Factorization connecting continuum and lattice TMDs

Jefferson Lab Theory Seminar

In collaboration with M. Ebert, S. Schindler and I. Stewart, based on arXiv: 2201.08401.
3D Tomography of the Proton (Hadrons)

Lattice QCD

Hard Scattering

Jefferson Lab 12 GeV

The Electron-Ion Collider
Outline

• Introduction to TMDs

• Lattice TMDs
  • LaMET and Quasi-TMDs
  • Lorentz-invariant approach (MHENS scheme)

• Relation between lattice and continuum TMDs

• First lattice results
  • Collins-Soper kernel for TMD evolution
  • Soft function
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High-energy proton structure and the parton model

The inner proton structure has been probed through high-energy scattering.

Feynman’s parton model (1969):

• When the proton travels at almost the speed of light, quarks and gluons are “frozen” in the transverse plane due to Lorentz contraction;

• During a hard collision, the struck quark/gluon (parton) appears to the probe that it does not interact with its surroundings.
Tomography in the 3D momentum space

Collinear parton distribution function (PDF) \( f_i(x) \)

- \( i = u, d, c, s, t, b \)
- \( i = g \)

Transverse-momentum dependent parton distribution (TMD) \( f_i(x, \vec{k}_\perp) \)

\[
f_i(x, \vec{b}_T) = \int d^2k_T \ e^{i\vec{k}_T \cdot \vec{b}_T} f_i(x, \vec{k}_T)
\]

TMD in the Fourier conjugate \( b_T \)-space:
Figure 5.6: Example of extracted (optimal) unpolarized TMD distributions. The color indicates the relative size of the uncertainty band. Plot from Ref. [324].

This functional form of was also used in [323]. It has five free parameters which grant sufficient flexibility in $G$-space as needed for the description of the precise LHC data. An example of distributions in the $G, b$ plane is presented in Fig. 5.6. Depending on the value of $G$, the $b$-behavior apparently changes. The authors of Ref. [324] observe (the same observation was made in Ref. [251]) that the unpolarized TMD FF gains a large $b$-term in the nonperturbative part. It could indicate non-trivial consequences of hadronization physics, or a tension between collinear and TMD distributions.

5.2.2 Drell-Yan and weak gauge boson production

Drell-Yan lepton pair production via either virtual photon or $W$/Z boson served in prior chapters of this handbook to set up the basic notation and concepts for TMD factorization. Factorized in terms of a convolution of two TMD PDFs from each incoming proton at the small transverse momentum $b$ as shown in Eq. (2.29a), Drell-Yan production in unpolarized proton-proton collisions is one of the most important processes for extracting unpolarized quark TMD PDFs.

There is a tremendous amount of experimental data for Drell-Yan production, ranging from lower energy Fermilab experiments to the highest energy data at the LHC. The lower-energy fixed-target Fermilab data include E605 [333] and E288 [334], while the higher-energy Fermilab data from collider Tevatron include CDF Run I [335] and Run II [336], and D0 Run I [337] and Run II [338, 339]. LHC data include forward $W$-production data from the LHCb experiment at 7 [340], 8 [341], and 13 [342] TeV, $W$-production data from the CMS experiment at 7 [343] and 8 [344] TeV, $W$-production data differential in rapidity from the ATLAS experiment at 7 [343] and 8 [345] TeV, and off-peak (low- and high-mass) Drell-Yan data from the ATLAS experiment at 8 TeV [345]. Finally, there is also preliminary $W$-production data from the STAR experiment at 510 GeV.

Earlier description of the small-$b$ Drell-Yan data from both fixed-target and collider Fermilab data within the Collins-Soper-Sterman (CSS) framework has been performed by several authors. The variation of color in the plot is due to variation of replicas and illustrates the uncertainty of the extraction. The nucleon polarization vector is along $\hat{H}$-direction. The figures are from Ref. [371].

Figure 5.11: Tomographic scan of the nucleon via the momentum space quark density function $\tau_1; b; \tau_2; H; \tau_3$ defined in Eq. (5.27) at $G = 0$. Panels are for $D$ and $S$ quarks.

The figures are from Ref. [358].

Figure 5.13: The density distribution of an unpolarized up and down quarks using Sivers functions from Ref. [18].

I. Scimemi and A. Vladimirov, JHEP 06 (2020).

Cammarota, Gamberg, Kang et al. (JAM Collaboration), PRD 102 (2020).
TMD from experiments

TMD processes:

**Semi-Inclusive DIS**

$$\sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T)$$

**Drell-Yan**

$$\sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T)$$

**Dihadron in e^+e^-**

$$\sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T)$$

- **HERMES, COMPASS, JLab, EIC, ...**
- **Fermilab, RHIC, LHC, ...**
- **Babar, Belle, BESIII, ...**
Differential cross section $\frac{d\sigma_{DY}}{dQdYd^2q_T}$:

- $Q$ and $Y$ uniquely determine $x_a$ and $x_b$ of initial state partons;
- $q_T$ is contributed from the transverse momenta of partons and radiated gluons.

\[ n_a = (1,0,0,1)/\sqrt{2}, \quad n_b = (1,0,0,-1)/\sqrt{2} \]

\[ q^\mu = (q^+, q^-, q^\perp), \quad Q^2 = q^2, \quad Y = \frac{1}{2} \ln \frac{n_a \cdot q}{n_b \cdot q} \]

\[ s = (P_A + P_B)^2, \quad q_T = |q^\perp| \]

\[ x_a = Q e^Y \sqrt{s}, \quad x_b = Q e^{-Y} \sqrt{s}, \quad Q, s \gg \Lambda_{QCD} \]
Differential cross section $\frac{d\sigma_{DY}}{dQdyd^2q_T}$:

- $Q$ and $Y$ uniquely determine $x_a$ and $x_b$ of initial state partons;
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TMD factorization for Drell-Yan processes

TMD factorization for \( q_T = |q_\perp| \ll Q \):

From J. Qiu's TMD School 2022 Lectures.
TMD factorization for Drell-Yan processes

TMD factorization for $q_T = |q_\perp| \ll Q$:

(Schematic) factorization formula:

$$\frac{d\sigma_{DY}}{dQdYd^2q_T} = H \otimes B \otimes B \otimes S$$

Hard factor

(Collinear) beam functions

Soft function

TMD Handbook, by TMD Collaboration
TMD factorization for \( q_T = |q_\perp| \ll Q \):

Separation of soft \((p_s)\) and collinear \((p_{n_a}, p_{n_b})\) modes:

\[
p_{n_a} \sim Q(\lambda^2, 1, \lambda), \quad p_{n_b} \sim Q(1, \lambda^2, \lambda), \quad p_s \sim Q(\lambda, \lambda, \lambda), \quad \lambda \ll 1
\]

\[
p_{n_a}^2 \sim p_{n_b}^2 \sim p_s^2 \sim q_T^2,
\]

rapidity: \( y = \frac{1}{2} \ln \frac{p^-}{p^+} \)

(Schematic) factorization formula:

\[
\frac{d\sigma_{\text{DY}}}{dQdYd^2q_T} = H \otimes B \otimes B \otimes S
\]

Hard factor \( (\text{Collinear}) \) beam functions \( \text{Soft function} \)
**TMD factorization for Drell-Yan processes**

TMD factorization for $q_T = |q_\perp| \ll Q$:

Separation of soft ($p_s$) and collinear ($p_{n_a}, p_{n_b}$) modes:

$$p_{n_a} \sim Q(\lambda^2, 1, \lambda), \ p_{n_b} \sim Q(1, \lambda^2, \lambda), \ p_s \sim Q(\lambda, \lambda, \lambda), \ \lambda \ll 1$$

$$p_{n_a}^2 \sim p_{n_b}^2 \sim p_s^2 \sim q_T^2,$$

rapidity: $y = \frac{1}{2} \ln \frac{p^-}{p^+}$

(Schematic) factorization formula:

$$\frac{d\sigma_{DY}}{dQdYd^2q_T} = H \otimes B \otimes B \otimes S$$

Hard factor

(Collinear) beam functions

Soft function

$$= H \otimes f^{TMD} \otimes f^{TMD}$$

Physical TMDs

$$f^{TMD} = B\sqrt{S}$$
TMD definition

• Beam function:

\[
 f_i^{\text{TMD}}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{\text{UV}}(\epsilon, \mu, xP^+) \lim_{\tau \to 0} B_i(x, \vec{b}_T, \epsilon, \tau, xP^+) \Delta_s^i(b_T, \epsilon, \tau)
\]

Collins-Soper scale: \( \zeta = 2(xP^+e^{-\gamma_n})^2 \)

• Soft function:

Ebert, Stewart and YZ, JHEP 09 (2019)
**TMD definition**

- Beam function: 

\[
    f_i^\text{TMD}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{\text{UV}}(\epsilon, \mu, xP^+) \lim_{\tau \to 0} B_i(x, \vec{b}_T, \epsilon, \tau, xP^+) \Delta_S(b_T, \epsilon, \tau)
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    \zeta = 2(xP^+ e^{-\gamma_n})^2
\]

- Soft function:

\[
    \text{Ebert, Stewart and YZ, JHEP 09 (2019)}
\]
TMD definition

Many different schemes for regulating rapidity divergences:

**Wilson lines on the light-cone (SCET):**
- Becher and Neubert, EPJC71 (2011);
- Echevarria, Idilbi and Scimemi, JHEP07 (2012), PLB726 (2013);
- Chiu, Jain, Neil and Rothstein, JHEP05 (2012), PRL108 (2012);
- Li, Neil and Zhu, NPB 960 (2020);
- Ebert, Moult, Stewart, Tackman and Vita, JHEP 04 (2019).

**Wilson lines off the light-cone:**

For reviews see also
- Ebert, Stewart and YZ, JHEP 09 (2019);

**Scheme-independent TMD factorization:**

\[
\frac{d\sigma_{DY}}{dQdYd^2q_T} = \sigma_0 \sum_{i,j} H_{ij}(Q, \mu) \int d^2b_T e^{ib_T \cdot \bar{q}_T} f_i^{\text{TMD}}(x_a, \bar{b}_T, \mu, \zeta_a) f_j^{\text{TMD}}(x_b, \bar{b}_T, \mu, \zeta_b) \left[ 1 + \mathcal{O}\left(\frac{q_T^2}{Q^2}, \frac{\Lambda^2_{\text{QCD}}}{Q^2}\right) \right] \\
\zeta_a \zeta_b = Q^4
\]
Many different schemes for regulating rapidity divergences:

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- Becher and Neubert, EPJC 71 (2011);
- Echevarria, Idilbi and Scimemi, JHEP 07 (2012), PLB 726 (2013);
- Chiu, Jain, Neil and Rothstein, JHEP 05 (2012), PRL 108 (2012);
- Li, Neil and Zhu, NPB 960 (2020);
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\]
\[
\zeta_a \zeta_b = Q^4
\]
- For $b_T \ll \Lambda_{\text{QCD}}^{-1}$, can be perturbatively matched onto collinear PDFs;
- For $b_T \sim \Lambda_{\text{QCD}}^{-1}$, becomes intrinsically non-perturbative, which motivates first-principles calculations.
Overview of the continuum and lattice schemes

Quasi

MHENS

Lattice schemes

P_\perp large, \eta \rightarrow \infty

Switch order of \epsilon \rightarrow 0, Y \rightarrow \infty

Continuum limits

LR

Matching relations

Change Wilson lines

Collins

JMY

Continuum schemes

Ebert, Schindler, Stewart and YZ, 2201.08401.
General correlator for the beam function

\[ \Omega_{q_i/h}^{[\Gamma]}(b, P, \epsilon, \eta v, \delta) = \left\langle h(P) \bigg| \bar{q}_i \left( \frac{b}{2} \right) \frac{\Gamma}{2} W_{\square}^F (b, \eta v, \delta) q_i \left( -\frac{b}{2} \right) \bigg| h(P) \right\rangle \]

\[ W_{\square}^R (b, \eta v, \delta) = \]

\[
\begin{align*}
\cosh \gamma_+ &= \frac{(\eta v \pm \delta/2) \cdot (b - \delta)}{|\eta v \pm \delta/2||b - \delta|} \\
\cosh \gamma_- &= \frac{(\eta v \mp \delta/2) \cdot (b + \delta)}{|\eta v \mp \delta/2||b + \delta|}
\end{align*}
\]
General correlator for the soft function

\[
S^R_R(b, \epsilon, \eta \nu, \bar{\eta} \bar{\nu}) = \frac{1}{d_R} \left\langle 0 \left| \right| Tr S^R_R(b, \eta \nu, \bar{\eta} \bar{\nu}) \right| 0 \right\rangle
\]

\[
S^R_R(b, \eta \nu, \bar{\eta} \bar{\nu}) = 
\]

\[|\vec{b_T}|.\]
Collins scheme

\[ b = (0, b^-, b_\perp) \quad \delta = b^- n_b \]

\[ \delta^\mu = (0, b^-, 0_\perp) \]

\[ v^\mu = n_B^{\mu}(y_B) \equiv n_b^\mu - e^{2y_B} n_a^\mu = (-e^{2y_B}, 1, 0_\perp) \]

\[ B_{q/h}^{C}(x, \vec{b}_T, \epsilon, y_P - y_B) = \int db^- \frac{e^{-ib^-(xP^+)}}{2\pi} \]

\[ \times \Omega_{q/h}^{[y^+]} \left[ b, P, \epsilon, -\infty n_B, b^- n_b \right] \]

\[ f_{i/l/h}^{C}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} \frac{Z_{UV}(\epsilon, \mu, \zeta)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}} \]

\[ \times \lim_{y_B \to -\infty} \frac{B_{i/l/h}^{C}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}} \]

Rapidity divergences

\[ \propto e^{-\gamma^\zeta(\vec{b}_T, \epsilon)(y_P - y_B)}, \quad e^{-\gamma^\zeta(\vec{b}_T, \epsilon)(2y_n - 2y_B)} \]

\[ \zeta = 2(xP^+ e^{-y_n})^2 = x^2 m_n^2 e^{2(y_P - y_n)} \]
Collins scheme

\[ b = (0, b^-, b^\perp) \quad \delta = b^- n_b \]

\[ \sqrt{S} \]

\[ n_B(y_B) \]

\[ n_B(y_B) \]

\[ n^\mu(y_n) \equiv (1, - e^{-2y_n}, 0_\perp) \]

\[ f^{C}_{ih}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{UV}(\epsilon, \mu, \zeta) \]

\[ \times \lim_{y_B \to -\infty} \frac{B^{C}_{ih}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}} \]

Rapidity divergences \[ \propto e^{-\gamma^g(b_T, \epsilon)(y_P - y_B)} \times e^{-\gamma^g(b_T, \epsilon)(2y_n - 2y_B)} \]

\[ \zeta = 2(xP^+ e^{-y_n})^2 = x^2 m_n^2 e^{2(y_P - y_n)} \]
Large Rapidity (LR) scheme

\[ b = (0, b^-, b_\perp) \quad \delta = b^- n_b \]

\[ B^C \]

\[ B^C \quad b_\perp \quad n_b \quad P \quad n(y_P) \]

\[ \sqrt{S} \]

\[ n_B(y_B) \quad n_B(y_B) \quad n(y_n') \quad n(y_n) \]

\[ n^\mu(y_n) \equiv (1, -e^{-2y_n}, 0_\perp) \]

Ebert, Schindler, Stewart and YZ, 2201.08401.

Reversed order of limits:

\[ f_{ilh}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) \]

\[ = \lim_{-y_B \gg 1} \lim_{\epsilon \to 0} \frac{Z_{UV}^B(\epsilon, \mu)}{\sqrt{Z_{UV}^S(\epsilon, \mu, 2y_n - 2y_B)}} \times \frac{B^C_{ilh}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S_C(b_T, \epsilon, 2(y_n - y_B))}} \]
Outline

• Introduction to TMDs

• **Lattice TMDs**
  
  • LaMET and Quasi-TMDs
  
  • Lorentz-invariant approach (MHENS scheme)

• Relation between lattice and continuum TMDs

• First lattice results
  
  • Collins-Soper kernel for TMD evolution
  
  • Soft function
Lattice QCD

Lattice gauge theory: a systematically improvable approach to solve non-perturbative QCD.

Simulating real-time dynamics has been extremely difficult due to the issue of analytical continuation. 😞
Large-Momentum Effective Theory (LaMET)

PDF $f(x)$:
Cannot be calculated on the lattice

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

$z + ct = 0, \ z - ct \neq 0$

$t = 0, \ z \neq 0$

X. Ji, PRL 110 (2013)
Large-Momentum Effective Theory (LaMET)

\[ z + ct = 0, \quad z - ct \neq 0 \]

PDF \( f(x) \):
Cannot be calculated on the lattice

Related by Lorentz boost

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PDF \( f(x) \):
Cannot be calculated on the lattice

\[ \lim_{P^z \to \infty} \tilde{f}(x, P^z) \overset{?}{=} f(x) \]

Quasi-PDF \( \tilde{f}(x, P^z) \):
Directly calculable on the lattice

Related by Lorentz boost

\[ t = 0, \quad z \neq 0 \]

X. Ji, PRL 110 (2013)
Large-Momentum Effective Theory (LaMET)

\[ z + ct = 0, \ z - ct \neq 0 \]

PDF \( f(x) \): Cannot be calculated on the lattice

\[ \lim_{P^z \to \infty} \tilde{f}(x, P^z) \neq f(x) \]

\[ X. \ Ji, \ PRL \ 110 \ (2013) \]

Quasi-PDF \( \tilde{f}(x, P^z) \): Directly calculable on the lattice

\[ t = 0, \ z \neq 0 \]
Large-Momentum Effective Theory (LaMET)

- Quasi-PDF: $P_z \ll \Lambda$;
  $\Lambda$: the ultraviolet lattice cutoff, $\sim 1/a$

- PDF: $P_z = \infty$, implying $P_z \gg \Lambda$.
  - The limits $P_z \ll \Lambda$ and $P_z \gg \Lambda$ are not usually exchangeable;
  - For $P_z \gg \Lambda_{QCD}$, the infrared (nonperturbative) physics is not affected, which allows for an effective field theory matching.

$$\tilde{f}(x, P_z, \Lambda) = C \left( x, \frac{P_z}{\mu}, \frac{\Lambda}{P_z} \right) \otimes f(x, \mu) + O\left( \frac{\Lambda_{QCD}^2}{P_z^2} \right)$$

- Perturbative matching
- Power corrections

- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
LaMET calculation of the collinear PDFs

A state-of-the-art calculation of the pion valence quark PDF with fine lattices, large momentum and NNLO matching:

Gao, Hanlon, Mukherjee, Petreczky, Scior, Syritsyn and YZ, 2112.02208.
Quasi-TMD

\[ \tilde{b}^\mu = (0, b_T^x, b_T^y, \tilde{b}^z) \quad \delta = \tilde{b}^z \hat{z} = (0, 0, 0, \tilde{b}^z) \]

Quasi-beam function:

\[ \tilde{B}_{llh}(x, \overrightarrow{b}_T, \alpha, \bar{n}, x \tilde{P}^z), \quad \tilde{P}^z \gg m_h \]

Quasi-soft function?

Naive quasi soft function:

Bent quasi soft function:

Neither can be boosted to the soft function in TMD factorization. 😞

- Ji, Jin, Yuan, Zhang and YZ, PRD99 (2019);

YONG ZHAO, 02/21/2022
**Quasi-TMD**

**Naive Quasi-TMD:**

\[
\tilde{f}_{i/h}(x, \vec{b}_T, \mu, x\vec{P}_z) = \lim_{a \to 0} \lim_{\tilde{\eta} \to \infty} \frac{Z_{uv}(a, \mu) \tilde{B}_{i/h}(x, \vec{b}_T, a, \tilde{\eta}, x\vec{P}_z)}{\sqrt{\tilde{S}_{\text{naive}}(b_T, a, \tilde{\eta})}}
\]

Conjectured factorization relation to the physical TMD:

\[
\tilde{f}_{\text{naive}}(x, \vec{b}_T, \mu, x\vec{P}_z) = C_{\text{ns}}(x\vec{P}_z, \mu) g^q_S(b_T, \mu) \exp \left[ \frac{1}{2} \gamma^q_S(b_T, \mu) \ln \frac{(2x\vec{P}_z)^2}{\zeta} \right] \times f_{\text{ns}}(x, \vec{b}_T, \mu, \zeta) \left\{ 1 + \mathcal{O}\left(\frac{1}{(x\vec{P}_z b_T)^2}, \frac{\Lambda^2_{\text{QCD}}}{(x\vec{P}_z)^2}\right) \right\}
\]

Matching coefficient

Non-perturbative factor

Rapidity evolution/ Collins-Soper kernel

Linear power divergence

\[\propto (2\tilde{\eta} + b_T)/a\]

Quasi-TMD

Proposal to calculate the soft sector from lattice:

\[
\frac{\tilde{f}_{\text{ns}}^{\text{naive}}(x, \vec{b}_T, \mu, \vec{P}_z)}{\sqrt{S_f^q(b_T, \mu)}} = C_{\text{ns}}(\mu, x\vec{P}_z) \exp\left[\frac{1}{2} \gamma^q_{\zeta}(\mu, b_T) \ln \frac{(2x\vec{P}_z)^2}{\zeta}\right] \\
\times f_{\text{ns}}(x, \vec{b}_T, \mu, \zeta) \left\{ 1 + \mathcal{O}\left[\frac{1}{(x\vec{P}_z b_T)^2}, \frac{\Lambda^2_{\text{QCD}}}{(x\vec{P}_z)^2}\right]\right\}
\]


Proof of factorization relation:

- Leading-region of Feynman diagrams
  

- Analysis of lattice-friendly TMD correlators
  

- Proof using effective field theory (LaMET).
  
  Ebert, Schindler, Stewart and YZ, 2201.08401.
Quasi-TMD

Desired quasi soft factor:

\[ \tilde{S}(b_T, a, \eta, y_A, y_B) = S^C \left[ b_\perp, a, -\tilde{\eta} \frac{n_A(y_A)}{|n_A(y_A)|}, -\tilde{\eta} \frac{n_B(y_B)}{|n_B(y_B)|} \right] \]

Ebert, Schindler, Stewart and YZ, 2201.08401.

Quasi-TMD:

\[ \tilde{f}_{ilh}(x, \overrightarrow{b}_T, \mu, \eta, x\hat{P}^z) = \lim_{a \to 0} Z_{uv}(a, \mu) \frac{\tilde{B}_{ilh}(x, \overrightarrow{b}_T, a, \eta, x\hat{P}^z)}{\sqrt{\tilde{S}^R(b_T, a, \eta, y_A, y_B)}} \]
Lorentz-invariant approach: MHENS scheme

**Musch-Engelhardt-Negele-Hägler-Schäfer (MHENS) scheme:**

\[
b = (0, b^-, b_\perp) \quad \delta = 0
\]

Can always use Lorentz invariance of \( P \cdot b \) to boost to a frame where

\[
\tilde{b}^\mu = (0, b_T^x, b_T^y, \tilde{b}^z)
\]

Ratios of TMDs can be calculated without the soft function.

\[
B_{q/h}^{\text{MHENS}} [\Gamma](x, \overrightarrow{b}_T, P, a, \eta, \nu)
\]

Hägler, Musch, Engelhardt, Negele, Schäfer, et al.,
Summary of the continuum and lattice schemes

<table>
<thead>
<tr>
<th></th>
<th>TMD</th>
<th>Beam function</th>
<th>Soft function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collins</td>
<td>( \lim_{\epsilon \rightarrow 0} Z_{\text{UV}}^R ) ( \lim_{y_B \rightarrow -\infty} \frac{\Omega_{i/h}}{\sqrt{S_R}} )</td>
<td>( \Omega^{[\gamma^+]}_{q/h} [b, P, \epsilon, -\infty n_B(y_B), b^{-} n_b] )</td>
<td>( S^R [b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)] )</td>
</tr>
<tr>
<td>LR</td>
<td>( \lim_{y_B \gg 1} \lim_{\epsilon \rightarrow 0} Z_{\text{UV}}^R ) ( \frac{\Omega_{i/h}}{\sqrt{S_R}} )</td>
<td>( \Omega^{[\gamma^+]}_{q/h} [b, P, \epsilon, -\infty n_B(y_B), b^{-} n_b] )</td>
<td>( S^R [b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)] )</td>
</tr>
<tr>
<td>Quasi</td>
<td>( \lim_{a \rightarrow 0} Z_{\text{UV}} ) ( \frac{B_{i/h}}{\sqrt{S_R}} )</td>
<td>( \Omega^{[\gamma^0, z]}_{q/h} (\tilde{b}, \tilde{P}, a, \tilde{n}_z, \tilde{b}^z \tilde{z}) )</td>
<td>( S^R \left[ b_\perp, a, -\tilde{n} \frac{n_A(y_A)}{</td>
</tr>
<tr>
<td>MHENS</td>
<td></td>
<td>( \Omega^{[\Gamma]}_{q/h} (b, P, a, \eta v, 0) )</td>
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<tr>
<td>( v^\mu )</td>
<td>((-e^{2y_B} 1, 0_\perp))</td>
<td>((0, 0, -1))</td>
<td>((0, v_x, v_y, v_z))</td>
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<tr>
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<tr>
<td>( P^\mu )</td>
<td>( \frac{m_h}{\sqrt{2}} (e^{y_P}, e^{-y_P}, 0_\perp))</td>
<td>( m_h (\cosh y_P, 0, 0, \sinh y_P))</td>
<td>( m_h \left( \cosh y_P, \frac{P_x}{m_h}, \frac{P_y}{m_h}, \sinh y_P \right) )</td>
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Outline

• Introduction to TMDs

• Lattice TMDs
  • LaMET and Quasi-TMDs
  • Lorentz-invariant approach (MHENS scheme)

• Relation between lattice and continuum TMDs

• First lattice results
  • Collins-Soper kernel for TMD evolution
  • Soft function
Lorentz-invariant TMD variables

Collins/LR/MHENS beam function

\[ B_{q/h}(x, \vec{b}_T, \epsilon, \eta, \ldots) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \Omega_{q/h}^{[\gamma^+]} = \int \frac{d(P \cdot b)}{2\pi} e^{-ix(P\cdot b)} \Phi_{q/h}(P \cdot b, b^2, \eta^2v^2, \ldots) \]

Quasi-/MHENS beam function

\[ \tilde{B}_{q/h}(x, \vec{b}_T, \epsilon, \tilde{\eta}, \ldots) = N_{\tilde{\Gamma}} \int \frac{d\tilde{b}^z}{2\pi} e^{ib^z(x\tilde{P}^z)} \Omega_{q/h}^{[\tilde{\Gamma}]} = \int \frac{d(\tilde{P} \cdot \tilde{b})}{2\pi} e^{-ix(\tilde{P}\cdot \tilde{b})} \Phi_{q/h}(\tilde{P} \cdot \tilde{b}, \tilde{b}^2, \tilde{\eta}^2\tilde{v}^2, \ldots) \]

- 10 Lorentz invariant scalars;
- Reduces to 6 when \( \delta = 0 \) in the MHENS scheme.
# Lorentz-invariant TMD variables

$$\sinh(y_P - y_B) = \sinh y_\tilde{p}$$

$$\Rightarrow y_\tilde{p} = y_P - y_B$$

$$y_B \to -\infty \Rightarrow y_\tilde{p} \to -\infty$$

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Relating LR and quasi-TMDs

- Collins/LR and quasi-beam functions are in the same class of correlators if

\[ y_P = y_P - y_B \]

- So we can define the quasi-TMD with the Collins soft function

\[
\tilde{f}_{q_i/h}(x, \tilde{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = \int \frac{d(\tilde{P} \cdot \tilde{b})}{2\pi} e^{-ix(\tilde{P} \cdot \tilde{b})} \lim_{\epsilon \to 0} Z^q_{uv}(\mu, \epsilon, y_n - y_B) \frac{\Omega_{q_i/h}(\tilde{b}, \tilde{P}, \epsilon, \tilde{\eta} \tilde{z}, \tilde{b}^z \tilde{z})}{\sqrt{\tilde{S}^q(b_T, \epsilon, \tilde{\eta}, 2y_n, 2y_B)}}
\]

\[ \tilde{\zeta} = x^2 m^2_{h} e^{2y_P + 2y_B - 2y_n} \quad \zeta \equiv x^2 m^2_{h} e^{2y_P - 2y_n} \]

Therefore,

\[
\lim_{y_B \to -1} \tilde{f}_{q_i/h}(x, \tilde{b}_T, \mu, \tilde{\eta}, \tilde{\zeta}, x\tilde{P}^z) = \lim_{y_B \to -1} f^{LR}_{q_i/h}(x, \tilde{b}_T, \mu, -\tilde{\eta} e^{-y_B}, \tilde{\zeta}, y_P - y_B)
\]
Relating LR and Collins TMDs

**Collins scheme:**
\[ f_{ilh}^C(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} \lim_{y_B \to -\infty} Z_{UV} \frac{B_{ilh}}{\sqrt{S_C}} \]

**LR scheme:**
\[ f_{ilh}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) = \lim_{-y_B \gg 1} \lim_{\epsilon \to 0} Z'_{UV} \frac{B_{ilh}}{\sqrt{S_C}} \]

- Large \(-y_B\) corresponds to a hard momentum scale \(\zeta_{LR} = 4x^2M^2 \sinh(y_P - y_B)\).
- Exchange of \(\epsilon \to 0\) and \(\zeta_{LR} \to \infty\) should not affect the infrared physics, so the difference between the orders of limits is compensated by perturbative matching!

\[ f_{ilh}^C(x, \vec{b}_T, \mu, \zeta) = C^{-1} \left( \frac{\zeta_{LR}}{\mu^2} \right) f_{ilh}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) + \mathcal{O}(y_B^{-k}e^{y_B}) \]

**Verified at 1-loop ✔**

- “LaMET”, Ji, PRL 110 (2013); SCPMA57 (2014); Ji, Liu, Liu, Zhang and YZ, RMP 93 (2021);
- Collins, 2011 book, Ch. 10, on Sudakov form factors.
Relating LR and Collins TMDs

Collins scheme:

\[ f^C_{ih}(x, \vec{b}_T, \mu, \zeta) = \left( \frac{\zeta}{\mu^2} \right) f^LR_{ih}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) + \mathcal{O}(y_B^{-1}) \]

LR scheme:

\[ f^LR_{ih}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) \]

The factor \( \tilde{Z}_A \) is to be adjusted so that we get the same results as in (10.94a). Now the non-uniformity of the limits \( n \to 4 \) and of infinite Wilson-line rapidities only concerns the limit of infinitely large transverse momentum; for \( n < 4 \), the limits can be exchanged. Thus the factor \( \tilde{Z}_A \) is a pure UV factor, and can be regarded as a kind of generalized UV renormalization factor, chosen to make a renormalization prescription that agrees with the combination of \( \overline{MS} \) renormalization and the opposite order of the limits.

- Large-\( \beta \) corresponds to a hard momentum scale.
- Exchange of \( A \) and \( B \) should not affect the infrared physics, so the difference between the orders of limits is compensated by perturbative matching.

\[ \zeta_{LR} = 4 \times \frac{2 \sinh(y_P - y_B)}{\sinh^2 \mu^2} \]

If we reversed the limits, we would need to compensate by an extra hard factor, e.g.,

\[ A_{\text{basic}} = \lim_{y_{u_2} \to -\infty} A_{\text{unsub}}(y_P - y_{u_2}) S(y_1 - y_{u_2}) \]

Verified at 1-loop ✔
Relating LR and Collins TMDs

Collins scheme:  
\[ f_{ilh}^C(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{UV} \lim_{y_B \to -\infty} \frac{B_{ilh}}{\sqrt{S_C}} \]

LR scheme:  
\[ f_{ilh}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) = \lim_{-y_B \gg 1} \lim_{\epsilon \to 0} Z_{UV} \frac{B_{ilh}}{\sqrt{S_C}} \]

• Large \(-y_B\) corresponds to a hard momentum scale \(\zeta_{LR} = 4x^2M^2 \sinh(y_P - y_B)\).

• Exchange of \(\epsilon \to 0\) and \(\zeta_{LR} \to \infty\) should not affect the infrared physics, so the difference between the orders of limits is compensated by perturbative matching!

\[ f_{ilh}^C(x, \vec{b}_T, \mu, \zeta) = C^{-1}\left(\frac{\zeta_{LR}}{\mu^2}\right) f_{ilh}^{LR}(x, \vec{b}_T, \mu, \zeta, y_P - y_B) + \mathcal{O}(y_B^{-k}e^{y_B}) \]

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- Collins, 2011 book, Ch. 10, on Sudakov form factors.

Verified at 1-loop ✔
Matching between quasi- and Collins TMDs

\[ \lim_{\tilde{\eta} \to \infty} \tilde{f}_{q/h}(x, \vec{b}_T, \mu, \zeta, x\tilde{P}^z, \tilde{\eta}) = C \left( \frac{\tilde{\zeta}}{\mu^2} \right) f_{q/h}^C(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(y_{\tilde{P}}^{-k} e^{-y_{\tilde{P}}}) \]

Moreover,

\[ \tilde{\zeta} = (2x\tilde{P}^z)^2 = \zeta_{LR} \]

Not directly calculable on the lattice

\[ \tilde{f}_{q/h} = \frac{B_{q/h}}{\sqrt{S_C(b_T, \mu, \tilde{\eta}, 2y_n, 2y_B)}} \]

Naive quasi soft function, lattice calculable

- Reduced soft function: \( S_r(b_T, \mu) = [g_S^q(b_T, \mu)]^2 \)
- Methods for calculation has been proposed and explored on the lattice.

- Ji, Liu and Liu, NPB 955 (2020);
- Q.-A. Zhang, et al. (LP Collaboration), PRL 125 (2020);
- Y. Li et al., PRL 128 (2022).
Factorization formulas

\[ \lim_{\tilde{\eta} \to \infty} \tilde{f}_{q/h}(x, \vec{b}_T, \mu, \tilde{\eta}, \zeta, x\bar{P}^z) = C\left(\frac{\tilde{\zeta}}{\mu^2}\right) f_{i/h}^C(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(y_P^{-k} e^{-y_P}) \]

\[ \frac{f_{i/h}^{\text{naive}}}{g_s^q(b_T, \mu)} = C\left(\frac{\tilde{\zeta}}{\mu^2}\right) e^{\frac{1}{2} \gamma^q_{b_T, \mu} \ln \frac{\tilde{\zeta}}{\tilde{\eta}}} f_{q/h}^C(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(y_P^{-k} e^{-y_P}) \]

\[ \mathcal{O}\left(\frac{b_T}{\tilde{\eta}}, \frac{1}{(xb_T\bar{P}^z)^2}, \frac{1}{\bar{P}^z\tilde{\eta}}, \frac{\Lambda_{QCD}^2}{(x\bar{P}^z)^2}\right) \]

- M. Ebert, I. Stewart and YZ, PRD99 (2019), JHEP09 (2019);
- A. Vladimirov and A. Schäfer, PRD 101 (2020);
- Ebert, Schindler, Stewart and YZ, 2201.08401.
Implications

• Same proof applies for the gluon quasi TMD and for all spin-dependent quasi TMDs;

Both gluon and singlet quark TMDs are calculable on the lattice!

• No mixing between quarks of different flavors, quark and gluon channels, or different spin structures;

Verified at 1-loop ✔

• The $P^z$-evolution;

Calculation of gluon TMDs easier than anticipated!

$$\frac{d}{d \ln(2x\tilde{P}^z)} \ln \lim_{\tilde{\eta} \gg b_T} B^{[\Gamma]}_{q/h}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z) = \gamma^q_{\zeta}(b_T, \mu) + \gamma^q_C(2x\tilde{P}^z, \mu)$$

Perturbative

Lattice calculation of the Collins-Soper kernel:

• Ji, Sun, Xiong and Yuan, PRD91 (2015);
• M. Ebert, I. Stewart, YZ, PRD99 (2019).

NLL resummation in the Wilson coefficient:

$$\gamma^q_{\zeta}(2x\tilde{P}^z, \mu) = \frac{d}{d \ln(2x\tilde{P}^z)} \ln C_{q}(x\tilde{P}^z, \mu)$$

• Ebert, Schindler, Stewart and YZ, 2201.08401.
Implications

- Ratios of TMDs and their $x$-moments should be calculated in the $x$-space;

$$
\lim_{\tilde{\eta} \to \infty} \frac{\tilde{B}_{q_i/h}^{[\tilde{\Gamma}_1]}(x, \tilde{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)}{\tilde{B}_{q_j/h'}^{[\tilde{\Gamma}_2]}(x, \tilde{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)} = \frac{f_{q_i/h}^{[\Gamma_1]}(x, \tilde{b}_T, \mu, \zeta)}{f_{q_j/h'}^{[\Gamma_2]}(x, \tilde{b}_T, \mu, \zeta)}
$$

- Factorization for the MHENS TMD.

Additional challenges beyond tree-level renormalization/matching:

- $b^z$-dependent renormalization
  
  Linear: $\propto (2 |\eta v| + \sqrt{\tilde{b}_T^2 + b^z_2})/a$
  
  Cusp: $\propto [3 - \frac{2\tilde{b}^z}{b_T} \tan^{-1} \frac{b_T}{\tilde{b}^z}] \ln(a)$

- $b^z$-dependent soft function?

- Ratios of the $x$-moments of TMDs can be calculated with MHENS beam functions at $\tilde{b}^z = 0$ with tree-level matching;

- With proper lattice renormalization and soft function subtraction, the MHENS scheme should be equivalent to the LR scheme.
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• First lattice results
  • Collins-Soper kernel for TMD evolution
  • Soft function
### Collins-Soper kernel

**Results by different groups with different systematics.**

- **SWZ20**, P. Shanahan, M. Wagman and **YZ**, PRD 102 (2020);
- **Regensburg/NMSU21**, Schlemmer, Vladimirov, Zimmerman, Engelhardt, Schäfer, JHEP 08 (2021);
- **LPC20**, Q.-A. Zhang, et al. (LPC), PRL 125 (2020);

**Comparison with phenomenology**

- **SV19**: I. Scimemi and A. Vladimirov, JHEP 06 (2020) 137
- **Pavia19**: A. Bacchetta et al., JHEP 07 (2020) 117

(Green points), P. Shanahan, M. Wagman and **YZ**, PRD 104 (2021).
Reduced soft function

With the quasi beam function, soft function, and Collins-Soper kernel calculable, we can obtain the full TMD from lattice QCD!
Conclusion

• New large-rapidity (LR) scheme;

• The quasi TMD is equivalent to the LR scheme through Lorentz invariance;

• The LR and Collins schemes differ by the order of UV renormalization and light-cone limits, so we can perturbatively match them;

• We derive the factorization formula for both quark and gluon quasi TMDs using such relations;

• There is no mixing between quarks of different flavors, quark and gluon channels, or different spin structures.
Drell-Yan production of lepton pair

Collinear factorization:

\[
\frac{d\sigma_{DY}}{dQ^2dY} = \sum_{i,j} \int_{x_a}^1 dx_a \int_{x_b}^1 dx_b \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2dY} \left[ 1 + \mathcal{O} \left( \frac{\Lambda_{QCD}^2}{Q^2} \right) \right]
\]
Collinear factorization:

$$\frac{d\sigma_{\text{DY}}}{dQ^2dY} = \sum_{i,j} \int_{x_a}^1 d\xi_a \int_{x_b}^1 d\xi_b \ f_{ilh_A}(\xi_a) f_{ilh_A}(\xi_b) \ \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2dY} \left[ 1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right) \right]$$
Ji-Ma-Yuan (JMY) scheme

\[ b^\mu = (0, b^-, b_\perp) \]
\[ \delta^\mu = (0, b^-, 0_\perp) \]
\[ \nu^\mu = (\nu^+, \nu^-, 0_\perp), \quad \nu^- \gg \nu^+ > 0 \]
\[ \tilde{\nu}^\mu = (\tilde{\nu}^+, \tilde{\nu}^-, 0_\perp), \quad \tilde{\nu}^+ \gg \tilde{\nu}^- > 0 \]
\[ | \eta | \rightarrow \infty \]

\[ B_{q/h}^{JMY}(x, \vec{b}_T, \mu, \zeta_\nu) = \int \frac{db^-}{2\pi} e^{-ib^- (xP^+)} \Phi_{q/h}[b, P, \mu, -\infty \nu, b^- n_b] \]
\[ S_{JMY}^R(b_T, \mu, y_A, y_B) = S^R[b_\perp, \mu, -\infty \nu, -\infty \tilde{\nu}] \]

**TMDPDF:**
\[ f_{i/lh}^{JMY}(x, \vec{b}_T, \mu, \zeta_\nu, \rho) = \frac{B_{i/lh}^{JMY}(x, \vec{b}_T, \mu, \zeta_\nu)}{\sqrt{S_{JMY}^R(b_T, \mu, \rho)}} \]
\[ \rho^2 = \frac{\nu^- \tilde{\nu}^+}{\nu^+ \tilde{\nu}^-} \]
\[ \zeta_\nu^2 = \frac{(2P \cdot \nu)^2}{v^2} = 2(P^+)^2 \frac{v^-}{v^+} \]

Analytically continuable from the LR scheme (y_P-y_B to q).