Pion and Kaon GPDs from Continuum Methods

Khépani Raya Montaño

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QCD: Basic Facts

- QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).
- Quarks and gluons not isolated in nature.
- Formation of colorless bound states: “Hadrons”
- 1-fm scale size of hadrons?

\[ L_{\text{QCD}} = \sum_{j=u,d,s,...} \bar{q}_j [\gamma_\mu D_\mu + m_j] q_j + \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}, \]
\[ D_\mu = \partial_\mu + ig \frac{1}{2} \lambda^a A_\mu^a, \]
\[ G^{\mu\nu}_{\mu\nu} = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a - g f^{abc} A_\mu^a A_\nu^b A_\rho^c, \]

Emergence of hadron masses (EHM) from QCD dynamics

Higgs mechanism

QCD dynamics

Proton Mass = $168 \times 10^{-26}$ g

\~ 1\% of proton mass

\~ 10 MeV

\~ 99\% of proton mass

\~ 928 MeV
QCD: Basic Facts

- QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).

Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD dynamics

![Graph showing dynamical chiral symmetry breaking and Higgs masses](image)

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\[ D_\mu = \partial_\mu + ig \frac{1}{2} \lambda^a A_\mu^a, \]

\[ G_{\mu\nu}^a = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c, \]

\[ S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma \cdot p + M_f(p^2)) \]
QCD: Basic Facts

QCD is characterized by two emergent phenomena: confinement and dynamical generation of mass (DGM).

Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD dynamics

Gluon and quark running masses

Schwinger

DCSB

Cui:2019dwv
**Mass Budgets**

- **Most** of the **mass** in the visible universe is contained within **nucleons**
  - Which remain **pretty massive** whether there is Higgs mechanism or not...

- In the absence of weak mass generation, these would be **massless**

- Both quark-antiquark **bound-state** and **Golstone Bosons**
  - Their mere **existence** is connected with **mass** generation in the **SM**
Pion Structure

➢ The experimental access to the pion (and Kaon) structure is via electromagnetic probes, yielding e.g.:

- Electromagnetic form factor
- Distribution functions

➢ Generalized parton distributions (GPDs) encode them both (and more):

\[ F_\pi(\Delta^2) = \int H_\pi^{u}(x, \xi, -\Delta^2; \zeta), \quad u_\pi(x; \zeta) = H_\pi^{u}(x, 0, 0; \zeta) \]

➢ But the experimental access and theoretical derivation is far more complicated.
Generalized Parton Distributions

- **Mathematically:**
  Non-local quark and gluon operators, evaluated between hadron states in **non-forward** kinematics and projected onto the **light front**.
  
  For example:
  
  \[
  H^q_\pi(x, \xi, t) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda} \langle \pi'(p') | \bar{\psi}^q(-\lambda n/2) \gamma^\mu \psi^q(\lambda n/2) | \pi(p) \rangle n_\mu
  \]
  (Leading-twist chiral-even valence-quark GPD of the pion)

- **Schematically:**
  The probe (**virtual photon**) interacts with a quark or a gluon within the hadron, which is then scattered off.
  
  The process can be separated into a **perturbative**, computable part, and a **non-perturbative** one (encoded by the GPD)

  For example: **DVCS**
GPDs: Kinematics

Kinematics: [Diehl:2003ny]

- **DGLAP** ($|x| > |\xi|$): Emits/takes a quark ($x > 0$) or antiquark ($x < 0$).
- **ERBL** ($|x| < |\xi|$): Emits pair quark-antiquark.

\[
(x + \xi) P \quad (x - \xi) P
\]

\[
h(p) \quad t = (p' - p)^2 \quad h(p')
\]

**P**: Average momentum of the hadron

**x**: Longitudinal momentum fraction

**\xi**: Longitudinal momentum transferred

**t = -\Delta^2**: Momentum transferred

-1 \leq x \leq -\xi

- \xi \leq x \leq \xi

\xi \leq x \leq 1

"Antiquark"

"Quark/antiquark pair"

"Quark"
GPDs: Physical pictures

- Connection to hadron structure:
  - **PDFs** are the forward limit of **GPDs**
  - Mellin moments of **GPDs** are connected to **electromagnetic** and **gravitational form factors** (FFs)
  - The so called **Compton FFs** parameterize the **DVCS** scattering amplitude
  - Impact parameter space GPDs (**IPS-GPDs**) sketch the likelihood of finding a parton with momentum fraction \( x \) at a given transverse position.

**GPDs: Properties**

- **Support:** \((x, \xi) \in [-1, 1] \times [-1, 1]\)  
  **Analyticity/Causality**

- **Polynomiality:**  
  Order-\(m\) Mellin moments are degree-(\(m+1\)) polynomials in \(\xi\)  
  \[
  \mathcal{M}_m(\xi, t) = \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (2\xi)^{2i} A_{i,m}^q(t) + \text{mod}(m, 2) (2\xi)^{m+1} C_{m+1}(t) 
  \]
  **Lorentz invariance**

- **Positivity:**  
  \[
  |H^q(x, \xi, t = 0)| \leq \sqrt{q \left( \frac{x + \xi}{1 + \xi} \right) q \left( \frac{x - \xi}{1 - \xi} \right)}, \quad \xi \leq |x| 
  \]
  **Hilbert Space Norm**

- **Soft-pion theorem:**  
  Low energy theorems and the connection of **GPDs** with distribution amplitudes (PDAs).  
  **PCAC/Axial-vector WGTI**

  e.g.  
  \[
  H_q^- (x, 1, 0) = \varphi \left( \frac{1 + x}{2} \right) 
  \]

  [Mezrag:2023nkp]
GPDs: Strategies

➢ Can we fulfill such mathematical requirements?
Here two strategies:

1. Overlap representation of the light-front wavefunction

\[ H^a_{\pi}(x, \xi, -\Delta^2; \zeta_H) = \int \frac{d^2 k_\perp}{16 \pi^3} \psi^{u*}_{\pi}(x_-, k_{\perp -}; \zeta_H) \psi^{a}_{\pi}(x_+, k_{\perp +}; \zeta_H) \]

- **Positivity** condition fulfilled by construction
- But, in principle, **restricted** to the DGLAP domain

2. Double distribution representation

\[ H^{\pi}_{u}(x, \xi, Q^2) = \int_{\Omega} d\beta d\alpha \delta(x - \beta - \xi \alpha) \left[ F(\beta, \alpha, Q^2) + \xi G(\beta, \alpha, Q^2) \right] \]

- **Polynomiality** can be explicitly proven
- **Positivity** is not guaranteed.
GPDs: Strategies

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2. Double distribution representation

\[ H^\pi_u(x, \xi, Q^2) = \int_\Omega d\beta d\alpha \delta(x - \beta - \xi \alpha) [F(\beta, \alpha, Q^2) + \xi G(\beta, \alpha, Q^2)] \]

   - **Polynomiality** can be explicitly proven
   - Cannot really **guarantee** the positivity

Following the so called “covariant extension”, this can be extended to the ERBL domain.

GPDs: Strategies

- Can we fulfill such mathematical requirements?
  Here two strategies:

1. Overlap representation of the light-front wavefunction

\[ H^u_\pi(x, \xi, -\Delta^2; \zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} \psi^*_\pi(x_-, k^2_\perp; \zeta_H) \psi^u_\pi(x_+, k^2_\perp; \zeta_H) \]

  ✔ Positivity condition fulfilled by construction

  ✗ But, in principle, restricted to the DGLAP domain

"3". Covariant/diagram approach

\[ 2H^\pi_u(x, \xi, Q^2) = N_c \text{tr} \int_q i\gamma_n M^\pi_u(q_-, p, k) \]

  ✔ In principle, all properties could be fulfilled

  ✗ But it relies on sensible construction of insertions/amplitudes.
Light-front wave functions

\[ \psi_{P}^{u}(x, k_{\perp}^{2}; \zeta) \]

“One ring to rule them all”
Goal: get a broad picture of the pion and Kaon structure.

The idea: Compute everything from the LFWF.

\[
\int dk_\perp, \quad \int dx, \quad t = 0, \xi = 0
\]
LFWF approach

- **Goal:** get a **broad picture** of the pion and Kaon structure.

**The idea:** Compute **everything** from the LFWF.

**The inputs:**

- **Solutions** from quark **DSE** and meson **BSE**.
  - Numerically **challenging**, but **doable**
  - Already on the market: PDAs, PDFs, Form factors...

  K. Raya *et al.*, arXiv: 1911.12941 [nucl-th]

\[ \psi^q_{M}(x, k^2) = \text{tr} \int \frac{d\mathbf{k}}{d\mathbf{k}} \, \delta^x_n(k_M) \gamma_5 \cdot n \, \chi_M(k_-, P) \]
Light-front wave functions

Goal: get a broad picture of the pion and Kaon structure.

The idea:
Compute everything from the LFWF.

The inputs:
Solutions from quark DSE and meson BSE.

The alternative inputs:
Construct BSWF from realistic DSE predictions.

BSWFs ➔ LFWFs ➔ PDAs

Project onto the light-front
Overlap representation

GPDs ➔ PDFs

Form Factors

Connection with Continuum Schwinger Methods (CSM)

PDA and PDF as benchmarks
A perturbation theory integral representation for the BSWF:

\[ n_K \chi_K(k^+_K, P_K) = \mathcal{M}(k, P) \int_{-1}^{1} dw \, \rho_K(w) \mathcal{D}(k, P) \]

1: Matrix structure (leading BSA):

\[ \mathcal{M}(k; P_K) = -\gamma_5 [\gamma \cdot P_K M_u + \gamma \cdot k(M_u - M_s) + \sigma_{\mu\nu} k_\mu P_K \nu] , \]

2: Spectral weight: Tightly connected with the meson properties.

3: Denominators: \n\[ \mathcal{D}(k; P_K) = \Delta(k^2, M_u^2) \Delta((k - P_K)^2, M_s^2) \hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2) , \]

where: \n\[ \Delta(s, t) = [s + t]^{-1}, \hat{\Delta}(s, t) = t \Delta(s, t) . \]
LFWF: PTIR approach

- Recall the expression for the LFWF:

\[
\psi^q_M(x, k^2_\perp) = \text{tr} \int \delta^x_n(k_M) \gamma_5 \gamma \cdot n \chi_M(k^2_-, P) \quad <x>_M^q := \int_0^1 dx \ x^m \psi^q_M(x, k^2_\perp)
\]

- Algebraic manipulations yield:

\[
\Rightarrow \psi^q_M(x, k^2) \sim \int dw \ \rho_M(w) \ldots
\]

- Uniqueness of Mellin moments

- Compactness of this result is a merit of the AM.

- Thus, \( \rho_M(w) \) determines the profiles of, e.g. PDA and PDF: (it also works the other way around)

\[
f_M^q(x; \zeta_H) = \int \frac{d^2 k^2_\perp}{16\pi^3} \psi^q_M(x, k^2_\perp; \zeta_H)
\]

\[
q_M(x; \zeta_H) = \int \frac{d^2 k^2_\perp}{16\pi^3} |\psi^q_M(x, k^2_\perp; \zeta_H)|^2
\]
LFWF: PTIR approach

More explicitly:

\[ \psi_M^q(x, k_\perp^2; \zeta_H) = 12 \left[ M_q(1 - x) + M_h x \right] X_M(x; \sigma_\perp^2) \]

\[ \sigma_\perp = k_\perp^2 + \Omega_P^2 \]

\[ X_M(x; \sigma_\perp^2) = \left[ \int_{-1}^{1-2x} dw \int_{1+\frac{2x}{w-1}}^1 dv + \int_{1-2x}^1 dw \int_{\frac{w-1+2x}{w+1}}^1 dv \right] \frac{\rho_M(w) \Lambda_M^2}{n_M \sigma_\perp^2} \]

\[ \Omega_M^2 = \nu M_q^2 + (1 - \nu) \Lambda_p^2 \]

\[ + (M_h^2 - M_q^2) \left( x - \frac{1}{2} [1 - w][1 - \nu] \right) \]

\[ + (x[x - 1] + \frac{1}{4} [1 - \nu][1 - w^2]) m_M^2 \]

Model parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( m_p )</th>
<th>( m_u )</th>
<th>( M_h )</th>
<th>( \Lambda_p )</th>
<th>( b_0^P )</th>
<th>( \omega_0^P )</th>
<th>( v_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>0.14</td>
<td>0.31</td>
<td>( M_u )</td>
<td>( M_h )</td>
<td>0.275</td>
<td>1.23</td>
<td>0</td>
</tr>
<tr>
<td>( K )</td>
<td>0.49</td>
<td>0.31</td>
<td>1.2M_u</td>
<td>3M_s</td>
<td>0.1</td>
<td>0.625</td>
<td>0.41</td>
</tr>
</tbody>
</table>

\[ \rho_p(\omega) = \frac{1 + \omega v_p}{2a_p b_0^P} \left[ \text{sech}^2 \left( \frac{\omega - \omega_0^P}{2b_0^P} \right) + \text{sech}^2 \left( \frac{\omega + \omega_0^P}{2b_0^P} \right) \right] \]
**LFWF: Factorized case**

- In the **chiral limit**, the PTIR reduces to:

\[
\psi^q_M(x, k^2_\perp; \zeta_H) \sim \tilde{f}(k^2_\perp) \phi^q_M(x; \zeta_H) \sim f(k^2_\perp) [q_M(x; \zeta_H)]^{1/2}
\]

**“Factorized model”**

\[
[\phi^q_M(x; \zeta_H)]^2 \sim q_M(x; \zeta_H)
\]

- Therefore:

\[
\psi^q_M(x, k^2_\perp; \zeta_H) = [q^M_M(x; \zeta_H)]^{1/2} \left[ 4\sqrt{3}\pi \frac{M_q^3}{(k^2_\perp + M_q^2)^2} \right]
\]

- Sensible assumption as long as:

\[
m_M^2 \approx 0, \quad M_h^2 - M_q^2 \approx 0, \quad \zeta_H
\]

  (meson mass) \hspace{1cm} (h-antiquark, q-quark masses)

- Produces **identical** results as PTIR model for pion

- Single parameter!

\[
M_q \sim r_M^{-1}
\]

(charge radius)

No need to determine the spectral weight!
A perturbation theory integral representation for the BSWF:

\[ n_{M\chi_M}(k^-, P) = \mathcal{M}_{q,\bar{h}}(k, P) \int_{-1}^{1} dw \tilde{\rho}_M^\nu(w) D_{q,\bar{h}}^\nu(k, P) \]

1: Matrix structure (leading BSA):

\[ \mathcal{M}_{q,\bar{h}}(k, P) \equiv -\gamma_5 \left[ M_q \gamma \cdot P + \gamma \cdot k (M_{\bar{h}} - M_q) + \sigma_{\mu\nu} k_\mu P_\nu - i (k \cdot p + M_q M_{\bar{h}}) \right] \]

2: Profile function:

\[ \tilde{\rho}_M^\nu(w) \equiv \rho_M(w) \Lambda_{w}^{2\nu} \]

3: Denominators:

\[ D_{q,\bar{h}}^\nu(k, P) \equiv \Delta \left( k^2, M_q^2 \right) \Delta \left( k_{w-1}^2, \Lambda_w^2 \right) \nu \Delta \left( p^2, M_{\bar{h}}^2 \right) \]

The crucial difference:

\[ \Lambda^2(w) := \Lambda_w^2 = M_q^2 - \frac{1}{4} (1 - w^2) m_M^2 + \frac{1}{2} (1 - w) (M_{\bar{h}}^2 - M_q^2) \]
LFWF: PTIR approach II

- Then a series of algebraic results follows.

1. For the **BSWF**:

   \[
   n_M \chi_M(k_-, P) = \mathcal{M}_{q, \bar{q}}(k, P) \int_0^1 d\alpha \mathcal{F}_M(\alpha, \sigma^{\nu+2}), \quad \sigma = (k - \alpha P)^2 + \Lambda^2_{1-2\alpha},
   \]

   \[
   \mathcal{F}_M(\alpha, \sigma^{\nu+2}) = 2^\nu(\nu + 1) \left[ \int_{-1}^{1-2\alpha} dw \left( \frac{\alpha}{1-w} \right)^\nu + \int_{1-2\alpha}^1 dw \left( \frac{1-\alpha}{1+w} \right)^\nu \right] \tilde{\rho}_M^\nu(w) / \sigma^{\nu+2}
   \]

2. **LFWF** in terms of **PDA/PDF**:

   \[
   f_M^q \phi_M^q(x; \zeta_H) = \int \frac{d^2k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H)
   \]

   \[
   \psi_M^q(x, k_\perp^2) = 16\pi^2 f_M \frac{\nu \Lambda^{2\nu}_{1-2x}}{(k_\perp^2 + \Lambda^2_{1-2x})^{\nu+1}} \phi_M^q(x)
   \]

\[
\Lambda_{1-2x}^2 = M_q^2 + x(M_{\bar{q}}^2 - M_q^2) - m_H^2 x(1-x)
\]

- **Flavor asymmetry**
- **Meson mass**

Encodes the breaking of **factorization**.

- Completely factorized in the **chiral limit**.
LFWFs and PDAs

\[ f_M \phi_M^q(x; \zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H) \]

\begin{itemize}
  \item **Smooth fall** at the endpoints
  \item **Broad** and concave functions of \( x \)
    \begin{itemize}
      \item Consequence of EHM
    \end{itemize}
  \item **Higgs** induced asymmetry for Kaon:
    \begin{itemize}
      \item Moduled by the difference \( M_s - M_u \)
    \end{itemize}
\end{itemize}
LFWFs and PDAs

Patterns supported by lattice:
Zhang:2020gaj

First CSM calculation of PDA:
Chang:2013pq

DILATION

\[
\phi(x) = \frac{1}{16\pi} \int \frac{d^3k}{\sqrt{M(x,k,k')H(x)}}
\]
LFWFs and GPDs

In the overlap representation, the valence-quark GPD reads as:

\[ H_M^q(x, \xi, t) = \int \frac{d^2 k_\perp}{16\pi^3} \psi_M^{q*}(x^-, (k_\perp^-)^2) \psi_M^q(x^+, (k_\perp^+)^2) \]

\[ \zeta_H \]
GPDs and PDFs

- The PDF is obtained from the **forward limit** of the GPD.

$$q(x) = H(x, 0, 0)$$

$$q_a(x) = 30x^2(1-x)^2$$

\[ u_\pi(x; \zeta_H) \quad \text{and} \quad u_K(x; \zeta_H) \]

\[ \langle x \rangle_{\pi}^u = 0.50 \]

\[ \langle x \rangle_{K}^u = 0.47 \]

- $\zeta_H$: meson properties determined by the fully-dressed valence-quarks.
- Broad + Higgs-induced asymmetry
Electromagnetic FFs

- Electromagnetic form factor is obtained from the \( t \)-dependence of the 0-th moment:

\[
F_M^q(-t = \Delta^2) = \int_{-1}^{1} dx \, H_M^q(x, \xi, t)
\]

Can safely take \( \xi = 0 \)

“Polinomiality”

\[
\mathcal{M}_m(\xi, t) = \sum_{i=0}^{[m/2]} (2\xi)^{2i} A_{i,m}^q(t) + \text{mod}(m, 2)(2\xi)^{m+1}C_{m+1}(t)
\]

\[
F_M^u(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2)
\]

Weighted by electric charges

- Isospin symmetry

\[
F_{\pi^+}(-t) = F_{\pi^+}^u(-t)
\]

Data: G.M. Huber et al. PRC 78 (2008) 045202

CSM: L. Chang et al. PRL 111 (2013) 14, 141802
Gravitational form factors are obtained from the \textit{t-dependence} of the \textbf{1-st moment}:

\[ J_M(t, \xi) = \int_{-1}^{1} dx \ x H_M(x, \xi, t) = \Theta^M_2(t) - \xi^2 \Theta^M_1(t) \]

- Directly obtained if $\xi = 0$
- Only DGLAP GPD is required
- \textbf{ERBL} GPD needed \textbf{D-term}

But a sound expression can be constructed:

\[
\theta_1^q(\Delta^2) = c_1^q \theta_2^q(\Delta^2) + \int_{-1}^{1} dx \ x [H^q_p(x, 1, 0)P_{Mq}(\Delta^2) - H^q_p(x, 1, -\Delta^2)]
\]

"Soft pion theorem"

\[ r_{\pi}^{\theta_1} = 0.78 \text{ fm} \quad (\text{mech radius}) \]

\[ r_{\pi}^{\theta_2} = 0.56 \text{ fm} \quad (\text{mass radius}) \]
**Kaon EFF**

Electromagnetic form factor: **charged** and **neutral** kaon

\[ r_K = 0.56 \text{ fm} \]

Kaon is more compressed

\[ r_K^j \approx 0.85 \ r_\pi^j \]

\( j = \text{mech, charge, mass} \)

CSM - \( K^+ \): Gao:2017mmp, Eichmann:2019bqf

CSM - \( K^0 \): Gao:2017mmp

Lattice: Davies:2018zav
Charge and mass distributions

\[ \rho_P(b) = \frac{1}{2\pi} \int_{0}^{\infty} d\Delta \Delta J_0(\Delta b) F_P(\Delta^2) \]

\[ F_P^E(\Delta^2) \rightarrow \rho_P^E(b) \]
\[ \theta_2^P(\Delta^2) \rightarrow \rho_P^M(b) \]

➢ Intuitively, we expect the meson to be localized at a finite space.

➢ Charge effect span over a larger domain than mass effects.

More massive hadron ➔ More compressed

\[ r_{\pi}^E = 0.68 \text{ fm} \]
\[ r_{\pi}^{\theta_2} = 0.56 \text{ fm} \]

\[ r_{K}^{j} \approx 0.85 \ r_{\pi}^j \]
Pressure distributions

\[ p^K_\nu(r) = \frac{1}{6\pi^2 r} \int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)} \sin(\Delta r) [\Delta^2 \theta_1^K(\Delta^2)], \]

\[ s^K_\nu(r) = \frac{3}{8\pi^2} \int_0^\infty d\Delta \frac{\Delta^2}{2E(\Delta)} j_2(\Delta r) [\Delta^2 \theta_1^K(\Delta^2)], \]

“Pressure” Quark attraction/repulsion

“Shear” Deformation QCD forces

(CONFINEMENT)
Impact parameter space GPDs

\[ u^p(x, b^2; \zeta_H) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta) H^u_\perp(x, 0, -\Delta^2; \zeta_H) \]

\[ \langle b^2_\perp(\zeta_H) \rangle^\pi_u = \frac{2}{3} r^2_\pi = \langle b^2_\perp(\zeta_H) \rangle^K_d, \]
\[ \langle b^2_\perp(\zeta_H) \rangle^K_u = 0.71 r^2_K, \langle b^2_\perp(\zeta_H) \rangle^K_s = 0.58 r^2_K. \]

\[ 2\pi b_\perp q_\pi(x, b_\perp^2) r_\pi \]

\[ 2\pi b_\perp q_K(x, b_\perp^2) r_K \]

- Likelihood of finding a valence-quark with momentum fraction \( x \), at position \( b \).
Evolved IPS-GPD: Pion Case

\[ u^p(x, b^2; \zeta_H) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta) H^\mu_p(x, 0, -\Delta^2; \zeta_H) \]

- **Likelihood** of finding a parton with LF momentum $x$ at transverse position $b$

- **Peaks broaden** and **maximum drifts**: 
  \[
  \text{max} : 3.29 \rightarrow 0.55
  \]
  \[
  (|x|, b) = (0.88, 0.13) \rightarrow (0.47, 0.23)
  \]
Evolved IPS-GPD: Kaon Case

\[ u^P(x, b^2; \zeta_H) = \int_0^{\infty} \frac{d\Delta}{2\pi} \Delta J_0(b \Delta) H_{P}^{\mu}(x, 0, -\Delta^2; \zeta_H) \]

- **Likelihood** of finding a parton with LF momentum \( x \) at transverse position \( b \)

- \( \zeta_H \rightarrow \zeta = 2 \text{ GeV} \)

\[
\text{max}(s,u) : (3.61, 2.38) \rightarrow (0.61, 0.49) \\
(x, b)_u = (0.84, 0.17) \rightarrow (0.41, 0.28) \\
(x, b)_s = (-0.87, 0.13) \rightarrow (-0.48, 0.22)
\]
The diagram approach

Mezrag:2023nkp
In the calculation of meson electromagnetic form factors, the triangle diagram is self consistent with the Rainbow-Ladder truncation.

So, as long as the QPV fulfills its own required symmetries, the computed EFF would be sensible.
For PDFs, there is a correspondence between the TD and the so called **HandBag** diagram, in the RL truncation.

\[
q \bar{q}' \Gamma_\mu \Gamma_\nu \sim q' \Gamma_\mu \Gamma_\nu \quad H(P,k)
\]

\[
q^\pi(x;\zeta_H) = N_{ctr} \int_{dk} i\Gamma_\pi(k, -P) \times S(k) \Gamma^n(k; x; \zeta_H) S(k) \Gamma_{\pi}(k, P) S(k)
\]

\[
i\Gamma^n(k; x; \zeta_H) = \delta^n(x)(k, n) \cdot \partial_{kHo} S^{-1}(k, n)
\]

\[
\Gamma^n_{\pi}(k, -P; \zeta_H) = n \cdot \partial_{kHo} \Gamma_\pi(k, -P; \zeta_H)
\]

Diagram approach

➢ For PDFs, there is a correspondence between the TD and the so-called HandBag diagram, in the RL truncation.

➢ Nonetheless, to fully preserve both baryon number and momentum conservation, the HandBag diagram must be supplemented. For the case of the PDF:

(in momentum-dependent interactions)

\[ q_{\text{A}}(x; \zeta_H) = N_c \text{tr} \int_{dk} i \Gamma_\pi(k_\eta, -P) \times S(k_\eta) i \Gamma_\pi(k_\eta, P) S(k_\eta) \]

\[ q_{\text{BC}}(x; \zeta_H) = N_c \text{tr} \int_{dk} \Gamma_\pi^n(k_\eta, -P; \zeta_H) \times S(k_\eta) \Gamma_\pi(k_\eta, P) S(k_\eta) \]

\[ i \Gamma_\pi^n(k; x; \zeta_H) = \delta_\pi^n(k_\eta)n \cdot \partial_{k_\eta} S^{-1}(k_\eta) \]

\[ \Gamma_\pi^n(k_\eta, -P; \zeta_H) = n \cdot \partial_{k_\eta} \Gamma_\pi(k_\eta, -P; \zeta_H) \]

Chang:2014lva, Ding:2019lwe
For **PDFs**, there is a correspondence between the TD and the so-called **HandBag** diagram, in the **RL** truncation.

Nonetheless, to fully preserve both **baryon number** and **momentum conservation**, the HandBag diagram must be **supplemented**. For the case of the **PDF**:

(in momentum-dependent interactions)

\[
q^\pi (x; \zeta_H) = N_c \text{tr} \int \delta^x_n (k_\eta) \\
\times \{ n \cdot \partial_{k_\eta} [\Gamma \pi (k_\eta, -P) S(k_\eta)] \} \Gamma \pi (k_\bar{\eta}, P) S(k_\bar{\eta})
\]

\[\text{Overlap} \quad \text{Full Result} \]

Symmetry-restoring terms

Diagram approach

➢ For PDFs, there is a correspondence between the TD and the so called HandBag diagram, in the RL truncation.

➢ Nonetheless, to fully preserve both baryon number and momentum conservation, the HandBag diagram must be supplemented. For the case of the PDF:

\[ \text{Mezrag:2014jka, Chang:2014lva, Ding:2019lwe} \]

\[ \text{in momentum-dependent interactions} \]

The GPD analogous has not been fully-solved in momentum-dependent interactions.
Contact Interaction model:
Some highlights
The quark gap equation in a symmetry-preserving contact interaction model (SCI):

\[
S^{-1}(p) = i\gamma \cdot p + m + \frac{16\pi}{3} \frac{\alpha_{\text{IR}}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S(q) \gamma_\mu
\]

Infrared strength \(\alpha_{\text{IR}} = 0.93\pi\)

Recall the quark gap equation:

\[
S_f^{-1}(p) = Z_2(i\gamma \cdot p + m_{f\text{bm}}) + \Sigma_f(p),
\]

\[
\Sigma_f(p) = \frac{4}{3} Z_1 \int_{dq}^A g^2 D_{\mu\nu}(p-q) \gamma_\mu S_f(q) \Gamma^f_\nu(p,q)
\]

Namely, SCI kernel is essentially RL + constant gluon propagator
Let us now consider the quark gap equation in a symmetry-preserving contact interaction model (SCI).

\[ S^{-1}(p) = i\gamma \cdot p + m + \frac{16\pi}{3} \frac{\alpha_{IR}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S(q) \gamma_\mu \]

- **Constant gluon propagator:**
  - Quark propagator, with constant mass function

  \[ Z_f(p^2)(i\gamma \cdot p + M_f(p^2))^{-1} \rightarrow S(p)^{-1} = i\gamma \cdot p + M \]

  - Non renormalizable
  - Needs regularization scheme:

\[
\frac{1}{s + M_f^2} = \int_0^\infty d\tau e^{-\tau(s+M_f^2)} \rightarrow \int_{\tau_{ir}}^{\tau_{uv}} d\tau e^{-\tau(s+M_f^2)}
\]

- \( \tau_{ir} = 1/0.24 \text{ GeV}^{-1} \): Ensures the absence of quark production thresholds (confinement)
- \( \tau_{uv} = 1/0.905 \text{ GeV}^{-1} \): UV cutoff. Sets the scale of all dimensioned quantities.
Let us now consider the quark gap equation in a symmetry-preserving contact interaction model (SCI).

\[ S^{-1}(p) = i \gamma \cdot p + m + \frac{16\pi}{3} \frac{\alpha_{\text{IR}}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S(q) \gamma_\mu \]

The meson Bethe-Salpeter equation:

\[ \Gamma(k; P) = -\frac{16\pi}{3} \frac{\alpha_{\text{IR}}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \chi(q; P) \gamma_\mu \]

The diquark Bethe-Salpeter equation:

\[ \Gamma_{qq}(k; P) = -\frac{8\pi}{3} \frac{\alpha_{\text{IR}}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \chi_{qq}(q; P) \gamma_\mu \]

Quark propagator, with constant mass function:

\[ S(p)^{-1} = i \gamma \cdot p + M \]

The interaction produces momentum independent BSAs:

\[ \Gamma_\pi(P) = \gamma_5 \left[ iE_\pi(P) + \frac{\gamma \cdot P}{M} F_\pi(P) \right] \]
\[ \Gamma_\sigma(P) = 1E_\sigma(P) , \]
\[ \Gamma_\rho(P) = \gamma^T E_\rho(P) , \]
\[ \Gamma_{a_1}(P) = \gamma_5 \gamma^T E_{a_1}(P) , \]

Recall a \( J^p \) diquark partners with an analogous \( J^p \) meson.
The quark-photon vertex:

\[
\Gamma_{\mu}^{\gamma}(Q) = \frac{Q_{\mu}Q_{\nu}}{Q^2} \gamma_{\nu} + \Gamma_{\mu}^{T}(Q)
\]

\[
\Gamma_{\mu}^{T}(Q) = P_{T}(Q^2)P_{\mu\nu}(Q)\gamma_{\nu}
\]

Introduces a vector meson pole in the timelike axis.

From a total of 12 tensor structures, the QPV is now characterized by just 2!

Fig. 9 Photon+quark vertex dressing function in Eq. (A.3). As in any symmetry preserving treatment of photon+quark interactions, \( P_{T}(Q^2) \) exhibits a pole at \( Q^2 = -m_{\rho}^2 \). Moreover, \( P_{T}(Q^2 = 0) = 1 = P_{T}(Q^2 \rightarrow \infty) \).

Albino:2021rvj
GPDs In the SCI

A fresh look at the generalized parton distributions of light pseudoscalar mesons

Zanbin Xing,1 Minghui Ding,2 Khépani Raya,3 and Lei Chang4

1School of Physics, Nankai University, Tianjin 300071, China
2Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany
3Department of Integrated Sciences and Center for Advanced Studies in Physics, Mathematics and Computation, University of Huelva, E-21071 Huelva, Spain.
(Dated: January 27, 2023)

Xing:2023eed, Xing:2022jtt, Xing:2022mvk
GPDs in the SCI

- Consider the following expression for the valence-quark pion GPD:

\[ 2H_u(x, \xi, Q^2) = N_c \text{tr} \int_q i\gamma_n \mathcal{M}_u^\pi(q-, p, k) \]

\( '\text{Bare'} \) insertion operator

\[ \gamma_n = \delta_n^x(q) \gamma \cdot n \]

\[ \delta_n^x(q) = \delta(n \cdot q - x n \cdot P) \]

- Rather than dressing the insertion operator, let’s take a look at the scattering amplitude:

\[ \mathcal{M}_u^\pi(q-, p, k) = \frac{q_- + k}{k} \frac{q_- + p}{p} \text{ ladder approximation} \]
GPDs in the SCI

- Consider the following expression for the valence-quark pion GPD:

\[ 2H_u^\pi(x, \xi, Q^2) = N_c \text{tr} \int q i \gamma_n M^\pi_u(q_-, p, k) \]

\[ \gamma_n = \delta_n^\pi(q) \gamma \cdot n \]

\[ \delta_n^\pi(q) = \delta(n \cdot q - x n \cdot P) \]

- Rather than dressing the insertion operator, let's take a look at the scattering amplitude:

\[ M^\pi_u(q_-, p, k) = \frac{q_- + k}{p} \]

Xing:2023eed
GPDs in the SCI

Therefore, we arrive at the following diagrams:

- Impulse approximation
- ‘Pole’ Contributions

It is now straightforward to compute Mellin moments:

\[ 2\langle x^m \rangle_u (\xi, Q^2) = N_c \text{tr} \int_q \frac{(n \cdot q)^m}{(n \cdot P)^{m+1}} i\gamma \cdot n M_u (q-, p, k) \]

Where two kinds of denominators appear:

\[ \frac{1}{D_{-Q/2D_Q/2D_P}} = \int_{\Omega_u} \frac{d u_1 d u_2}{[(q + \alpha Q - \beta P)^2 + \omega_3]^3} \] \hspace{1cm} (5a)

\[ \frac{1}{D_{-Q/2D_Q/2}} = \int_{\Omega_u} \frac{d u_1 d u_2 \delta(\beta)}{[(q + \alpha Q)^2 + \omega_2]^2} \] \hspace{1cm} (5b)

where \( D_X = D(q - X) \), \( \alpha = u_1 - u_2 \), \( \beta = 1 - u_1 - u_2 \)
and \( \Omega_u = \{ (u_1, u_2) | 0 < u_1 < 1, 0 < u_2 < 1 - u_1 \} \)

Already suggested in chiral quark effective theories!

Here, this term arises from its DSE!

Polyakov:1999gs, Broniowski:2007si
Therefore, we arrive at the following diagrams:

- Impulse approximation
- ‘Pole’ Contributions

It is now straightforward to compute **Mellin moments**:

\[
2\langle x^m \rangle \pi_\alpha^u (\xi, Q^2) = N_c \text{tr} \int_q \frac{(n \cdot q)^m}{(n \cdot P)^{m+1}} i \gamma \cdot n M_u(q^-, p, k)
\]

Where two kinds of denominators appear:

\[
\frac{1}{D_{Q/2}D_{Q/2}D_P} = \int_{\Omega_u} \frac{du_1 du_2}{[\omega_3^2 + \alpha Q - \beta P]^3},
\]

\[
\frac{1}{D_{Q/2}D_{Q/2}} = \int_{\Omega_u} \frac{du_1 du_2 \delta(\beta)}{[\omega_2^2 + \alpha Q]^2},
\]

where \( D_X = D(q - X), \alpha = u_1 - u_2, \beta = 1 - u_1 - u_2 \) and \( \Omega_u = \{(u_1, u_2)|0 < u_1 < 1, 0 < u_2 < 1 - u_1\} \)

|\alpha| + |\beta| \leq 1  
Domain of DD variables… almost there!
GPDs in the SCI

- Therefore, we arrive at the following diagrams:

- Concerning the highlighted part... Let \( \alpha := (\beta + \xi \alpha)^m \)
  \[ m(\cdot)^{m-1} \]
  Can be written as derivatives of \( \beta \) or \( \xi \alpha \)
  So that, we can turn \( m(\cdot)^{m-1} \rightarrow (\cdot)^m \)

- Nonetheless, the arbitrariness in the differentiation is unavoidable.

- After changes of variables, regularization, ...

\[
\langle x^m \rangle_{u,1} = \int_\Omega d\beta d\alpha \left[ (\beta + \xi \alpha)^m (h_{a0} + \xi h_{a1}) + m(\beta + \xi \alpha)^{m-1} (h_{b0} + \xi h_{b1} + \xi^2 h_{b2}) \right]
\]

\[
\langle x^m \rangle_{u,2} = \int_\Omega d\beta d\alpha (\beta + \xi \alpha)^m (h_{c0} + \xi h_{c1}) \delta(\beta),
\]

\(|\alpha| + |\beta| \leq 1 \quad \text{Domain of DD variables… almost there!}\)
Finally, the GPD Mellin moments acquire the form:

$$\langle x^m \rangle_u = \int d\beta d\alpha (\beta + \xi \alpha)^m \left[ \mathcal{F}(\beta, \alpha, Q^2) + \xi \mathcal{G}(\beta, \alpha, Q^2) \right]$$

This is nothing that the Mellin moments of the DD representation:

$$H_u^\pi(x, \xi, Q^2) = \int d\beta d\alpha \delta(x - \beta - \xi \alpha) \left[ \mathcal{F}(\beta, \alpha, Q^2) + \xi \mathcal{G}(\beta, \alpha, Q^2) \right]$$

- **Polinomiality**

- Mellin moments are even functions of $\xi$

- The larger power of $\xi$, at most $m+1$

$$\mathcal{F}(\beta, \alpha, Q^2) = \mathcal{F}(\beta, -\alpha, Q^2)$$

$$\mathcal{G}(\beta, \alpha, Q^2) = -\mathcal{G}(\beta, -\alpha, Q^2)$$
Finally, the GPD Mellin moments acquire the form:

\[ \langle x^m \rangle_{u}^{\pi} = \int_{\Omega} d\beta d\alpha (\beta + \xi \alpha)^m [F(\beta, \alpha, Q^2) + \xi G(\beta, \alpha, Q^2)] \]

This is nothing that the Mellin moments of the DD representation:

\[ H_{u}^{\pi}(x, \xi, Q^2) = \int_{\Omega} d\beta d\alpha \delta(x - \beta - \xi \alpha) [F(\beta, \alpha, Q^2) + \xi G(\beta, \alpha, Q^2)] \]

\[ F(\beta, \alpha, Q^2) = h_{a0} - \frac{\partial h_{b0}}{\partial \beta} - \frac{\partial h_{b1}}{\partial \alpha} + h_{c0} \delta(\beta) + h_{b0} [\delta(\bar{\beta} - |\alpha|) - \delta(\beta)] + \eta h_{b1} [\delta(\bar{\beta} - \alpha) - \delta(\bar{\beta} + \alpha)] \]

\[ G(\beta, \alpha, Q^2) = h_{a1} - \eta \frac{\partial h_{b1}}{\partial \beta} - \frac{\partial h_{b2}}{\partial \alpha} + h_{c1} \delta(\beta) + \eta h_{b1} [\delta(\bar{\beta} - |\alpha|) - \delta(\beta)] + h_{b2} [\delta(\bar{\beta} - \alpha) - \delta(\bar{\beta} + \alpha)] \]

- Positive definite GPDs!
- Discontinuous at \( x = \pm \xi \)
- Non-vanishing at \( x = 1 \)

(Last two are drawbacks of the SCI)
GPDs in the SCI

- The **moments** can be conveniently expressed as:

\[
\mathcal{M}_m(Q^2, \xi) = \mathcal{M}_m^{(0)}(Q^2, \xi) + \mathcal{M}_m^{s,v}(Q^2, \xi)
\]

\(\mathcal{M}_m^{s,v}(Q^2, \xi) \int_0^1 du (\xi(2u-1))^m I_0(u)[\xi T_s(Q^2)(1-2u) + T_v(Q^2)u(1-u)]\)

**Sum Rules**

\[
\langle x^0 \rangle_f^{PS}(\xi, Q^2) = F_f^{em,PS}(Q^2)
\]

\[
\langle x^1 \rangle_f^{PS}(\xi, Q^2) = A_f^{PS}(Q^2) + \xi^2 D_f^{PS}(Q^2)
\]

\(A = \theta_2, D = -\theta_1\)

Scalar pole is crucial to get:

**[Soft-pion theorem]**

\[
D = -\frac{E_\pi - 6F_\pi}{3(E_\pi - 2F_\pi)} - \frac{2E_\pi}{3(E_\pi - 2F_\pi)} = -1
\]

Impulse approximation

‘Pole’ Contributions

This one contains a vector meson pole!

This is pole-free

This one contains a scalar pole
GPDs in the SCI

- The contributions to the GPD:

Impulse approximation

‘Pole’ Contributions

\[ H^{c,1}_{u,1}(x, \xi, 0) = \frac{(\xi + x)E_{PS} - 2\xi F_{PS}}{2\xi(E_{PS} - 2F_{PS})} \theta(\xi - x, x + \xi) + \theta(1 - x, x - \xi), \]

\[ H^{c,1}_{u,2}(x, \xi, 0) = \frac{-xE_{PS}}{2\xi(E_{PS} - 2F_{PS})} \theta(\xi - x, x + \xi), \]

\[ H^{l=1}_\pi(2u - 1, 1, 0) = \frac{1}{2} \varphi_{\pi}(u), \quad u \in [0, 1] \]

- Soft pion theorem:

\[ H^{c,1}_u(x, \xi, 0) = \frac{1}{2} \theta(\xi - x, x + \xi) + \theta(1 - x, x - \xi) \]

- In this limit, both SCI-derived PDA and PDF are constants

- The above expression also ensures the PDF is the forward limit of the GPD
Conclusions and Scope
Conclusions and Scope

➢ We discussed two different approaches to address the pion and Kaon GPDs: Overlap of LFWF and diagram-based calculation.

➢ The LFWF, by construction, fulfills the requirement of positivity. The drawback is that it is limited to the DGLAP domain.

➢ Charge, Mass, Spatial and Momentum distributions are already within the reach of DGLAP domain.

➢ In this domain, we can also evolve the GPDs to disentangle valence, glue and sea content.

➢ Sophisticated covariant extensions to the ERBL domain are known.

(Still, D-term is not fixed unambiguously)
Conclusions and Scope

- We discussed two different approaches to address the pion and Kaon GPDs: Through the overlap of the LFWF and following a diagrammatic representation.

- The LFWF, by construction, fulfills the requirement of positivity. The Drawback is that it is limited to the DGLAP domain.
  (Recall many distributions are readily available within this domain.. and we can extend it)

- The diagram approach, in principle, could give you everything. But building up the expressions, is not an easy task.

- An alternative is proposed on the grounds of deriving the scattering amplitude by means of its inhomogeneous BSE.

- Positive outcomes arise naturally, thus revealing the potential of this approach, and motivating a more sophisticated calculation.
Obtaining the Scattering amplitude
The **Scattering Amplitude** is written as:

\[
F(k, P_1, P_2) = F_0(k, P_1, P_2) + \Sigma^F(k, P_1, P_2)
\]

The **self-energy**:

\[
\Sigma^F(k, P_1, P_2) = -\frac{4}{3} \int_q g_{\alpha\beta}(k-q)\gamma_\alpha S(q + P_1) \\
\times F(q, P_1, P_2)S(q - P_2)\gamma_\beta.
\]

In the **SCI**, solving this equation is equivalent to solving:

\[
\Sigma^F(P_1, P_2) = \Sigma^{F_0}(P_1, P_2) \\
- \Delta_g \int_q \gamma_\alpha S(q_1) \Sigma^F(P_1, P_2)S(q_2)\gamma_\alpha.
\]
The Scattering Amplitude is written as:

\[ F(k, P_1, P_2) = F_0(k, P_1, P_2) + \Sigma^F(k, P_1, P_2) \]

The self-energy:

\[ \Sigma^F(k, P_1, P_2) = -\frac{4}{3} \int_q G_{\alpha \beta}(k-q) \gamma_\alpha S(q + P_1) \times F(q, P_1, P_2) S(q - P_2) \gamma_\beta. \]

In the SCI, solving this equation is equivalent to solving:

\[ \Sigma^F(P_1, P_2) = \Sigma^{F_0}(P_1, P_2) \]

\[ -\Delta_g \int_q \gamma_\alpha S(q_1) \Sigma^F(P_1, P_2) S(q_2) \gamma_\alpha \]

We can build up a basis for:

\[ \Sigma^F(P_1, P_2) = \sum_{i=1}^{4} t_i T_i \]

Where only two structures survive:

\[ T_1 = \mathbb{1}, \]
\[ T_2 = \frac{-i}{M} \mathcal{K} \]

We proceed similarly with the F0 part (let’s call b1 and b2 to their dressing functions)
The following solutions are found:

\[ t_1(Z^2) = \frac{b_1(Z^2)}{1 + \Delta_g f_s(Z^2)} , \quad f_s(Z^2) = \text{tr} \int_q S(q_1)S(q_2) , \]

\[ t_2(Z^2) = \frac{b_2(Z^2)}{1 + \Delta_g f_v(Z^2)} , \quad f_v(Z^2) = \frac{K_\mu K_\nu}{2K^2} \text{tr} \int_q i\gamma^T_\mu S(q_1)i\gamma^T_\nu S(q_2) . \]

Where the b dressing functions are:

\[ b_1(Z^2) = -\frac{1}{4}\Delta_g \text{tr} \int_q T_1 \gamma_\alpha S(q_1)F_0(q, P_1, P_2)S(q_2)\gamma_\alpha , \]

\[ b_2(Z^2) = \frac{M^2}{4K^2}\Delta_g \text{tr} \int_q T_2 \gamma_\alpha S(q_1)F_0(q, P_1, P_2)S(q_2)\gamma_\alpha . \]

Recalling the expressions for the scalar and vector form factors:

\[ iF_s(Z^2) = N_c \text{tr} \int_q i\Gamma_\mu(Z)S(q_1)i^2F_0(q, P_1, P_2)S(q_2) , \]

\[ 2K_\mu F_{em}(Z^2) = N_c \text{tr} \int_q \Gamma_\mu(Z)S(q_1)i^2F_0(q, P_1, P_2)S(q_2) , \]

\[ \Sigma^F(P_1, P_2) = \Sigma^{F_0}(P_1, P_2) - \Delta_g \int_q \gamma_\alpha S(q_1)\Sigma^F(P_1, P_2)S(q_2)\gamma_\alpha , \]

\[ \Sigma^F(P_1, P_2) = \sum_{i=1}^{4} t_i T_i , \]

\[ \Sigma^{F_0}(P_1, P_2) = \sum_{i=1}^{4} b_i T_i . \]

\[ N_c b_1(Z^2) = \Delta_g F^b_s(Z^2) , \]

\[ N_c b_2(Z^2) = - M \Delta_g F^b_{em}(Z^2) , \]

\[ N_c \Sigma^F(P_1, P_2) = F^b_s(Z^2)\Delta_s(Z^2) + iK F^b_{em}(Z^2)\Delta_v(Z^2) . \]