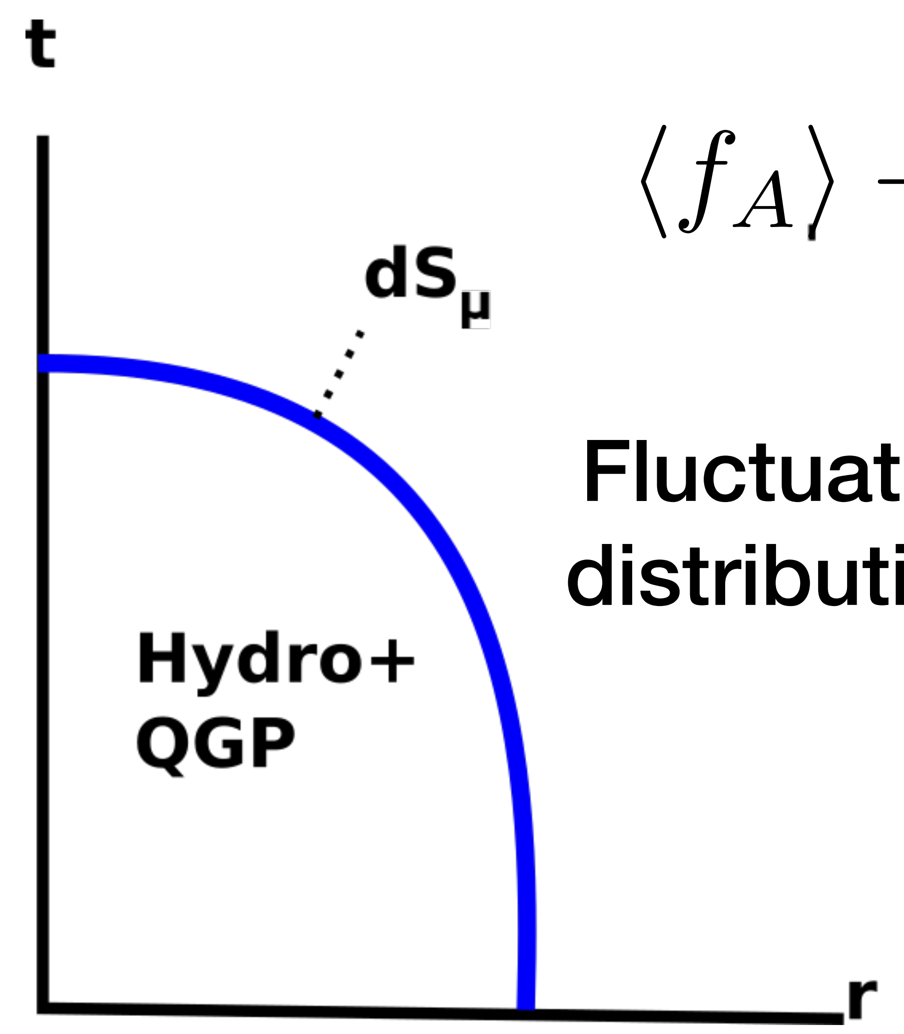
- How are the hydro fluctuations translated into the *measured* cumulants of particle multiplicities at freeze-out?

Freeze-out in heavy-ion collisions



- Evolution of fireball in the r-t plane in the center of mass frame
- Assume there is a hyper-surface in space-time where both the descriptions are valid

Freeze-out : Transition from hydrodynamics to hadron gas



$\langle f_A \rangle + \delta f_A$

Fluctuating particle distribution function

Hydrodynamic mean densities

$$\{\langle \epsilon u^{\mu} \rangle \,,\, \langle n \rangle\} \equiv \Psi^a$$

Conserved energy, momentum and charge densities and their correlations

Hydrodynamic correlations

$$\Psi^a,\, \langle \delta\Psi^a \delta\Psi^b \rangle \equiv H^{ab}, \ldots\, H^{abc\ldots}$$

Particle distribution function at freeze-out

$$\langle f_A \rangle = \bar{f}_A \,,\, \langle \delta f_A \delta f_B \rangle = G_{AB} \,,\, \langle \delta f_A \delta f_B \delta f_C \rangle = G_{ABC} \,\ldots$$

f_A **Is the phase space distribution function for species A**

Matching conditions at freeze-out

$$\langle \epsilon \, u^\mu \rangle = \sum_A \int_{p_A} \bar{f}_A \, p_A^\mu \,, \quad \langle n \rangle = \sum_A q_A \int_{p_A} \bar{f}_A$$

$$\Psi^a = \sum_A \int_{p_A} \bar{f}_A P_A^a$$

$$P_A = \begin{bmatrix} p_A^\mu \\ q_A \end{bmatrix}$$

$$H^{abc\ldots} = \sum_{A,B,C,\ldots} \int_{p_A p_B p_C \ldots} G_{ABC\ldots} P_A^a P_B^b P_C^c \ldots$$

- Matching conditions for averages of conserved densities
- Infinitely many sets of distribution functions that satisfy these matching conditions
- Freeze-out prescription corresponds to choosing one of these sets - **How to choose**?

Maximum entropy approach to freeze-out

Given only the information about the hydrodynamic densities on the freeze-out hypersurface, what is the *the least biased* ensemble of free streaming particles after freeze-out that obeys the conservation laws?

Hydro

- EoS
- Conserved Densities and/or their Higher Point Correlations

HRG

- Masses, degeneracies
- Quantum numbers of the hadrons

Maximum entropy freeze-out of ordinary hydrodynamics

$$\langle \epsilon\, u^\mu \rangle = \sum_A \int_{p_A} \bar{f}_A\, p_A^\mu\,, \quad \langle n \rangle = \sum_A q_A \int_{p_A} \bar{f}_A$$

$$S_A[f_A] = f_A\,[1 - \log f_A]$$ (Classical statistics)

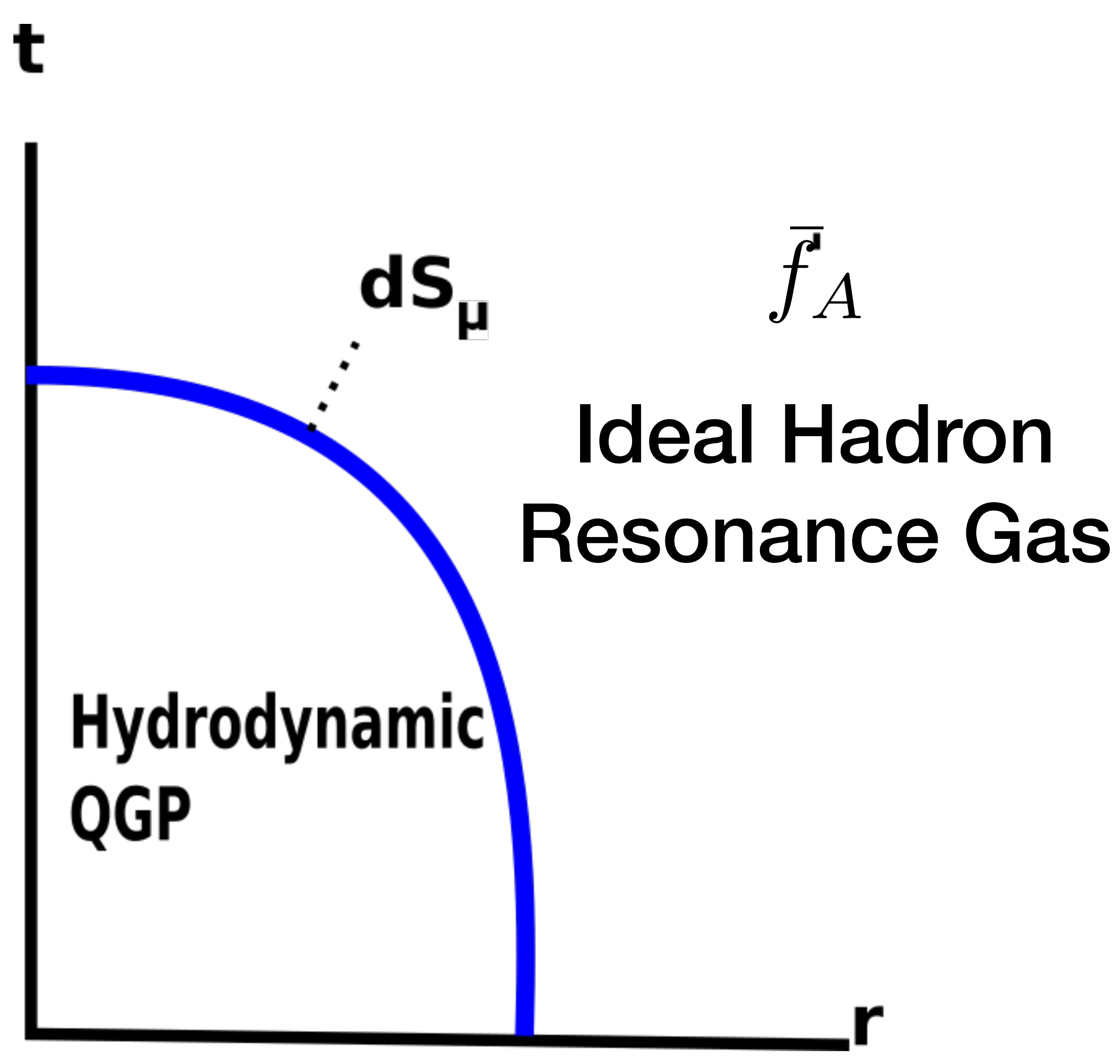
Maximize thermodynamic entropy subject to conservation equations

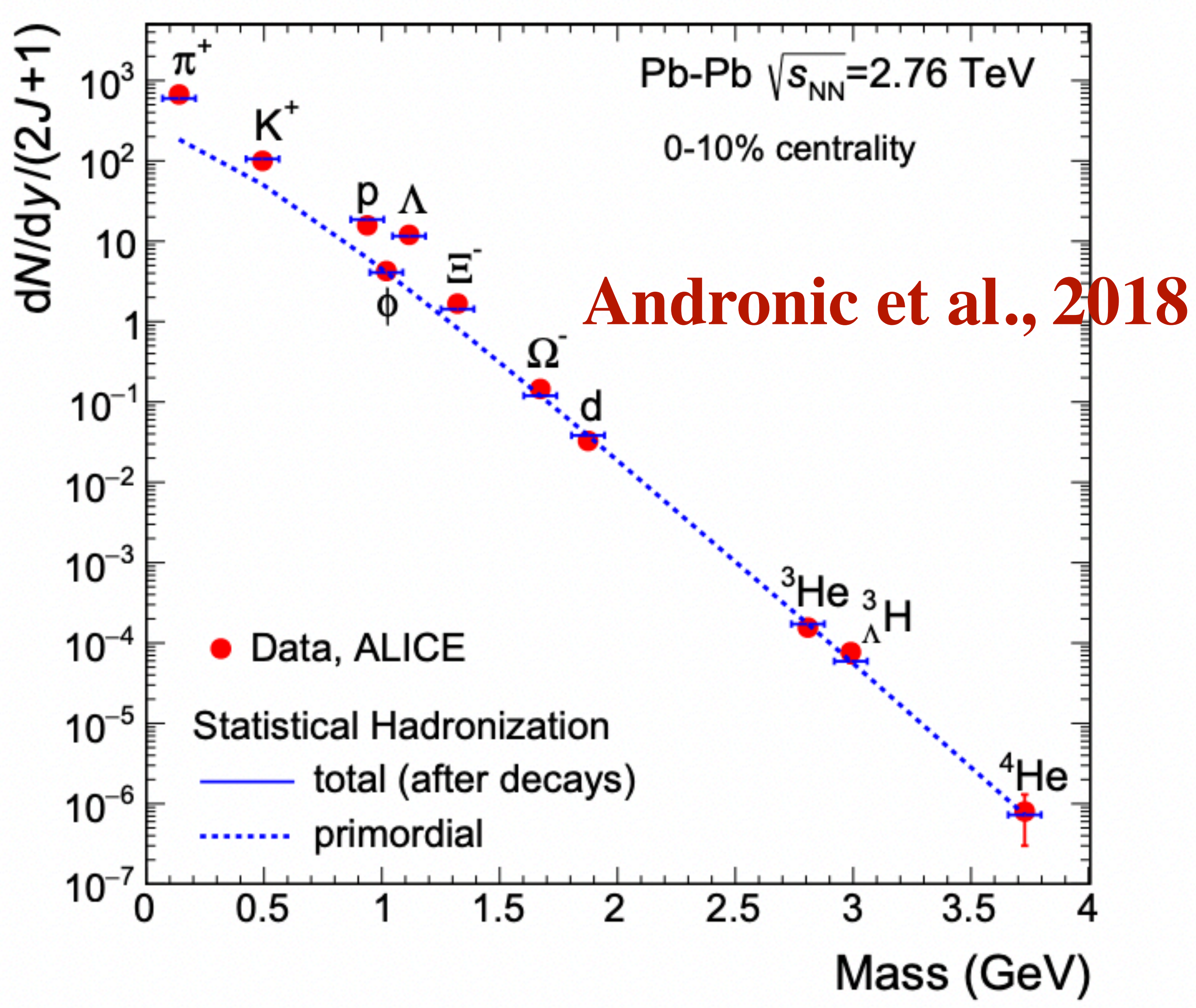
Solution - well known Cooper-Frye freeze-out (74) $\bar{f}_A \sim e^{-(u^\mu p_\mu - q_A \mu_B)/T}$

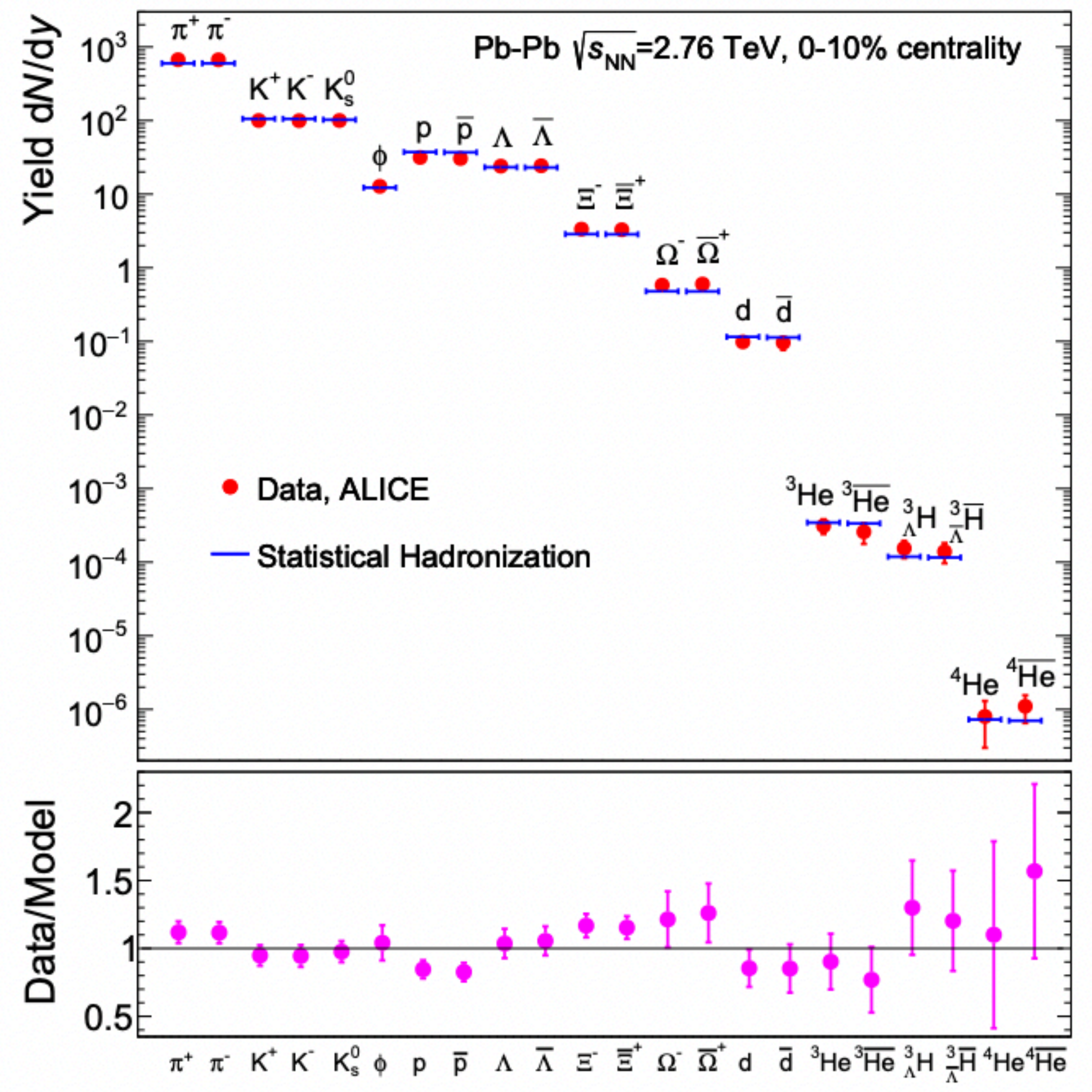
Hydrodynamic system freezes out into ideal hadron-resonance gas with the same values of conserved densities

Cooper-Frye freeze-out

Cooper, Frye, 74







Cooper-Frye procedure doesn't freeze-out of fluctuations

$$\langle N_A \rangle = \int dS_\mu \int Dp\, p^\mu\, \bar{f}_A(x,p)$$

Averages of conserved densities are matched

Maximum entropy approach to freeze-out hydrodynamic fluctuations

MP, Stephanov, 23

More entropy = Less information(assumptions) we have(make) about the system

- Generalization to Cooper-Frye freeze-out to include fluctuations
- Maximum entropy principle gives the least biased ensemble of hadron resonance gas
- Matches all the information about the conserved densities and their correlations
- Can be employed to freeze-out non-Gaussian correlations of all kinds of hydrodynamic densities

Matching correlations of conserved densities

Hydrodynamic correlations

$$\Psi^a, \left\langle \delta\Psi^a \delta\Psi^b \right\rangle \equiv H^{ab}, \ldots H^{abc\ldots}$$

Particle distribution function at freeze-out

$$\langle f_A \rangle = \bar{f}_A \,, \langle \delta f_A \delta f_B \rangle = G_{AB} \,, \langle \delta f_A \delta f_B \delta f_C \rangle = G_{ABC} \ \ldots$$

$$H^{abc\ldots} = \sum_{A,B,C,\ldots} \int_{p_A p_B p_C \ldots} G_{ABC\ldots} P_A^a P_B^b P_C^c \ldots$$

$$P_A = \begin{bmatrix} p_A^\mu \\ q_A \end{bmatrix}$$

More degrees of freedom on the kinetic side

Infinitely many solutions for the conservation equations

Equilibrium

$$S[\bar{f}]$$

Which entropy to maximize?

Generalized

$$S[P(f)]$$

$$S[P(f)] \; = \; S_0 \left[\bar{f}\right] + \int_f P(f) \log \frac{P_{\rm eq}(f)}{P(f)}$$

$$S[\bar{f}, G_2, G_3, \ldots]$$

G s are the correlation functions in the Hadron Gas description

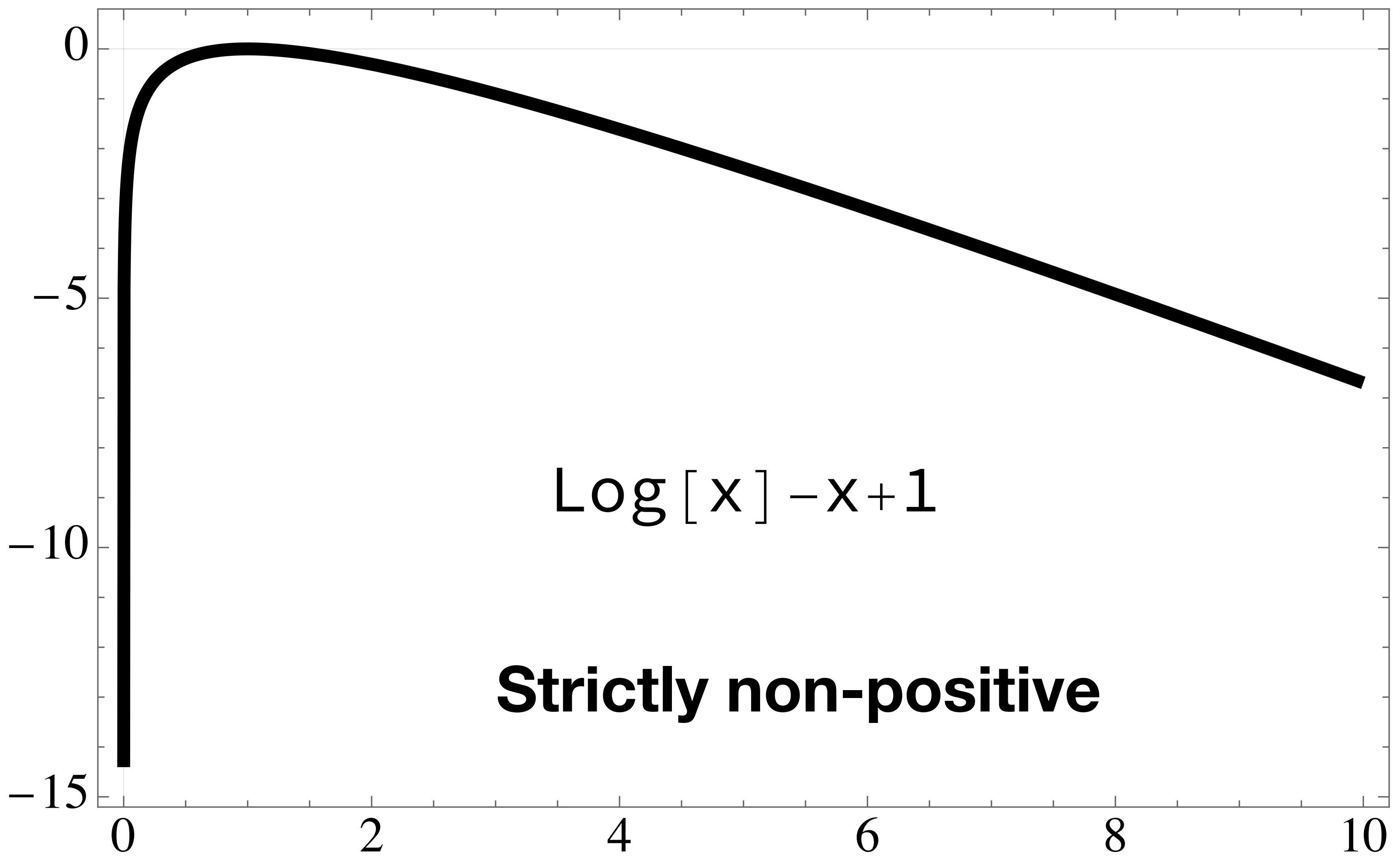
$$S_0[\bar{f}] = -\int_f P_{\rm eq}(f) \log P_{\rm eq}(f)$$

- Maximize the relative entropy when correlations are out of equilibrium
- Constraints from matching conditions

Entropy to describe out-of equilibrium two-point correlations in ideal HRG

$$S_2 = S_0 + \frac{1}{2}\mathrm{Tr}\left[\log G\bar{G}^{-1} - G\bar{G}^{-1} + 1\right] \ ,$$

Equilibrium



Similar 2-PI action

Berges, 04, Stephanov, Yin, 17…

2-PI entropy

Upon maximizing the 2PI entropy, subject to constraints of conservation

$$G^{-1}_{AB} = \bar{G}^{-1}_{AB} + (H^{-1} - \bar{H}^{-1})^{ab}(\bar{H}^{-1}P\bar{G})^a_A(\bar{H}^{-1}P\bar{G})^b_B$$

When all but two-point correlations are in equilibrium, the solution given above is exact.

Linearizing,

$$G_{AB\ldots} = \bar{G}_{AB} + \Delta H_{ab}(\bar{H}^{-1}P\bar{G})^a_A(\bar{H}^{-1}P\bar{G})^b_B + \ldots,$$

Self correlations

Contribution of self correlations to hydrodynamics is subtracted

$$\Delta H^{ab} = H^{ab} - \bar{H}^{ab}$$

Contribution of self correlations to hydrodynamics

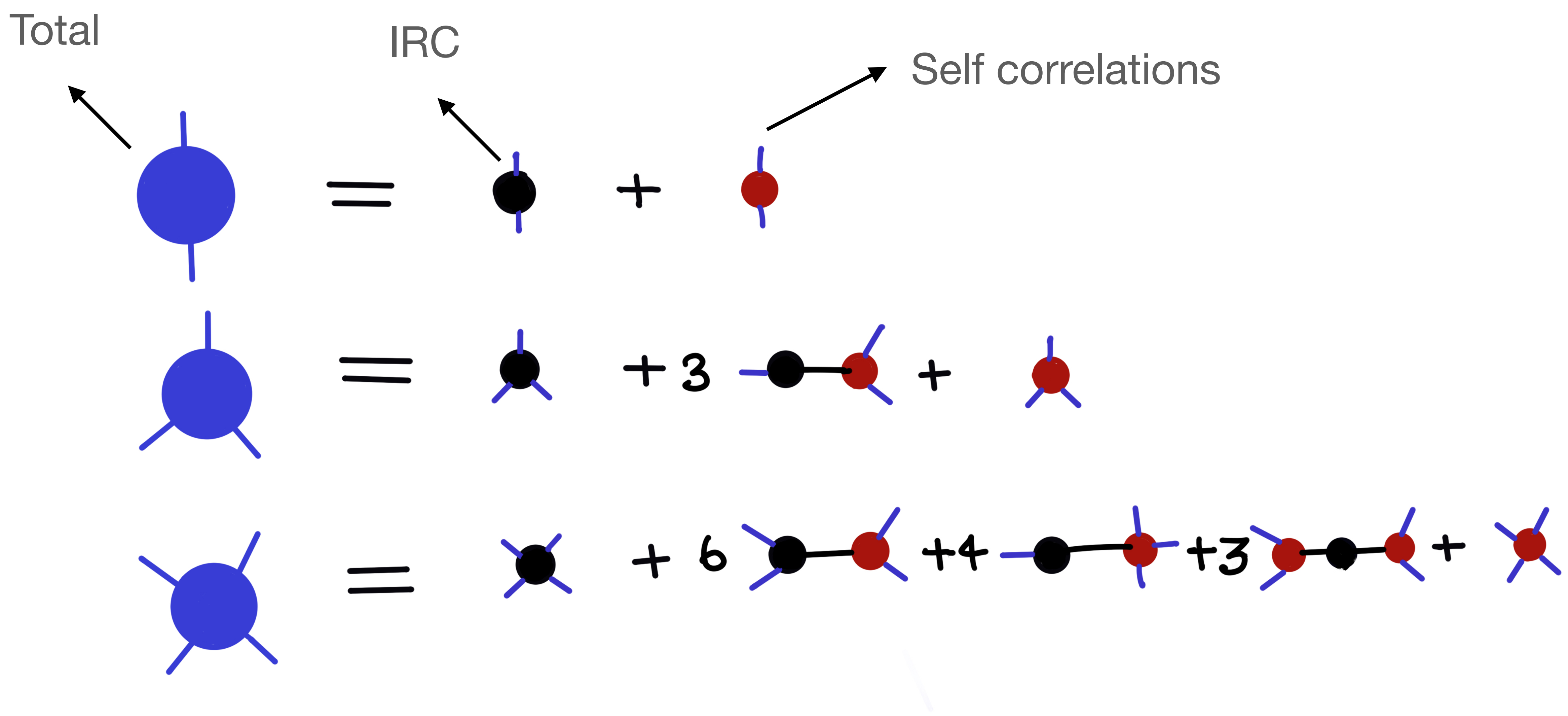
$$\bar{H}_{ab} = \sum_{A,B}\int \bar{G}_{AB}P^a_A P^b_B$$

Generalization to Non-Gaussian Correlations

IRC — Equilibrium — IRC

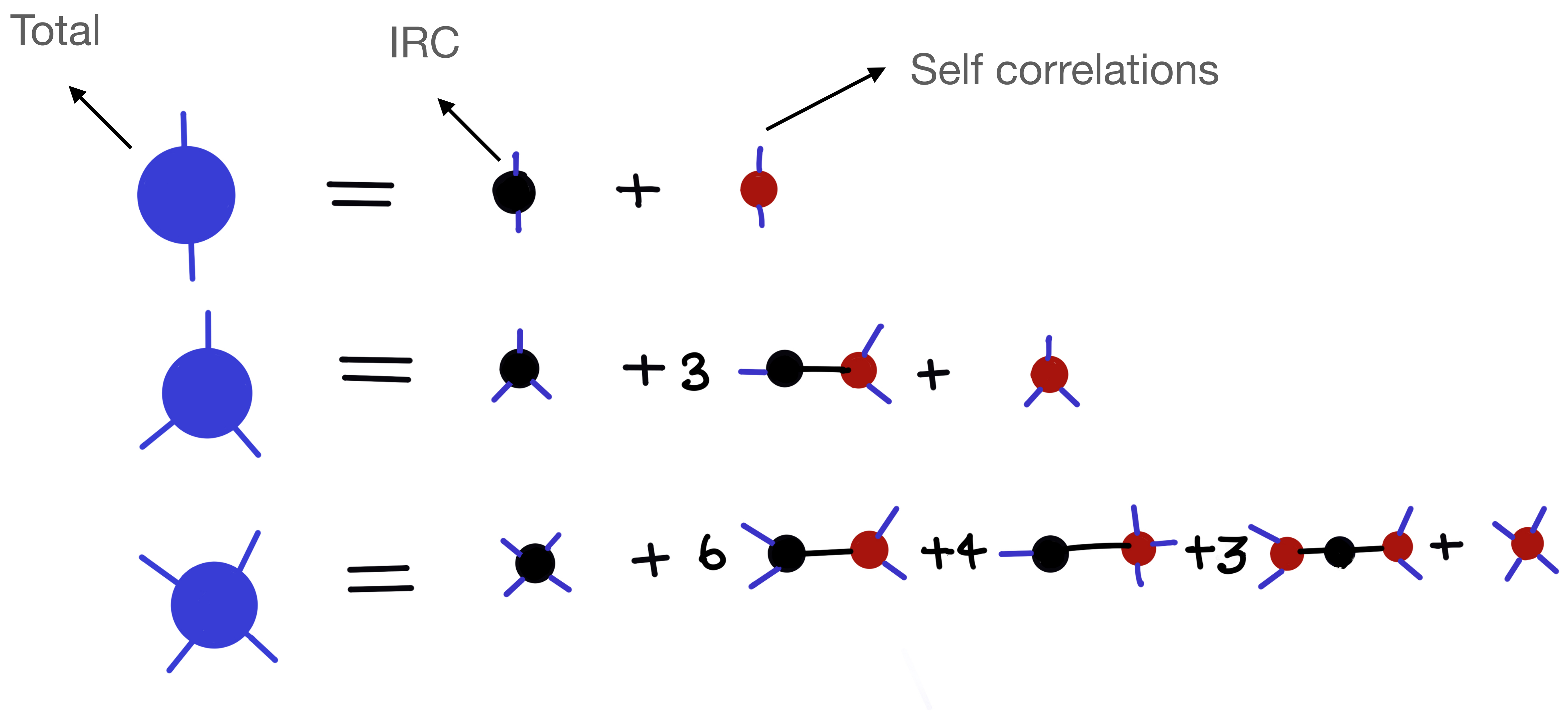
$$\hat{\Delta} G_{ABC\ldots} = \mathcal{F}(\bar{H}, \bar{G})\, \hat{\Delta} H_{ABC\ldots}$$

For classical gas, irreducible relative cumulants (IRCs) reduce to so called "factorial cumulants".



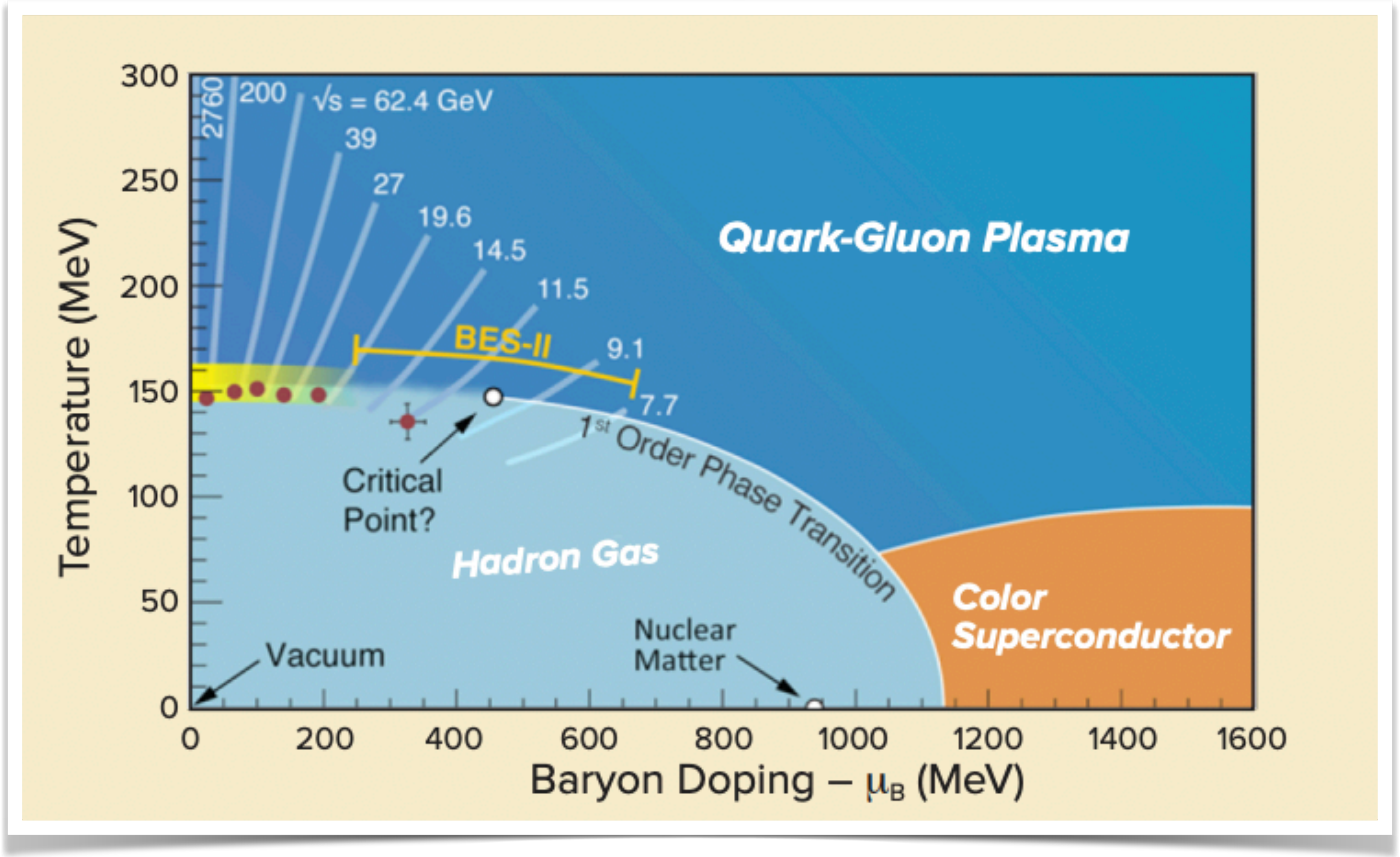
Irreducible relative cumulants

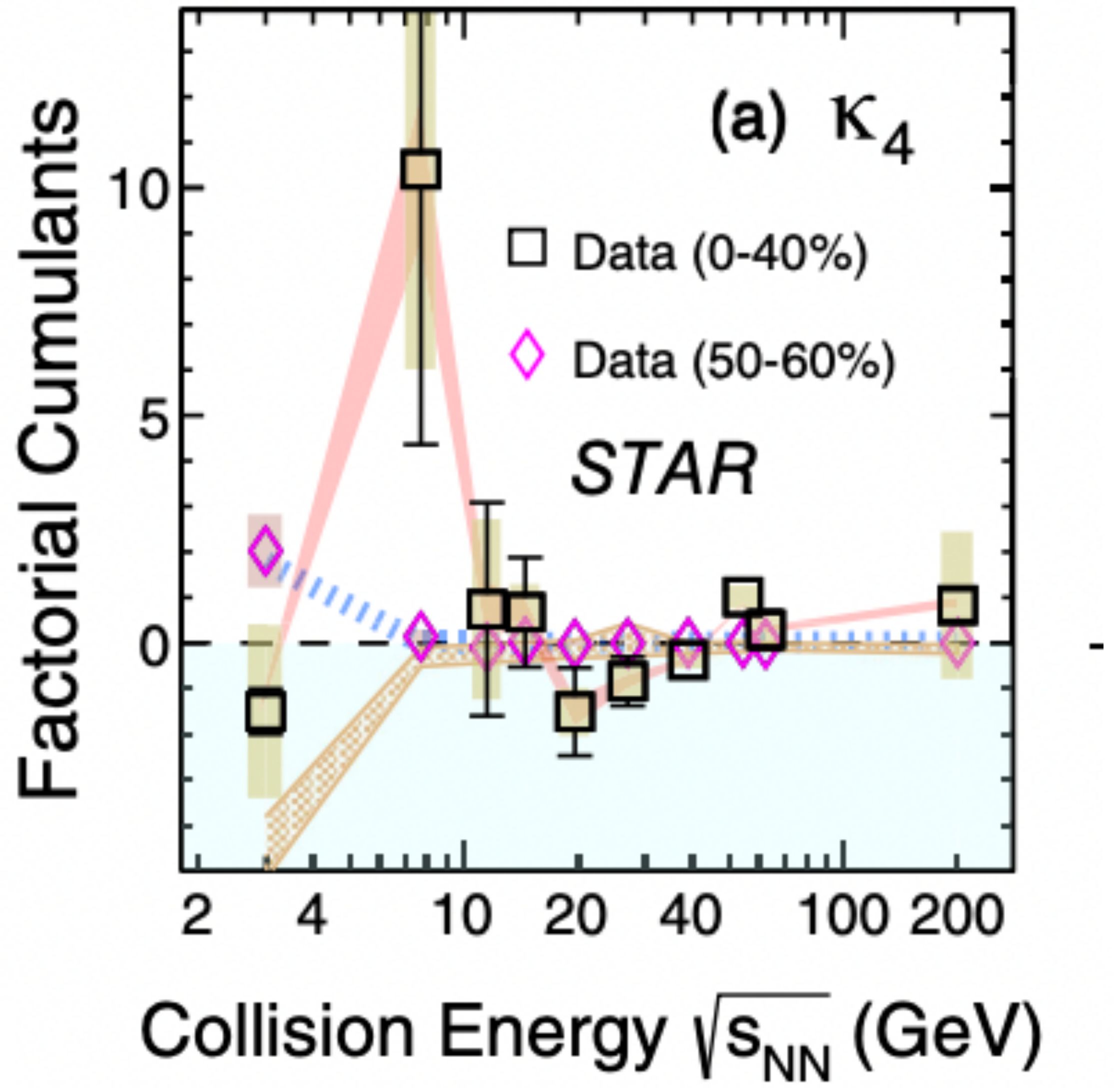
For classical gas, irreducible relative cumulants (IRCs) reduce to so called “factorial cumulants”.



- For gases obeying different statistics, IRCs quantify the non-trivial correlations
- Non-trivial correlations relative to any specified baseline distribution

Returning to the bigger picture : Is there a critical point in the QCD phase diagram?





Fourth factorial cumulants for proton multiplicity

Intriguing results from Beam Energy Scan - I (STAR: PRL 130, 082301 (2023))

If there is a CP in the QCD phase diagram,

Build a theoretical framework for describing fluctuations in HICs.

- How ***large*** are the cumulants of particle multiplicities in ***equilibrium***?
- In a realistic evolution in HICs, how large can they grow?

Crucial elements of a theoretical description of thermodynamic fluctuations for BES program

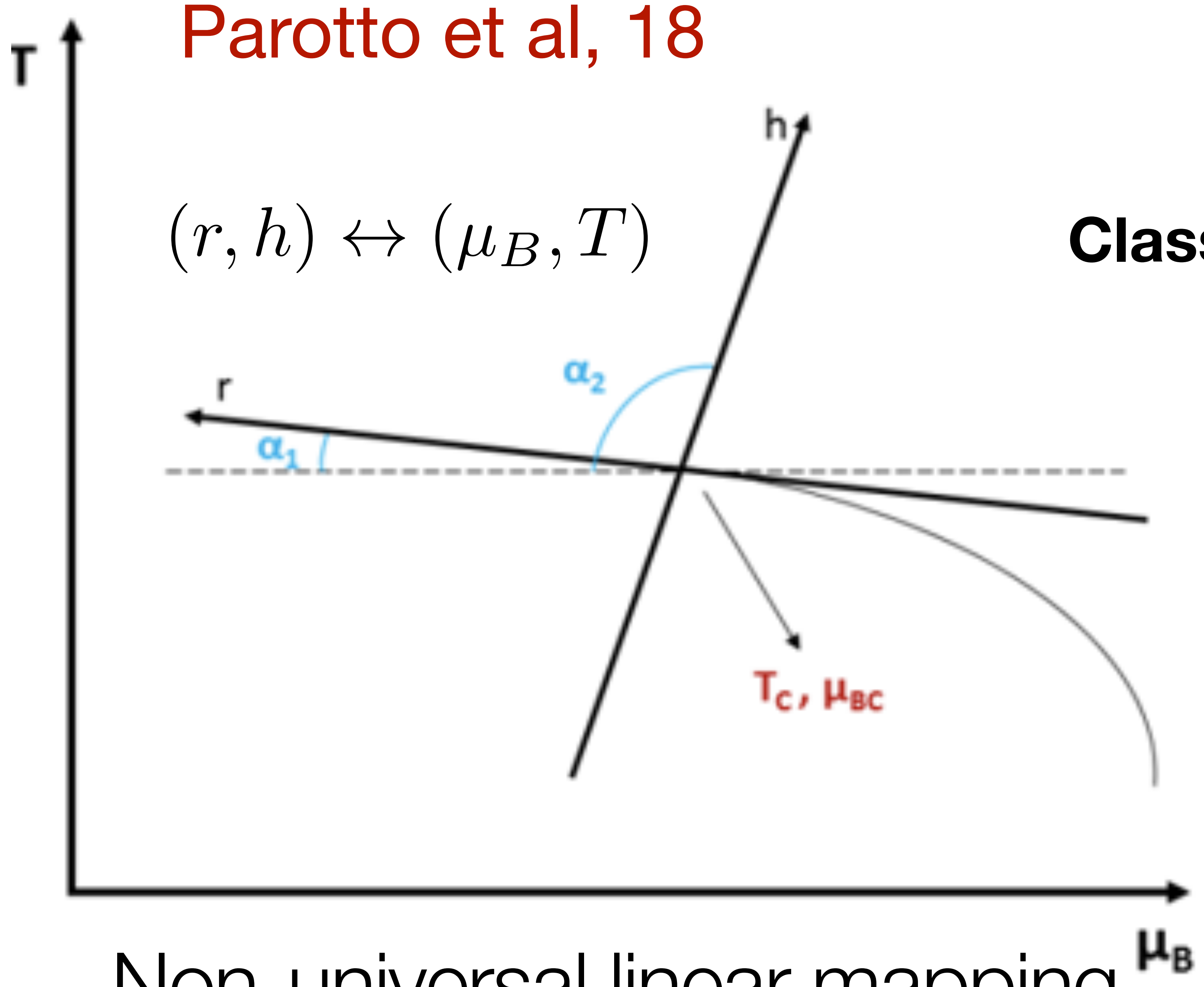
- QCD EoS
- Evolve the hydrodynamic fluctuations - until freeze-out —Hydro+
- Freeze-out procedure

Hydrodynamic correlation functions in equilibrium

QCD EoS is unknown from first principles

$$P_{\mathrm{QCD}}(\mu,T) = P_{\mathrm{bg}}(\mu,T) + A\,\overbrace{G(r,h)}^{\text{universal}}$$

Parotto et al, 18

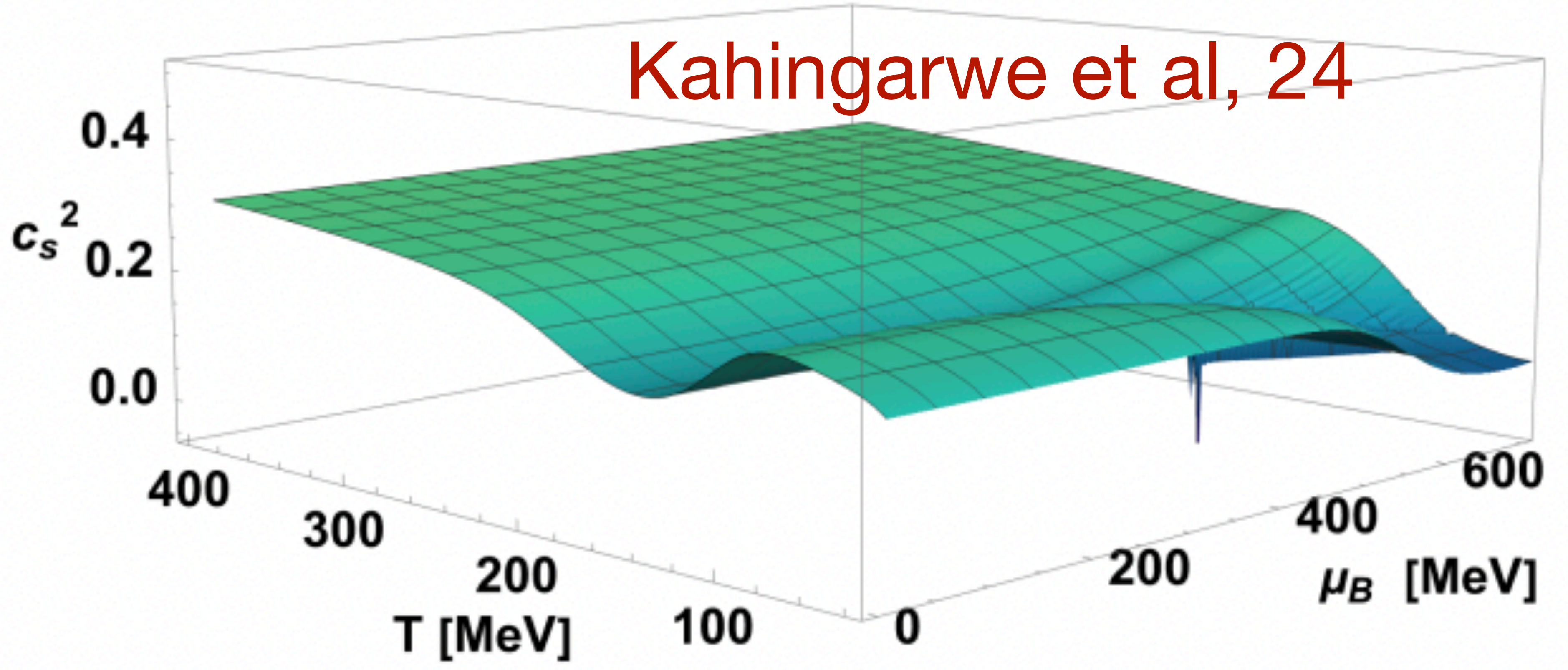


Non-universal linear mapping

Class of QCD EoS with a CP

- **r and h are linear functions of μ and T**

Kahingarwe et al, 24



$$\frac{\langle \delta N_A^k \rangle_{\mathrm{eq}}}{\langle N_A \rangle} = 1 + \#_A(\text{Mapping Parameters})\partial^k G$$

Karthein, MP, Stephanov, Rajagopal, Yin (In preparation)

Realistic scenario

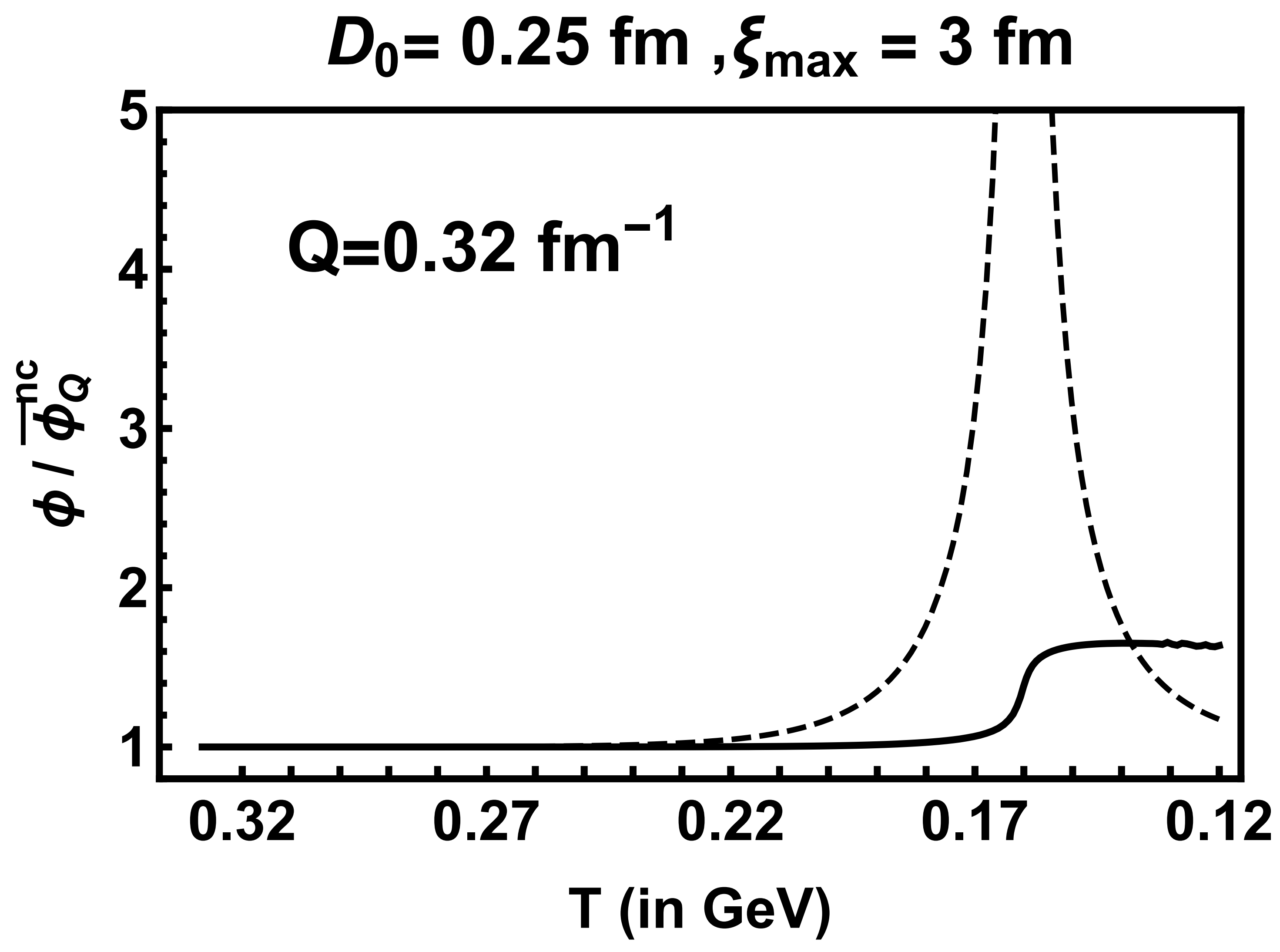
$$\frac{\langle \delta N_A^k \rangle}{\langle N_A \rangle} = 1 + \int_{Q_1 \dots Q_{k-1}} \#_A(\text{Mapping Parameters}; Q_1 \dots Q_{k-1}) \Delta \tilde{H}^k (Q_1, \dots Q_{k-1})$$

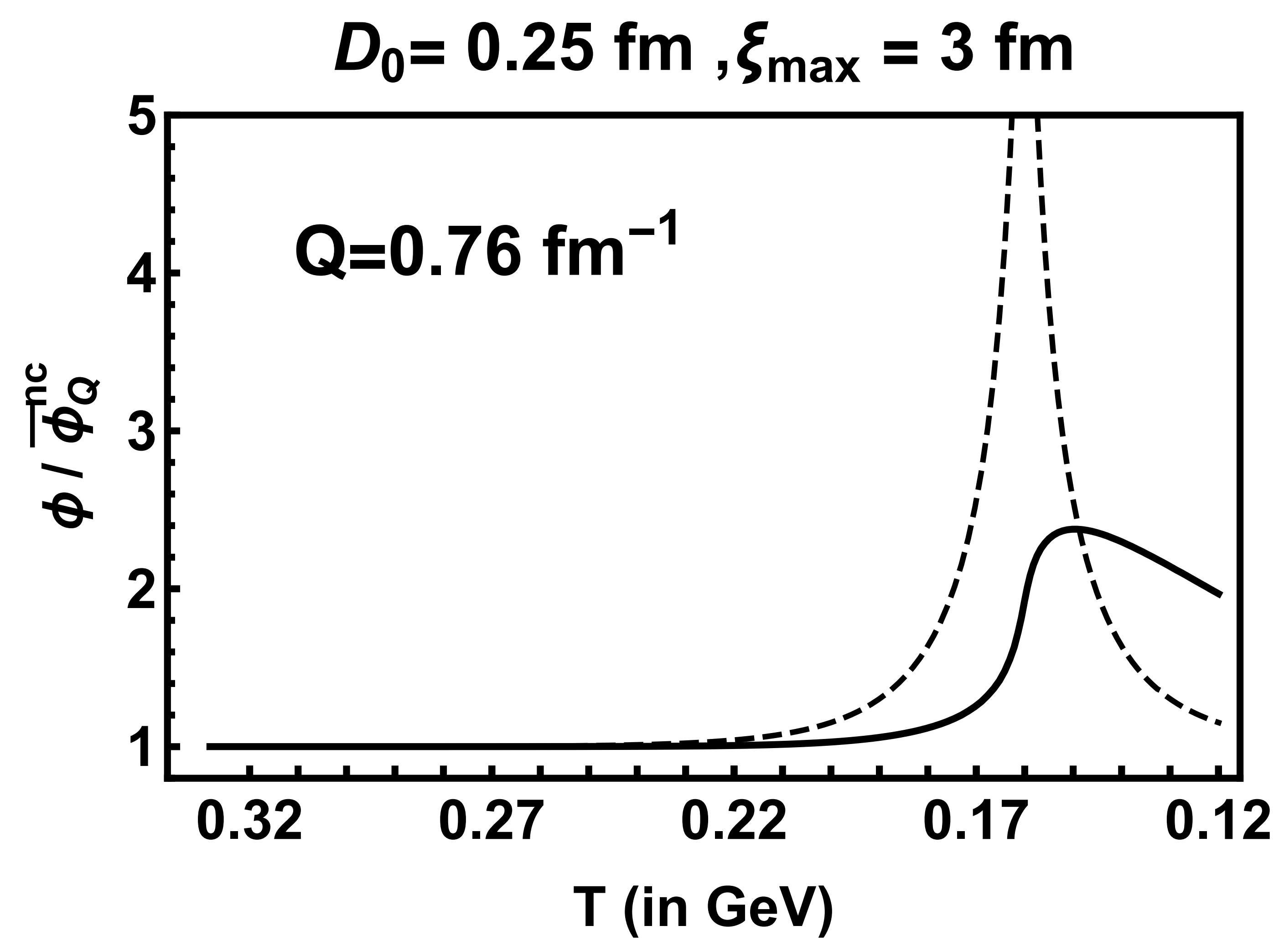
$$u \cdot \partial \phi_{\mathbf{Q}} = -\Gamma(|Q|\xi) \left(\phi_{\mathbf{Q}} - \bar{\phi}_{\mathbf{Q}} \right)$$

$$\Gamma(\mathbf{Q}\xi) \approx \frac{2D_0 \xi_0}{\xi} \mathbf{Q}^2$$

Stephanov, Yin, 17

$$\langle \delta \hat{s}(x_+) \delta \hat{s}(x_-) \rangle = \int e^{i\mathbf{Q}\cdot\mathbf{\Delta x}} \phi_{\mathbf{Q}} \,, \quad \mathbf{\Delta x} = x_+ - x_-$$

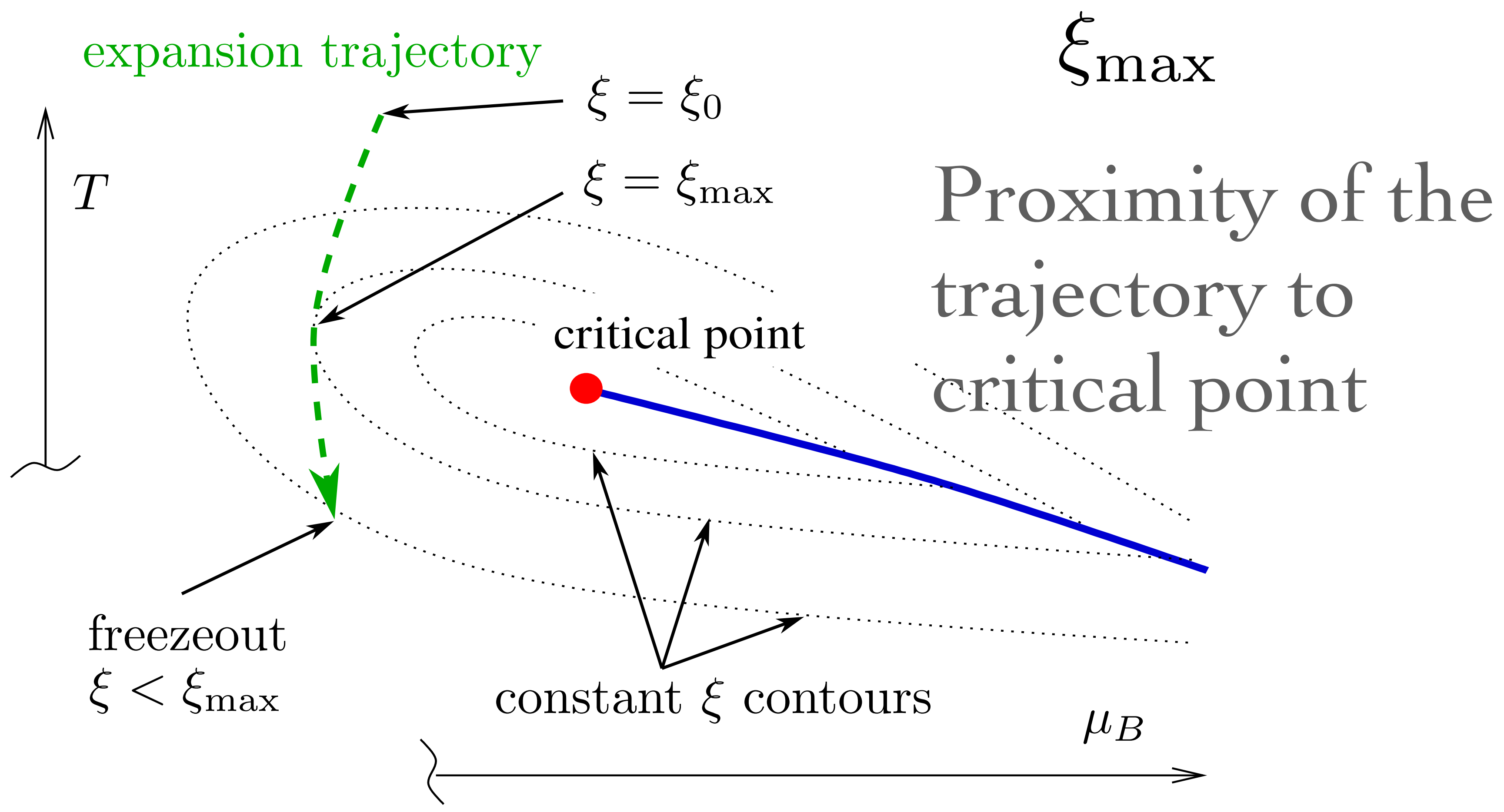


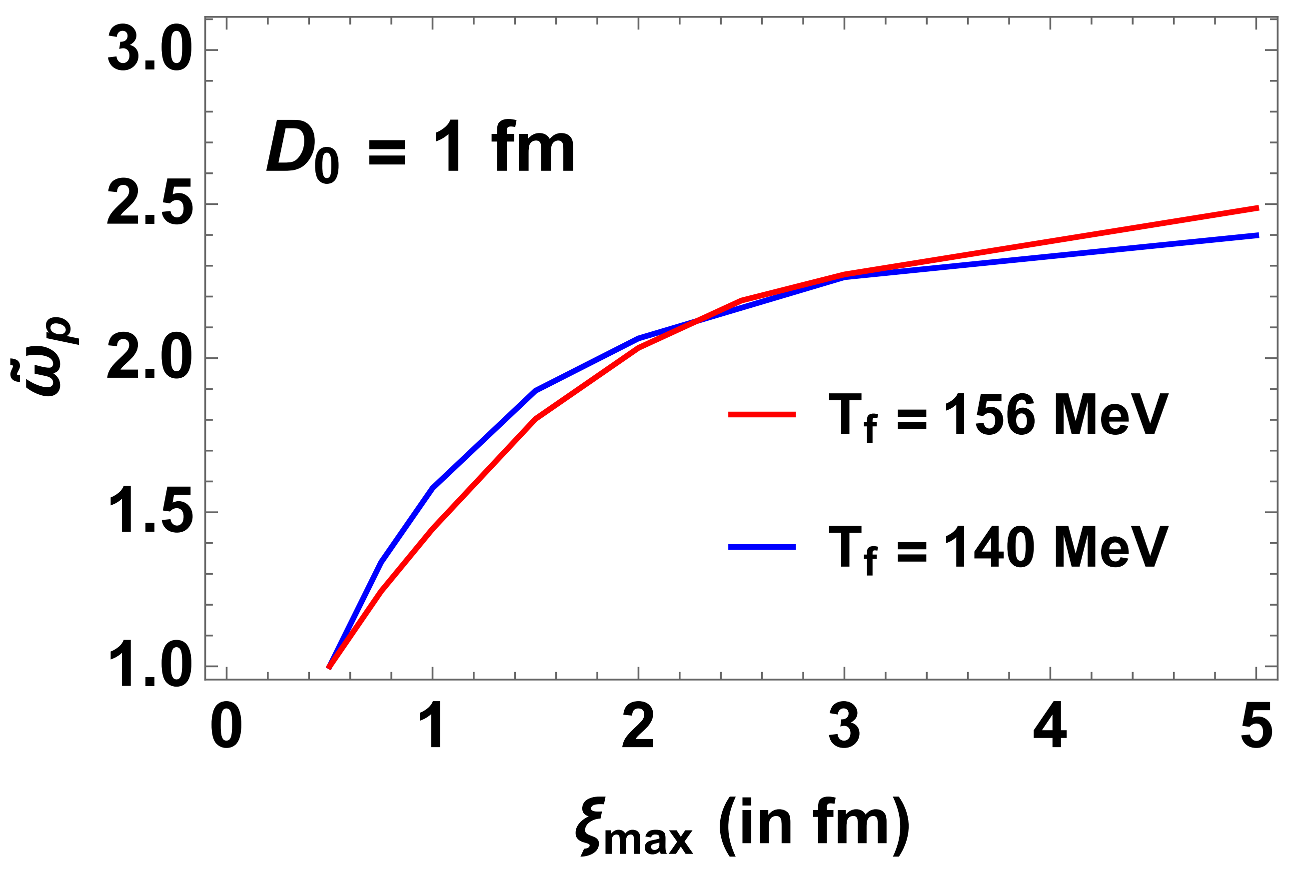


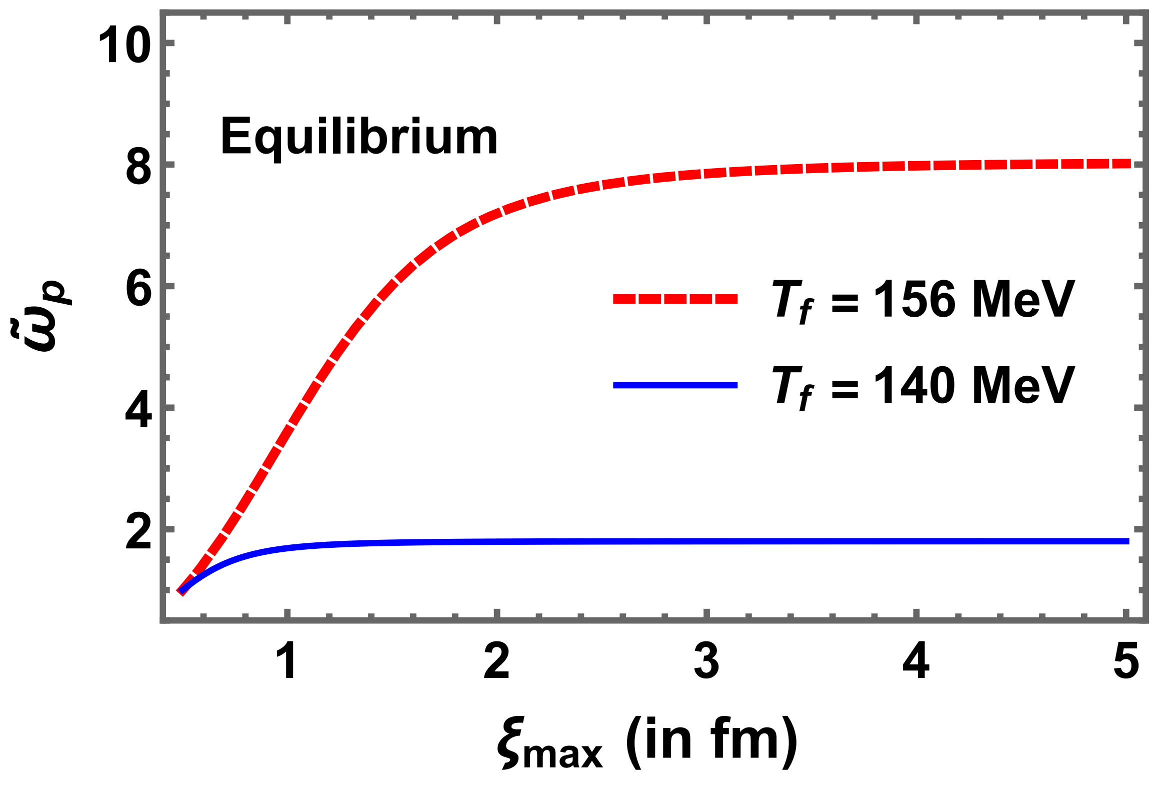
Critical Slowing down

MP, Stephanov, Rajagopal, Yin, 22

Cumulants out of equilibrium







MP, Stephanov, Rajagopal, Yin, 22

- The fluctuations are **reduced relative to equilibrium** value (conservation laws)
- Fluctuations increase with D_0 (faster diffusion)
- Compared to the equilibrium scenario, the fluctuations are **less sensitive to freeze-out temperature**

Numerical Implementation of Maximum Entropy Freezeout of semi-realistic cumulants of proton multiplicity
Karthein, MP, Stephanov, Rajagopal, Yin (In preparation)

Summary

- Fluctuations are inevitable in any realistic system, enhanced near CP
- Dynamics of fluctuations have important consequences for their magnitude at freeze-out , it also reduces the sensitivity to the freeze-out location
- We have developed a general prescription for freezing out hydrodynamic fluctuations- Maximum entropy approach

Ongoing…

- Test in ***simplified*** scenarios
- Identify the most ***consequential*** parameters/***time scales*** in the paradigm
- Identify possible ***challenges***
- Make ***estimates***

BEST EoS

↓

Hydro+ evolution

↓

ME freezeout

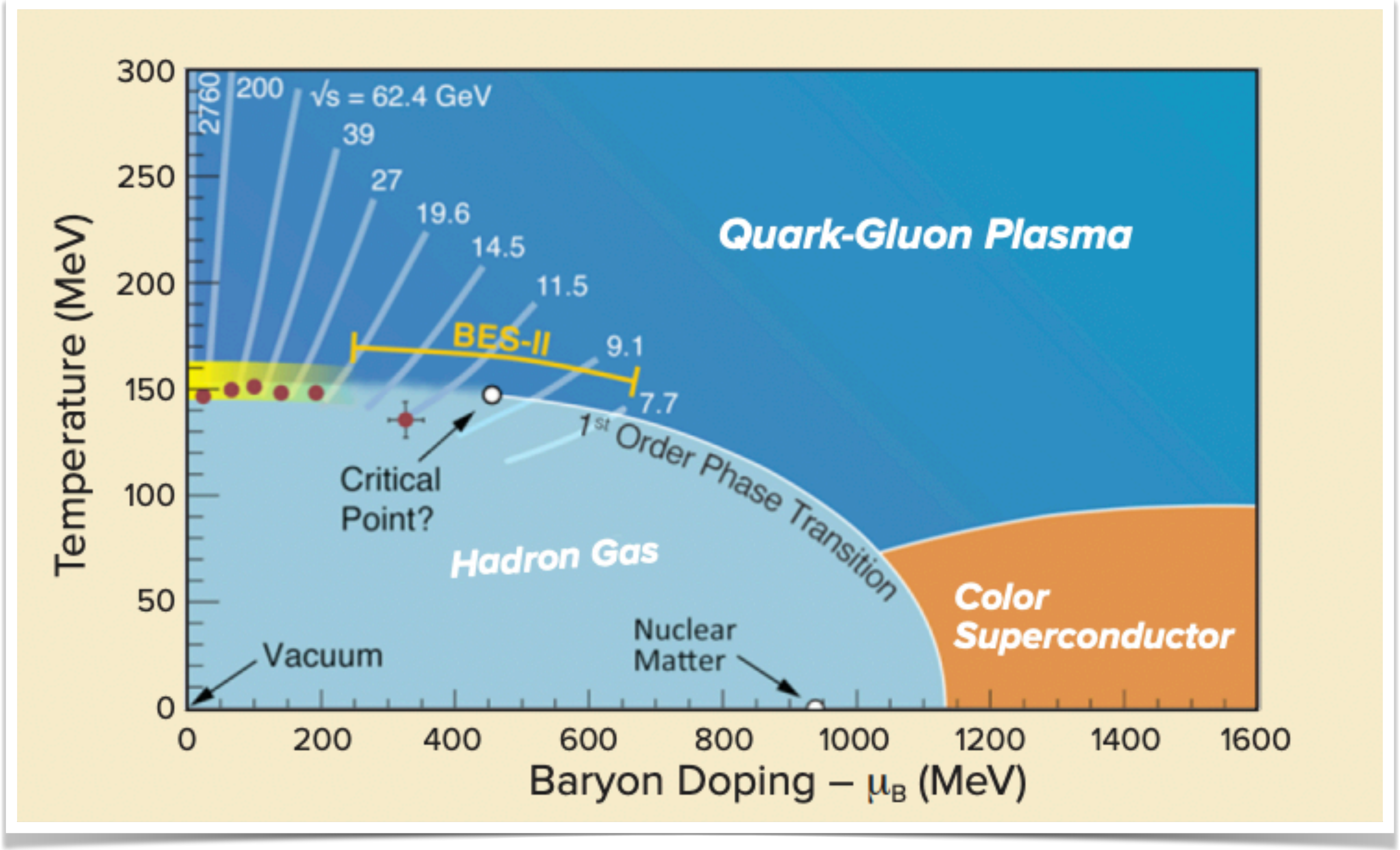
Future….

- Simulate realistic scenarios
- Bayesian analysis to make comparisons to experiment

Thank you!

Fluctuations are key observables in critical point search

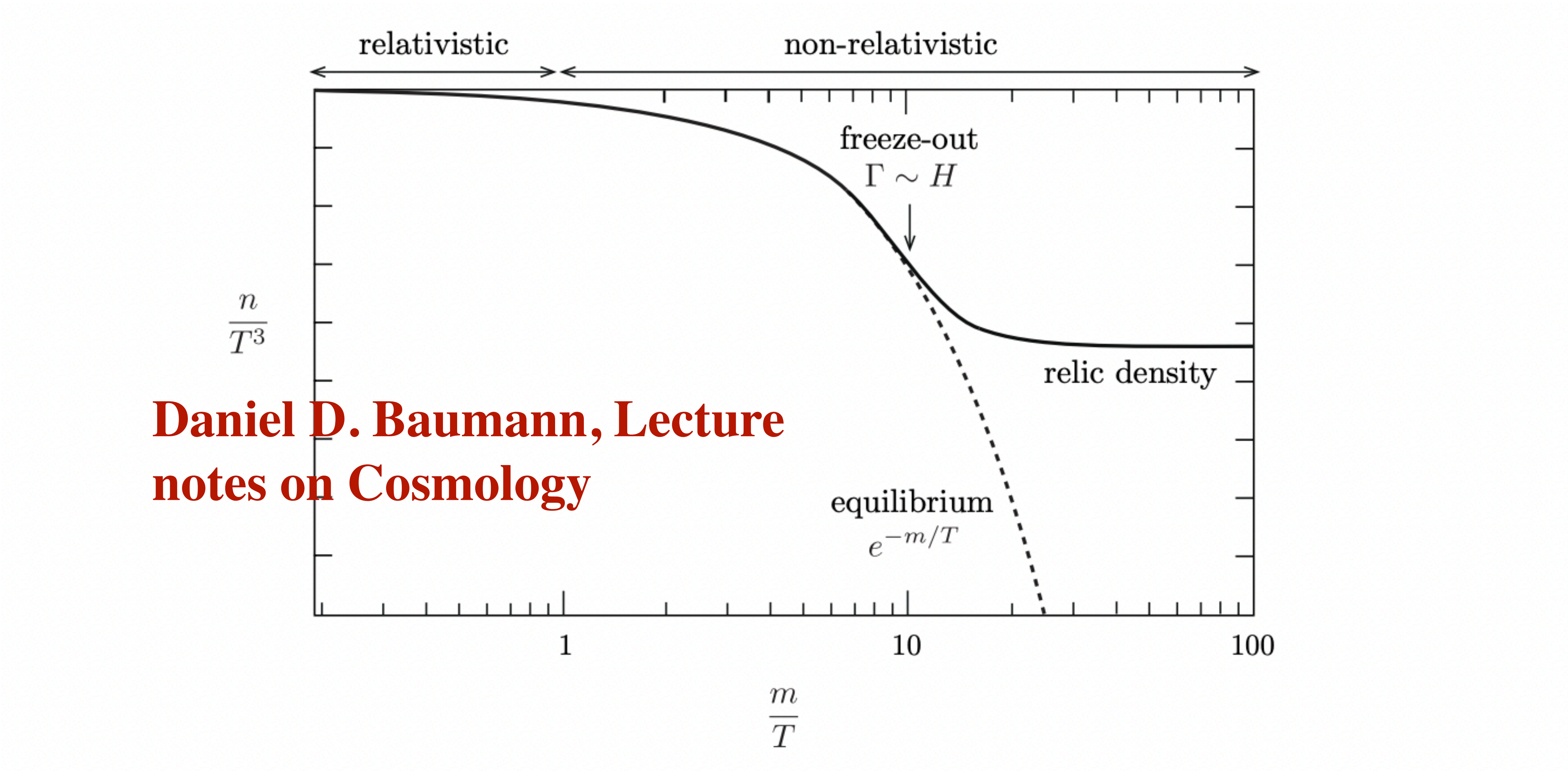
$$\langle \delta n^k \rangle_c \propto \frac{\partial^k P}{\partial \mu^k}$$



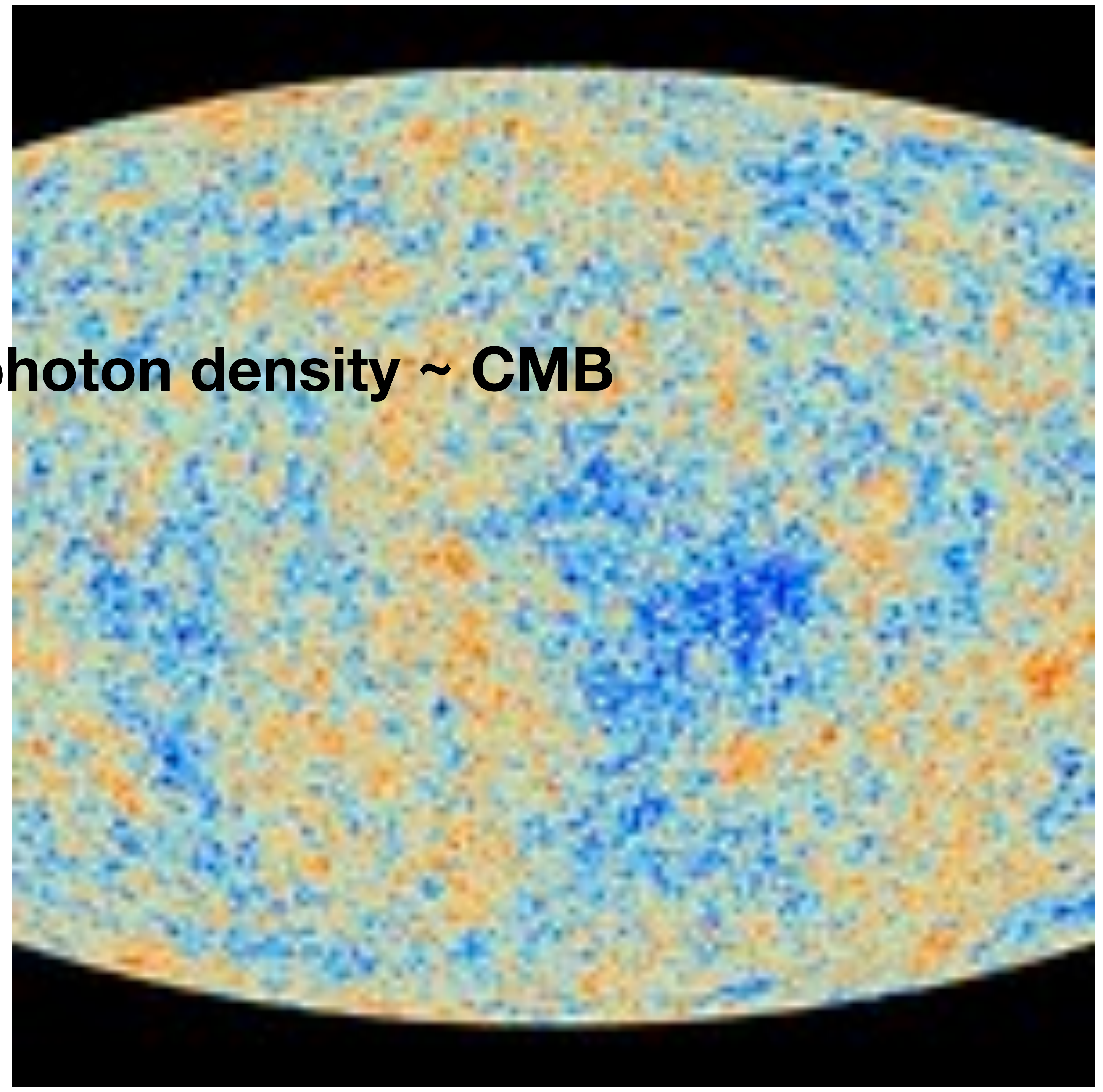
If there is a CP in the QCD phase diagram,

- Thermodynamic fluctuations are enhanced near the critical point ***equilibrium***
- Evolution of fluctuations as the fireball expands and cools
- Does the enhancement in the thermodynamic fluctuations survive till ***freeze-out***?

Freeze-out in cosmology

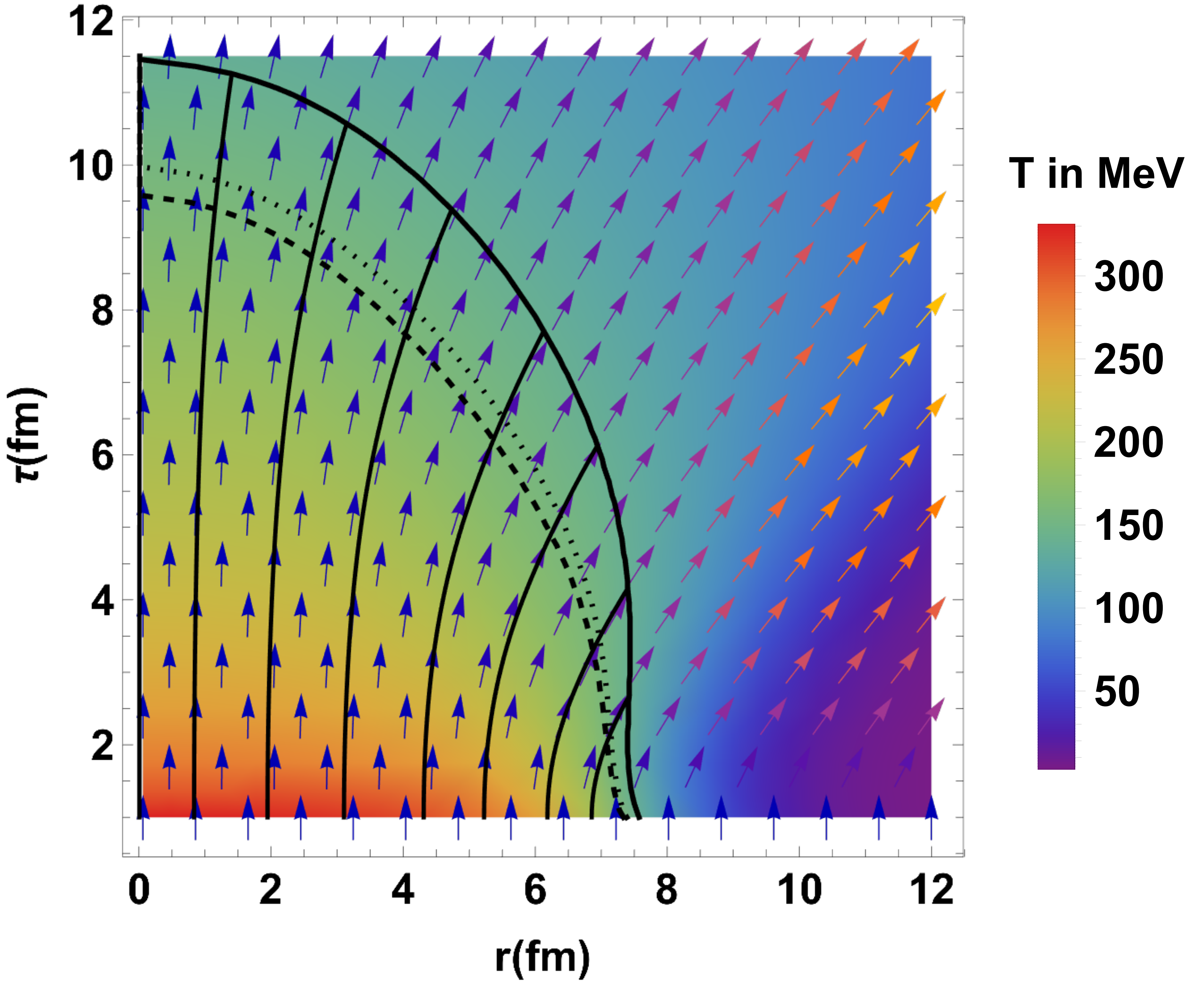


Daniel D. Baumann, Lecture notes on Cosmology



Relic photon density ~ CMB

- Particle distribution functions in phase space get fixed at freeze-out
- Obseved CMB reveals the relic density of photons at the instant of last scattering



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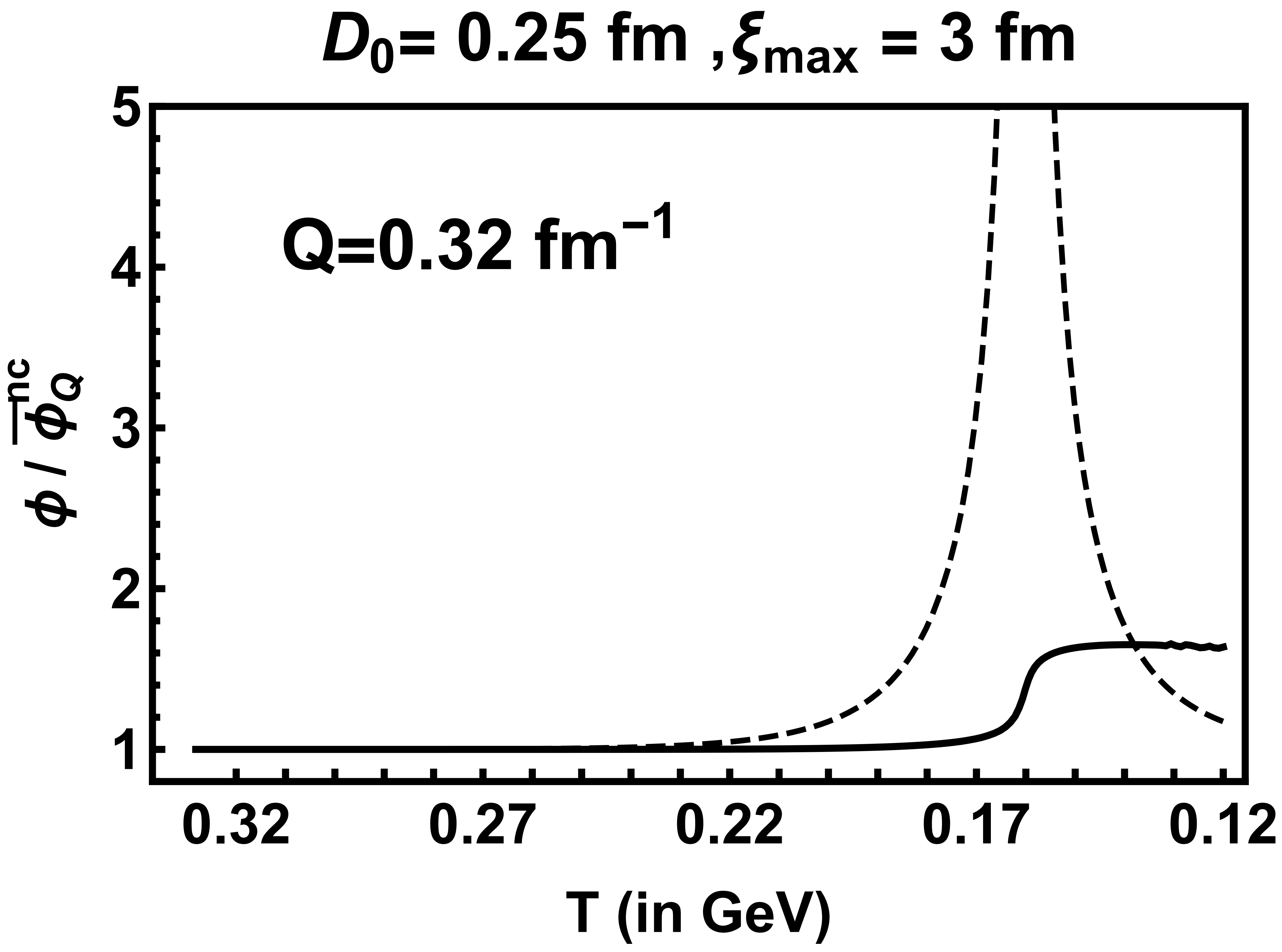

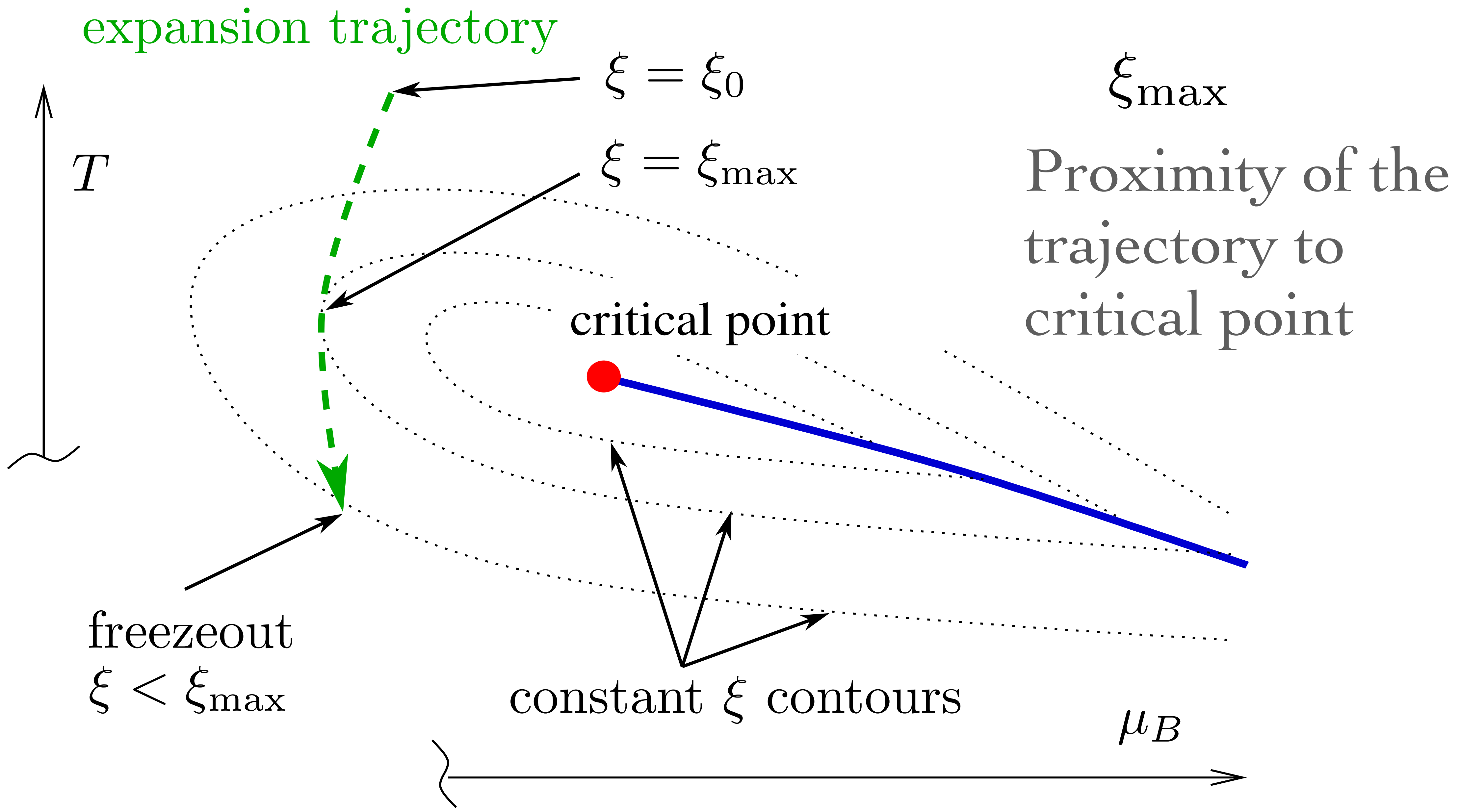

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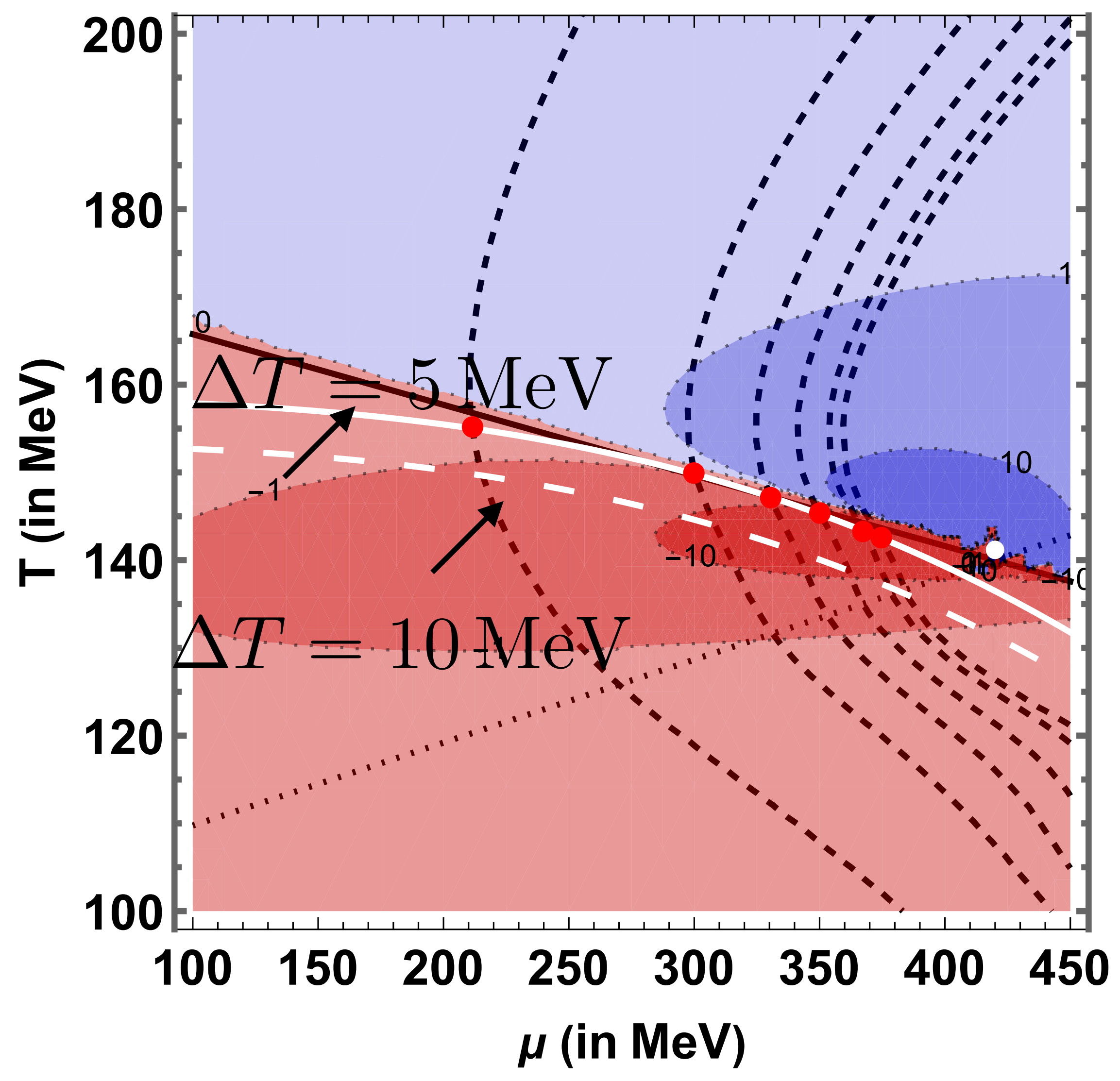
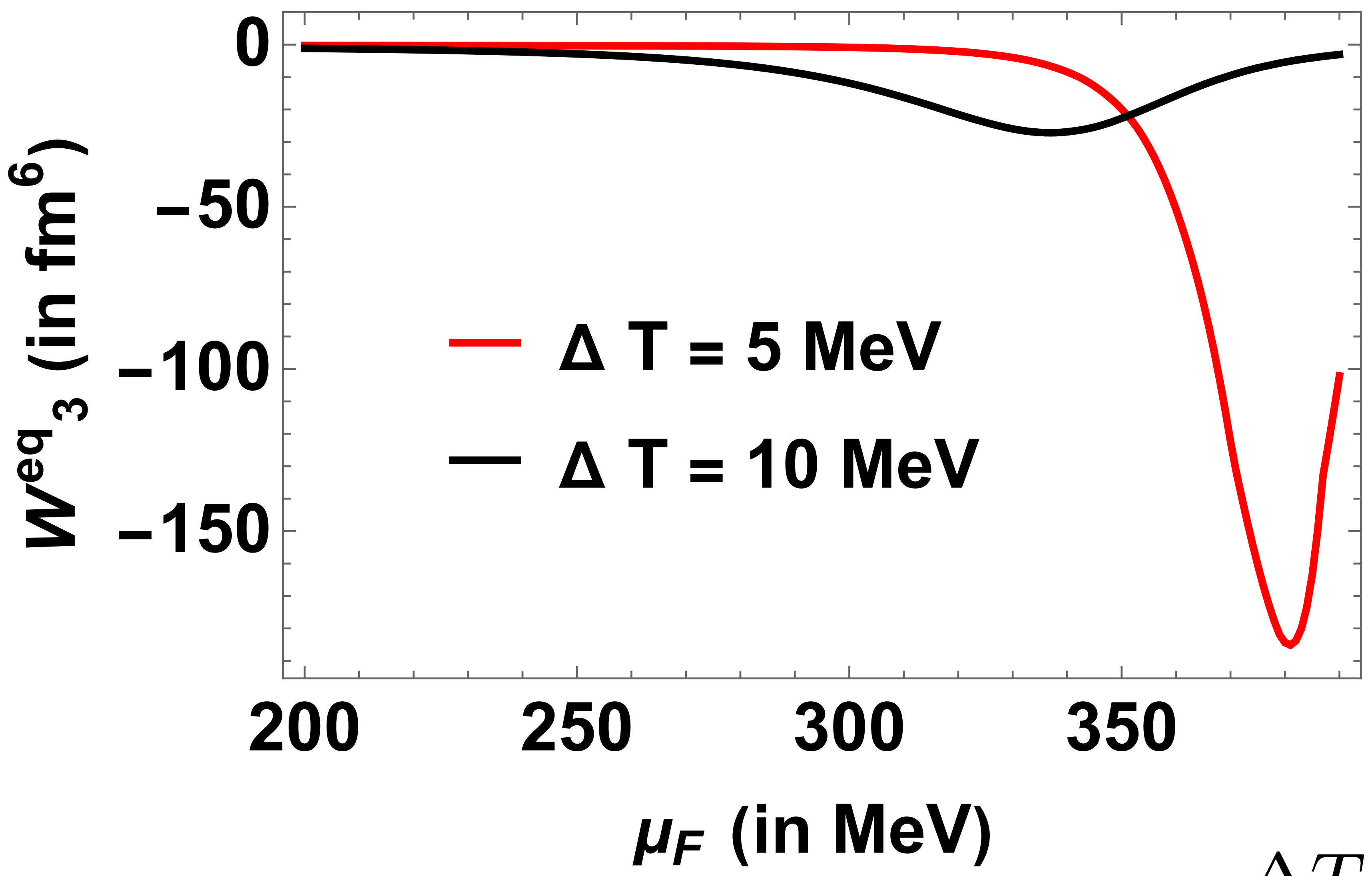
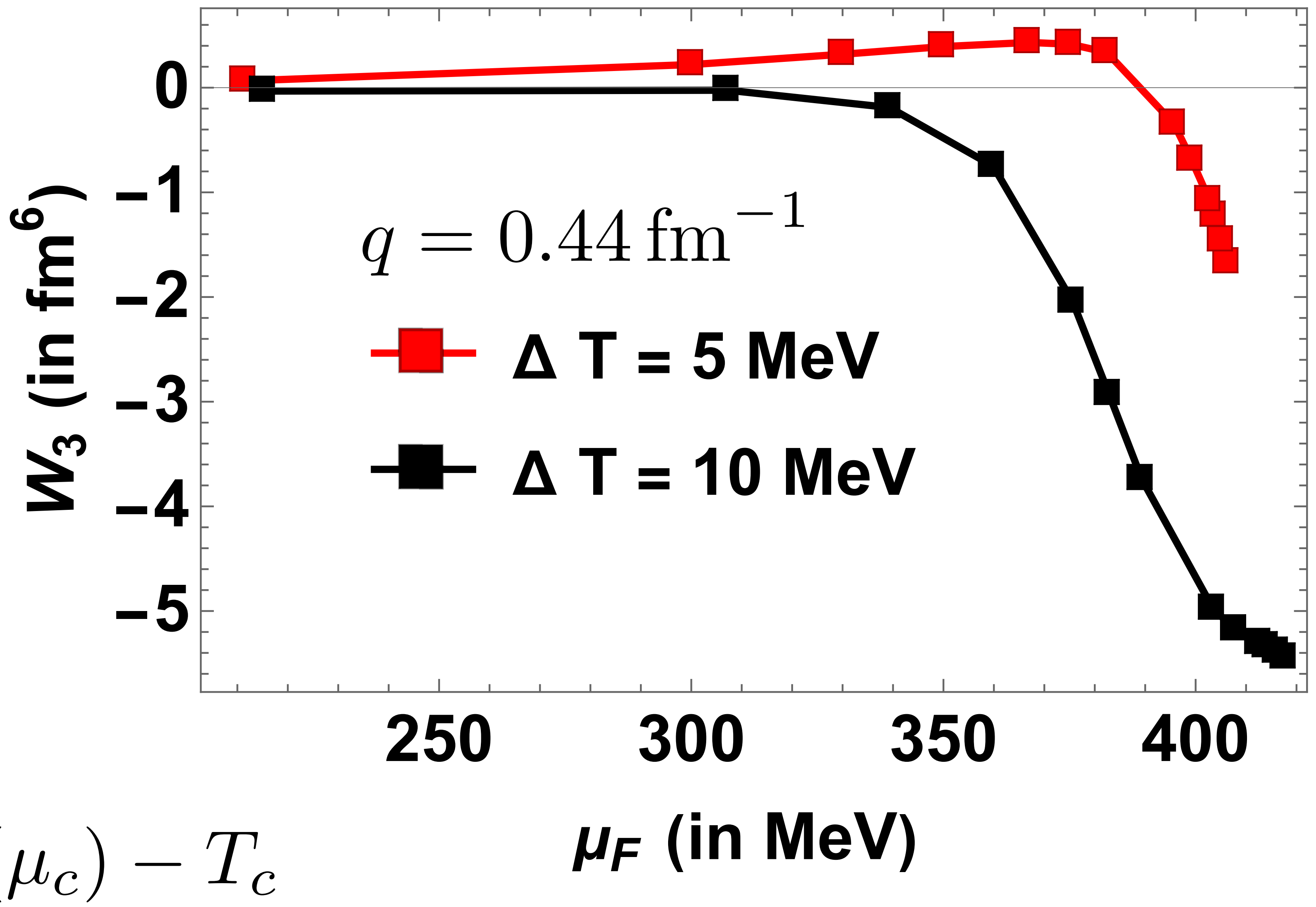
Replace with Newer Pictures using BEST EoS







$$\Delta T = T_f(\mu_c) - T_c$$

Karthein, MP, Stephanov, Rajagopal, Yin (in preparation)

Maximum entropy freeze-out of two point fluctuations

$$G_{AB} = \bar{G}_{AB} + \Delta G_{AB}$$

$$H^{ab} = \sum_{A,B,\ldots} \int_{p_A, p_B, \ldots} G_{AB} P_A^a P_B^b = \bar{H}^{ab} + \Delta H^{ab}$$

$$\bar{H}_{ab} = \sum_{A,B} \int \bar{G}_{AB} P_A^a P_B^b$$

Entropy to describe out-of equilibrium two-point correlations in ideal HRG

$$S_2 = S_0 + \frac{1}{2}\mathrm{Tr}\left[\log G\bar{G}^{-1} - G\bar{G}^{-1} + 1\right] ,$$

Similar 2-PI action

Berges, 04, Stephanov, Yin, 17…

2-PI entropy