Loop-String-Hadron on a Maximal Tree

A tale of gauge invariance and flowers

Based on work with Christian Bauer

Iván Mauricio Burbano Aldana
Advised by Christian Bauer and Raúl Briceño
Introduction
Why am I here?
To learn about “classical” lattice QCD!


Why Quantum Computers?
Because doing scattering on Euclidean time is getting hard

We could do this at arbitrarily high energies if we could compute:

\[ \langle \vec{p}_f | j_M(t) j(0) | \vec{p}_i \rangle \]

In Minkowski space!

Raul A. Briceño, Marco A. Carrillo, Juan V. Guerrero, Maxwell T. Hansen, and Alexandru M. Sturzu, 2112.01968.
Or we can be more literal…

Roland C. Farrell, Marc Illa, Anthony N. Ciavarella, and Martin J. Savage 2401.08044.
Lattice QCD on Quantum Computers
Real Time Evolution

\[ \int \mathcal{D}\phi e^{-S_{E}(\phi)} = \langle \varphi_{f} | e^{-\beta H} | \varphi_{i} \rangle \]

\[ \int \mathcal{D}\phi e^{iS(\phi)} = \langle \varphi_{f} | e^{-iHt} | \varphi_{i} \rangle \]
Primer on Quantum Computation

Shut up and evolve!

\[ |\psi\rangle \]

Prepare

Time evolve
\[ e^{-iHt} \]

Measure
Due to errors inherent in the current available chips, even starting with a gauge invariant state we can drift into the unphysical sector of the Hilbert space.
Comparison Between Formulations
Everything here comes with caveats!

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>gauge Invariant</td>
<td>Non-Abelian per vertex</td>
<td>Abelian per link</td>
<td>Abelian per link</td>
<td>Non-Abelian on a single vertex</td>
<td>✓</td>
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<tr>
<td>low coupling</td>
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Why do we need gauge “symmetry”?
Proposal: because of locality

\[
\int_{\mathcal{C}} A = \int_{\Sigma} F
\]

Let’s give it up!
Redundancy
Yang-Mills Background
Hamiltonian Lattice Yang-Mills
Kogut-Susskind: Classical kinematics

\[ U = P \exp \int_e A \in G = U(1), SU(2), SU(3), SU(N), \ldots \]

Gauge Transformation: \( U \leftrightarrow gUh^{-1} \)

For me link = edge
Hamiltonian Lattice Yang-Mills
Kogut-Susskind: Quantum kinematics

Magnetic basis: (analogous to traditional lattice QCD)

States $|\psi\rangle$ are wavefunctions that assign to each configuration of Wilson lines $U, \ldots$ a probability amplitude density $\langle U, \ldots | \psi \rangle$

Electric Operators: $E = E_a T^a$

$\langle U, \ldots | e^{-i\omega^a E^a} | \psi \rangle = \langle g^{-1} U, \ldots | \psi \rangle$

Gauss’ Law: $0 = E_1 + E_2 + E_3 + E_4 \leftrightarrow \int_{S^2} d\vec{S} \cdot \vec{E}$
Hamiltonian $SU(2)$ Yang-Mills

Electric Basis

**Peter-Weyl Theorem:** The space of wavefunctions is spanned by the matrix elements of every representation.

$$\langle U | \psi \rangle := (-1)^{j-m_1} \sqrt{2j + 1} \langle j, -m_1 | R^{(j)}(U) | j, m_2 \rangle$$

New way of thinking of states:

\[ |j_1, m_1\rangle + \sqrt{2j + 1} |j_2, m_2\rangle \]

Abelian Gauss’ Law: $j_1 = j_2$
Conclusions

Spoiler alert!

1. We can (almost) fully gauge fix using a maximal tree into a flower

2. We can split the flower into a branch to avoid Mandelstam constraints

Christian W. Bauer, Irian D’Andrea, Marat Freytsis, and Dorota M. Grabowska [2307.11829].
Conclusions

3. Leafs induce special functions that translate between electric and magnetic bases

4. The system can be described through $3((\# \text{ of plaquettes}) - 1)$ quantum numbers

A single constraint:

$$d > \max\{r - l, r - l + l - r\}$$
Hamiltonian $SU(2)$ Yang-Mills
Prepotential Formulation

Let us think of the spin 1/2 states as different bosonic species

$$|j, m\rangle \propto (a_{\uparrow}^\dagger)^{j+m} (a_{\downarrow}^\dagger)^{j-m} |0\rangle$$

Can recover all other states in any representation
Hamiltonian $SU(2)$ Yang-Mills

What have we gained?

Indices 😊

\[
\begin{pmatrix}
\alpha_A^+ \\
\alpha_i^+
\end{pmatrix} = \begin{pmatrix}
\alpha_i^+ \\
\alpha_i^-
\end{pmatrix}
\]

\[\epsilon^{AB}\psi_A\phi_B = \psi_A\phi^A = \psi\phi\]

Gauge invariance $\leftrightarrow$ No free indices!
Hamiltonian $SU(2)$ Yang-Mills
Crazy Repackaging

\[ A^A_B = \begin{cases} a^A & B = - \\ a^\dagger_A & B = + \end{cases} \]

Properties:

\[ [A^A_B, A^C_D] = \epsilon^{AC} \epsilon_{BD} \]

\[ (A^\dagger)^A_B = A^A_B \]

\[ AA^\dagger = N + 1 \]
Hamiltonian $SU(2)$ Yang-Mills
Loop-String-Hadron (LSH) formulation

Loop operators:

$$\mathcal{L}_{AB}(h_1, h_2) = A_{CA}(h_2)A^C_B(h_1)$$

Creation: $A = B = +$

Number: $A = +, B = -, h_1 = h_2$

Dynamics

These are the plaquettes with a timelike direction

\[ H_E = \frac{g^2}{4} \sum_{h} E(h)^2 \]

\[ H_B = \frac{1}{2g^2} \sum_{\gamma} \text{tr}(U(\gamma)) \]

strong coupling

weak coupling
These can be written in LSH terms

\[ E_a = a^\dagger \sigma_a a \]

\[ E(h)^2 = \frac{\mathcal{L}_{+-}(h, h)}{2} \left( \frac{\mathcal{L}_{+-}(h, h)}{2} + 1 \right) \]

\[ U = \frac{1}{\sqrt{N + 1}} A \]

\[ U^A_B(h_1, h_2) = U^A_C(h_2)(\sigma_x)_D^C U^D_B(h_1) \]
Manipulating Graphs
Virtual Point-Splitting
Mandelstam Constraints

In the presence of vertices with valency higher than 4, the LSH creation operators don’t create linearly independent states.
Virtual Point-Splitting

Solution

This is not an equivalence at face value!
Virtual Point-Splitting
Equivalence up to gauge invariance
Virtual Point-Splitting

Electric behavior

\[ u h w \]

\[ S \]

\[ u h w \]

\[ \vec{E}(h) \]

Parallel Transport

\[ W^{-1} \vec{E}(h) W \]

\[ u w g^{-1} \]

\[ S \]

\[ u w g^{-1} w^{-1} \]
**Theorem:** The theories are equivalent (even at the quantum level) as long as
I) $S$ is surjective (which almost always happens) and
II) all configurations with the same coarsening are gauge equivalent to one another

The example above is **not** an equivalence (think about the orbit of any configuration with the identity at the loop).
Maximal Tree Gauge-Fixing

\[ u \leftarrow u^{-1} \rightarrow v \leftarrow v^{-1} \]

Only one single gauge degree of freedom remaining
Enter Flowers
Coarse graining through maximal trees
Flower Hamiltonian

Magnetic Hamiltonian

Electric Hamiltonian

\[ U = U^{-1} U \]

\[ \vec{E}^2 \implies \left( \vec{E}(f) + \vec{E}(f) - \vec{E}(i) - \vec{E}(i) \right)^2 \]

Parallel Transport to
Point-Splitting Flowers
Correcting for high-valency vertices
Yang-Mills on the Flower
Magnetic Degrees of Freedom

Orientation of $\vec{\omega}$

Delocalized at low coupling

Localized at low coupling (i.e. near the continuum limit)

Christian W. Bauer, Irian D’Andrea, Marat Freytsis, and Dorota M. Grabowska

2307.11829.
LSH on a Leaf

\[ \mathcal{L} = a^\dagger(r)a^\dagger(s) \]

\[ \mathcal{L} = (a^\dagger(s)a^\dagger(t))(a^\dagger(r)a^\dagger(t)) \]

\[ |n, N\rangle \propto \ell^n \mathcal{L}^N |0\rangle \]

\[ \langle e^{-i\omega J_z} | n, N \rangle = \sqrt{N + n + 1} \sum_{m=-\frac{n+N}{2}}^{\frac{n+N}{2}} e^{-i\omega m} \langle N, 0 | \left( \left| \frac{n+N}{2}, -m \right\rangle \otimes \left| \frac{n+N}{2}, m \right\rangle \right) \]
Special Function

\[ \text{Re}\langle e^{-i\omega J_z}|1, 2\rangle \]

\[ \text{Re}\langle e^{-i\omega J_z}|1, 10\rangle \]

\[ \text{Re}\langle e^{-i\omega J_z}|4, 2\rangle \]

\[ \text{Im}\langle e^{-i\omega J_z}|4, 11\rangle \]
A single constraint:  \[ d > \max\{r - l, r - l + l - r\} \]
Conclusions

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Thanks!

Raúl Briceño

Christian Bauer

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Jesse Stryker

Anthony Ciavarella

Irian D’Andrea

Marco Carrillo
My other work…
Recovering Scattering Amplitudes from Monte Carlo Simulations
O(3) Model Primer

Spacetime or some material

\[ S = \frac{1}{2g^2} \int d^2 x \left( \partial \vec{\phi} \right)^2 \]

Properties:
1. Asymptotic freedom: at \( E = \infty \) the theory has two free particles.
2. Confinement: at any finite \( E \) the two particles above are confined into an \( O(3) \) vector multiplet of particles.
3. Dynamical mass generation: classical conformal invariance is broken at the quantum level.
4. Integrable

A very interesting model to study Yang-Mills and QCD!
One Particle

We can recover the one-particle spectrum from the two-point function

\[
\langle \vec{\phi}(0, -p) \cdot \vec{\phi}(t, p) \rangle = \langle 0 | \vec{\phi}(t, -p) \cdot \vec{\phi}(0, p) | 0 \rangle = \sum_n e^{-E_n t} | \langle 0 | \vec{\phi}(0, -p) | n \rangle |^2
\]
Generalized Eigenvalue Problem

Problem: Given a state $|n\rangle$ and a list of operators $\{O_a\}$, which linear combination $|\psi\rangle = \sum_{a} \nu^a O_a |0\rangle$ best approximates $|n\rangle$?

Idea: Two point functions give us

$$
\sum_{a,b} \nu^a \nu^b \langle O_a(t)O_b^*(0) \rangle = \langle \psi | e^{-tH} | \psi \rangle = \sum_{n} e^{-tE_n} | \langle \psi | n \rangle |^2
$$

Minimize keeping $\langle \psi | e^{-t_0H} | \psi \rangle$ constant

$$
\langle O_a(t)O_b^*(0) \rangle \nu^b = \lambda \langle O_a(t_0)O_b^*(0) \rangle \nu^b
$$

$e^{-tE_n}$

“Dual to $\langle 0 | O_a | n \rangle$”
Two Particle

Operator list: \( \{ O^A(t, P, p) = P^A_{ab} \phi^a(t, P + p/2) \phi^b(t, P - p/2) \}_p \)
Lüscher Quantization Condition
Recovering scattering from finite-volume spectrum

\[ \det(\mathcal{M} + F^{-1}) = 0 \]

In 1+1 D

\[ \delta + \gamma L k^* = (2n + \eta)\pi \]
One Particle Form Factor

\[ \langle p, a \mid J^b_\mu(0) \mid q, c \rangle = e^{abc}(p + q)_\mu F(Q^2) \]

LSZ-like arguments

\[ \left| F(Q^2) \right|^2 = \frac{E_pE_q}{(p + k)_\mu^2} \frac{\left| e^{abc}\langle \phi^a_\mu(p, T) J^b_\mu(t, -p - q) \phi^b_\mu(0, q) \rangle \right|^2}{\langle \vec{\phi}(2T, p) \cdot \vec{\phi}(2t, -p) \rangle \langle \vec{\phi}(2t, q) \cdot \vec{\phi}(0, -q) \rangle} \]