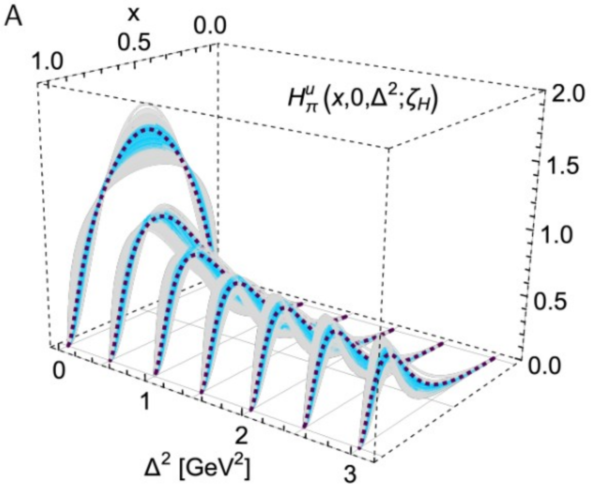
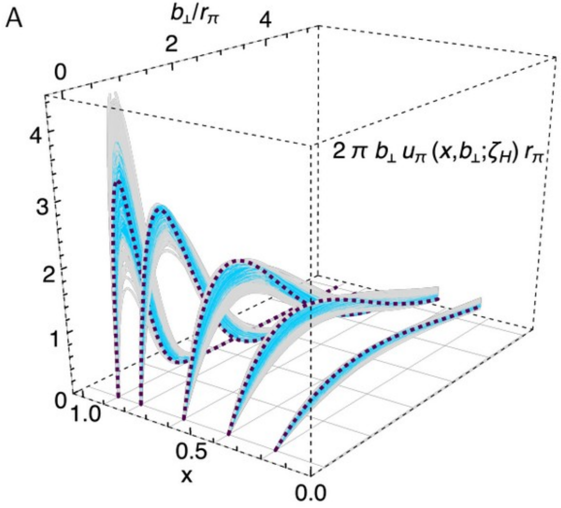
Pion and Kaon GPDs from Continuum Methods

Khépani Raya Montaño





Universidad de Huelva

Theory Seminar

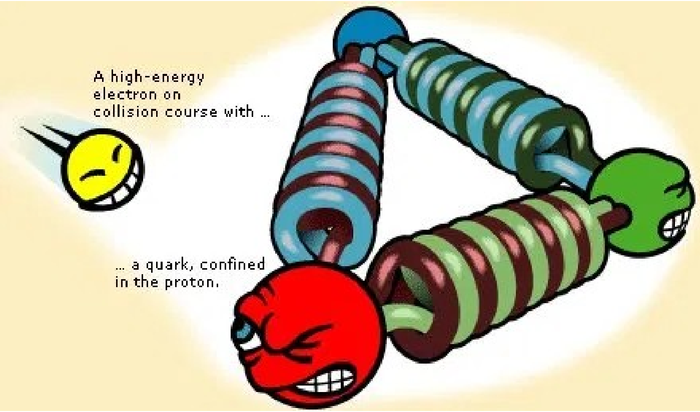
May 15, 2023. JLab (Online)

QCD: Basic Facts

➢ QCD is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
$$D_\mu = \partial_\mu + ig\frac{1}{2}\lambda^a A^a_\mu,$$
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc} A^b_\mu A^c_\nu},$$

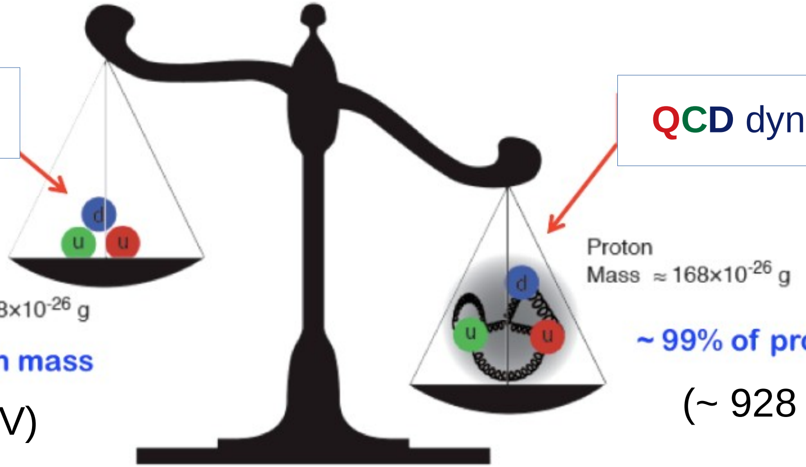
- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?



- Emergence of hadron masses (EHM) from QCD **dynamics**

Higgs mechanism

QCD dynamics



~ 1% of proton mass

(~ 10 MeV)

~ 99% of proton mass

(~ 928 MeV)

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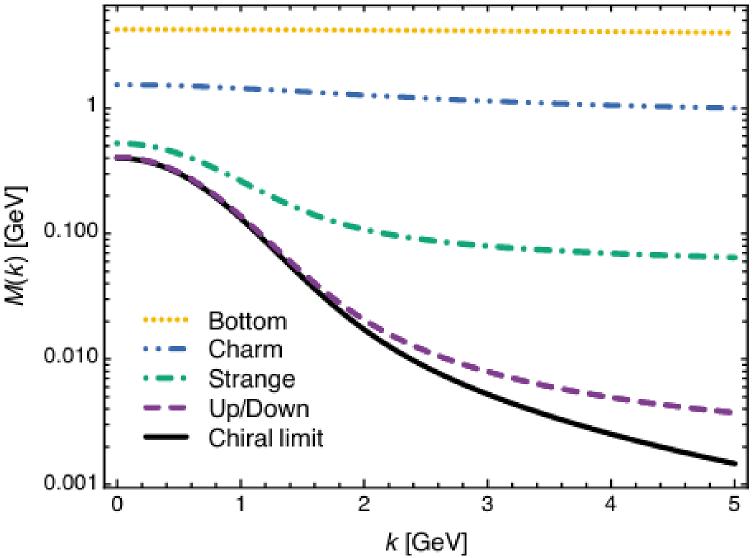
Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD **dynamics**

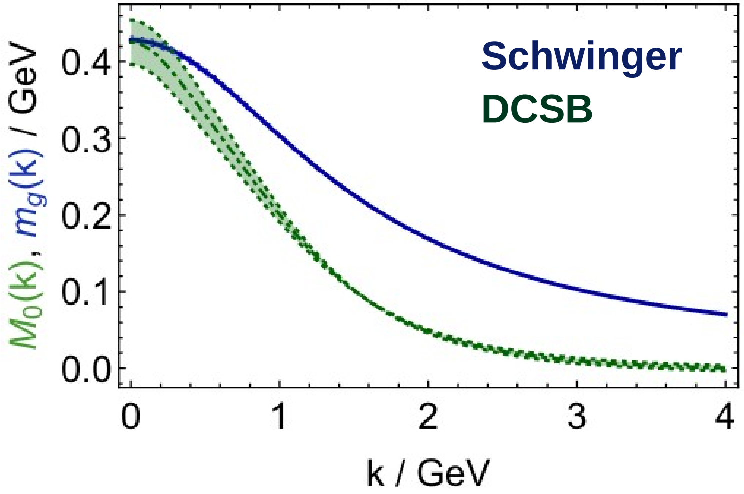
Dynamical masses

(**D**ynamical **C**hiral **S**ymmetry **B**reaking)



"Higgs" masses

$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathbf{M_f(p^2)})$$



Gluon and quark *running masses*

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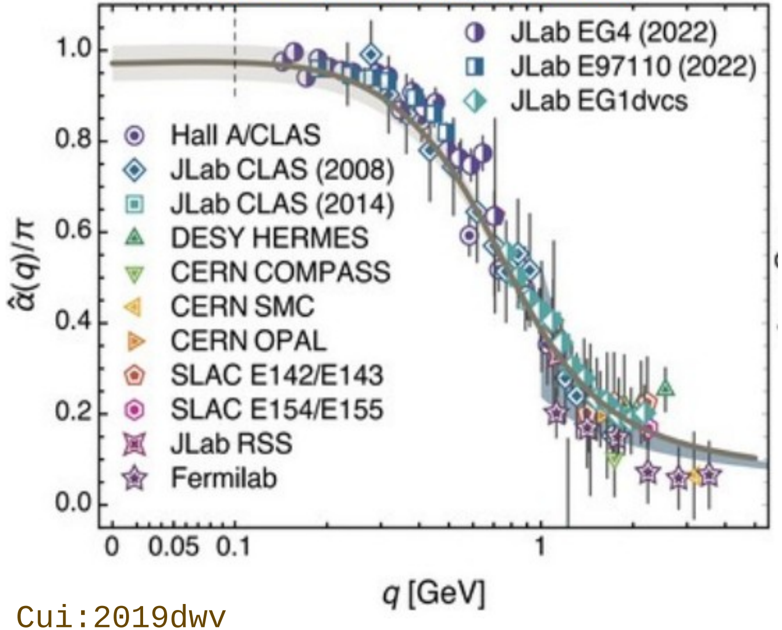
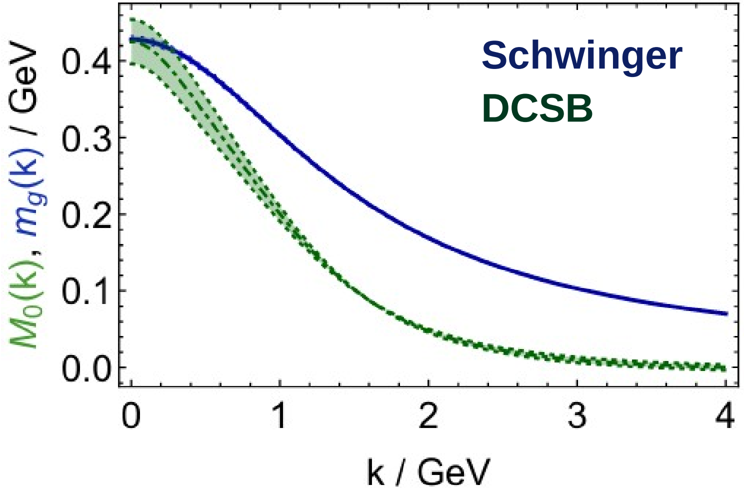
Can we trace them down to fundamental d.o.f?

(figure: D. Binosi's courtesy!)



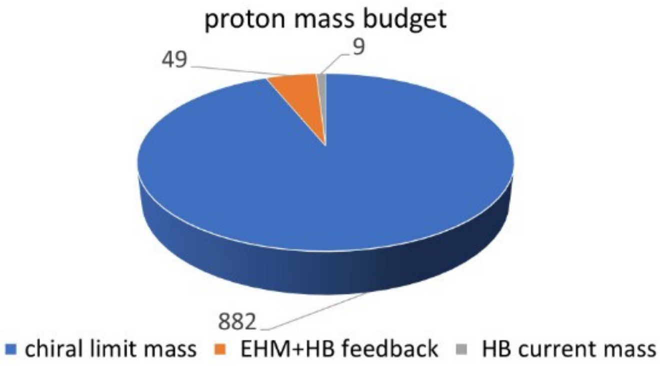
Cui:2019dwv

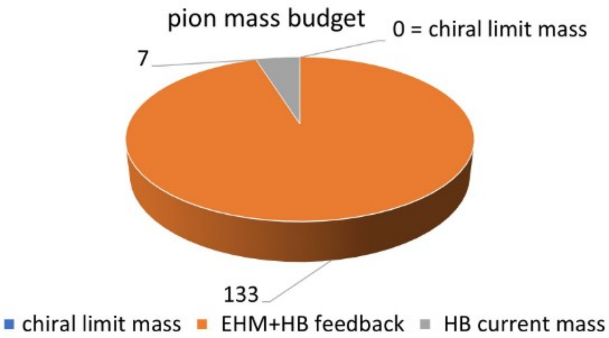
◆ Emergence of hadron masses (EHM) from QCD **dynamics**

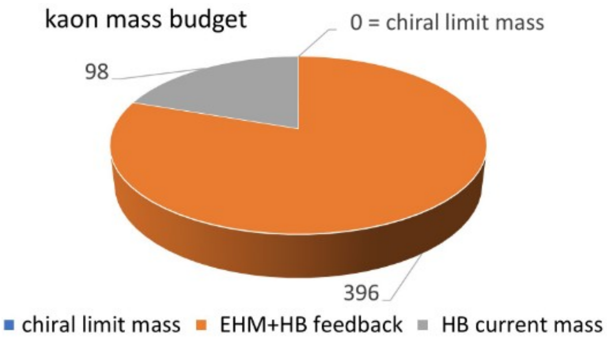


Gluon and quark *running masses*

Mass **Budgets**





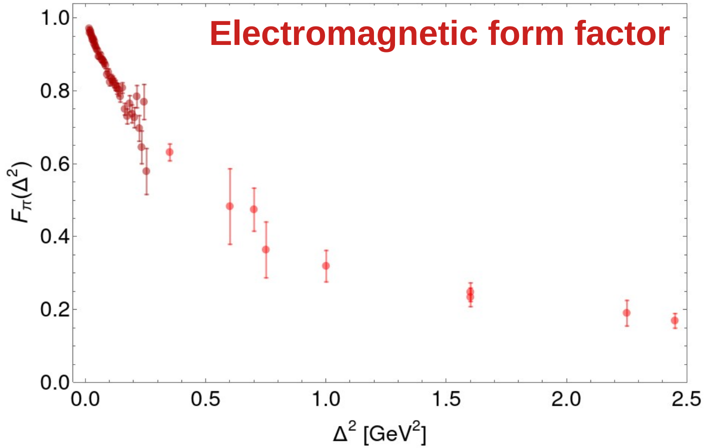


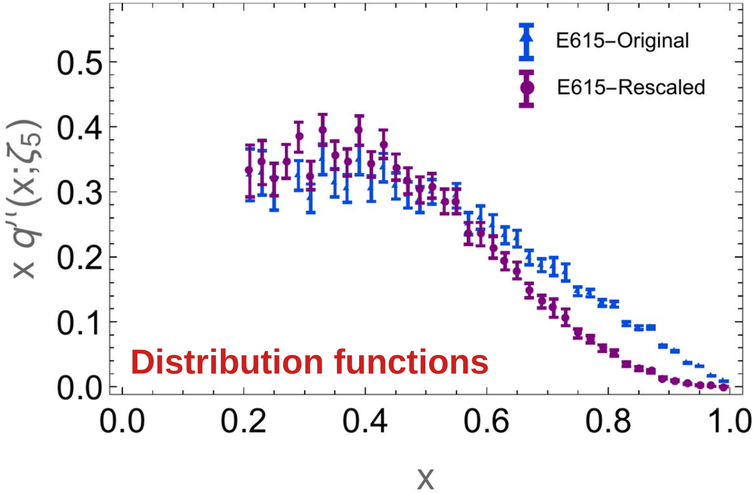
- **Most** of the **mass** in the visible universe is contained within **nucleons**
  - Which remain **pretty massive** whether there is Higgs mechanism or not...

- In the absence of weak mass generation, these would be **massless**
- Both quark-antiquark **bound-state** and **Golstone Bosons**
  - Their mere **existence** is connected with **mass** generation in the **SM**

Pion Structure

- The experimental access to the pion (and Kaon) structure is via electromagnetic probes, yielding e.g.:





- Generalized parton distributions (**GPDs**) encode them **both** (and more):

$$F_\pi(\Delta^2) = \int H_\pi^u(x,\xi,-\Delta^2;\zeta)\,,\quad u^\pi(x;\zeta) = H_\pi^u(x,0,0;\zeta)$$

→ But the experimental access and theoretical derivation is **far more complicated**.

Generalized Parton Distributions

- **Mathematically**:
  Non-local quark and gluon operators, evaluated between hadron states in **non-forward** kinematics and projected onto the **light front**.

[Muller:1994ses, Radyushkin:1996nd, Ji:1996nm]

For example:

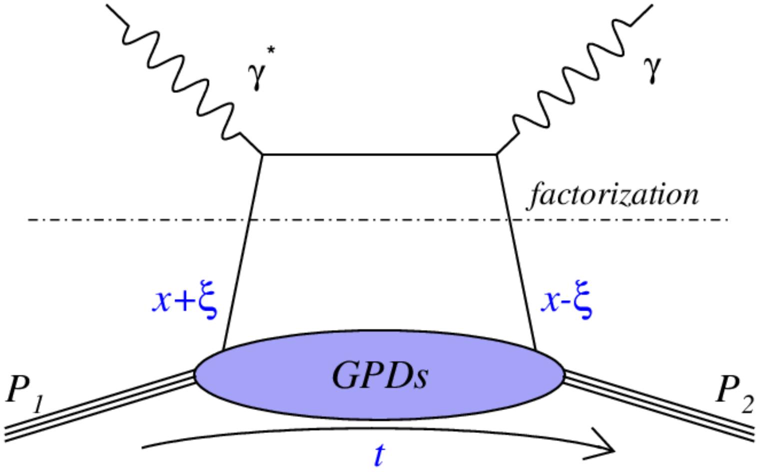
$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle \pi(p')|\, \bar{\psi}^q(-\lambda n/2)\, \gamma^\mu \psi^q(\lambda n/2)\, |\pi(p)\rangle\, n_\mu$$

(Leading-twist chiral-even valence-quark GPD of the pion)

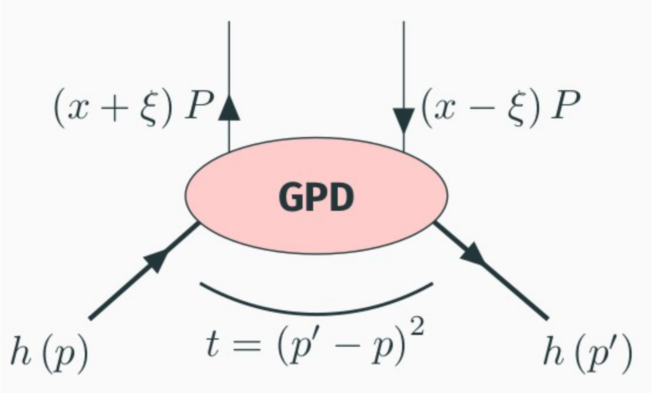
- **Schematically**:
  The probe *(virtual photon)* interacts with a quark or a gluon within the hadron, which is then scattered off.

  The process can be separated into a **perturbative**, computable part, and a **non-perturbative** one (encoded by the GPD)

For example: DVCS



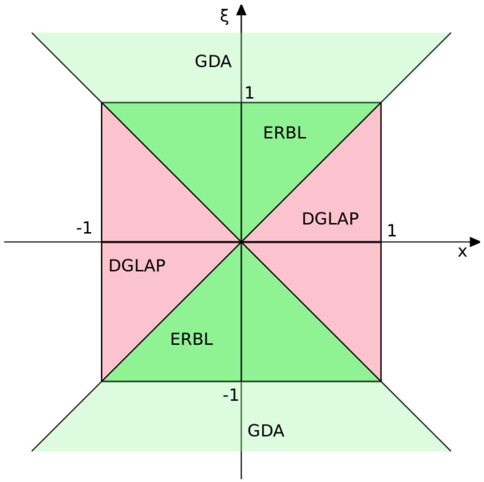
GPDs: Kinematics

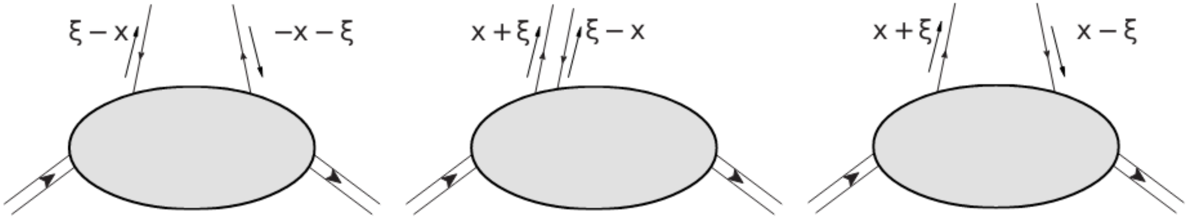


P : Average momentum of the hadron
x : Longitudinal momentum fraction
ξ : Longitudinal momentum transfered
$t=-\Delta^2$: Momentum transfered

Kinematics: [Diehl:2003ny]

- **DGLAP** $(|x|>|\xi|)$: Emits/takes a quark $(x>0)$ or antiquark $(x<0)$.
- **ERBL:** $(|x|<|\xi|)$: Emits pair quark-antiquark.





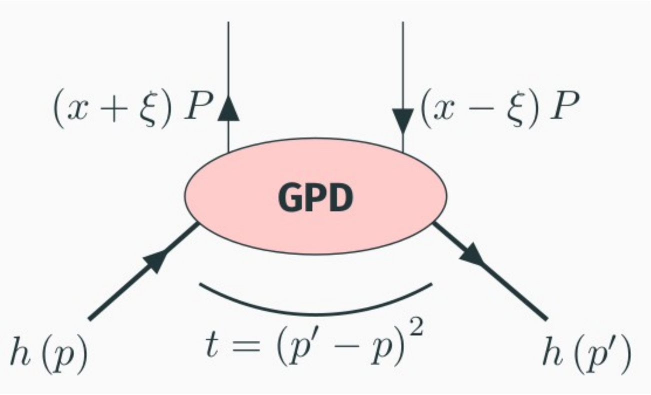
$-1 \le x \le -\xi$ "Antiquark"

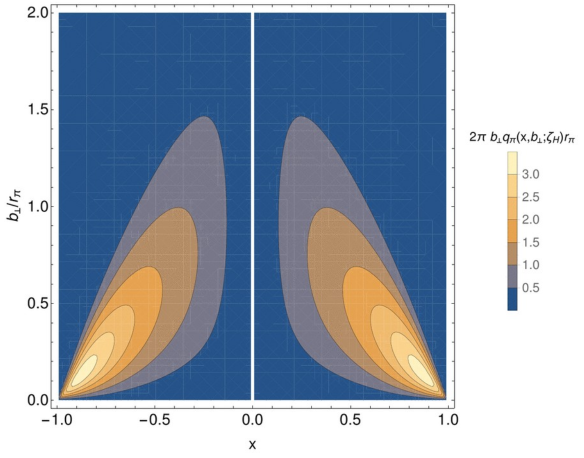
$-\xi \le x \le \xi$ "Quark/antiquark pair"

$\xi \le x \le 1$ "Quark"

GPDs: Physical pictures

- **Connection to hadron structure:**
  - **PDFs** are the forward limit of **GPDs**
  - Mellin moments of **GPDs** are connected to **electromagnetic** and **gravitational form factors (FFs)**
  - The so called **Compton FFs** parameterize the **DVCS** scattering amplitude
  - Impact parameter space GPDs (**IPS-GPDs**) sketch the likelihood of finding a parton with momentum fraction *x* at a given transverse position.





[Burkardt:2000za, Belitsky:2005qn, Polyakov:2018zvc, Chavez:2021koz]

GPDs: Properties

- **Support:** $(x,\xi) \in [-1,1] \times [-1,1]$

Analyticity/Causality

- **Polinomiality:**

Order-*m* Mellin moments are degree-*(m+1)* polynomials in ξ

Lorentz invariance

$$\mathcal{M}_m(\xi,t) = \sum_{i=0}^{\left[\frac{m}{2}\right]} (2\xi)^{2i} A^q_{i,m}(t) + \mathrm{mod}(m,2)(2\xi)^{m+1} C_{m+1}(t)$$

$$\int_{-1}^{1} \mathrm{d}y\, y^m D(y,t) = (2)^{m+1} C_{m+1}(t).$$

(D-term)

- **Positivity:**

(DGLAP region)

$$|H^q(x,\xi,t=0)| \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}\ ,\ \xi \leq |x|$$

Hilbert Space Norm

- **Soft-pion theorem:**

Low energy theorems and the connection of **GPDs** with distribution amplitudes (**PDAs**).

PCAC/Axial-vector WGTI

e.g. $$H_q^-(x,1,0) = \varphi\left(\frac{1+x}{2}\right)$$

[Mezrag:2023nkp]

GPDs: Strategies

- **Can we fulfill such mathematical requirements?**
  Here two strategies:

1. Overlap representation of the **light-front wavefunction** [Diehl:2000xz]

$$H_\pi^u(x,\xi,-\Delta^2;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi_\pi^{u*}(x_-,k_{\perp-}^2;\zeta_H)\psi_\pi^u(x_+,k_{\perp+}^2;\zeta_H)$$

- ✔ **Positivity** condition fulfilled by construction
- ✗ But, in principle, restricted to the **DGLAP** domain

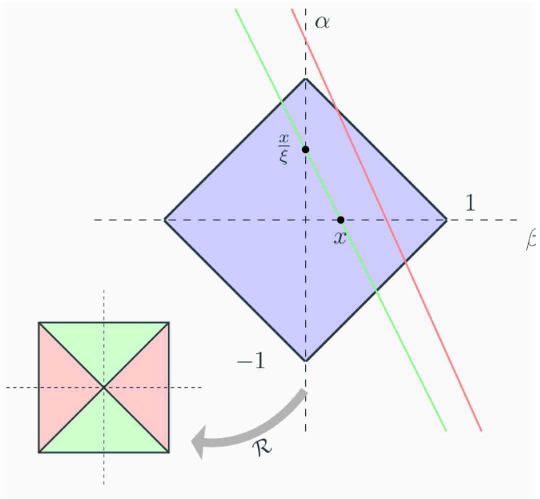
2. **Double distribution** representation

$$H_u^\pi(x,\xi,Q^2) = \int_\Omega d\beta d\alpha\, \delta(x-\beta-\xi\alpha)\left[\mathcal{F}(\beta,\alpha,Q^2)+\xi\mathcal{G}(\beta,\alpha,Q^2)\right]$$

- ✔ **Polinomiality** can be explicitly proven
- ✗ **Positivity** is **not guaranteed**.

[Muller:1994ses]

$|\alpha|+|\beta|\leq 1$



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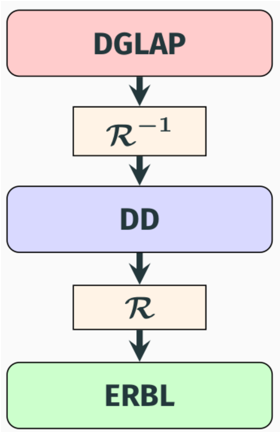
Following the so called **"covariant extension"**, this can be extended to the **ERBL** domain.

Chouika:2017dhe, Chouika:2017rzs, Chavez:2021koz, Chavez:2021llq



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  - ✔ **Polinomiality** can be explicitly proven
  - ✗ Cannot really **guarantee** the positivity

GPDs: Strategies

- **Can we fulfill such mathematical requirements?**
  Here two strategies:

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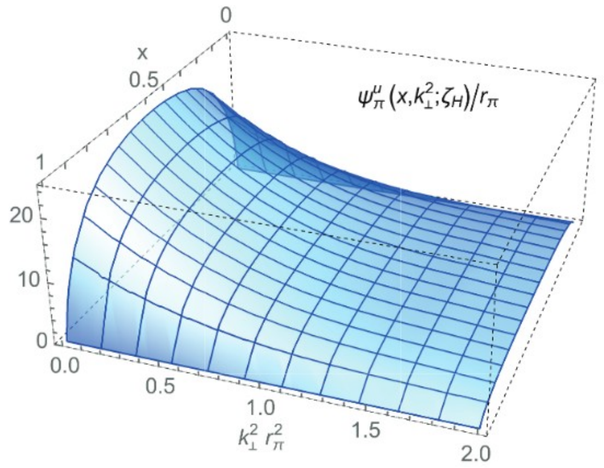
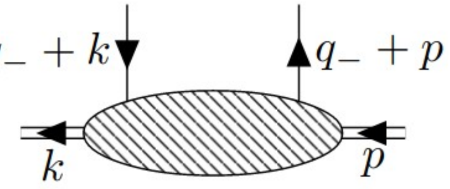


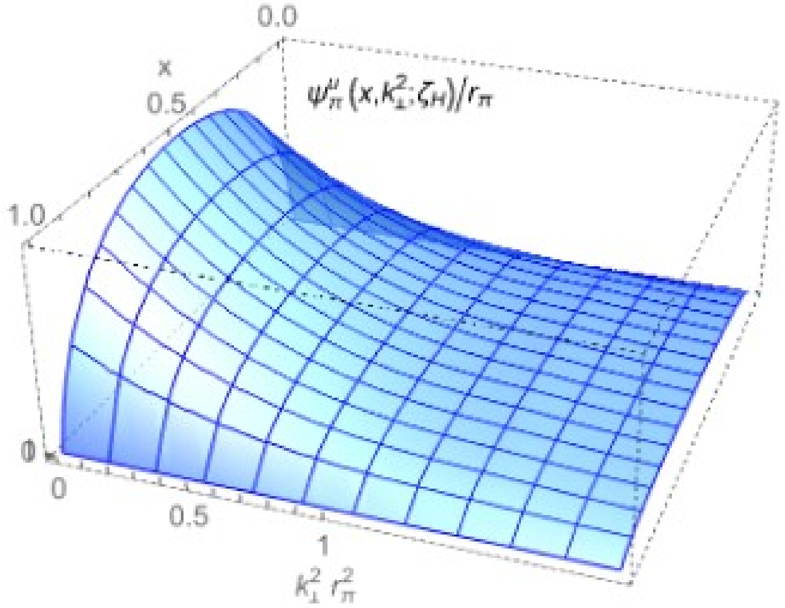
"3". **Covariant/diagram** approach

$$2H_u^\pi(x,\xi,Q^2) = N_c \text{tr}\int_q i\gamma_n \mathcal{M}_u^\pi(q_-,p,k)$$

$\mathcal{M}_u^\pi(q_-,p,k) =$ 

- ✔ In principle, **all properties** could be fulfilled
- ✗ But it relies on sensible construction of insertions/**amplitudes**.

Light-front wave functions

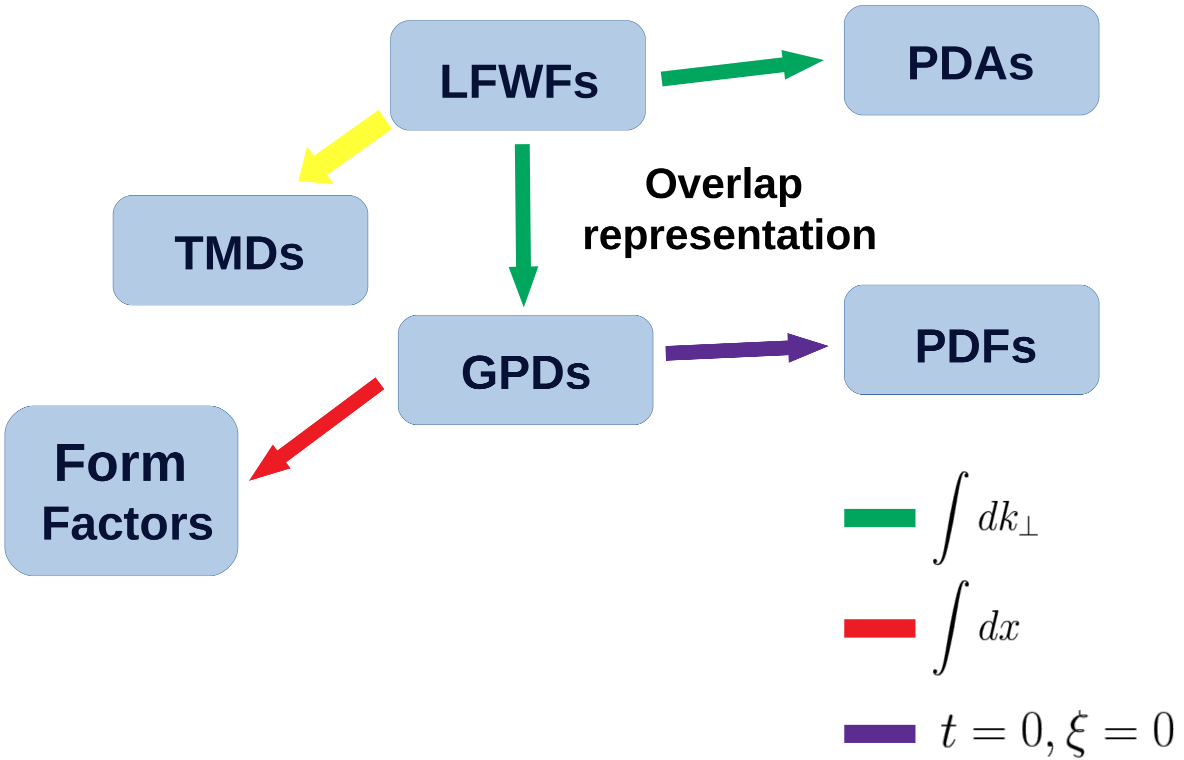


$$\psi_{\mathsf{P}}^u(x,k_\perp^2;\zeta)$$

"One ring to rule them all"

Light-front wave functions

- **Goal:** get a **broad picture** of the pion and Kaon structure.



The idea:
Compute ***everything*** from the **LFWF.**

LFWF approach

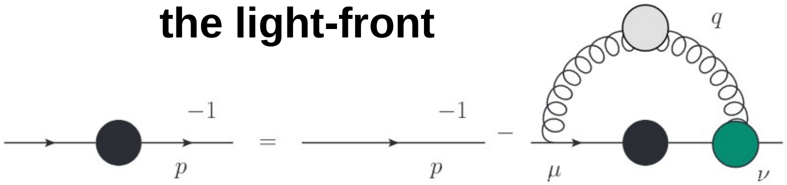
$$\psi_{\mathrm{M}}^{q}\left(x, k_{\perp}^{2}\right)=\operatorname{tr} \int_{d k_{\|}} \delta_{n}^{x}\left(k_{\mathrm{M}}\right) \gamma_{5} \gamma \cdot n\, \chi_{\mathrm{M}}\left(k_{-}, P\right)$$

- **Goal:** get a **broad picture** of the pion and Kaon structure.

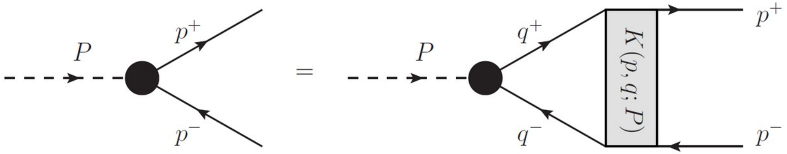
BSWFs → **LFWFs**

Project onto the light-front

Connection with Continuum Schwinger Methods (**CSM**)



Quark DSE



Meson BSE

Truncation required!

The idea:

Compute ***everything*** from the **LFWF.**

The inputs:

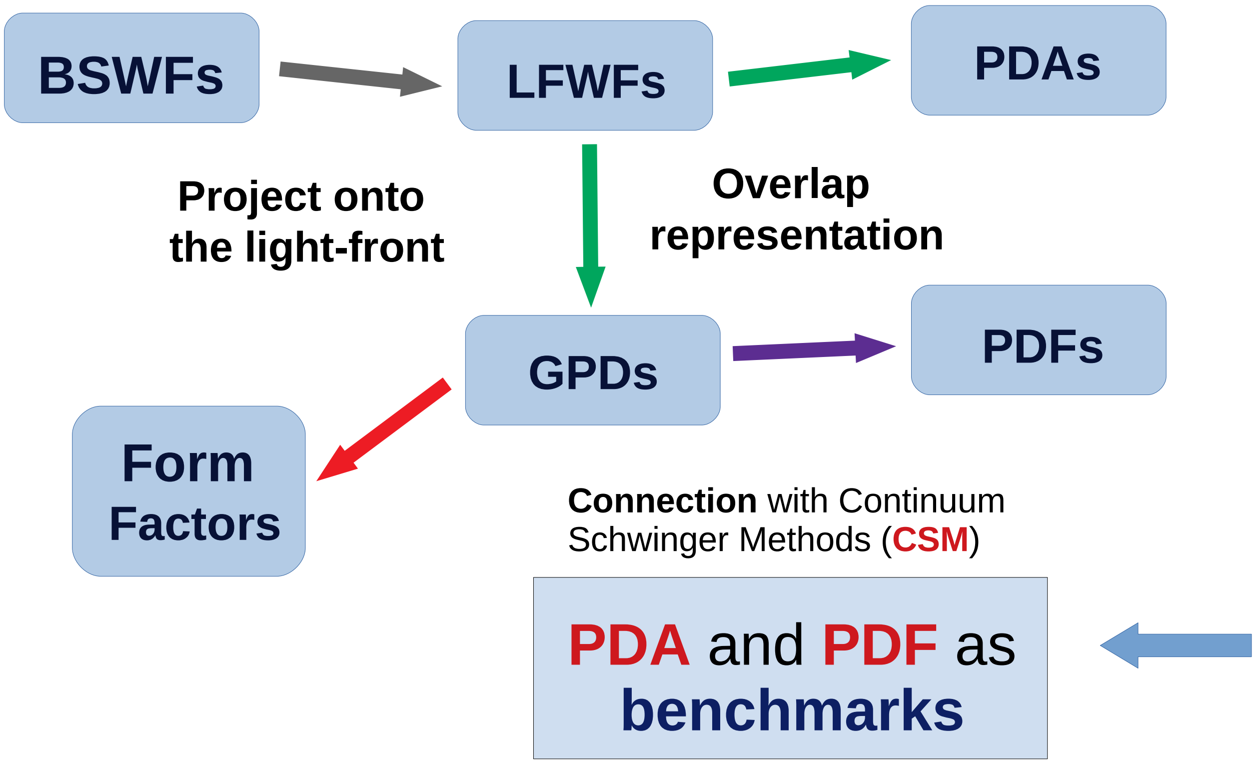
Solutions from quark **DSE** and meson **BSE**.

- ✓ Numerically challenging, but doable
- ✓ Already on the market: PDAs, PDFs, Form factors...

K. Raya *et al.*,
arXiv: 1911.12941 [nucl-th]

Light-front **wave functions**

- **Goal:** get a **broad picture** of the pion and Kaon structure.



The idea:
Compute ***everything*** from the **LFWF.**

The inputs:
Solutions from quark **DSE** and meson **BSE**.

The alternative inputs:
Construct **BSWF** from realistic DSE ***predictions***.

LFWF: PTIR approach

- A perturbation theory integral representation for the **BSWF**:

$$n_K \chi_K(k_-^K, P_K) = \underbrace{\mathcal{M}(k,P)}_{1} \int_{-1}^{1} dw\, \underbrace{\rho_K(w)}_{2} \underbrace{\mathcal{D}(k,P)}_{3}$$

(Kaon as example)

1: Matrix structure (leading BSA):

$$\mathcal{M}(k;P_K) = -\gamma_5[\gamma\cdot P_K M_u + \gamma\cdot k(M_u - M_s) + \sigma_{\mu\nu}k_\mu P_{K\nu}]\,,$$

2: Spectral weight: Tightly connected with the meson properties.

3: Denominators: $\mathcal{D}(k;P_K) = \Delta(k^2, M_u^2)\Delta((k-P_K)^2, M_s^2)\hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2)\,,$

where: $\Delta(s,t) = [s+t]^{-1},\ \hat{\Delta}(s,t) = t\Delta(s,t)\,.$

Raya:2021zrz
Raya:2022eqa

LFWF: PTIR approach

- Recall the expression for the **LFWF**:

$$\psi_{\rm M}^{q}\left(x,k_{\perp}^{2}\right) = \text{tr}\int_{dk_{||}} \delta_{n}^{x}(k_{\rm M})\gamma_5\gamma\cdot n\,\chi_{\rm M}(k_-,P)$$

$$<x>_{\rm M}^{q} := \int_0^1 dx\; x^m \psi_{\rm M}^{q}(x,k_{\perp}^{2})$$

- Algebraic manipulations yield:

+ Uniqueness of Mellin moments

$$\Rightarrow \psi_{\rm M}^{q}(x,k_{\perp}) \sim \int dw\; \rho_{\rm M}(w)\cdots$$

✓ Compactness of this result is a merit of the AM.

- Thus, $\rho_{\rm M}(w)$ determines the profiles of, *e.g.* **PDA** and **PDF**: (it also works the **other way around**)

$$f_{\rm M}\phi_{\rm M}^{q}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi_{\rm M}^{q}(x,k_\perp;\zeta_H)$$

$$q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}|\psi_{\rm M}^{q}(x,k_\perp;\zeta_H)|^2$$

LFWF: PTIR approach

Raya:2021zrz
Raya:2022eqa

- More **explicitly**:

$$\psi_{\rm M}^{q}(x,k_\perp^2;\zeta_H) = 12\left[M_q(1-x)+M_{\bar h}x\right]X_{\rm P}(x;\sigma_\perp^2)$$

$$\sigma_\perp = k_\perp^2 + \Omega_{\rm P}^2$$

$$X_{\rm M}(x;\sigma_\perp^2) = \left[\int_{-1}^{1-2x} dw \int_{1+\frac{2x}{w-1}}^{1} dv + \int_{1-2x}^{1} dw \int_{\frac{w-1+2x}{w+1}}^{1} dv\right] \frac{\rho_{\rm M}(w)}{n_{\rm M}}\frac{\Lambda_{\rm M}^2}{\sigma_\perp^2}$$

$$\Omega_{\rm M}^2 = vM_q^2 + (1-v)\Lambda_{\rm P}^2 + (M_{\bar h}^2 - M_q^2)\left(x - \tfrac{1}{2}[1-w][1-v]\right) + \left(x[x-1] + \tfrac{1}{4}[1-v][1-w^2]\right) m_{\rm M}^2$$

- Model **parameters**:

P	$m_{\rm P}$	M_u	M_h	$\Lambda_{\rm P}$	$b_0^{\rm P}$	$\omega_0^{\rm P}$	$v_{\rm P}$
π	0.14	0.31	M_u	M_u	0.275	1.23	0
K	0.49	0.31	$1.2M_u$	$3M_s$	0.1	0.625	0.41

$$\rho_{\rm P}(\omega) = \frac{1+\omega\, v_{\rm P}}{2a_{\rm P}b_0^{\rm P}}\left[\operatorname{sech}^2\left(\frac{\omega-\omega_0^{\rm P}}{2b_0^{\rm P}}\right) + \operatorname{sech}^2\left(\frac{\omega+\omega_0^{\rm P}}{2b_0^{\rm P}}\right)\right]$$

LFWF: Factorized case

- In the **chiral limit**, the **PTIR** reduces to:

$$\psi_{\rm M}^q(x,k_\perp^2;\zeta_H) \sim \tilde{f}(k_\perp)\phi_{\rm M}^q(x;\zeta_H) \sim f(k_\perp)[q_{\rm M}(x;\zeta_H)]^{1/2}$$

"Factorized model"

$$[\phi_{\rm M}^q(x;\zeta_H)]^2 \sim q_{\rm M}(x;\zeta_H)$$

- ✓ Sensible assumption as long as:

$m_{\rm M}^2 \approx 0$ (meson mass) $\quad M_{\bar{h}}^2 - M_q^2 \approx 0$ (*h*-antiquark, *q*-quark masses) $\quad \zeta_H$

→ Produces **identical** results as PTIR model for pion

- Therefore:

$$\psi_{\rm M}^q(x,k_\perp^2;\zeta_H) = [q^{\rm M}(x;\zeta_H)]^{1/2}\left[4\sqrt{3}\pi\frac{M_q^3}{(k_\perp^2+M_q^2)^2}\right]$$

Single parameter!

$M_q \sim r_{\rm M}^{-1}$

(charge radius)

No need to determine the spectral weight !

LFWF: PTIR approach II

Albino:2022gzs

- A perturbation theory integral representation for the **BSWF**:

$$n_{\mathrm{M}}\chi_{\mathrm{M}}(k_-,P)=\underbrace{\mathcal{M}_{q,\bar{h}}(k,P)}_{1}\int_{-1}^{1}dw\,\underbrace{\tilde{\rho}_{\mathrm{M}}^{\nu}(w)}_{2}\underbrace{\mathcal{D}^{\nu}_{q,\bar{h}}(k,P)}_{3}$$

(Meson M)

1: Matrix structure (leading BSA): $\mathcal{M}_{q,\bar{h}}(k,P)\equiv-\gamma_5\left[M_q\gamma\cdot P+\gamma\cdot k(M_{\bar{h}}-M_q)+\sigma_{\mu\nu}k_\mu P_\nu-i\left(k\cdot p+M_qM_{\bar{h}}\right)\right]$

2: Profile function: $\tilde{\rho}_{\mathrm{M}}^{\nu}(w)\equiv\rho_{\mathrm{M}}(w)\Lambda_w^{2\nu}\,.$

3: Denominators: $\mathcal{D}^{\nu}_{q,\bar{h}}(k,P)\equiv\Delta\left(k^2,M_q^2\right)\Delta\left(k_{w-1}^2,\Lambda_w^2\right)^{\nu}\Delta\left(p^2,M_{\bar{h}}^2\right)\,.$

The crucial difference:

$$\Lambda^2(w):=\Lambda_w^2=M_q^2-\frac{1}{4}\left(1-w^2\right)m_{\mathrm{M}}^2+\frac{1}{2}\left(1-w\right)\left(M_{\bar{h}}^2-M_q^2\right)\,.$$

LFWF: PTIR approach II

- Then a series of algebraic results follows.

$$q(x;\zeta_H) \sim \frac{\phi_q^2(x;\zeta_H)}{\Lambda_{1-2x}^2}$$

1. For the **BSWF**:

$$n_M \chi_M(k_-, P) = \mathcal{M}_{q,\bar{h}}(k,P) \int_0^1 d\alpha \mathcal{F}_M(\alpha, \sigma^{\nu+2}) \, , \quad \sigma = (k-\alpha P)^2 + \Lambda_{1-2\alpha}^2 ,$$

$$\mathcal{F}_M(\alpha, \sigma^{\nu+2}) = 2^\nu (\nu+1) \Big[\int_{-1}^{1-2\alpha} dw \left(\frac{\alpha}{1-w} \right)^\nu + \int_{1-2\alpha}^{1} dw \left(\frac{1-\alpha}{1+w} \right)^\nu \Big] \frac{\tilde{\rho}_M^\nu(w)}{\sigma^{\nu+2}}$$

Albino:2022gzs

2. **LFWF** in terms of **PDA/PDF:**

$$f_M \phi_M^q(x;\zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H)$$

$$\psi_M^q(x, k_\perp^2) = 16\pi^2 f_M \frac{\nu \Lambda_{1-2x}^{2\nu}}{(k_\perp^2 + \Lambda_{1-2x}^2)^{\nu+1}} \phi_M^q(x)$$

$$\Lambda_{1-2x}^2 = M_q^2 + x(M_{\bar{h}}^2 - M_q^2) - m_H^2 x(1-x)$$

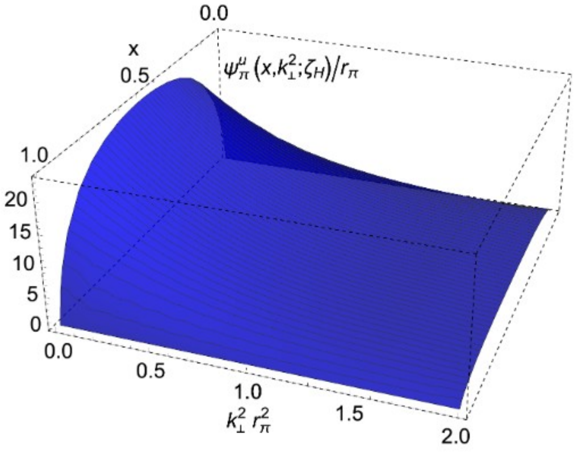
Flavor asymmetry   Meson mass

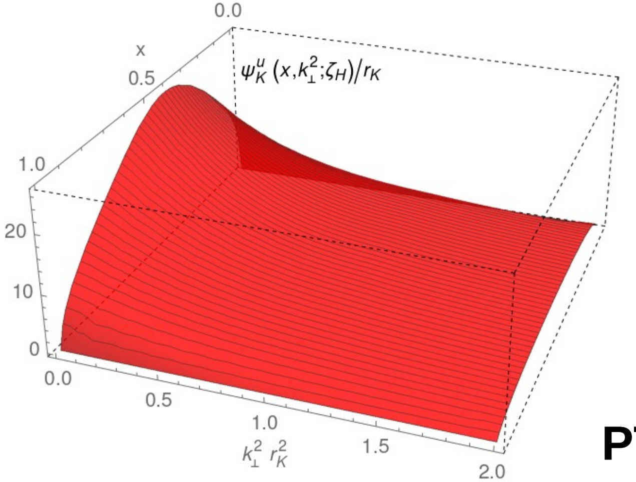
Encodes the breaking of **factorization.**

- Completely factorized in the chiral limit.

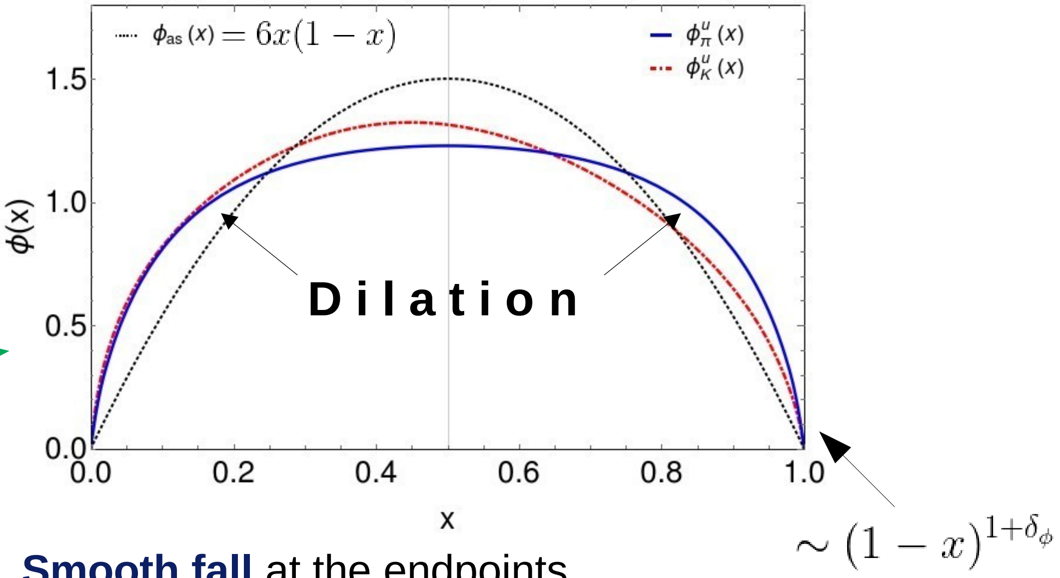
LFWFs and **PDAs**

$$f_{\rm M}\phi^q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi^q_{\rm M}(x,k_\perp;\zeta_H)$$





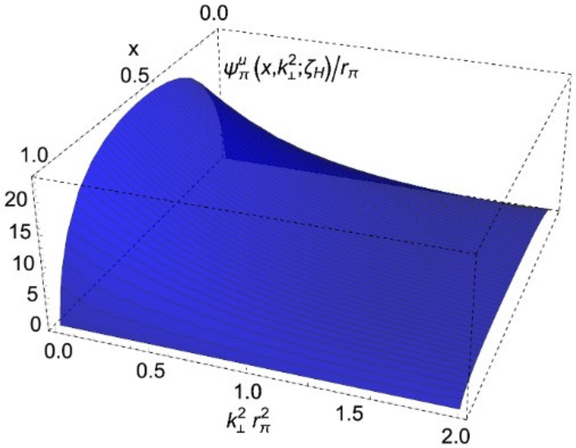
PTIR

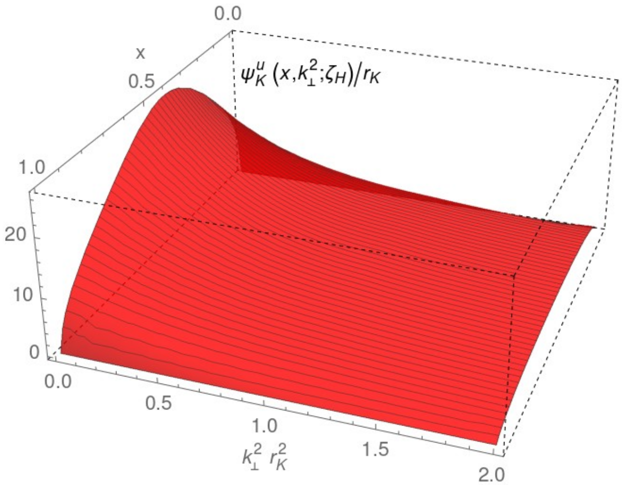


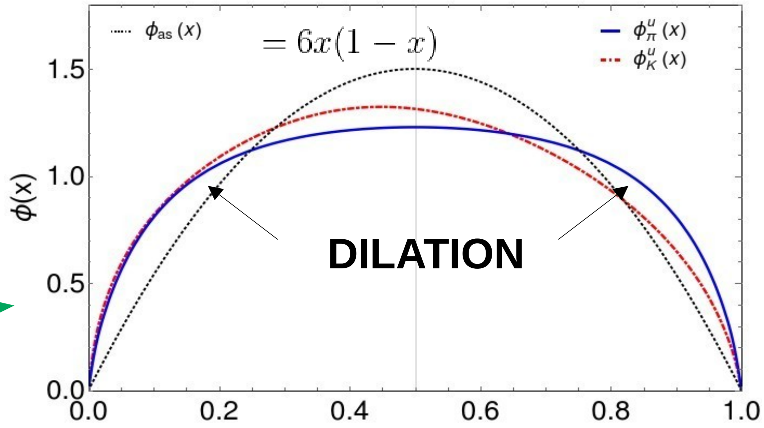
- **Smooth fall** at the endpoints
- **Broad** and concave functions of ***x***
  - Consequence of **EHM**
- **Higgs** induced asymmetry for **Kaon**:
  - Moduled by the difference M_s - M_u

LFWFs and PDAs

$$f_{\rm M}\phi^q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi^q_{\rm M}(x,k_\perp;\zeta_H)$$

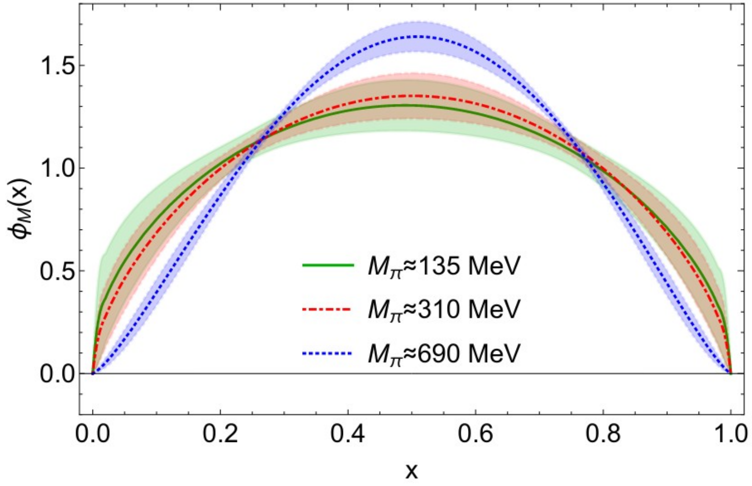






First **CSM** calculation of pion **PDA**:

Chang:2013pq



Patterns supported by **lattice**:

Zhang:2020gaj

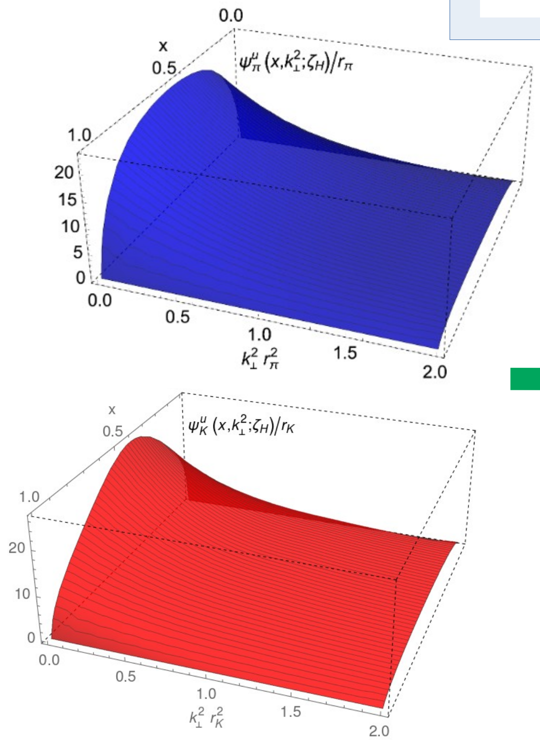
LFWFs and GPDs

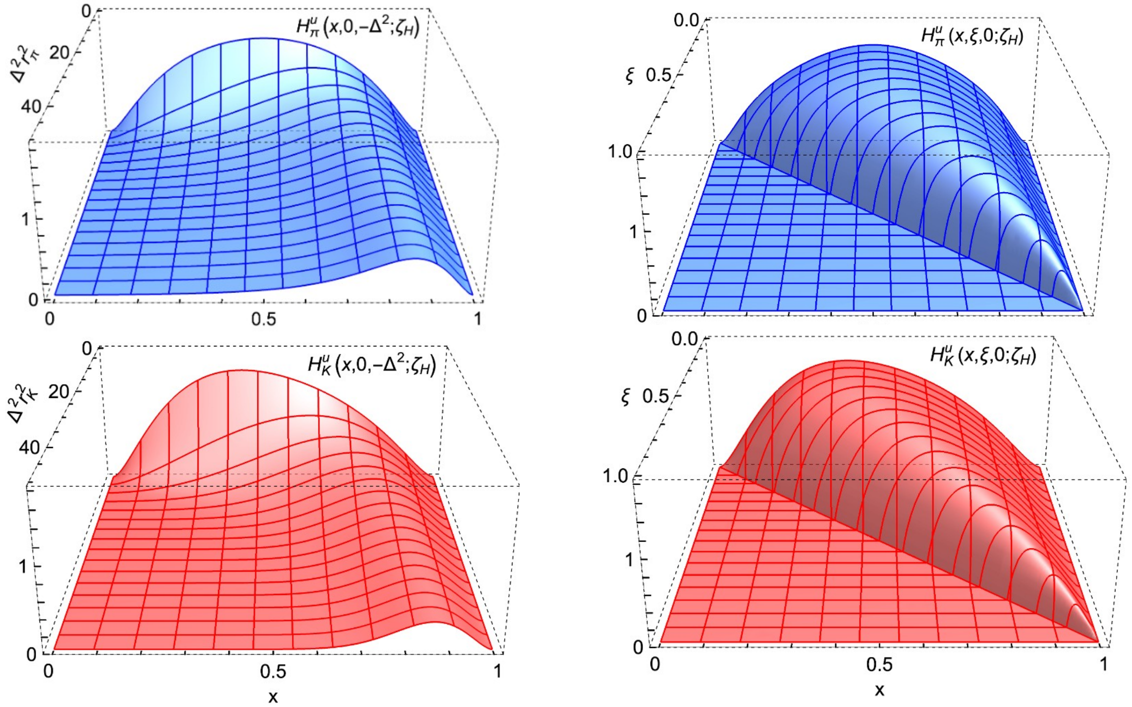
LFWFs → GPDs

- In the **overlap representation**, the valence-quark **GPD** reads as:

$$H^q_{\rm M}(x,\xi,t) = \int \frac{d^2 k_\perp}{16\pi^3} \psi^{q*}_{\rm M}\left(x^-, (\mathbf{k}^-_\perp)^2\right) \psi^q_{\rm M}\left(x^+, (\mathbf{k}^+_\perp)^2\right)$$

ζ_H



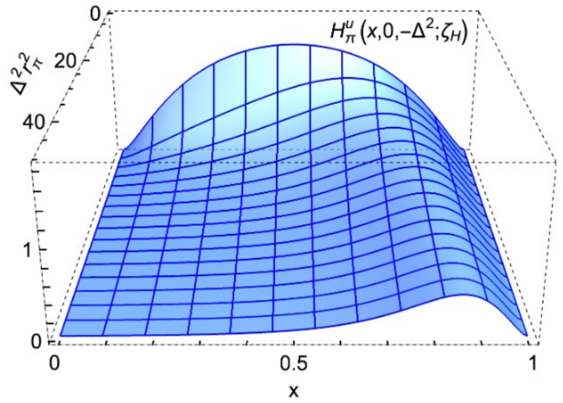


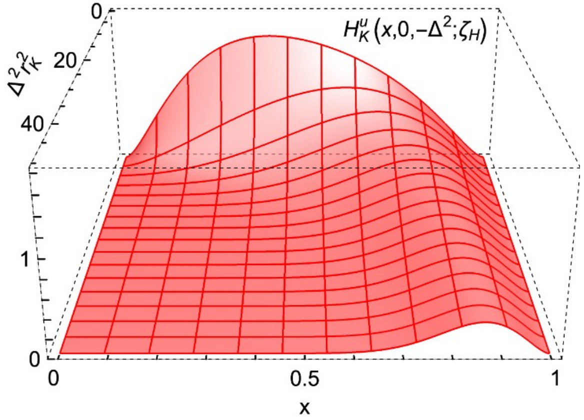
GPDs and PDFs

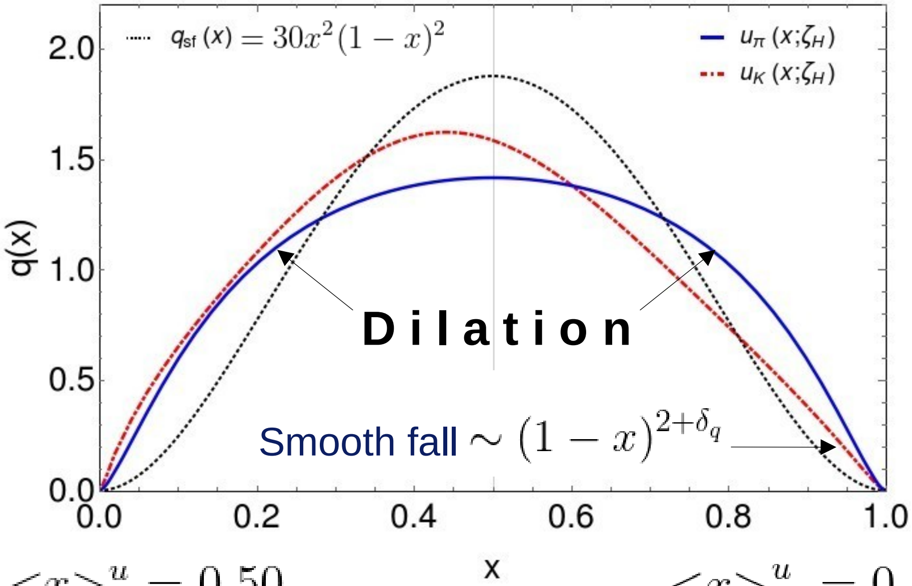
GPD → PDF

- The **PDF** is obtained from the **forward limit** of the **GPD**.

$$q(x) = H(x,0,0)$$







$<x>^u_\pi = 0.50$ $<x>^u_K = 0.47$

- ζ_H: meson properties determined by the fully-dressed valence-quarks.
- **Broad** + **Higgs**-induced *asymmetry*

Electromagnetic FFs

GPD → FFs

- **Electromagnetic** form factor is obtained from the **t-dependence** of the **0-th moment**:

$$F_M^q(-t=\Delta^2) = \int_{-1}^{1} dx\; H_M^q(x,\xi,t)$$

Can safely take **ξ = 0**

"Polinomiality"

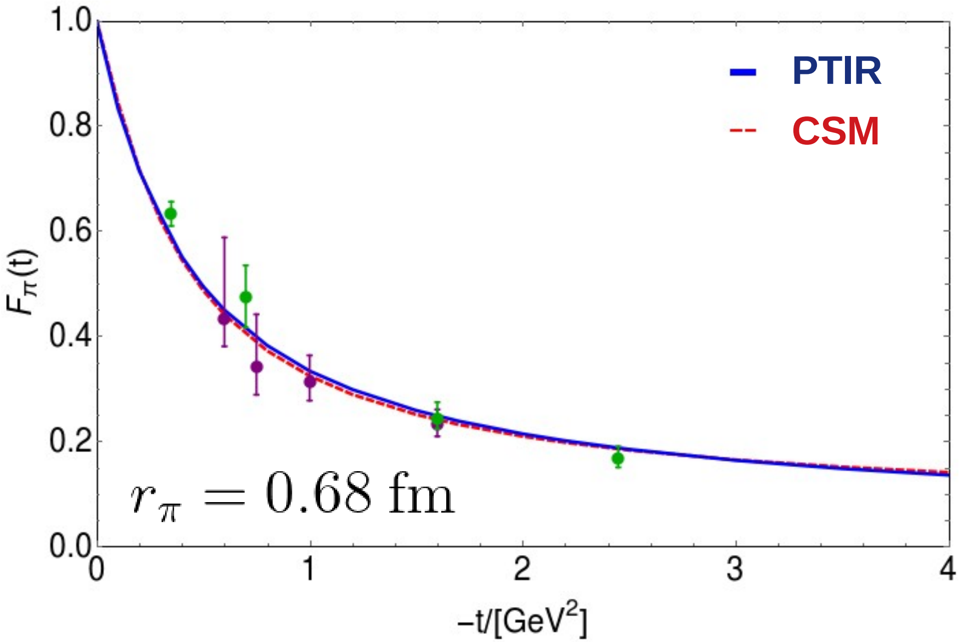
$$\mathcal{M}_m(\xi,t) = \sum_{i=0}^{\left[\frac{m}{2}\right]} (2\xi)^{2i} A_{i,m}^q(t) + \mathrm{mod}(m,2)(2\xi)^{m+1} C_{m+1}(t)$$

$$F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_{\bar{f}} F_M^{\bar{f}}(\Delta^2)$$

Weighed by electric charges

- **Isospin symmetry**

⇒ $F_{\pi^+}(-t) = F_{\pi^+}^u(-t)$



Data: G.M. Huber *et al*. PRC 78 (2008) 045202

CSM: L. Chang *et al*. PRL 111 (2013) 14, 141802

Pion Gravitational FFs

GPD → FFs

- **Gravitational** form factors are obtained from the **t-dependence** of the **1-st moment**:

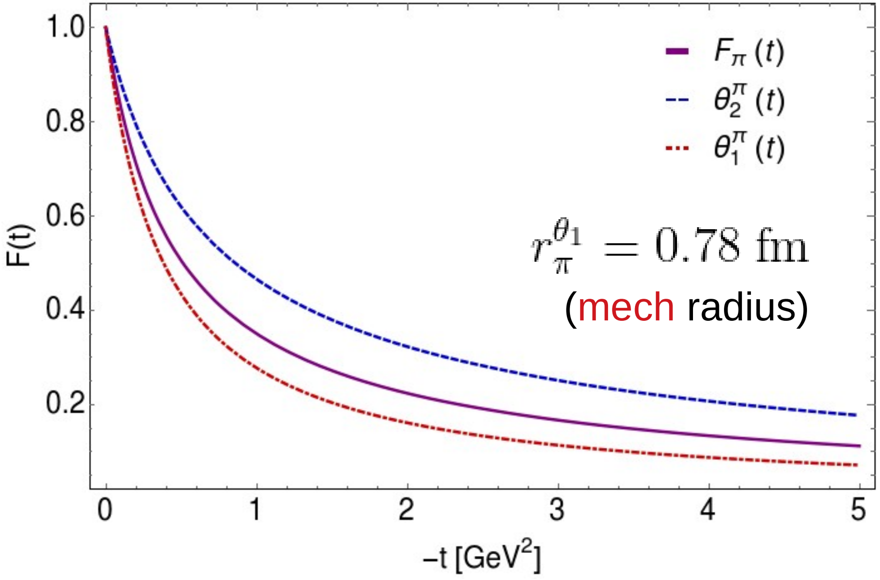
$$J_M(t,\xi) = \int_{-1}^{1} dx\, x H_M(x,\xi,t) = \Theta_2^M(t) - \xi^2 \Theta_1^M(t)$$

  - ✓ Directly obtained if **ξ = 0**
  - ✓ Only **DGLAP** GPD is required
  - ✗ **ERBL** GPD needed
    D-term

- But a sound expression can be constructed:

"Soft pion theorem"

$$\theta_1^{P_q}(\Delta^2) = c_1^{P_q}\theta_2^{P_q}(\Delta^2) + \int_{-1}^{1} dx\, x\left[H_P^q(x,1,0)P_{M_q}(\Delta^2) - H_P^q(x,1,-\Delta^2)\right]$$



$r_\pi^{\theta_1} = 0.78$ fm (mech radius)

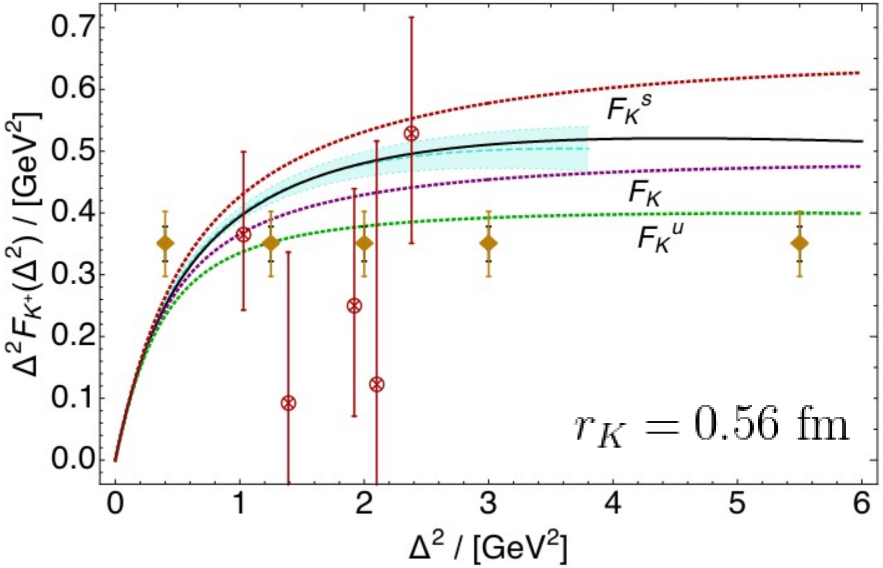
$r_\pi^E = 0.68 \text{ fm}\,,\quad r_\pi^{\theta_2} = 0.56 \text{ fm}$

(charge radius) (mass radius)

Kaon EFF

GPD → FFs

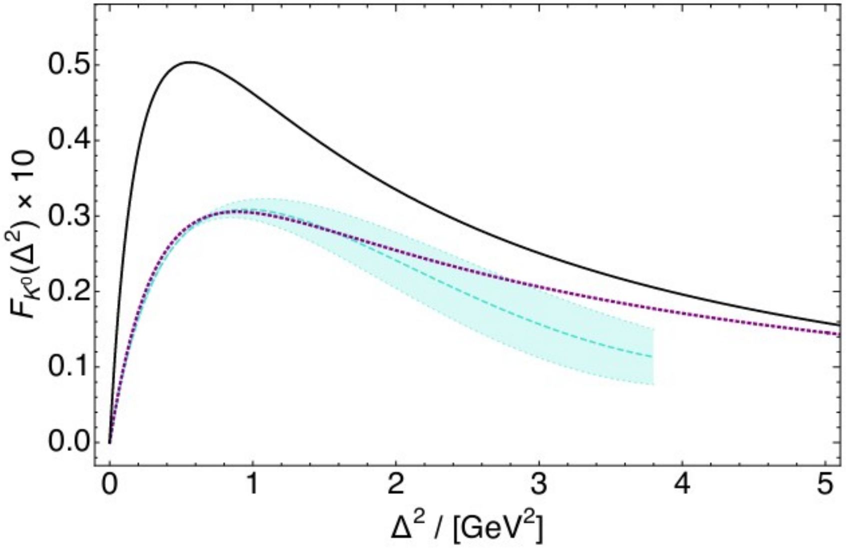
- **Electromagnetic** form factor: **charged** and **neutral** kaon



CSM - K^+: `Gao:2017mmp, Eichmann:2019bqf`

CSM - K^0: `Gao:2017mmp`

Lattice: `Davies:2018zav`



Kaon is more **compressed**

$$r_K^j \approx 0.85\, r_\pi^j$$

j = *mech, charge, mass*

Charge and mass distributions

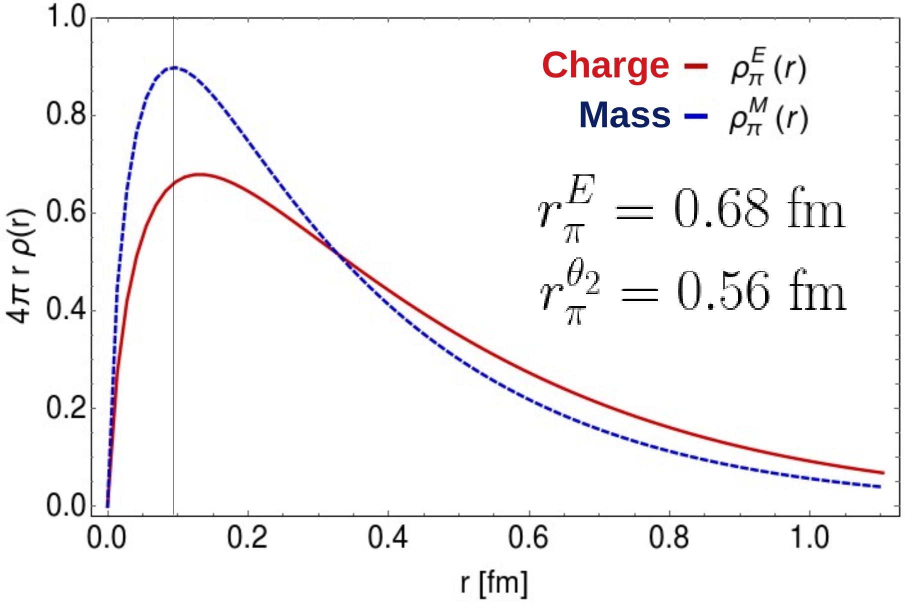
$$\rho_{\rm P}(b) = \frac{1}{2\pi}\int_0^\infty d\Delta\, \Delta J_0(\Delta\, b) F_{\rm P}(\Delta^2)$$

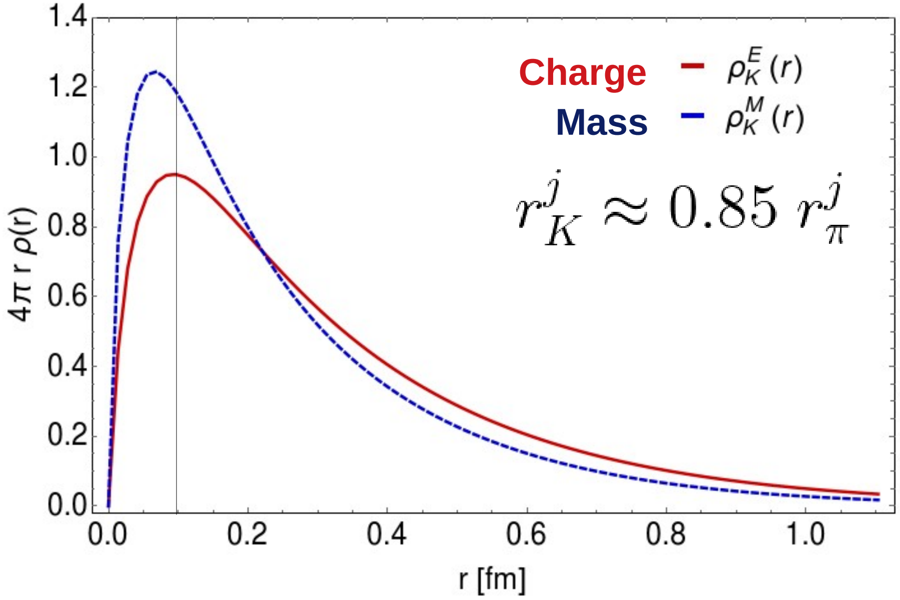
$$F_{\rm P}^E(\Delta^2) \to \rho_{\rm P}^E(b)$$

$$\theta_2^{\rm P}(\Delta^2) \to \rho_{\rm P}^M(b)$$

- **Intuitively**, we expect the meson to be localized at a **finite space**.
- **Charge** effect span over a **larger domain** than **mass** effects.

More **massive** hadron ➔ More **compressed**





Pressure distributions

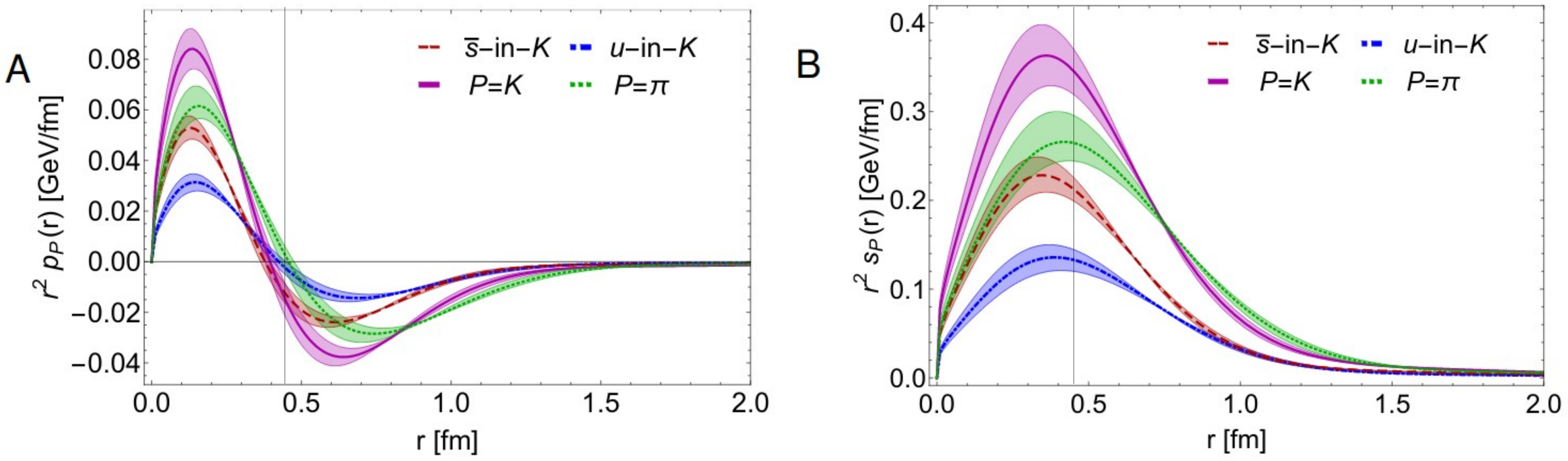
$$p_K^u(r) = \frac{1}{6\pi^2 r}\int_0^\infty d\Delta \, \frac{\Delta}{2E(\Delta)} \, \sin(\Delta r)[\Delta^2\theta_1^{K_u}(\Delta^2)] \, ,$$

$$s_K^u(r) = \frac{3}{8\pi^2}\int_0^\infty d\Delta \, \frac{\Delta^2}{2E(\Delta)} j_2(\Delta r)\,[\Delta^2\theta_1^{K_u}(\Delta^2)] \, ,$$

"Pressure" Quark attraction/repulsion

CONFINEMENT

"Shear" *Deformation* QCD forces



Impact parameter space GPDs

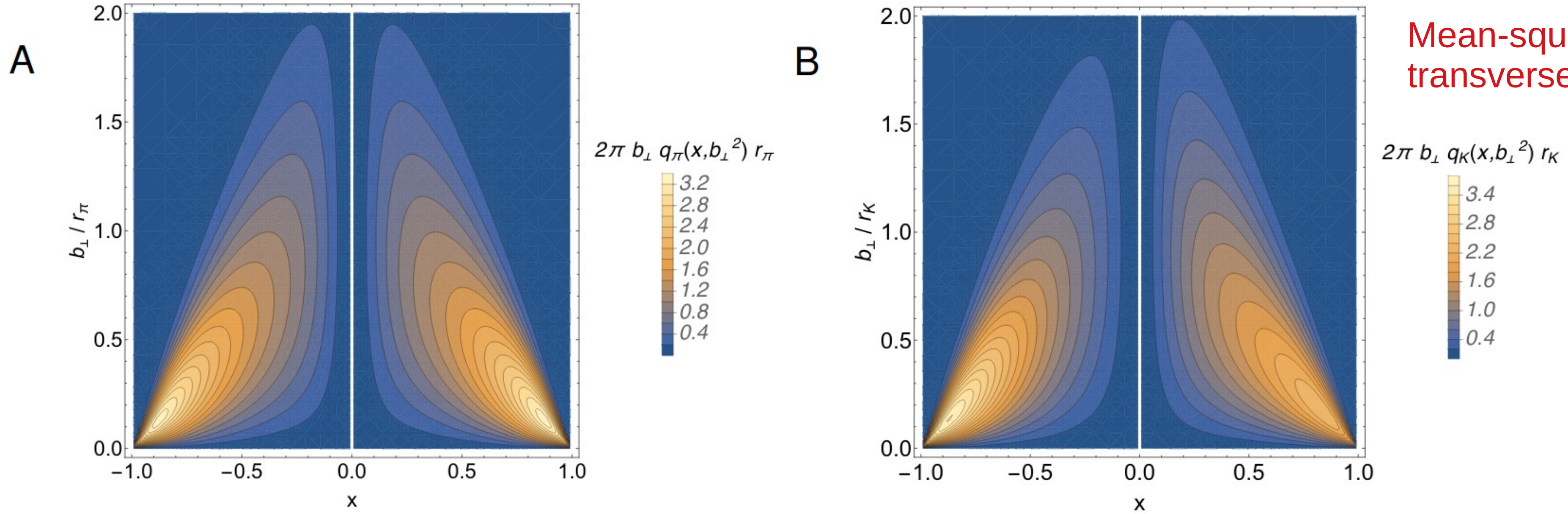
$$u^{\mathsf{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathsf{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$

Algebraic derivation!

$$\langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_u^\pi = \boxed{\frac{2}{3}} r_\pi^2 = \langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_{\bar{d}}^\pi ,$$

$$\langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_u^K = 0.71 r_K^2 , \langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_{\bar{s}}^K = 0.58 r_K^2 .$$

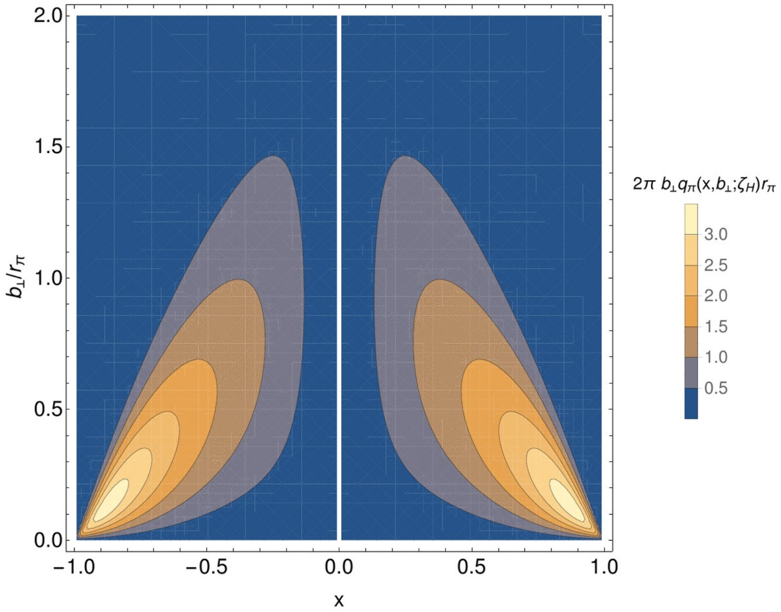
Mean-square transverse extent



- Likelihood of finding a valence-**quark** with **momentum fraction x**, at **position b**.

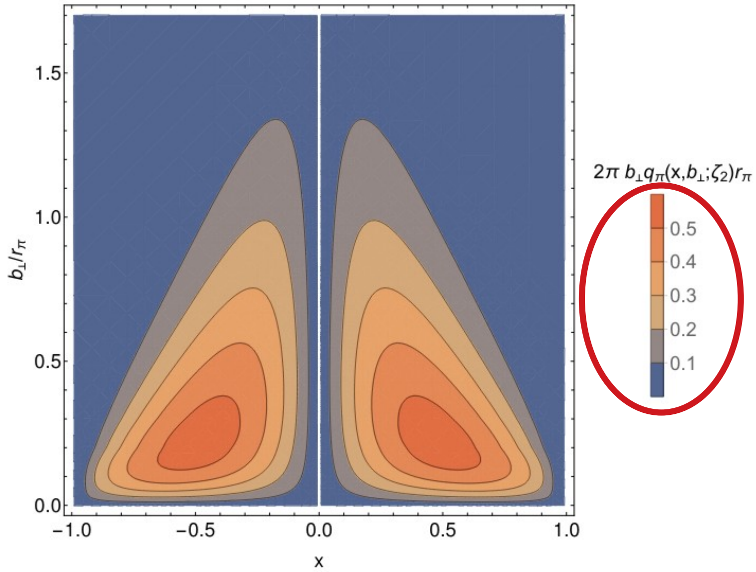
Evolved IPS-GPD: Pion Case

$$u^{\mathrm{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathrm{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$



- **Likelihood** of finding a parton with LF momentum **x** at transverse position **b**

$$\zeta_H \to \zeta = 2\ \mathrm{GeV}$$

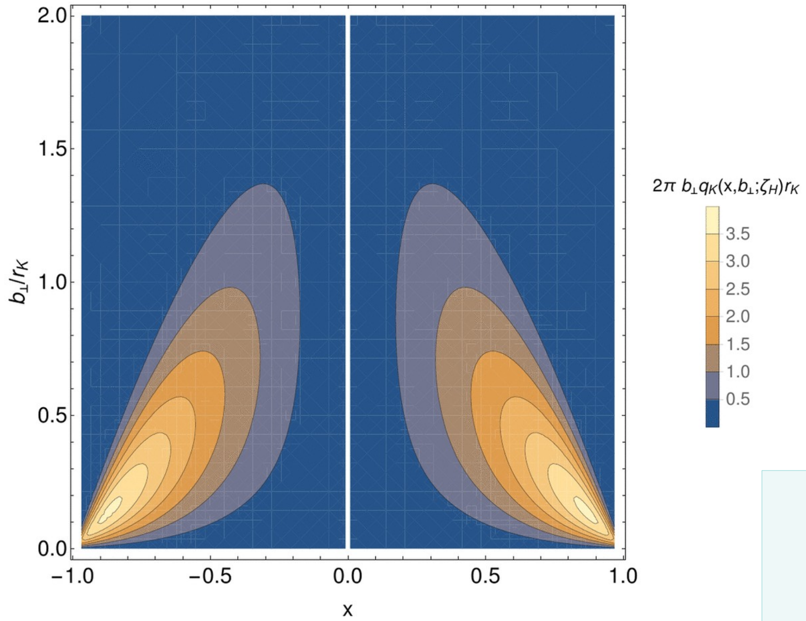


- Peaks **broaden** and **maximum drifts:**

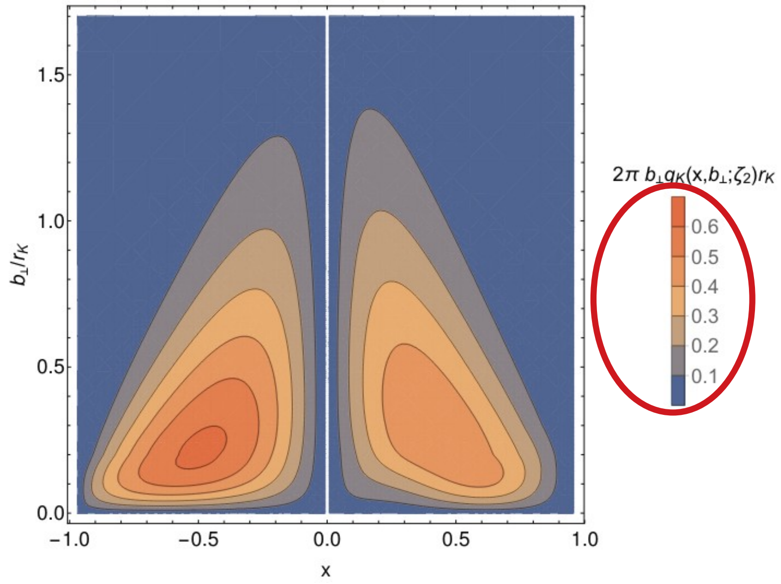
$$\max:\ 3.29 \to 0.55$$
$$(|x|, b) = (0.88, 0.13) \to (0.47, 0.23)$$

Evolved IPS-GPD: Kaon Case

$$u^{\mathsf{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathsf{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$



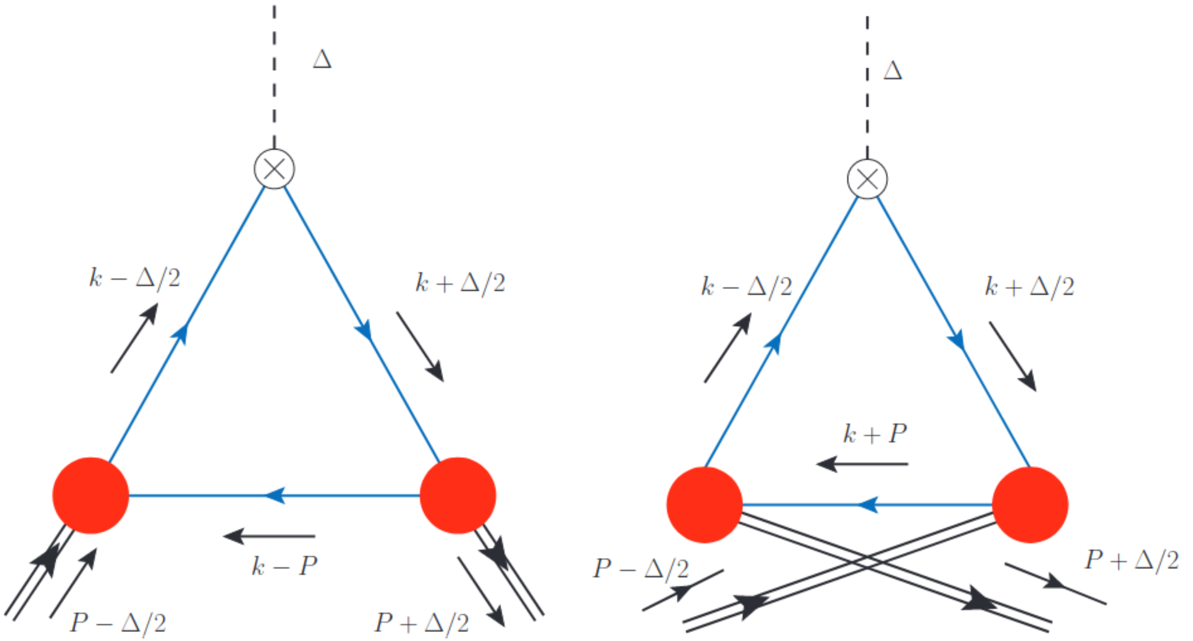
$\zeta_H \to \zeta = 2$ GeV



- **Likelihood** of finding a parton with LF momentum **x** at transverse position **b**

$$\max_{(s,u)} : (3.61, 2.38) \to (0.61, 0.49)$$
$$(x, b)_u = (0.84, 0.17) \to (0.41, 0.28)$$
$$(x, b)_s = (-0.87, 0.13) \to (-0.48, 0.22)$$

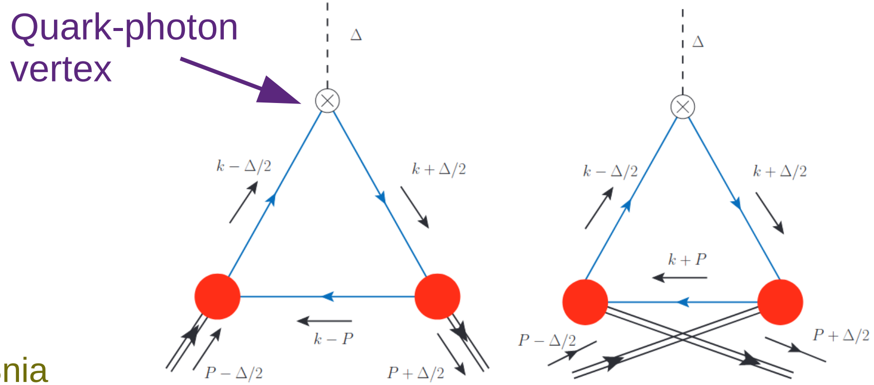
The diagram approach



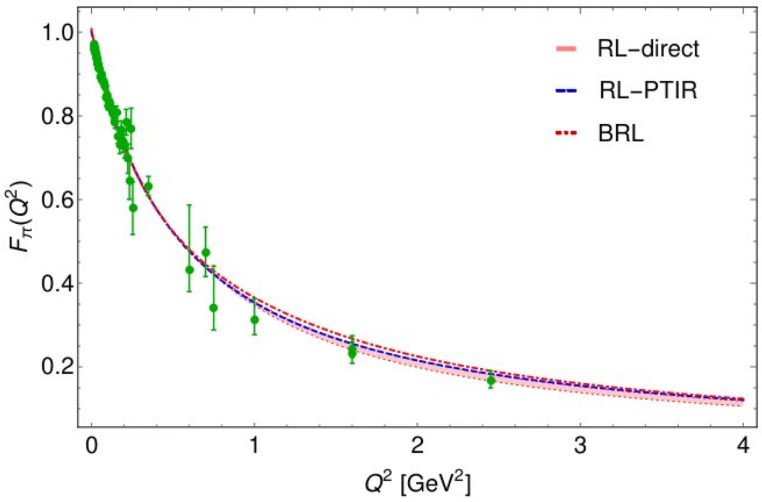
Mezrag:2023nkp

Diagram **approach**

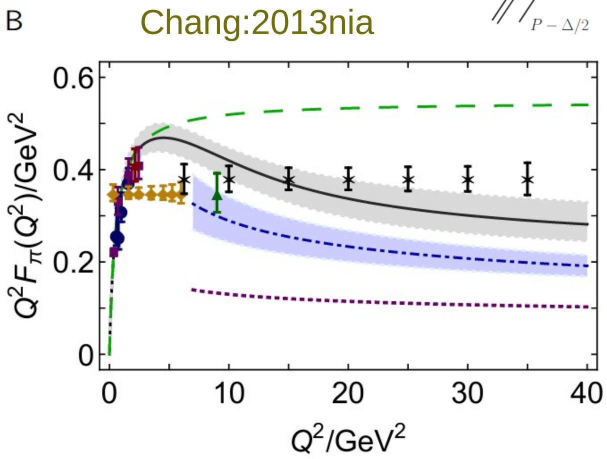
- In the calculation of meson **electromagnetic form factors**, the triangle diagram is self consistent with the **Rainbow-Ladder** truncation.
  - So, as long as the **QPV** fulfills its own required symmetries, the computed **EFF** would be sensible.



Miramontes:2021exi



Chang:2013nia



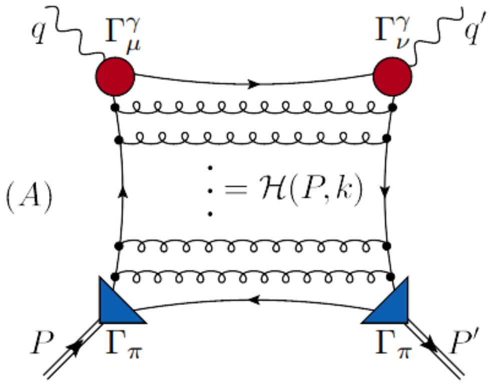
DSE calculation

HS formula

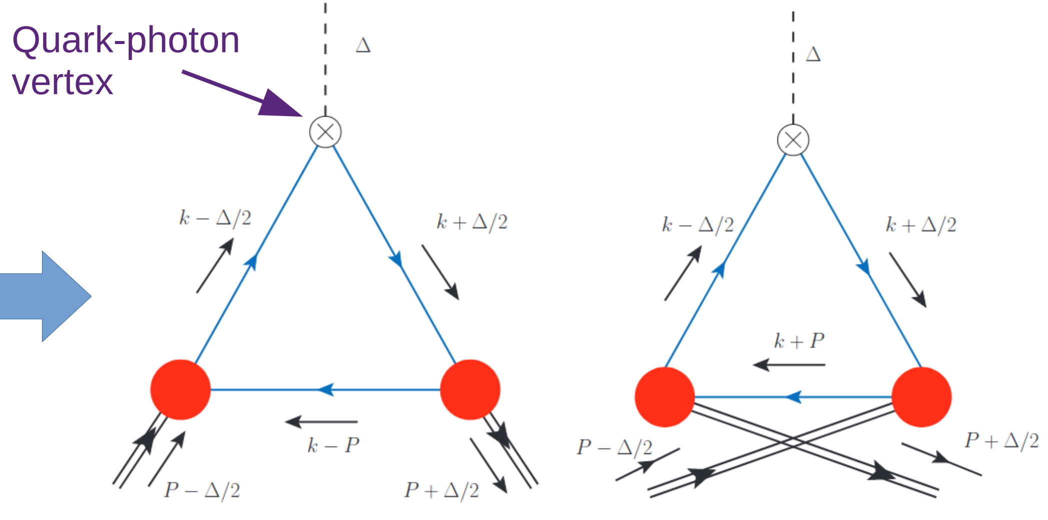
Asymptotic EFF

Diagram **approach**

- For **PDFs**, there is a correspondence between the **TD** and the so called **HandBag** diagram, in the **RL** truncation.



$$q_A^\pi(x;\zeta_H) = N_c \mathrm{tr}\int_{dk} i\Gamma_\pi(k_\eta,-P)$$
$$\times\ S(k_\eta)\, i\Gamma^n(k;x;\zeta_H)\, S(k_\eta)\, i\Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})$$



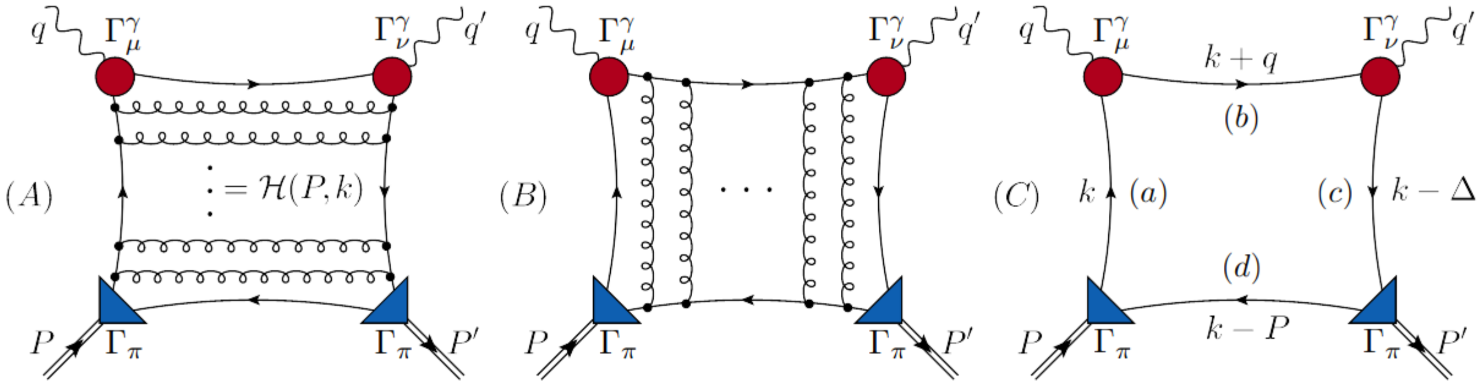
$$i\Gamma^n(k;x;\zeta_H) = \delta_n^x(k_\eta)\, n\cdot\partial_{k_\eta} S^{-1}(k_\eta)$$
$$\Gamma_\pi^n(k_\eta,-P;\zeta_H) = n\cdot\partial_{k_\eta}\Gamma_\pi(k_\eta,-P;\zeta_H)$$

Mezrag:2014jka, Chang:2014lva, Ding:2019lwe

Diagram **approach**

- For **PDFs**, there is a correspondence between the **TD** and the so called **HandBag** diagram, in the **RL** truncation.
- Nonetheless, to fully preserve both **baryon number** and **momentum conservation**, the HandBag diagram must be **supplemented**. For the case of the **PDF**:

  (in momentum-dependent interactions)



$$q_A^\pi(x;\zeta_H) = N_c \text{tr}\int_{dk} i\Gamma_\pi(k_\eta,-P) \times S(k_\eta)\, \underline{i\Gamma^n(k;x;\zeta_H)}\, S(k_\eta)\, i\Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})$$

$$q_{BC}^\pi(x;\zeta_H) = N_c \text{tr}\int_{dk} \Gamma_\pi^n(k_\eta,-P;\zeta_H) \times S(k_\eta)\Gamma_\pi(k_{\bar\eta},P)\, S(k_{\bar\eta})$$

$$i\Gamma^n(k;x;\zeta_H) = \delta_n^x(k_\eta)\, n\cdot\partial_{k_\eta} S^{-1}(k_\eta)$$

$$\Gamma_\pi^n(k_\eta,-P;\zeta_H) = n\cdot\partial_{k_\eta}\Gamma_\pi(k_\eta,-P;\zeta_H)$$

Chang:2014lva, Ding:2019lwe

Diagram **approach**

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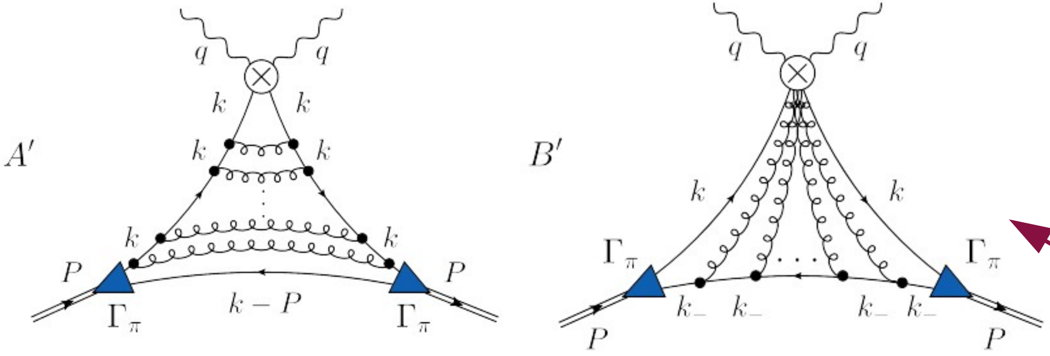
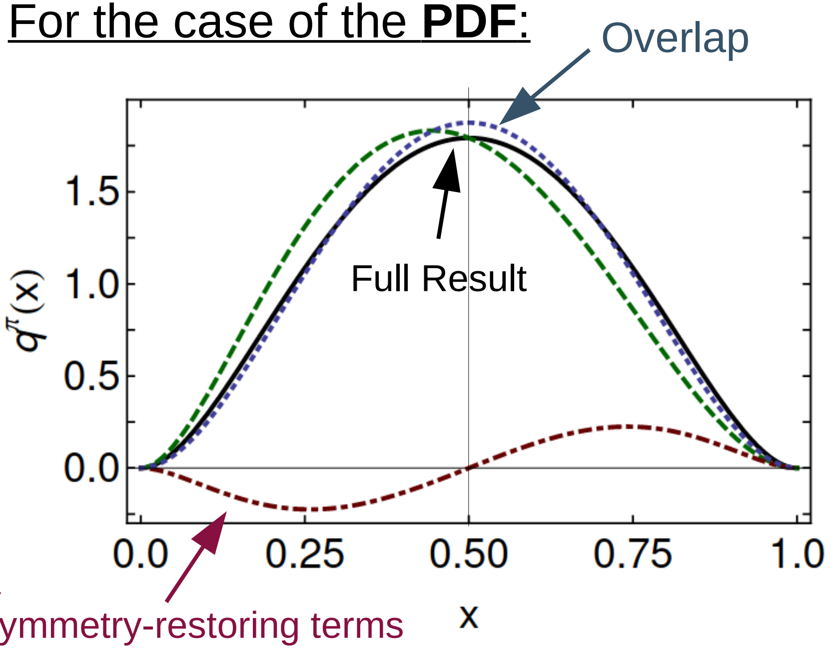
- Nonetheless, to fully preserve both **baryon number** and **momentum conservation**, the HandBag diagram must be **supplemented**. For the case of the **PDF**:

(in momentum-dependent interactions)

$$q^{\pi}(x;\zeta_H) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \times \{n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)]\} \Gamma_\pi(k_{\bar{\eta}}, P)\, S(k_{\bar{\eta}})$$





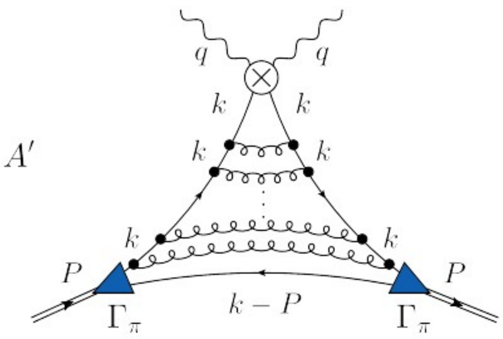
Symmetry-restoring terms

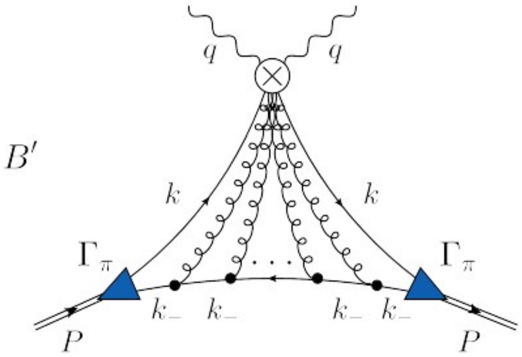
Mezrag:2014jka, Chang:2014lva, Ding:2019lwe

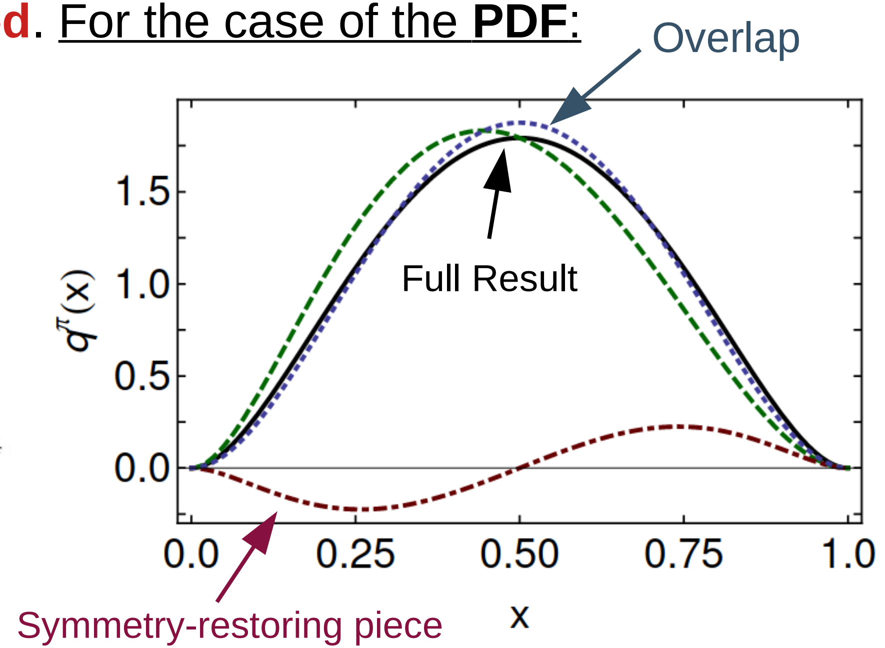
Diagram **approach**

- For **PDFs**, there is a correspondence between the **TD** and the so called **HandBag** diagram, in the **RL** truncation.
- Nonetheless, to fully preserve both **baryon number** and **momentum conservation**, the HandBag diagram must be **supplemented**. For the case of the **PDF**:

(in momentum-dependent interactions)







The GPD analogous has not been *fully-solved* in momentum-dependent interactions.

Mezrag:2014jka, Chang:2014lva, Ding:2019lwe

Contact Interaction model:
Some highlights

Contact Interaction

Roberts:2010rn

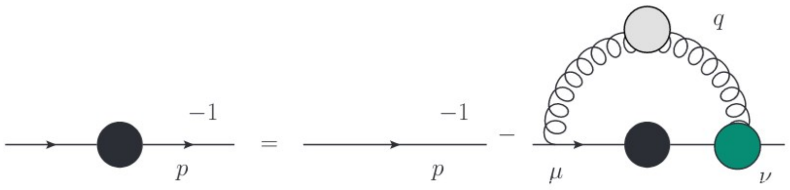
Gutierrez-Guerrero:2010waf

- The quark gap equation in a symmetry-preserving **contact interaction** model (**SCI**):

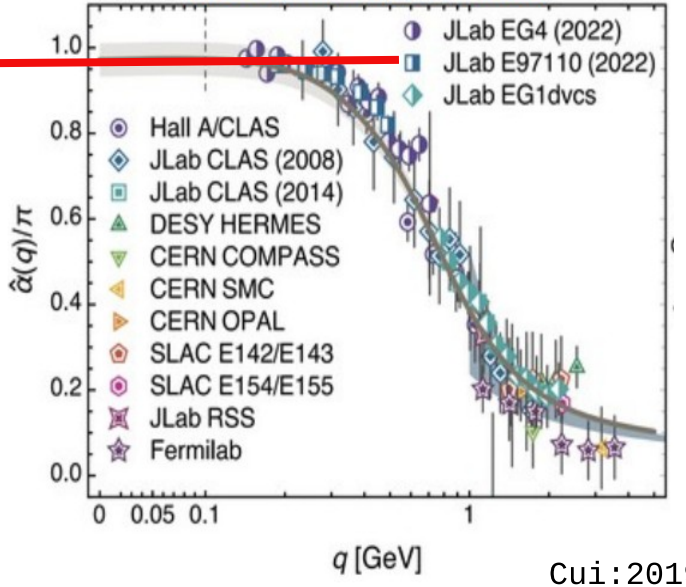
$$S^{-1}(p) = i\gamma\cdot p + m + \frac{16\pi}{3}\boxed{\frac{\alpha_{\rm IR}}{m_G^2}}\int\frac{d^4q}{(2\pi)^4}\,\gamma_\mu\, S(q)\,\gamma_\mu$$

Infrared strength $\alpha_{\rm IR} = 0.93\pi$

Compatible with modern computations.



(figure: D. Binosi's courtesy!)



Cui:2019dwv

> - **Recall** the quark **gap equation**:
>
> $$S_f^{-1}(p) = Z_2(i\gamma\cdot p + m_f^{\rm bm}) + \Sigma_f(p)\,,$$
>
> $$\Sigma_f(p) = \frac{4}{3}Z_1\int_{dq}^{\Lambda} g^2 D_{\mu\nu}(p-q)\gamma_\mu S_f(q)\Gamma_\nu^f(p,q)$$

- Namely, **SCI kernel** is essentially **RL** + **constant** gluon propagator

Contact **Interaction**

- Let us now consider the quark gap equation in a symmetry-preserving **contact interaction** model (**SCI**)

Roberts:2010rn

Gutierrez-Guerrero:2010waf

$$S^{-1}(p) = i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q)\gamma_\mu$$

- **Constant gluon propagator:**
  - Quark propagator, with **constant mass function**

$$S_f(p) = Z_f(p^2)(i\gamma\cdot p + M_f(p^2))^{-1} \Rightarrow S(p)^{-1} = i\gamma\cdot p + M$$

- **Non renormalizable**
- **Needs regularization scheme:**

$$\frac{1}{s+M_f^2} = \int_0^\infty d\tau e^{-\tau(s+M_f^2)} \to \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau e^{-\tau(s+M_f^2)}$$

input: current masses				output: dressed masses			
m_0	m_u	m_s	m_s/m_u	M_0	M_u	M_s	M_s/M_u
0	0.007	0.17	24.3	0.36	0.37	0.53	1.43

$\tau_{ir} = 1/0.24\,\text{GeV}^{-1}$: Ensures the absence of quark production thresholds (confinement)

$\tau_{uv} = 1/0.905\,\text{GeV}^{-1}$: UV cutoff. Sets the scale of all dimensioned quantities.

Contact **Interaction**

- Let us now consider the quark gap equation in a symmetry-preserving **contact interaction** model (**SCI**)

$$S^{-1}(p) = i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q)\gamma_\mu$$

- Quark propagator, with **constant mass function**

$$S(p)^{-1} = i\gamma\cdot p + M$$

- The **meson** Bethe-Salpeter equation:

$$\Gamma(k;P) = -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\chi(q;P)\gamma_\mu$$

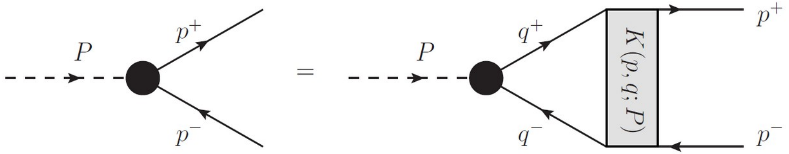
- The **diquark** Bethe-Salpeter equation:

$$\Gamma_{qq}(k;P) = -\frac{8\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\chi_{qq}(q;P)\gamma_\mu$$

  - Recall a J^{p} **diquark** partners with an analogous J^{-p} **meson**.

- The interaction produces **momentum independent** BSAs:

$$\begin{aligned}
\Gamma_\pi(P) &= \gamma_5\left[iE_\pi(P) + \frac{\gamma\cdot P}{M}F_\pi(P)\right]\\
\Gamma_\sigma(P) &= \mathbb{1}E_\sigma(P)\,,\\
\Gamma_\rho(P) &= \gamma^T E_\rho(P)\,,\\
\Gamma_{a_1}(P) &= \gamma_5\gamma^T E_{a_1}(P)\,,
\end{aligned}$$



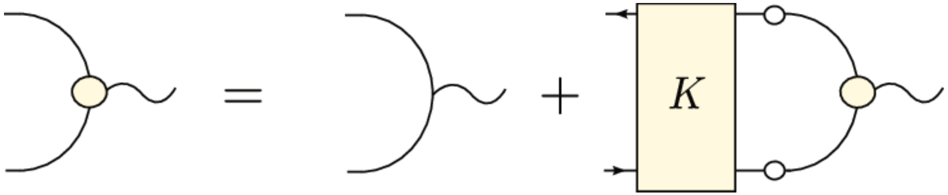
Contact Interaction

- The **quark-photon** vertex:

$$\Gamma^\gamma_\mu(Q) = \frac{Q_\mu Q_\nu}{Q^2}\gamma_\nu + \Gamma^{\rm T}_\mu(Q)$$

$$\Gamma^{\rm T}_\mu(Q) = P_{\rm T}(Q^2)\mathcal{P}_{\mu\nu}(Q)\gamma_\nu$$

Introduces a vector meson pole in the timelike axis.



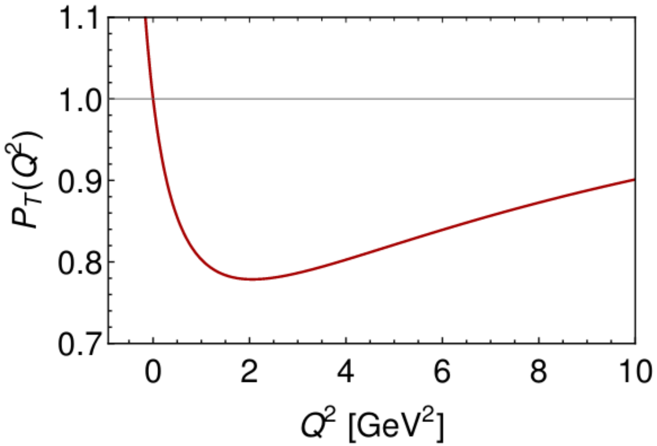
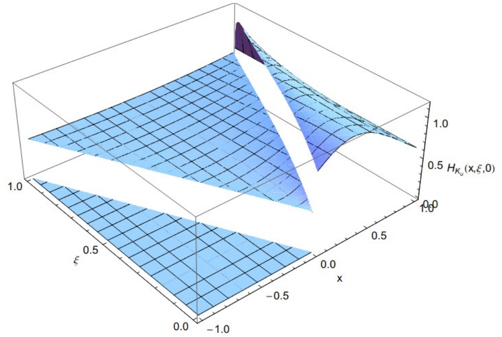


Fig. 9 Photon+quark vertex dressing function in Eq. (A.3). As in any symmetry preserving treatment of photon+quark interactions, $P_{\rm T}(Q^2)$ exhibits a pole at $Q^2 = -m_\rho^2$. Moreover, $P_{\rm T}(Q^2=0) = 1 = P_{\rm T}(Q^2 \to \infty)$.

➔ From a total of **12** tensor structures, the **QPV** is now characterized by just **2**!

Albino:2021rvj

GPDs In the SCI



A fresh look at the generalized parton distributions of light pseudoscalar mesons

Zanbin Xing,[1] Minghui Ding,[2] Khépani Raya,[3] and Lei Chang[1]

[1] *School of Physics, Nankai University, Tianjin 300071, China*
[2] *Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany*
[3] *Department of Integrated Sciences and Center for Advanced Studies in Physics, Mathematics and Computation, University of Huelva, E-21071 Huelva, Spain.*

(Dated: January 27, 2023)

Xing:2023eed, Xing:2022jtt, Xing:2022mvk

GPDs in the SCI

- Consider the following expression for the **valence-quark** pion **GPD**:

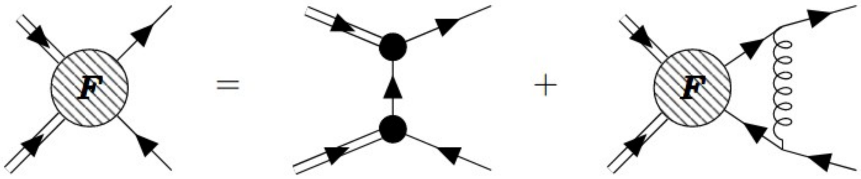
$$2H_u^\pi(x,\xi,Q^2) = N_c \mathrm{tr} \int_q i\gamma_n \mathcal{M}_u^\pi(q_-,p,k)$$

Off-forward scattering amplitude

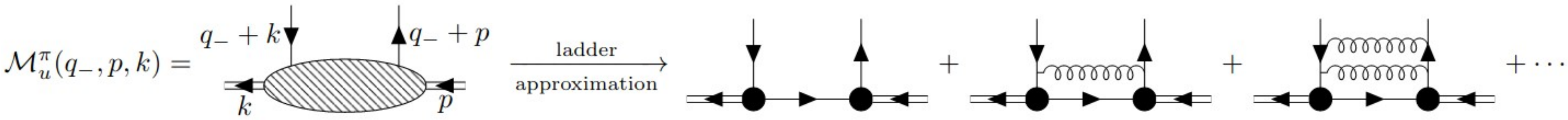
'Bare' insertion operator

$$\gamma_n = \delta_n^x(q)\gamma \cdot n$$
$$\delta_n^x(q) = \delta(n\cdot q - xn\cdot P)$$

Xing:2022mvk



- Rather than dressing the **insertion operator**, let's take a look at the **scattering amplitude**:



Xing:2023eed

GPDs in the SCI

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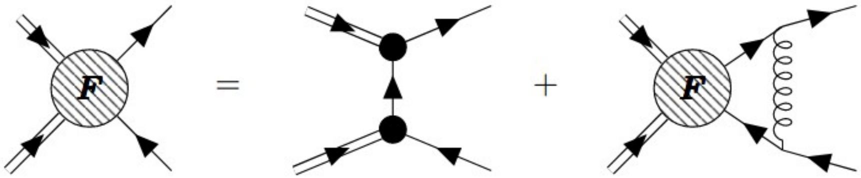
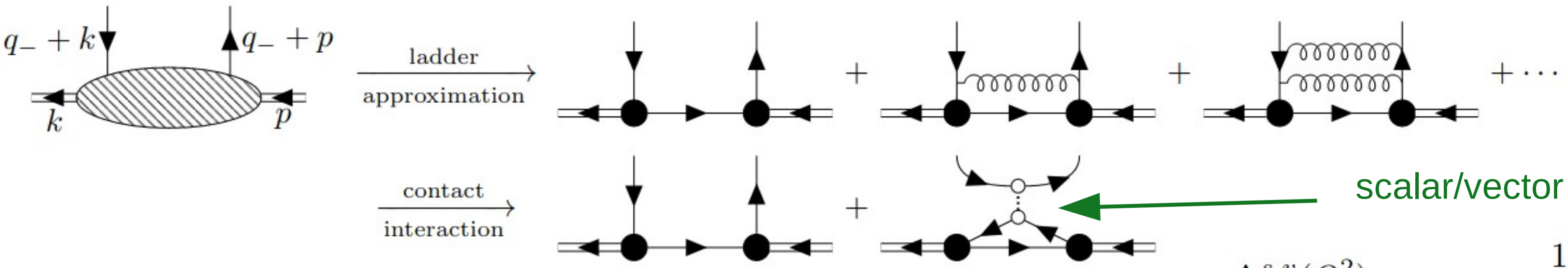
Xing:2022mvk



- Rather than dressing the **insertion operator**, let's take a look at the **scattering amplitude**:

$$\mathcal{M}_u^\pi(q_-,p,k) =$$



scalar/vector poles

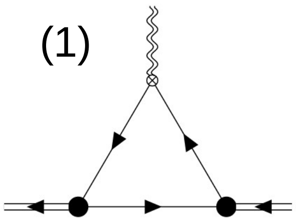
$$\Delta^{s,v}(Q^2) = \frac{1}{3m_G^2/4 + f^{s,v}(Q^2)}$$

Xing:2023eed

GPDs in the SCI

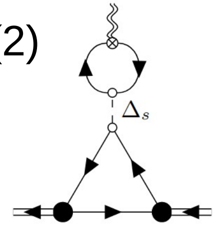
- Therefore, we arrive at the following diagrams:

(1)



Impulse approximation

(2)



'Pole' Contributions

- It is now straightforward to compute **Mellin moments**:

$$2\langle x^m\rangle_u^\pi(\xi,Q^2) = N_c \mathrm{tr}\int_q \frac{(n\cdot q)^m}{(n\cdot P)^{m+1}} i\gamma\cdot n \mathcal{M}_u(q_-,p,k)$$

- Where two kinds of denominators appear:

$$\frac{1}{D_{-Q/2}D_{Q/2}D_P} = \int_{\Omega_u}\frac{du_1du_2}{\left[(q+\alpha Q-\beta P)^2+\omega_3\right]^3}, \quad (5a)$$

$$\frac{1}{D_{-Q/2}D_{Q/2}} = \int_{\Omega_u}\frac{du_1du_2\delta(\beta)}{\left[(q+\alpha Q)^2+\omega_2\right]^2}, \quad (5b)$$

where $D_X = D(q-X)$, $\alpha = u_1 - u_2$, $\beta = 1-u_1-u_2$ and $\Omega_u = \{(u_1,u_2)|0<u_1<1, 0<u_2<1-u_1\}$

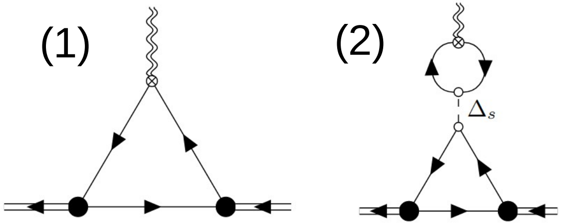
Domain of integration of Feynman parameters

- Already suggested in **chiral quark effective theories!**
- Here, this term **arises** from its **DSE**!

Polyakov:1999gs, Broniowski:2007si

GPDs in the SCI

- Therefore, we arrive at the following diagrams:



Impulse approximation — ‘Pole’ Contributions

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where $D_X = D(q-X)$, $\alpha = u_1-u_2$, $\beta = 1-u_1-u_2$ and $\Omega_u = \{(u_1,u_2)|0<u_1<1, 0<u_2<1-u_1\}$

Domain of integration of Feynman parameters

- After changes of variables, regularization, ...

$$\langle x^m\rangle_{u,1}^\pi = \int_\Omega d\beta d\alpha\,[(\beta+\xi\alpha)^m(h_{a0}+\xi h_{a1}) + m(\beta+\xi\alpha)^{m-1}(h_{b0}+\xi h_{b1}+\xi^2 h_{b2})]$$

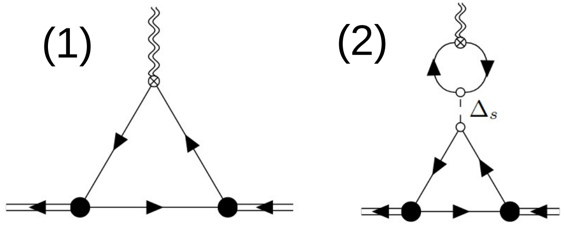
$$\langle x^m\rangle_{u,2}^\pi = \int_\Omega d\beta d\alpha\,(\beta+\xi\alpha)^m(h_{c0}+\xi h_{c1})\delta(\beta),$$

$|\alpha|+|\beta|\leq 1$

Domain of DD variables… almost there!

GPDs in the SCI

- Therefore, we arrive at the following diagrams:



Impulse approximation — 'Pole' Contributions

- After changes of variables, regularization, ...

$$\langle x^m\rangle^{\pi}_{u,1} = \int_{\Omega} d\beta d\alpha\,[(\beta+\xi\alpha)^m(h_{a0}+\xi h_{a1}) + m(\beta+\xi\alpha)^{m-1}(h_{b0}+\xi h_{b1}+\xi^2 h_{b2})]$$

$$\langle x^m\rangle^{\pi}_{u,2} = \int_{\Omega} d\beta d\alpha\,(\beta+\xi\alpha)^m(h_{c0}+\xi h_{c1})\delta(\beta)\,,$$

$|\alpha|+|\beta|\leq 1$

Domain of DD variables… almost there!

- Concerning the highlighted part... Let $(.) := (\beta+\xi\alpha)^m$

  $m(.)^{m-1}$ Can be written as derivatives of **β or ξα**

  So that, we can turn $m(.)^{m-1} \to (.)^m$

- Nonetheless, the **arbitrariness** in the differentiation is **unavoidable.** $\frac{\partial(\cdot)^m}{\partial\beta}$ or $\frac{\partial(\cdot)^m}{\xi\partial\alpha}$

  Let $\eta+\bar{\eta}=1$ Characterize such arbitrariness.

- Thus:

$$m(\cdot)^{m-1}(h_{b0}+\xi h_{b1}+\xi^2 h_{b2}) = \frac{\partial\left[(\cdot)^m(h_{b0}+\eta\xi h_{b1})\right]}{\partial\beta} - (\cdot)^m\frac{\partial(h_{b0}+\eta\xi h_{b1})}{\partial\beta} + \frac{\partial\left[(\cdot)^m(\bar{\eta}h_{b1}+\xi h_{b2})\right]}{\partial\alpha} - (\cdot)^m\frac{\partial(\bar{\eta}h_{b1}+\xi h_{b2})}{\partial\alpha}$$

GPDs in the SCI

- Finally, the **GPD** Mellin **moments** acquire the form:

$$\langle x^m \rangle_u^\pi = \int_\Omega d\beta d\alpha (\beta + \xi\alpha)^m \left[\mathcal{F}(\beta,\alpha,Q^2) + \xi\mathcal{G}(\beta,\alpha,Q^2)\right]$$

$$\begin{aligned}\mathcal{F}(\beta,\alpha,Q^2) &= h_{a0} - \frac{\partial h_{b0}}{\partial\beta} - \bar{\eta}\frac{\partial h_{b1}}{\partial\alpha} + h_{c0}\delta(\beta) \\ &+ h_{b0}\left[\delta(\bar{\beta} - |\alpha|) - \delta(\beta)\right] \\ &+ \bar{\eta} h_{b1}\left[\delta(\bar{\beta} - \alpha) - \delta(\bar{\beta} + \alpha)\right] , \\ \mathcal{G}(\beta,\alpha,Q^2) &= h_{a1} - \eta\frac{\partial h_{b1}}{\partial\beta} - \frac{\partial h_{b2}}{\partial\alpha} + h_{c1}\delta(\beta) \\ &+ \eta h_{b1}\left[\delta(\bar{\beta} - |\alpha|) - \delta(\beta)\right] \\ &+ h_{b2}\left[\delta(\bar{\beta} - \alpha) - \delta(\bar{\beta} + \alpha)\right] .\end{aligned}$$

- This is nothing that the Mellin moments of the **DD** representation**:**

$$H_u^\pi(x,\xi,Q^2) = \int_\Omega d\beta d\alpha\, \delta(x - \beta - \xi\alpha)\left[\mathcal{F}(\beta,\alpha,Q^2) + \xi\mathcal{G}(\beta,\alpha,Q^2)\right]$$

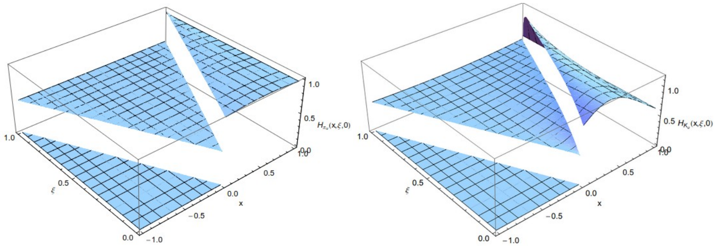
✔ **Polinomiality**

$$\mathcal{F}(\beta,\alpha,Q^2) = \mathcal{F}(\beta,-\alpha,Q^2)$$

$$\mathcal{G}(\beta,\alpha,Q^2) = -\mathcal{G}(\beta,-\alpha,Q^2)$$

- Mellin moments are **even** functions of ξ
- The larger power of **ξ**, at most **m+1**

$$\mathcal{M}_m(\xi,t) = \sum_{i=0}^{\left[\frac{m}{2}\right]} (2\xi)^{2i} A_{i,m}^q(t) + \mathrm{mod}(m,2)(2\xi)^{m+1} C_{m+1}(t)$$



GPDs in the SCI

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$$\langle x^m \rangle^\pi_u = \int_\Omega d\beta d\alpha (\beta + \xi\alpha)^m \left[\mathcal{F}(\beta,\alpha,Q^2) + \xi \mathcal{G}(\beta,\alpha,Q^2)\right]$$

- This is nothing that the Mellin moments of the **DD** representation**:**

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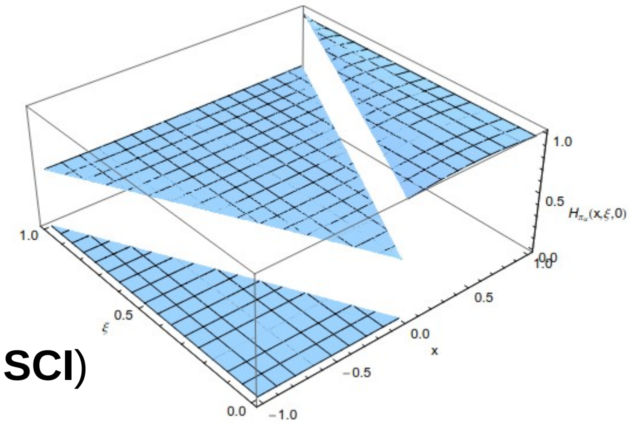
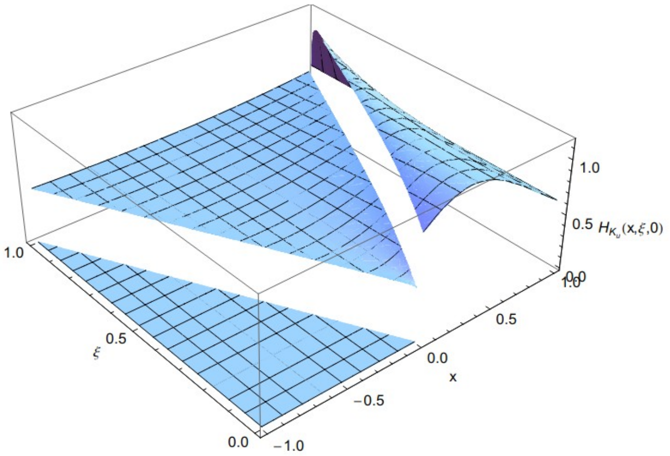
$$\begin{aligned}
\mathcal{F}(\beta,\alpha,Q^2) &= h_{a0} - \frac{\partial h_{b0}}{\partial \beta} - \bar{\eta}\frac{\partial h_{b1}}{\partial \alpha} + h_{c0}\delta(\beta) \\
&+ h_{b0}\left[\delta(\bar{\beta}-|\alpha|) - \delta(\beta)\right] \\
&+ \bar{\eta} h_{b1}\left[\delta(\bar{\beta}-\alpha) - \delta(\bar{\beta}+\alpha)\right] , \\
\mathcal{G}(\beta,\alpha,Q^2) &= h_{a1} - \eta\frac{\partial h_{b1}}{\partial \beta} - \frac{\partial h_{b2}}{\partial \alpha} + h_{c1}\delta(\beta) \\
&+ \eta h_{b1}\left[\delta(\bar{\beta}-|\alpha|) - \delta(\beta)\right] \\
&+ h_{b2}\left[\delta(\bar{\beta}-\alpha) - \delta(\bar{\beta}+\alpha)\right] .
\end{aligned}$$

- ✔ **Positive** definite **GPDs**!
- ✗ **Discontinuous** at **x = ± ξ**
- ✗ **Non-vanishing** at **x = 1**

(Last two are drawbacks of the **SCI**)





GPDs in the SCI

$$2\langle x^m\rangle_u^\pi(\xi,Q^2) = N_c \mathrm{tr}\int_q \frac{(n\cdot q)^m}{(n\cdot P)^{m+1}} i\gamma\cdot n \mathcal{M}_u(q_-,p,k)$$

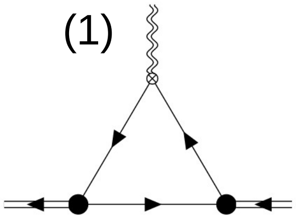
- The **moments** can be conveniently expressed as:

$$\mathcal{M}_m(Q^2,\xi) = \mathcal{M}_m^{(0)}(Q^2;\xi) + \mathcal{M}_m^{s,v}(Q^2;\xi)$$

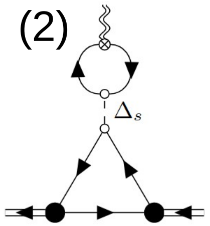
Pole-free part

scalar vector

$$\mathcal{M}_m^{s,v}(Q^2;\xi)\ \int_0^1 du\,(\xi(2u-1))^m I_0(u)[\xi T_s(Q^2)(1-2u)+T_v(Q^2)u(1-u)]$$



Impulse approximation



'Pole' Contributions

- **Sum Rules**

$$A=\theta_2\,,\ D=-\theta_1$$

$$\langle x^0\rangle_f^{\mathrm{PS}}(\xi,Q^2) = F_f^{\mathrm{em,PS}}(Q^2)$$

This one contains a vector meson pole!

$$\langle x^1\rangle_f^{\mathrm{PS}}(\xi,Q^2) = A_f^{\mathrm{PS}}(Q^2) + \xi^2 D_f^{\mathrm{PS}}(Q^2)$$

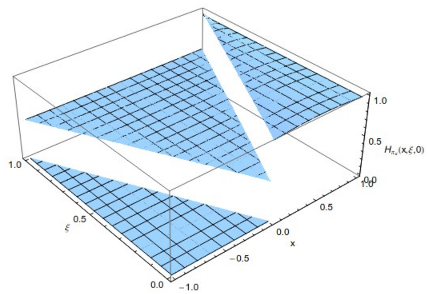
This is pole-free

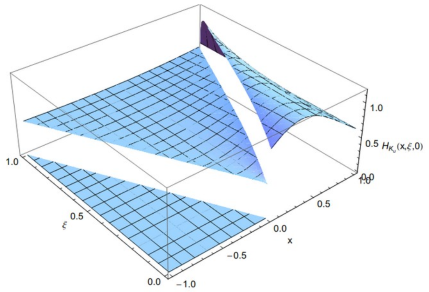
This one contains a scalar pole

Scalar pole is crucial to get: [Xing:2022mvk]

(Soft-pion theorem)

$$D = -\frac{E_\pi - 6F_\pi}{3(E_\pi-2F_\pi)} - \frac{2E_\pi}{3(E_\pi-2F_\pi)} = -1$$

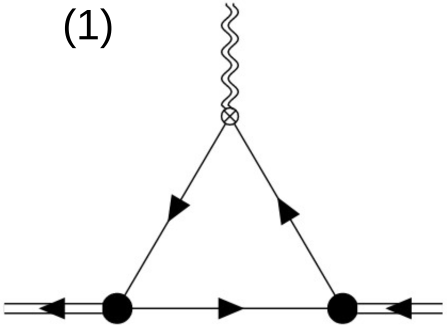




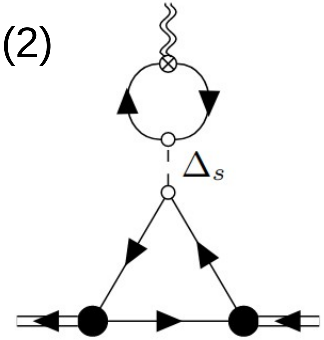
GPDs in the SCI

- The **contributions** to the **GPD**:

(1)

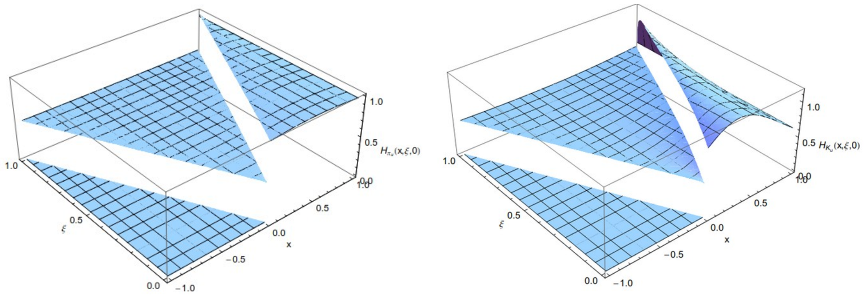


(2)



Impulse approximation

'Pole' Contributions



$$H_{\pi}^{I=1}(2u-1,1,0)=\frac{1}{2}\varphi_{\pi}(u)\,,\;u\in[0,1]$$

$$H_{u,1}^{\text{c.l.}}(x,\xi,0)=\frac{(\xi+x)E_{\text{PS}}-2\xi F_{\text{PS}}}{2\xi(E_{\text{PS}}-2F_{\text{PS}})}\theta(\xi-x,x+\xi)$$
$$+\theta(1-x,x-\xi)\,,$$

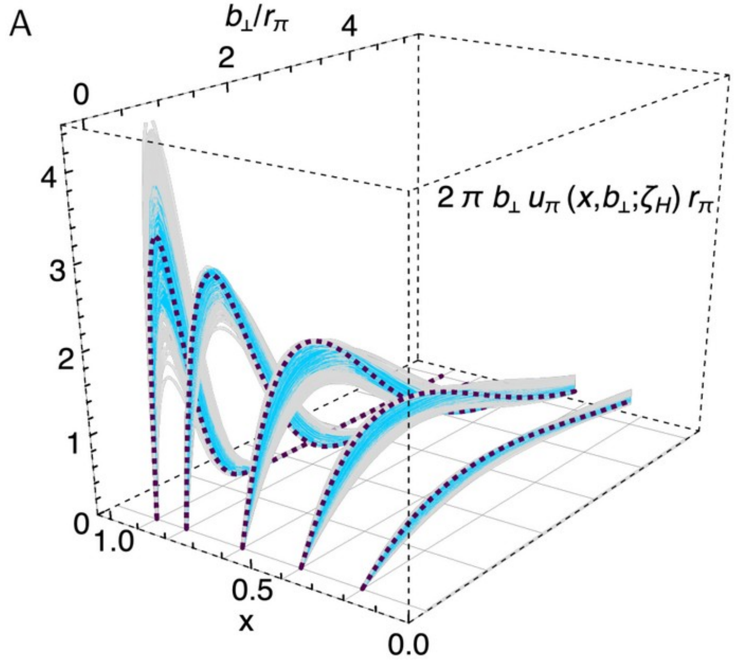
$$H_{u,2}^{\text{c.l.}}(x,\xi,0)=\frac{-xE_{\text{PS}}}{2\xi(E_{\text{PS}}-2F_{\text{PS}})}\theta(\xi-x,x+\xi)\,,$$

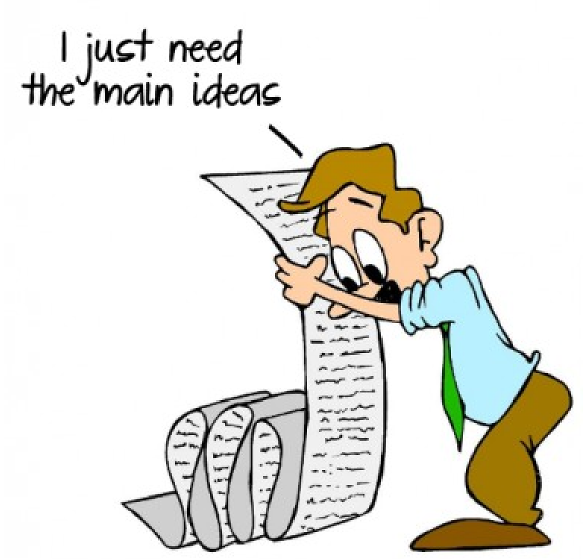
- **Soft pion theorem:**

$$H_{u}^{\text{c.l.}}(x,\xi,0)=\frac{1}{2}\theta(\xi-x,x+\xi)+\theta(1-x,x-\xi)$$

- In this limit, both SCI-derived **PDA** and **PDF** are constants
- The above expression also ensures the **PDF** is the forward limit of the **GPD**

Conclusions and Scope



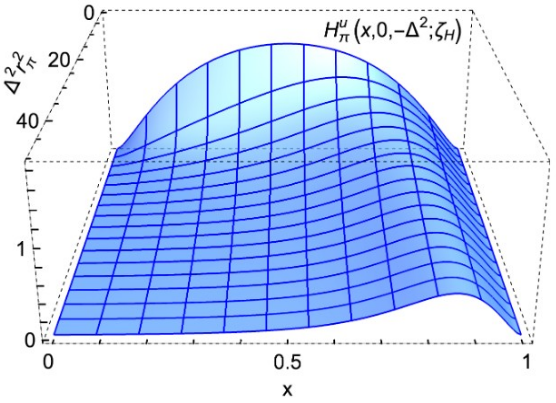


Conclusions and Scope

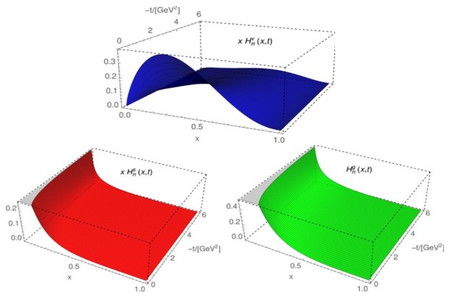
- We discussed two different approaches to address the **pion** and **Kaon** **GPDs**: Overlap of **LFWF** and **diagram-based** calculation.

- The **LFWF**, by construction, fulfills the requirement of positivity. The drawback is that it is limited to the **DGLAP** domain.

  - **Charge**, **Mass**, **Spatial** and **Momentum** distributions are already within the reach of **DGLAP** domain.
  - In this domain, we can also evolve the **GPDs** to disentangle **valence**, **glue** and sea **content**.
  - Sophisticated covariant extensions to the **ERBL** domain are known.



(Still, D-term is not fixed unambiguously)



Conclusions and Scope

- We discussed two different approaches to address the **pion** and **Kaon** **GPDs**:
  Through the overlap of the **LFWF** and following a **diagrammatic** representation.

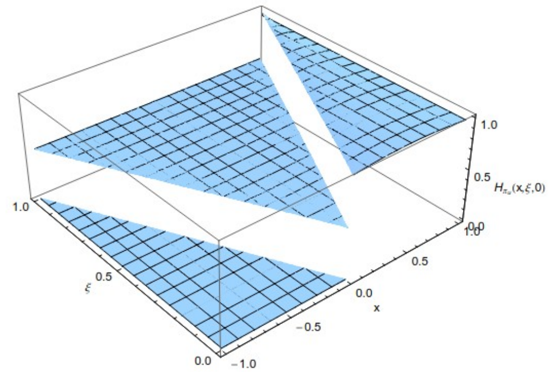
- The **LFWF**, by construction, fulfills the requirement of positivity. The Drawback is that it is limited to the **DGLAP** domain.

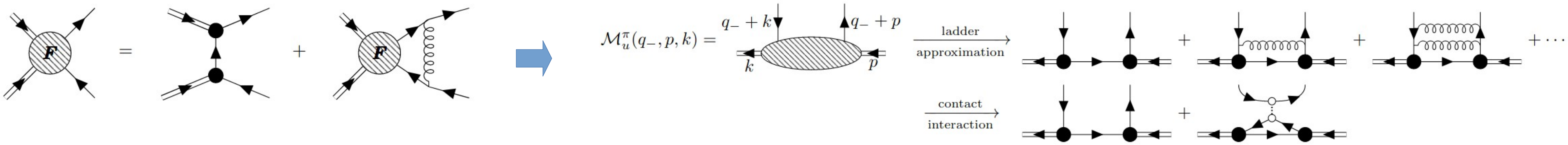
  (Recall many distributions are readily available within this domain.. and we can extend it)

- The **diagram** approach, in principle, could give you *everything*. But building up the expressions, is not an easy task.

- An alternative is proposed on the grounds of deriving the **scattering amplitude** by means of its inhomogeneous **BSE**.

- Positive outcomes arise **naturally**, thus revealing the **potential** of this approach, and motivating a more sophisticated calculation.

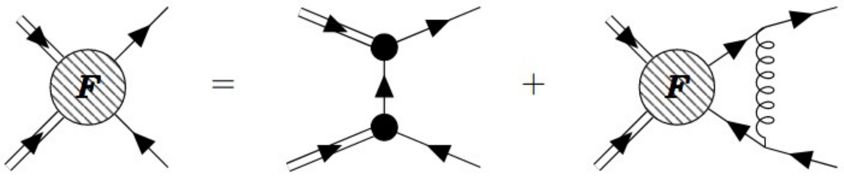




Obtaining the Scattering amplitude

Scattering **Amplitude**

- The **Scattering Amplitude** is written as:



$$F(k,P_1,P_2) = F_0(k,P_1,P_2) + \Sigma^F(k,P_1,P_2)$$

- The **self-energy**:

$$\Sigma^F(k,P_1,P_2) = -\frac{4}{3}\int_q \mathcal{G}_{\alpha\beta}(k-q)\gamma_\alpha S(q+P_1) \times F(q,P_1,P_2)S(q-P_2)\gamma_\beta\,.$$

- In the **SCI**, solving this equation is equivalent to solving:

$$\Sigma^F(P_1,P_2) = \Sigma^{F_0}(P_1,P_2) - \Delta_g \int_q \gamma_\alpha S(q_1)\Sigma^F(P_1,P_2)S(q_2)\gamma_\alpha$$

Xing:2022mvk

Scattering Amplitude

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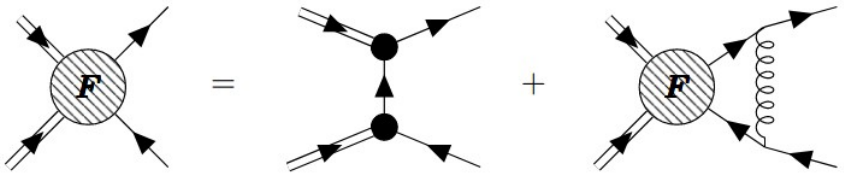

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- We can build up a basis for:

$$\Sigma^F(P_1, P_2) = \sum_{i=1}^{4} t_i T_i$$

- Where only two structures survive:

$$T_1 = \mathbb{1}\,,$$
$$T_2 = \frac{-i}{M}\not{K}$$

We proceed similarly with the F0 part (let's call b1 and b2 to their dressing functions)

Xing:2022mvk

Scattering Amplitude

- The following solutions are found:

$$t_1(Z^2) = \frac{b_1(Z^2)}{1+\Delta_g f_s(Z^2)}\,, \qquad f_s(Z^2) = \mathrm{tr}\int_q S(q_1)S(q_2)\,,$$

$$t_2(Z^2) = \frac{b_2(Z^2)}{1+\Delta_g f_v(Z^2)}\,, \qquad f_v(Z^2) = \frac{K_\mu K_\nu}{2K^2}\mathrm{tr}\int_q i\gamma_\mu^T S(q_1) i\gamma_\nu^T S(q_2)$$

- Where the b dressing functions are:

$$b_1(Z^2) = -\frac{1}{4}\Delta_g \mathrm{tr}\int_q T_1\gamma_\alpha S(q_1)F_0(q,P_1,P_2)S(q_2)\gamma_\alpha\,,$$

$$b_2(Z^2) = \frac{M^2}{4K^2}\Delta_g \mathrm{tr}\int_q T_2\gamma_\alpha S(q_1)F_0(q,P_1,P_2)S(q_2)\gamma_\alpha\,.$$

- Recalling the expressions for the scalar and vector form factors:

$$iF_s(Z^2) = N_c \mathrm{tr}\int_q i\Gamma_I(Z)S(q_1)i^2F_0(q,P_1,P_2)S(q_2)\,,$$

$$2K_\mu F_{\mathrm{em}}(Z^2) = N_c \mathrm{tr}\int_q i\Gamma_\mu(Z)S(q_1)i^2F_0(q,P_1,P_2)S(q_2)\,,$$

$$\Sigma^F(P_1,P_2) = \Sigma^{F_0}(P_1,P_2) - \Delta_g\int_q \gamma_\alpha S(q_1)\Sigma^F(P_1,P_2)S(q_2)\gamma_\alpha$$

$$\Sigma^F(P_1,P_2) = \sum_{i=1}^{4} t_i T_i$$

$$\Sigma^{F_0}(P_1,P_2) = \sum_{i=1}^{4\to 2} b_i T_i.$$

$$N_c b_1(Z^2) = \Delta_g F_s^b(Z^2)\,,$$

$$N_c b_2(Z^2) = -M\Delta_g F_{\mathrm{em}}^b(Z^2)\,,$$

$$N_c\Sigma^F(P_1,P_2) = F_s^b(Z^2)\Delta_s(Z^2) + i\not{K}F_{\mathrm{em}}^b(Z^2)\Delta_v(Z^2)$$

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