A novel approach to the global analysis of proton form factors in elastic *ep* scattering

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†in collaboration with Craig McRae (MSc thesis)

C. McRae and PGB, PRC **109**, 015503 (2024)

Generic problem: In a Global Analysis, how to combine data sets from different experiments with different overall normalization (multiplicative) uncertainties?

A simple example   Barlow, NIMPR **A 987**, 164864 (2021)

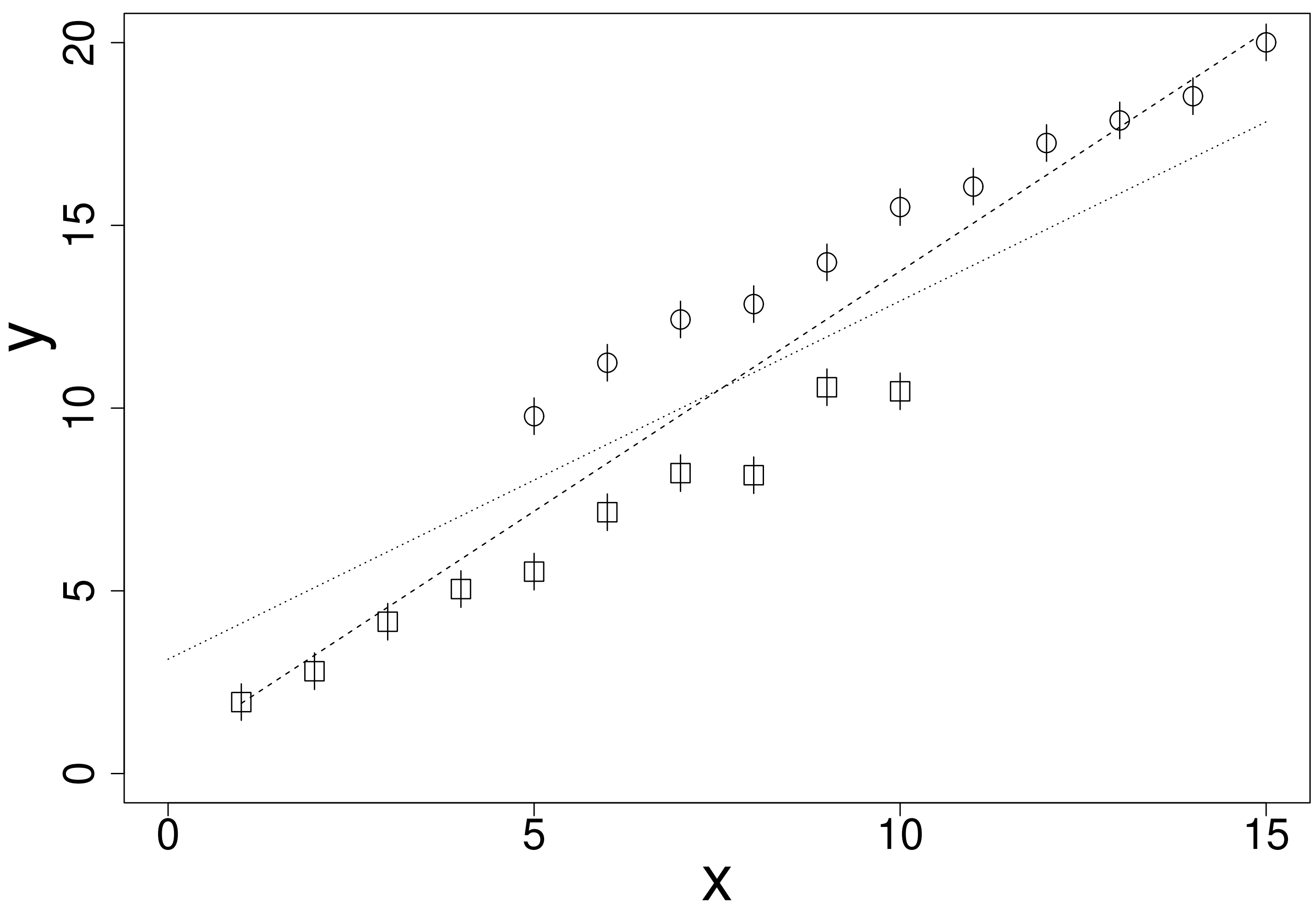
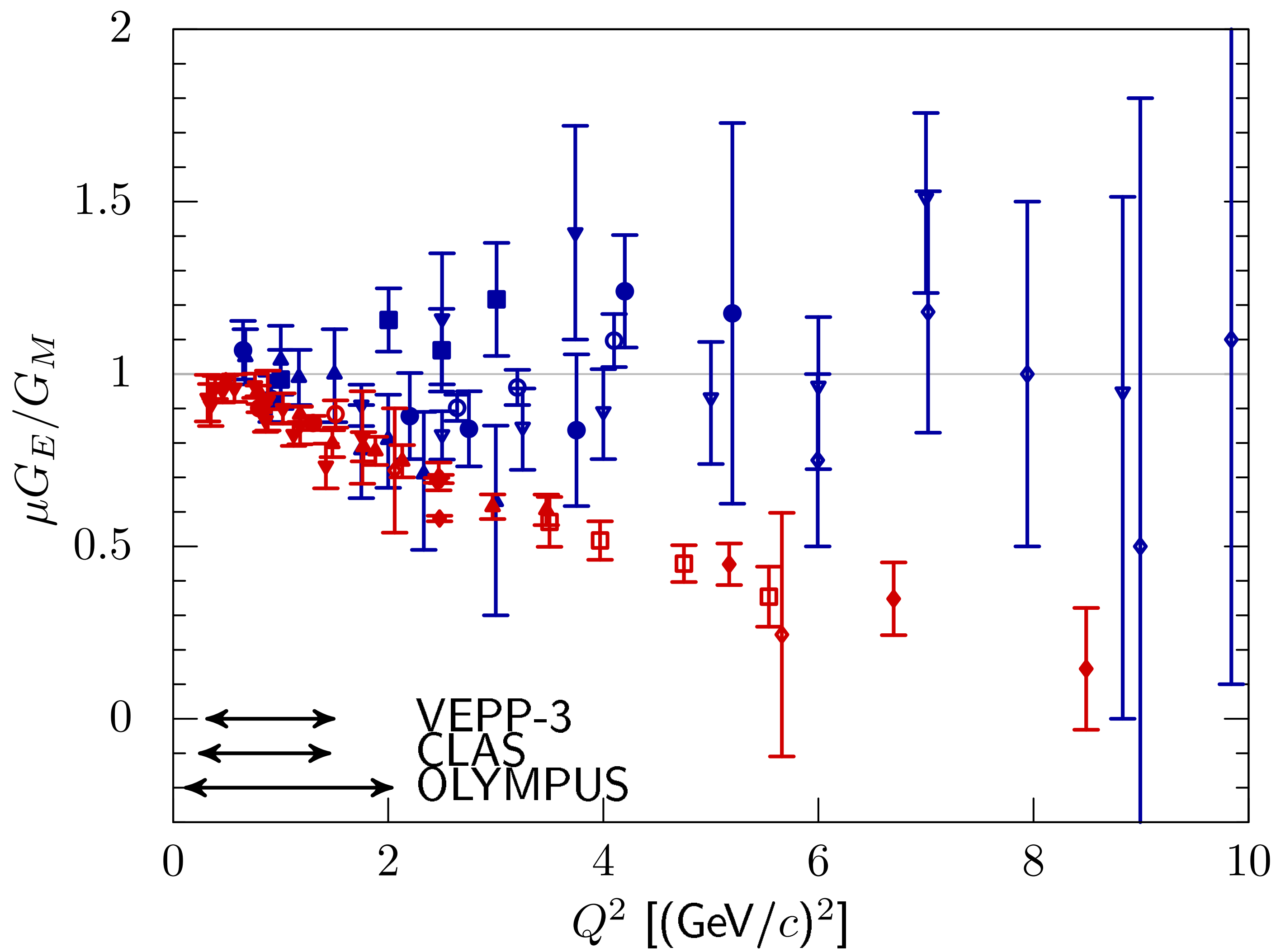


Fig. 1. Data from two different experiments.

- Form factors and the G_E/G_M ratio discrepancy



"Rosenbluth" separation (LT)

$$\frac{1}{\tau}\sigma_{\mathrm{red}} = G_M^2 + \frac{\varepsilon}{\tau} G_E^2$$

Polarization Transfer (PT)

Two Photon Exchange (TPE) leading candidate to explain the discrepancy

- Proton radius discrepancy: muonic H vs electron scattering

$$\langle r^2 \rangle = -6 \frac{dG_E(Q^2)}{dQ^2}\bigg|_{Q^2 \to 0}$$

Outline

- Conventional approach: "Penalty Trick"
- "t_0 method" of NNPDF collaboration Ball *et al.*, JHEP **05**, 075 (2010)
- Modification for multiple kinematics ("IMF method") us
- Global fits and extraction of $G_E(Q^2)$ and $G_M(Q^2)$
  - IMF vs penalty trick (linear model): reanalysis of SLAC data (48 points) for $Q^2 > 1\ \mathrm{GeV}^2$ by Gramolin & Nikolenko, PRC **93**, 055201 (2016)
  - IMF vs penalty trick (nonlinear model): SLAC + other + new data (121 points) by Gmp12 collaboration Christy *et al.*, PRL **128**, 102002 (2022)
- The importance of correlations
- Inclusion of TPE in reconciling G_E/G_M discrepancy

Penalty Trick

- $\mathcal{N}$ experiments labelled $r = 1, 2 \ldots \mathcal{N}$ (indices $r, s, t \ldots$)
- data $y_{r,j}$ for each experiment, labelled $j = 1, 2 \ldots N_r$ (indices $i, j, k, \ldots$) has both additive ($\Delta y_{r,j}$) and overall multiplicative (Δn_r) uncertainties
- model $\mathrm{M} = \sigma_{\mathrm{red}} = \tau G_M^2(Q^2) + \varepsilon G_E^2(Q^2)$; $\tau \equiv Q^2/(4M^2)$, $0 < \varepsilon < 1$
- normalization parameters n_r assumed to be normally distributed around 1

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Find model parameters plus the normalization parameters by minimizing chi-square

$$\chi_{\mathrm{p}}^2 = \sum_{r=1}^{\mathcal{N}} \left[\frac{(n_r - 1)^2}{(\Delta n_r)^2} + \sum_{j=1}^{N_r} \frac{(n_r y_{r,j} - \mathrm{M}_{r,j})^2}{(\Delta y_{r,j})^2} \right]$$

Updated penalty trick scales uncertainties as well (avoids D'Agostini bias)

$$\chi_{\mathrm{p}}^2 = \sum_{r=1}^{\mathcal{N}} \left[\frac{(n_r - 1)^2}{(\Delta n_r)^2} + \sum_{j=1}^{N_r} \frac{(y_{r,j} - \mathrm{M}_{r,j}/n_r)^2}{(\Delta y_{r,j})^2} \right]$$

Likelihood function $\exp(-\chi_p^2/2)$ no longer Gaussian in normalization parameters n_r

t_0 method (a more sound alternative?)

Seminal paper by NNPDF collaboration Ball *et al.*, JHEP **05**, 075 (2010)

- demonstrate the statistical biases in usual approach when combining multiple data sets
- construct a Monte Carlo method that accounts for normalization uncertainty without compromising the integrity of the fit

$$(y_{r,j} \pm \Delta y_{r,j})\,(1 \pm \Delta n_r) \longrightarrow y_{r,j} \pm \sqrt{(\Delta y_{r,j})^2 + (y_{r,j}\Delta n_r)^2}$$

» Build a chi-square for model M (likelihood no longer Gaussian in the data)

$$\chi^2 = \sum_{r=1}^{\mathcal{N}} \left[\sum_{j=1}^{N_r} \frac{(y_{r,j} - \mathrm{M}_{r,j})^2}{(\Delta y_{r,j})^2 + (y_{r,j}\Delta n_r)^2} \right]$$

» Use model as a stand-in for data in the denominator (likelihood not Gaussian in the model parameters)

$$\chi^2 = \sum_{r=1}^{\mathcal{N}} \left[\sum_{j=1}^{N_r} \frac{(y_{r,j} - \mathrm{M}_{r,j})^2}{(\Delta y_{r,j})^2 + (\mathrm{M}_{r,j}\,\Delta n_r)^2} \right]$$

» Replace model by "best guess" $\hat{\mathrm{M}}$ (achieved by iterating until model converges)

$$\chi^2 = \sum_{r=1}^{\mathcal{N}} \left[\sum_{j=1}^{N_r} \frac{(y_{r,j} - \mathrm{M}_{r,j})^2}{(\Delta y_{r,j})^2 + \left(\hat{\mathrm{M}}_{r,j}\Delta n_r\right)^2} \right]$$

Iterated Model Fit (IMF) method

Generalize the method to allow for multiple kinematics, using full covariance matrix

$$\mathrm{cov}_{r,jk} \;\longrightarrow\; \mathrm{cov}^{\mathrm{IMF}}_{r,jk} = \mathrm{cov}_{r,jk} + (\Delta n_r)^2\, \hat{\mathrm{M}}_{r,j}\, \hat{\mathrm{M}}_{r,k}$$

Monte Carlo procedure:

- For each datum generate replica data $y^R_{r,j}$ from a normal distribution around the point's central value and uncertainty
- Generate a normalization factor n^R_r around 1 with uncertainty Δn^R_r
- The final chi-square to minimize is given in matrix form as

$$\chi^2_{\mathrm{IMF}} = \sum_{r=1}^{\mathcal{N}} \left(n^R_r y^R_r - \mathrm{M}_r\right)^{\mathsf{T}} (\mathrm{cov}^{\mathrm{IMF}}_r)^{-1} \left(n^R_r y^R_r - \mathrm{M}_r\right)$$

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- For $\mathcal{R}$ replica's (we use $\mathcal{R} = 1,000$) we have $\mathcal{R}$ best-fit models and $\mathcal{R}$ sets of best-fit parameters
- These fits are averaged together to produce an updated best-fit model $\hat{\mathrm{M}}$, which is then used to update $\mathrm{cov}^{\mathrm{IMF}}_r$
- Iterate until convergence

Form factors and the G_E/G_M ratio

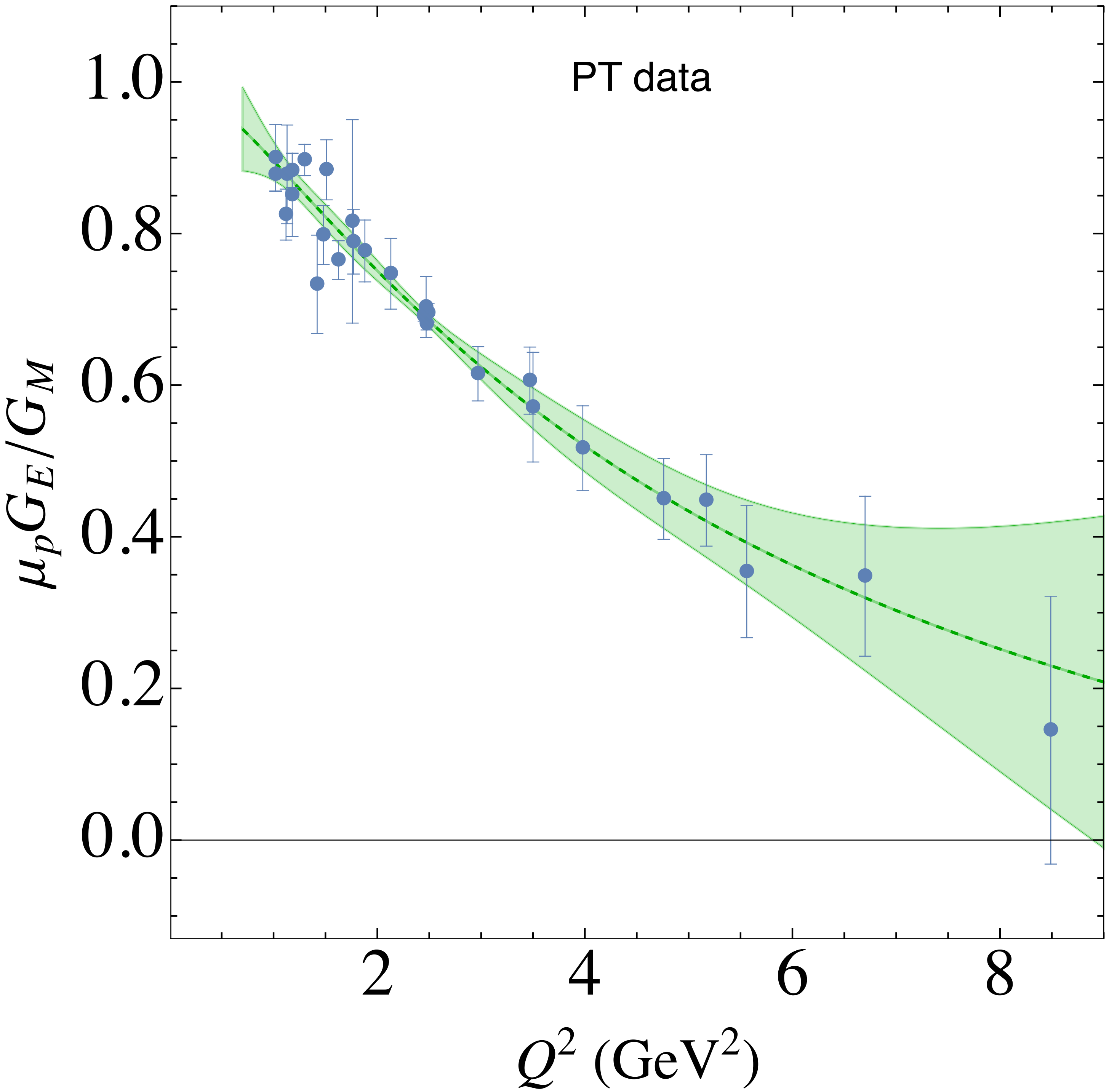
SLAC E140/NE11 LT: Walker *et al*, PRD **49**, 5671 (1994); Andivahis *et al*, PRD **50**, 5491 (1994)

Super Rosenbluth LT: Qattan *et al*, PRL **94**, 142301 (2005)

Polarization Transfer (PT): (various)

$$1\ \mathrm{GeV}^2 \leq Q^2 \leq 8.83\ \mathrm{GeV}^2$$

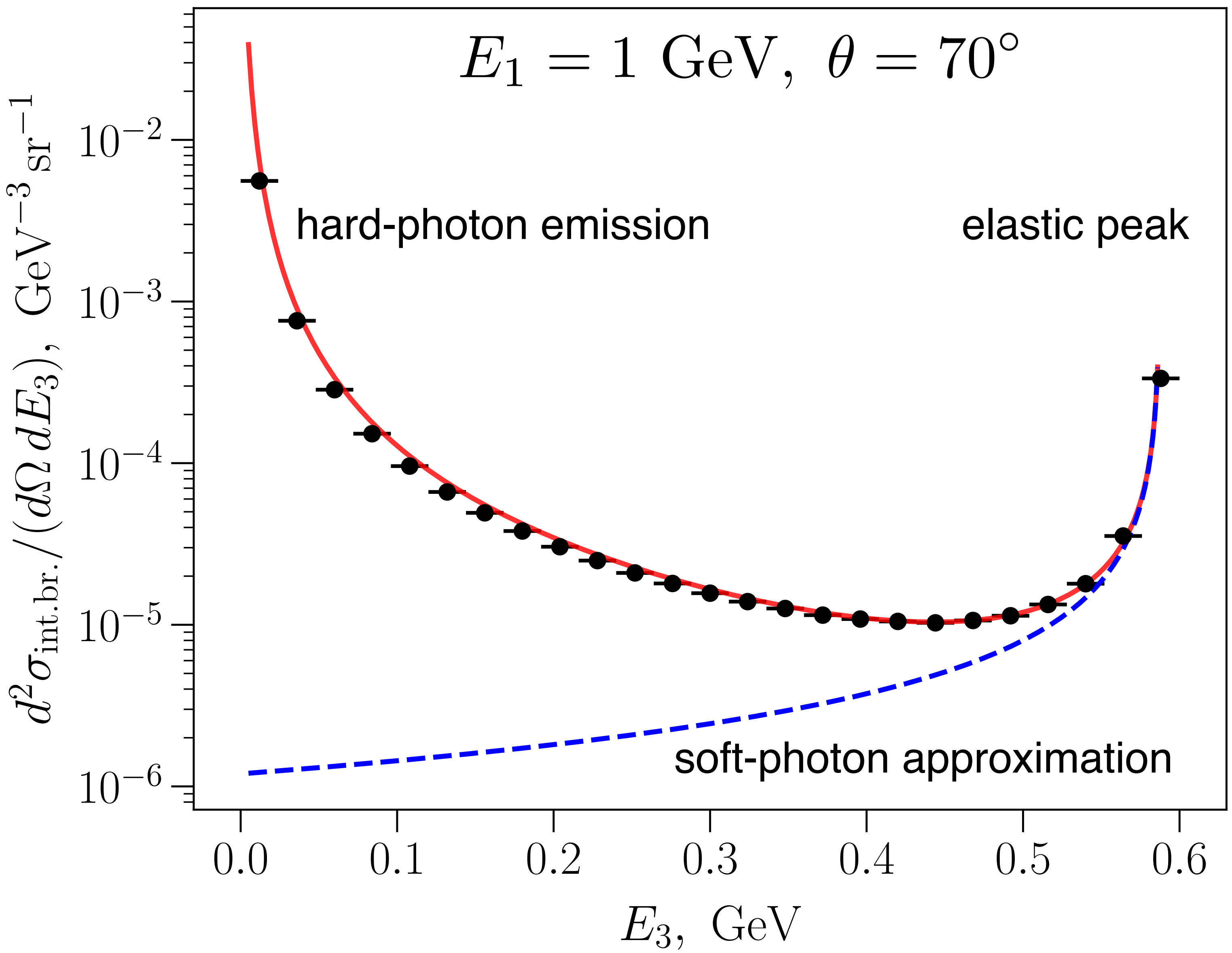
- To extract G_E and G_M from LT measurements we should correct the **data** for TPE at the same level as other RCs.
- SLAC: all details of RC are published
- Super Rosenbluth: no RC details are published, not even cross sections!



- Band is at 90% confidence interval

Radiative correction improvements: Gramolin & Nikolenko, PRC **93**, 055201 (2016)

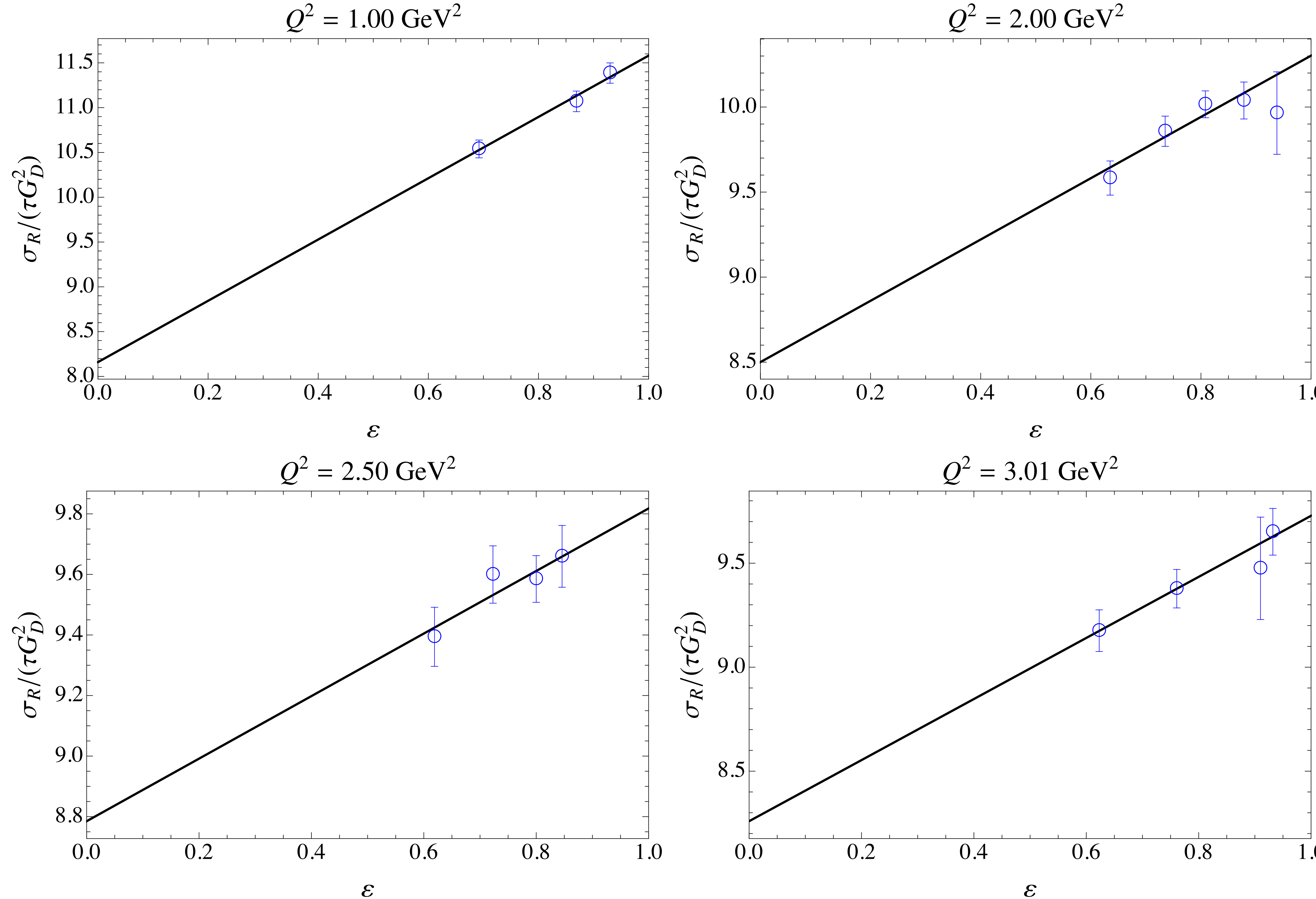
- Reanalyze old SLAC data (48 points)
- Use Maximon-Tjon instead of Mo-Tsai (no difference at order Z^0)
- Improvements to hard internal and external δ_{brem} bremsstrahlung
- Minor improvements to VP and ionization factors
- No inclusion of TPE



"Penalty trick" with linear model

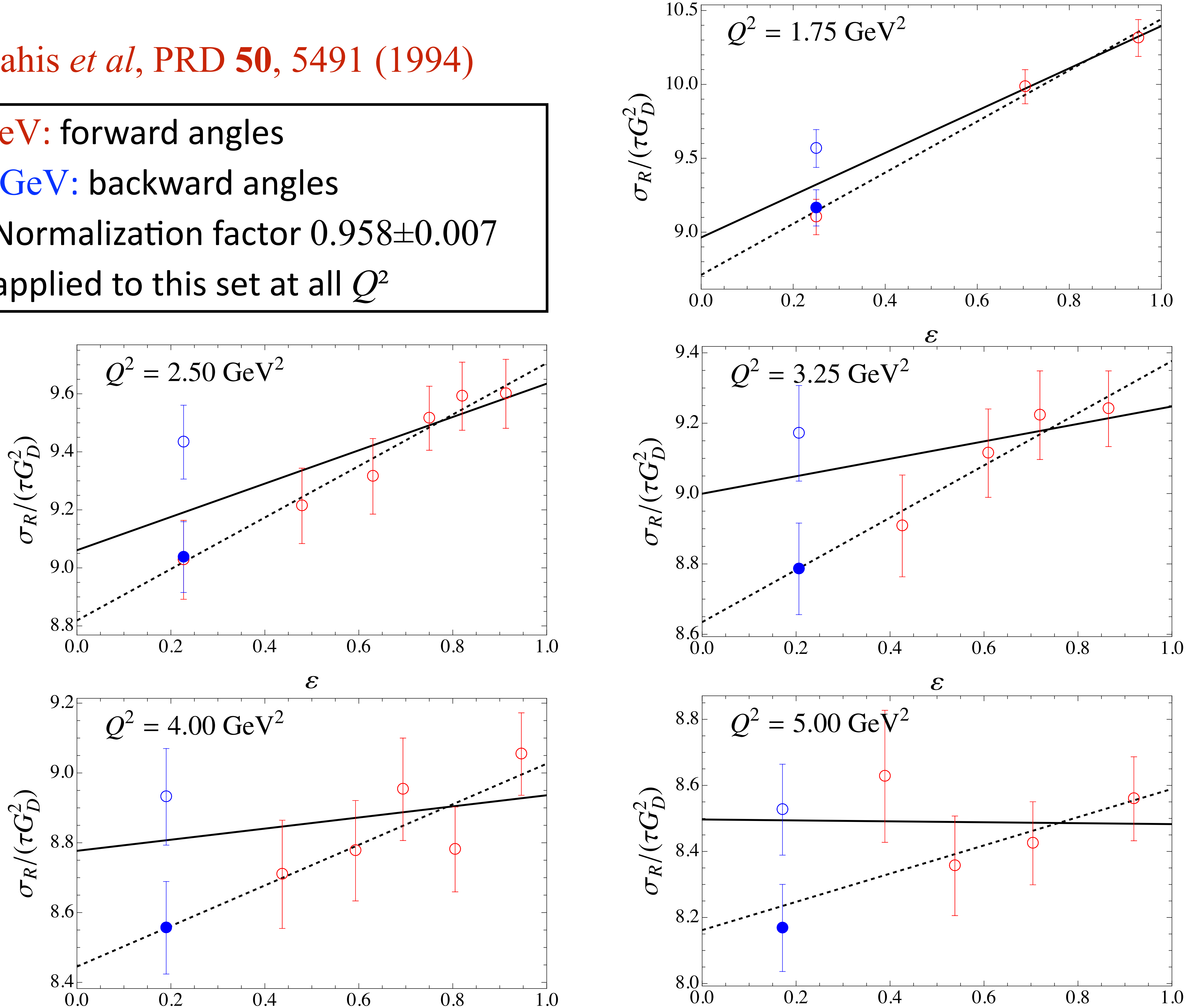
$$G_E^2(Q^2; \boldsymbol{a}) = \left(1 - a_1\tau - a_2\tau^2 - a_3\tau^3\right) G_{\text{dip}}^2$$
$$G_M^2(Q^2; \boldsymbol{b}) = \mu_p^2 \left(1 - b_1\tau - b_2\tau^2 - b_3\tau^3\right) G_{\text{dip}}^2$$

Walker: Reduced cross sections $\frac{1}{\tau}\sigma_{\rm red} = G_M^2 + \frac{\varepsilon}{\tau}G_E^2$



Andivahis *et al*, PRD **50**, 5491 (1994)

- 8 GeV: forward angles
- 1.6 GeV: backward angles
  - Normalization factor 0.958±0.007 applied to this set at all Q^2



IMF vs Penalty trick (linear model)

$$G_E^2(Q^2;\boldsymbol{a}) = \left(1 - a_1\tau - a_2\tau^2 - a_3\tau^3\right) G_{\rm dip}^2$$
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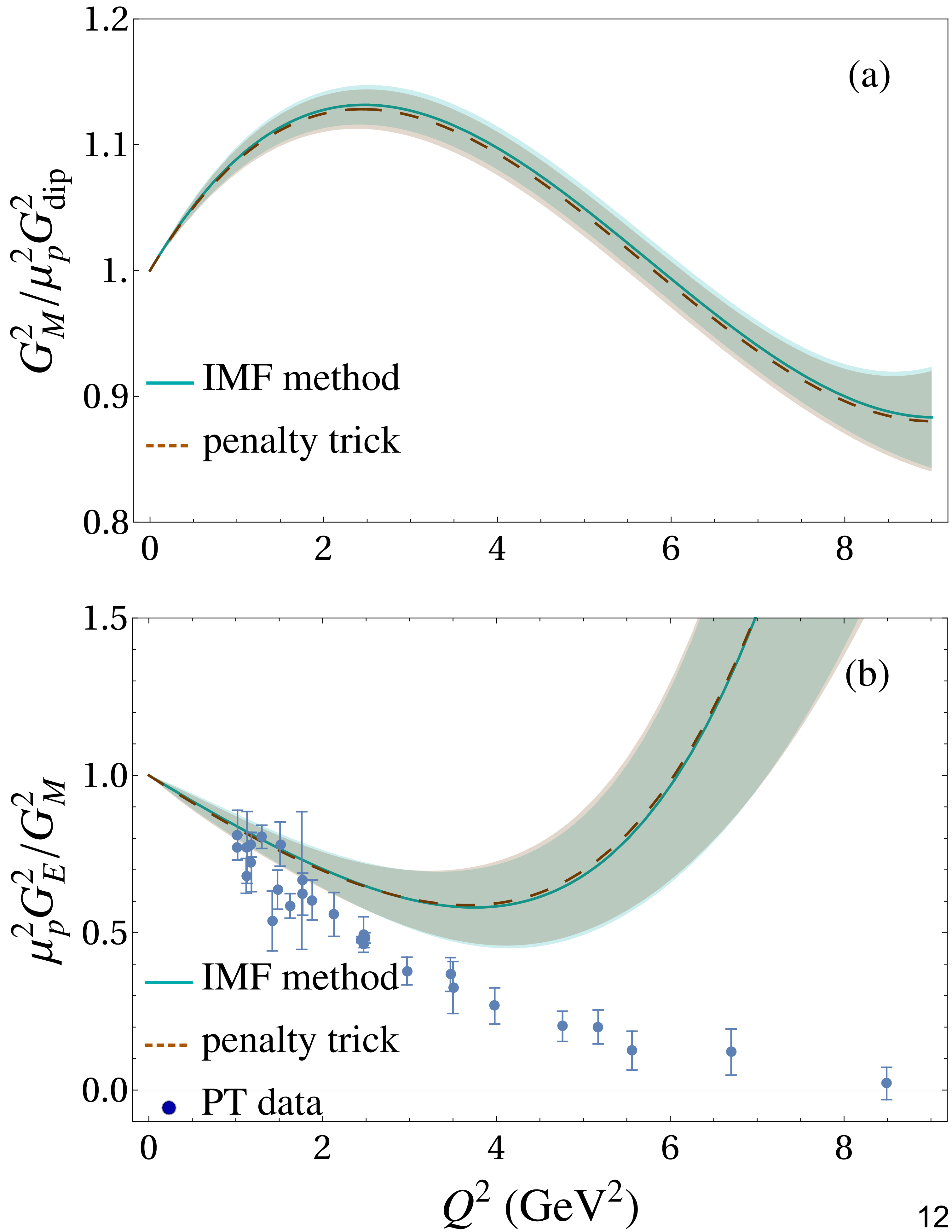
	Penalty trick	IMF
a_1	0.23 ± 0.22	0.17 ± 0.22
a_2	0.47 ± 0.44	0.55 ± 0.44
a_3	-0.35 ± 0.21	-0.38 ± 0.21
b_1	-0.407 ± 0.045	-0.413 ± 0.045
b_2	0.373 ± 0.045	0.375 ± 0.046
b_3	-0.076 ± 0.013	-0.077 ± 0.013
n_1	1.008 ± 0.012	–
n_2	1.008 ± 0.012	–
n_3	0.963 ± 0.013	–
χ^2/d.o.f.	28.3/39	28.1/42

1. Walker (16 points)
2. Andivahis 8 GeV (24 points)
3. Andivahis 1.6 GeV (8 points)



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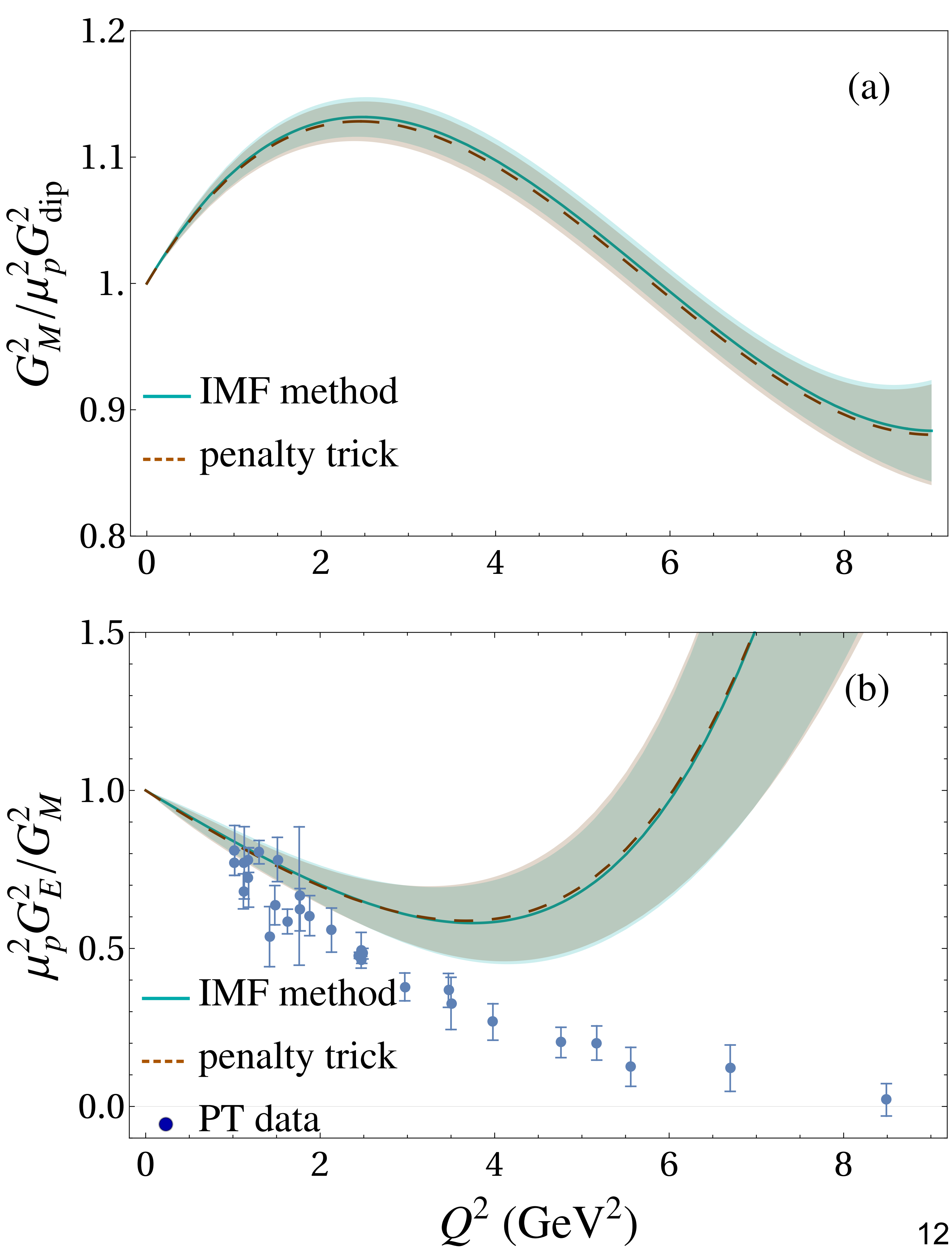

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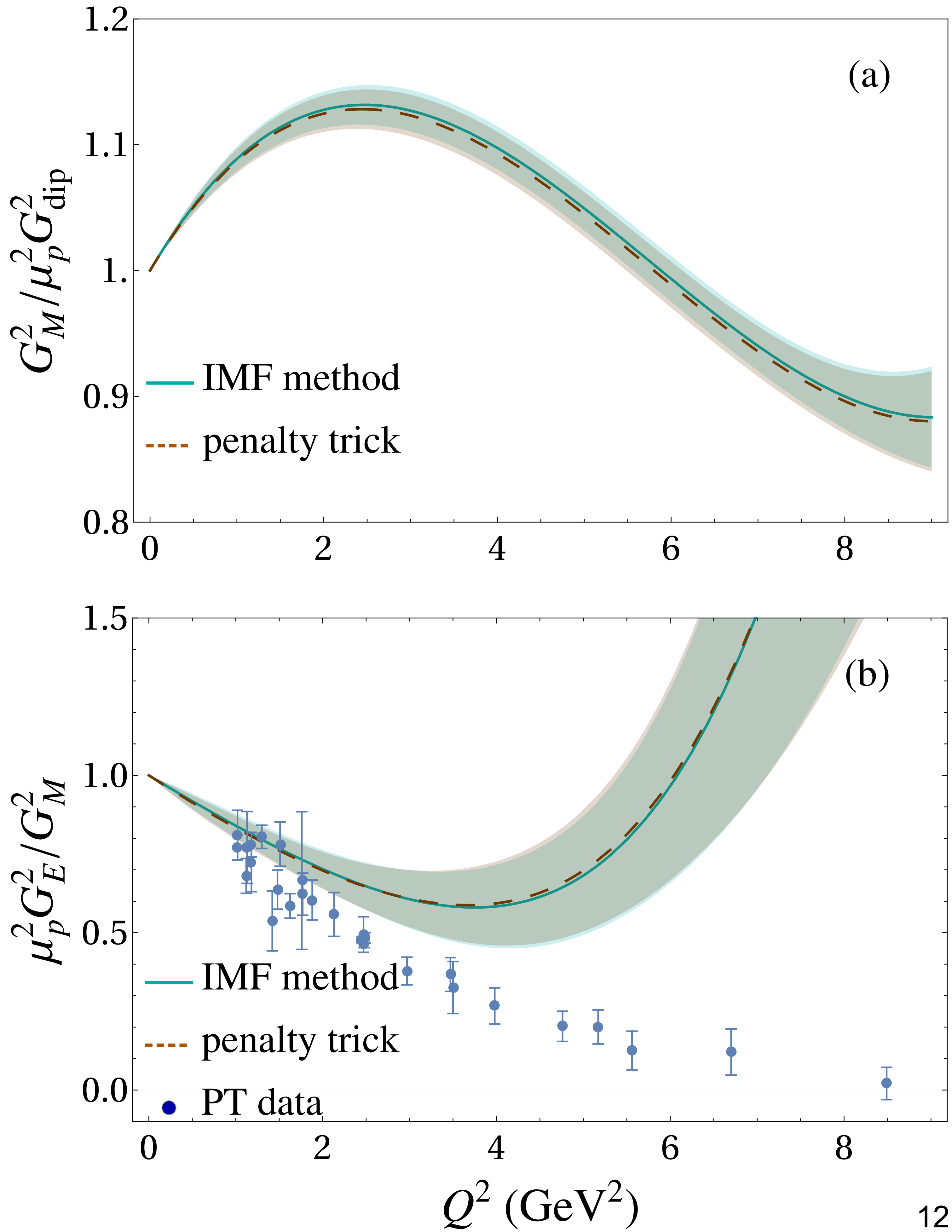
Achieves same quality fit without the need for explicit normalization parameters



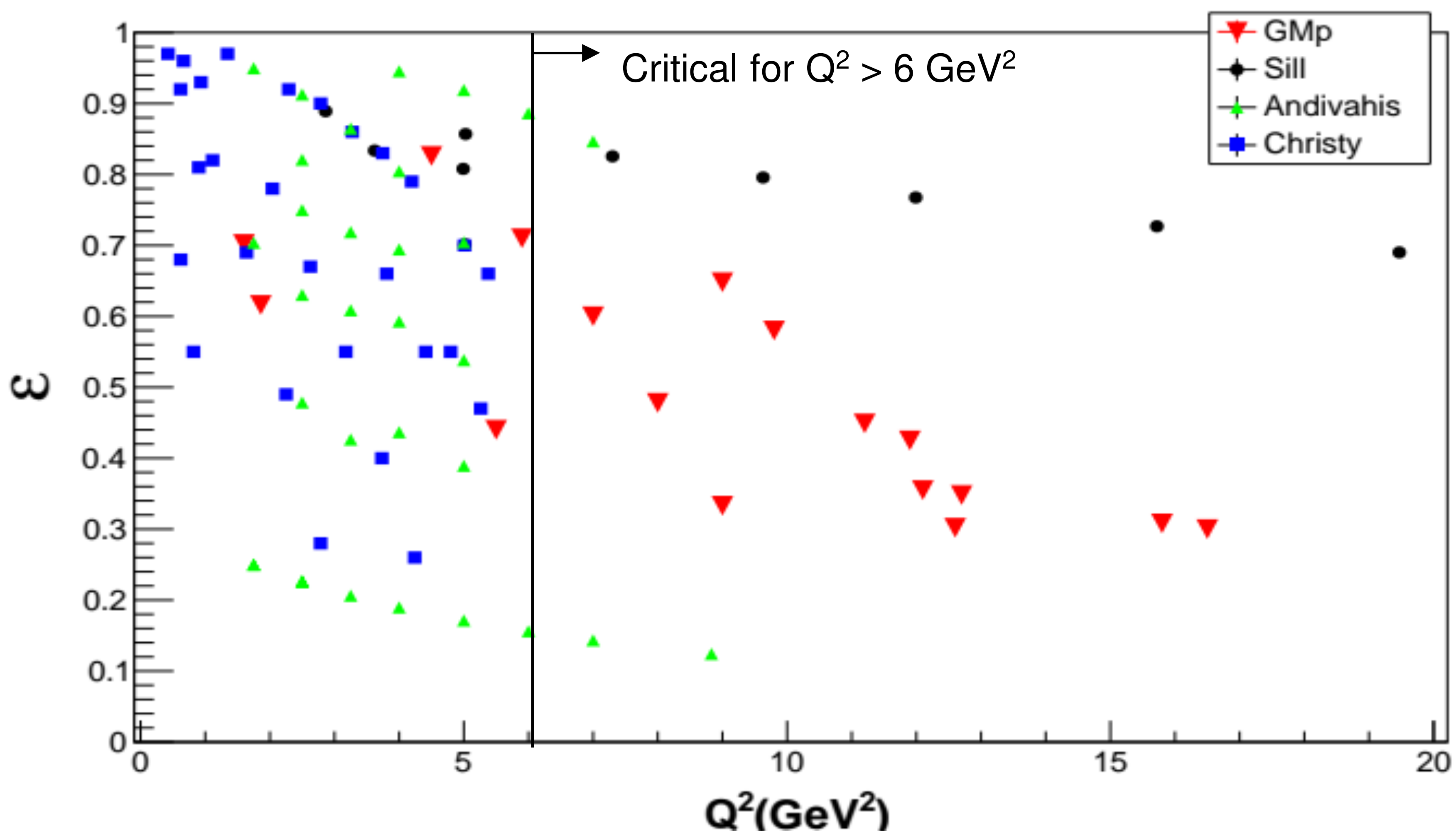
GMp12 experiment

Goals of GMp-12

Christy *et al.*, PRL **128**, 102002 (2022)

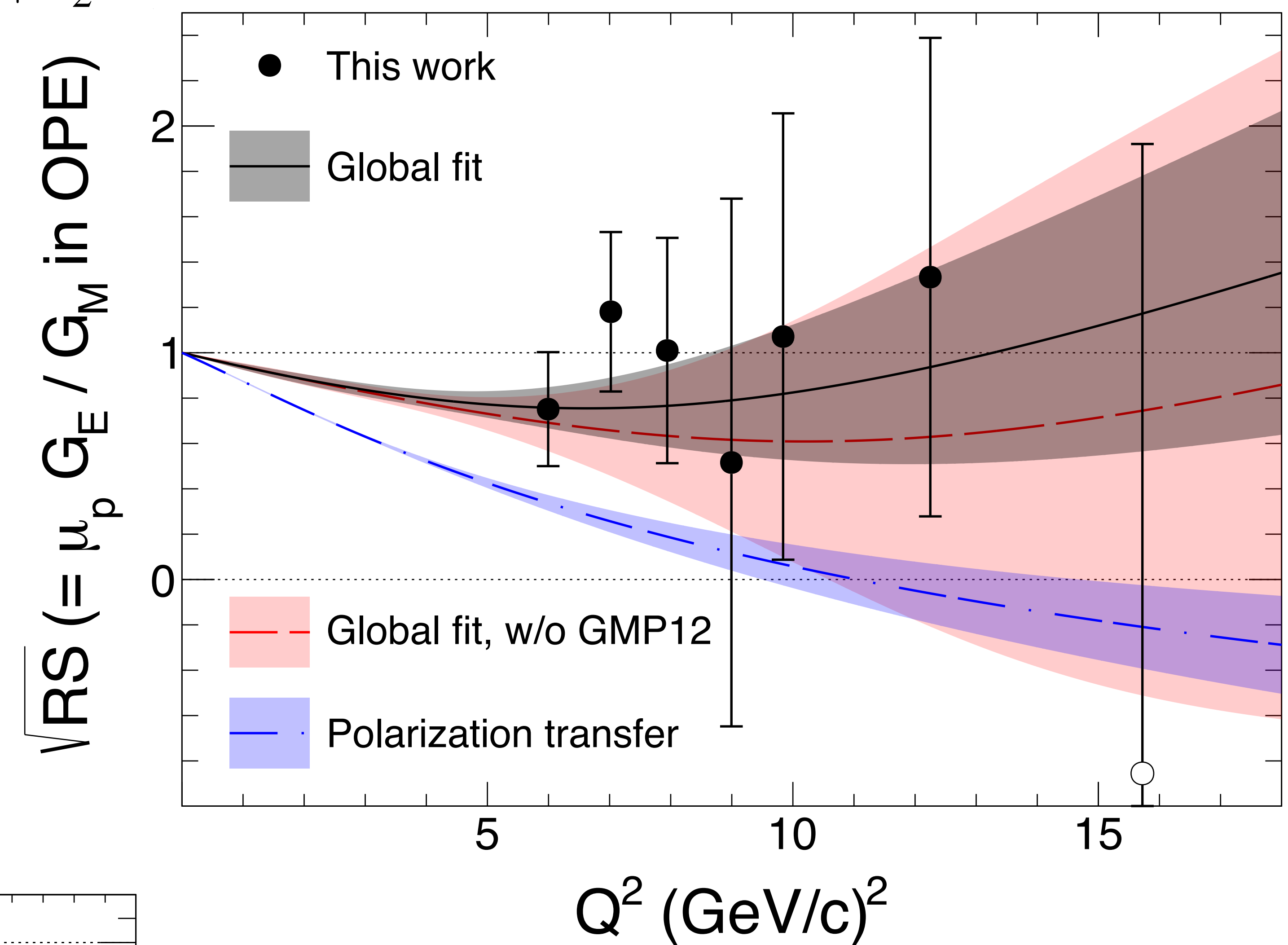
- Precise e-p elastic cross section over wide range of JLab-12 kinematics
- Expand measurements of TPE to higher Q^2 values
- Improved extraction of GMp at high Q^2

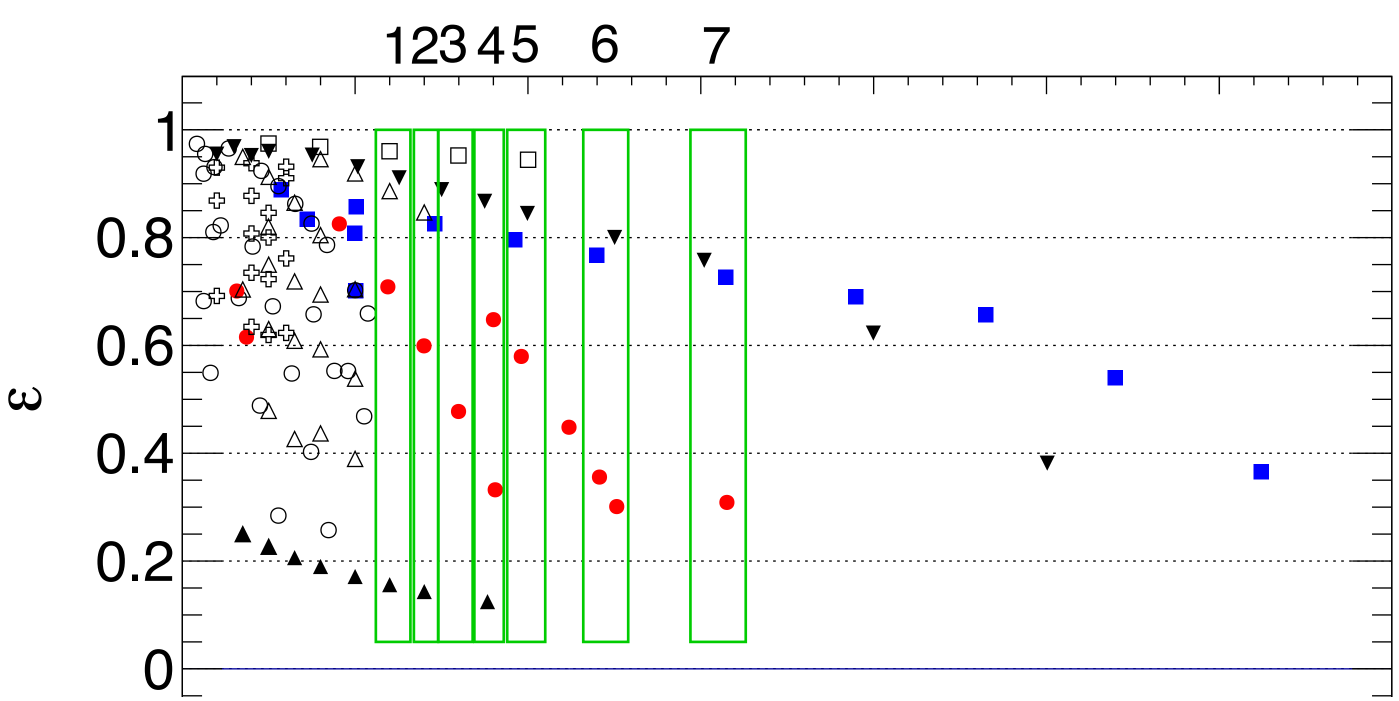
- Performed global analysis of new data plus 6 other experiments (121 points) with $Q^2 > 1\ \mathrm{GeV}^2$
- Apply updated radiative corrections to old data



$$G_M = \mu_p \frac{1+a_1\tau}{1+b_1\tau+b_2\tau^2+b_3\tau^3}; \quad RS = 1+c_1\tau+c_2\tau^2$$

Experiment	$n_i \pm \Delta n_i$	Scale uncertain
Kirk [7]	1.023 ± 0.008	4
Rock [8]	1.068 ± 0.006	3
Sill [9]	1.015 ± 0.014	3.6
Christy [6]	1 (fixed)	1.7
Andivahis (8 GeV) [3]	1.006 ± 0.004	1.77
Andivahis (1.6 GeV) [3]	0.960 ± 0.007	2.7
Walker [2]	1.008 ± 0.004	1.9
This work (LHRS)	0.982 ± 0.006	1.6
This work (RHRS)	1.006 ± 0.011	2





Black dots from direct LT separations (interpolate to a common Q^2)

Is this meaningful?

IMF method

Improved parametrization of form factors: z-expansion model

$$G_E\left(z;\boldsymbol{a}\right) = \sum_{k=0}^{k_{\max}} a_k z^k, \quad G_M\left(z;\boldsymbol{b}\right) = \mu_p \sum_{k=0}^{k_{\max}} b_k z^k,$$

$$z(Q^2) = \frac{\sqrt{t_{\mathrm{cut}} + Q^2} - \sqrt{t_{\mathrm{cut}} - t_0}}{\sqrt{t_{\mathrm{cut}} + Q^2} + \sqrt{t_{\mathrm{cut}} - t_0}}$$

$$t_{\mathrm{cut}} = 4m_\pi^2, \quad t_0 = t_{\mathrm{cut}}\left(1 - \sqrt{1 + Q^2_{\max}/t_{\mathrm{cut}}}\right)$$

$$\sum_{k=n}^{k_{\max}} \frac{k!}{(k-n)!} a_k = 0, \quad n = 0, 1, 2, 3 \longrightarrow G_E(Q^2) \sim 1/Q^4 \text{ as } Q^2 \to \infty$$

Choosing $k_{\max} = 8$ leaves 4 free parameters for each of G_E and G_M

IMF for nonlinear models

Potential difficulty: average of a functional form $F(x, \boldsymbol{\alpha})$ will not be of the same form when not linear in the parameters $\boldsymbol{\alpha}$, ruining iterative steps for best fit model.

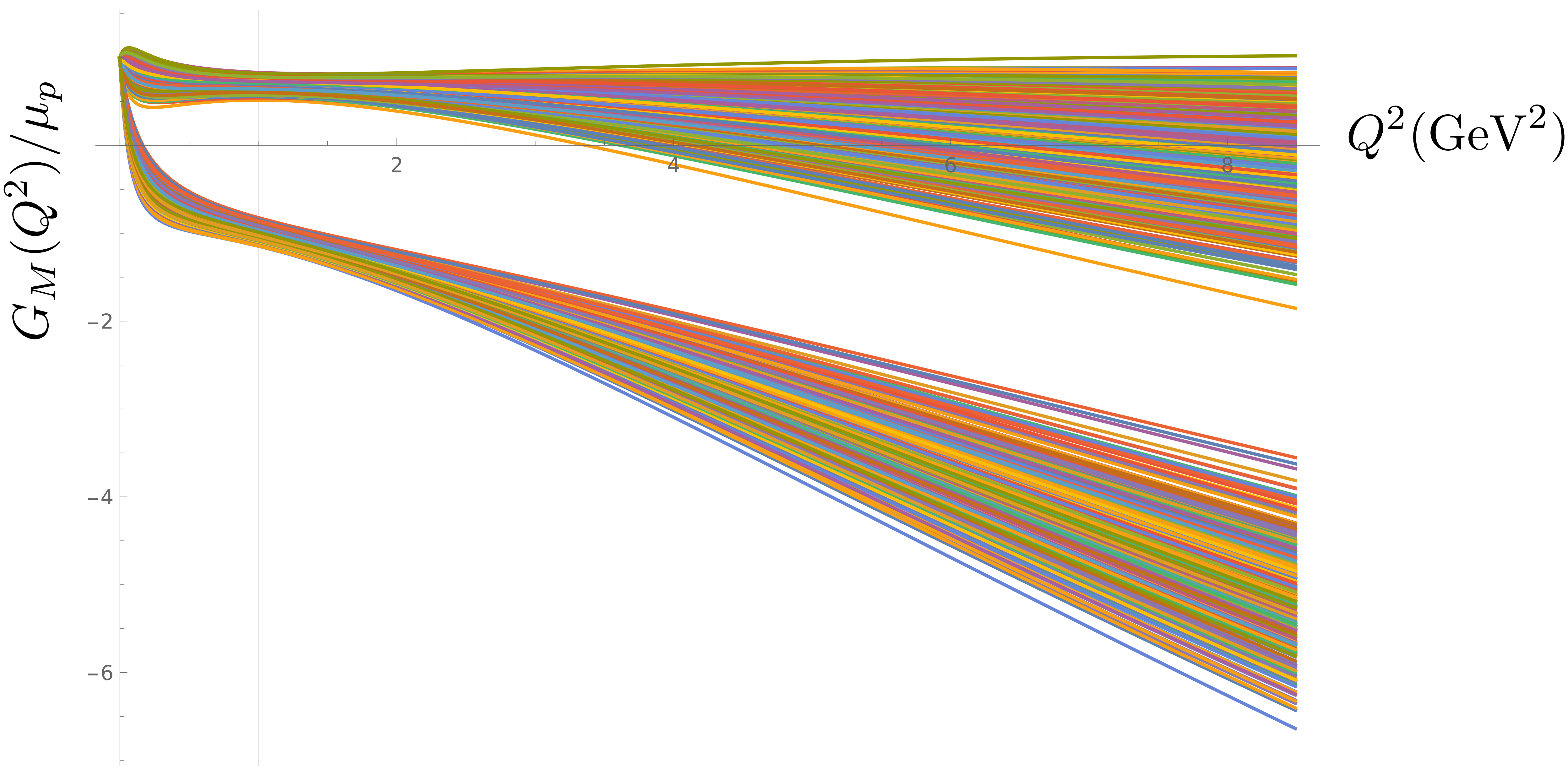
Two possible solutions:

1. Find parameters $\boldsymbol{\alpha}$ that are closest to the average of the replica fits $\langle F_R(x) \rangle$ in the least squares sense:

$$\frac{\partial}{\partial \boldsymbol{\alpha}} \int_{x_{\min}}^{x_{\max}} dx \left(F(x, \boldsymbol{\alpha}) - \langle F_R(x) \rangle \right)^2 = \mathbf{0}.$$

2. Proceed regardless and hope for convergence of the iterative process (convergence is all that's needed to be statistically sound).

- We use $\mathcal{R} = 1,500$ replicas per iteration; convergence in 5 iterations.
- We manually exclude $\sim 10\%$ of replica form factors that change sign for $Q^2 < 1$ GeV^2 (model depends on form factors squared).



IMF vs Penalty trick (nonlinear model)

	Penalty trick	IMF
a_0	-0.847 ± 0.028	-0.847 ± 0.028
a_1	1.269 ± 0.065	1.269 ± 0.067
a_2	0.47 ± 0.34	0.45 ± 0.34
a_3	-2.36 ± 0.49	-2.31 ± 0.51
b_0	-0.821 ± 0.015	-0.822 ± 0.015
b_1	1.374 ± 0.058	1.377 ± 0.057
b_2	-0.513 ± 0.086	-0.516 ± 0.083
b_3	-1.082 ± 0.056	-1.085 ± 0.059
χ^2/d.o.f.	76.9/98	76.5/107

Experiment	$n_i \pm \Delta n_i$	Scale uncertainty(%)
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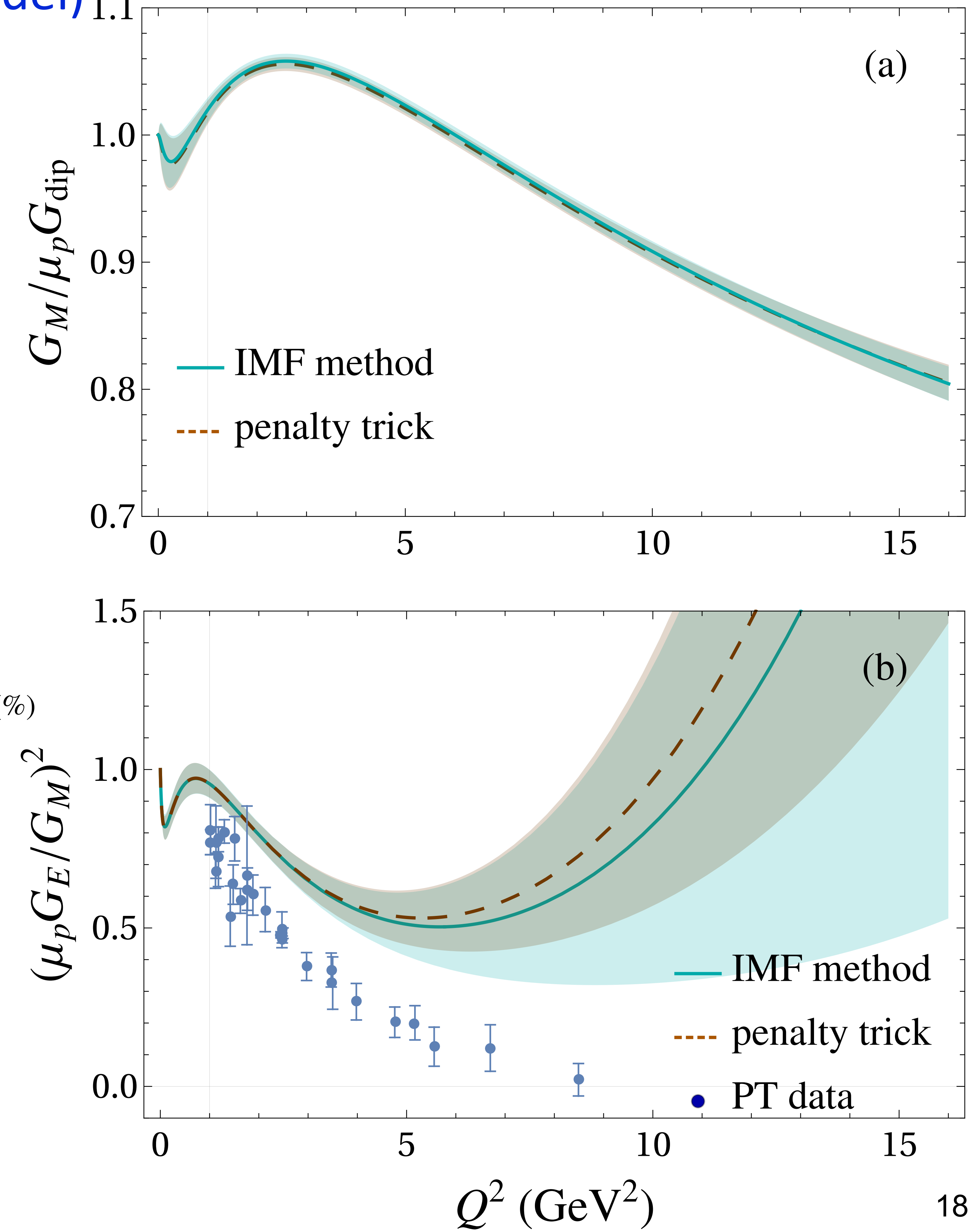
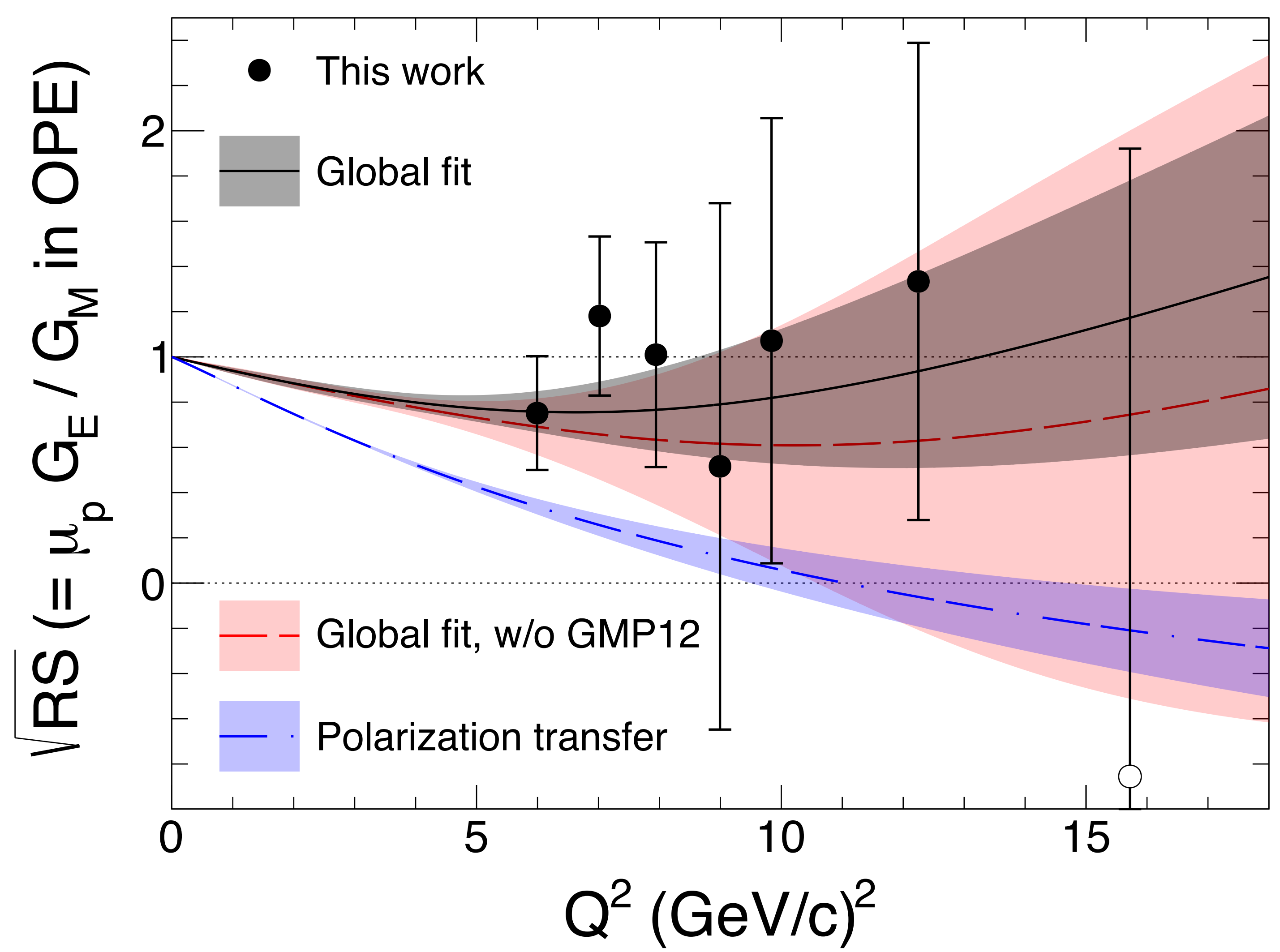


The importance of correlations

- IMF method achieves fits of the same quality as penalty trick without the need for normalization parameters n_i.
- The additional parameters n_i are a means to approximate the correlated nature of the data. They are not physically meaningful on their own!
- Hence we do not perform LT separations using the IMF method to subsets of the data at fixed Q^2, as this is blind to covariance of data at different Q^2.
- With the penalty trick fits there is a temptation to treat the normalization factors n_i as a pseudo model-independent means to bring data sets into agreement (*e.g.* Andivahis 1.6 GeV).
- This was done in Gmp12 analysis to facilitate comparison LT separation fits with those from global analysis.
- Fitted normalizations are designed to pull cross section data toward the model, so it's no surprise that fits are similar.

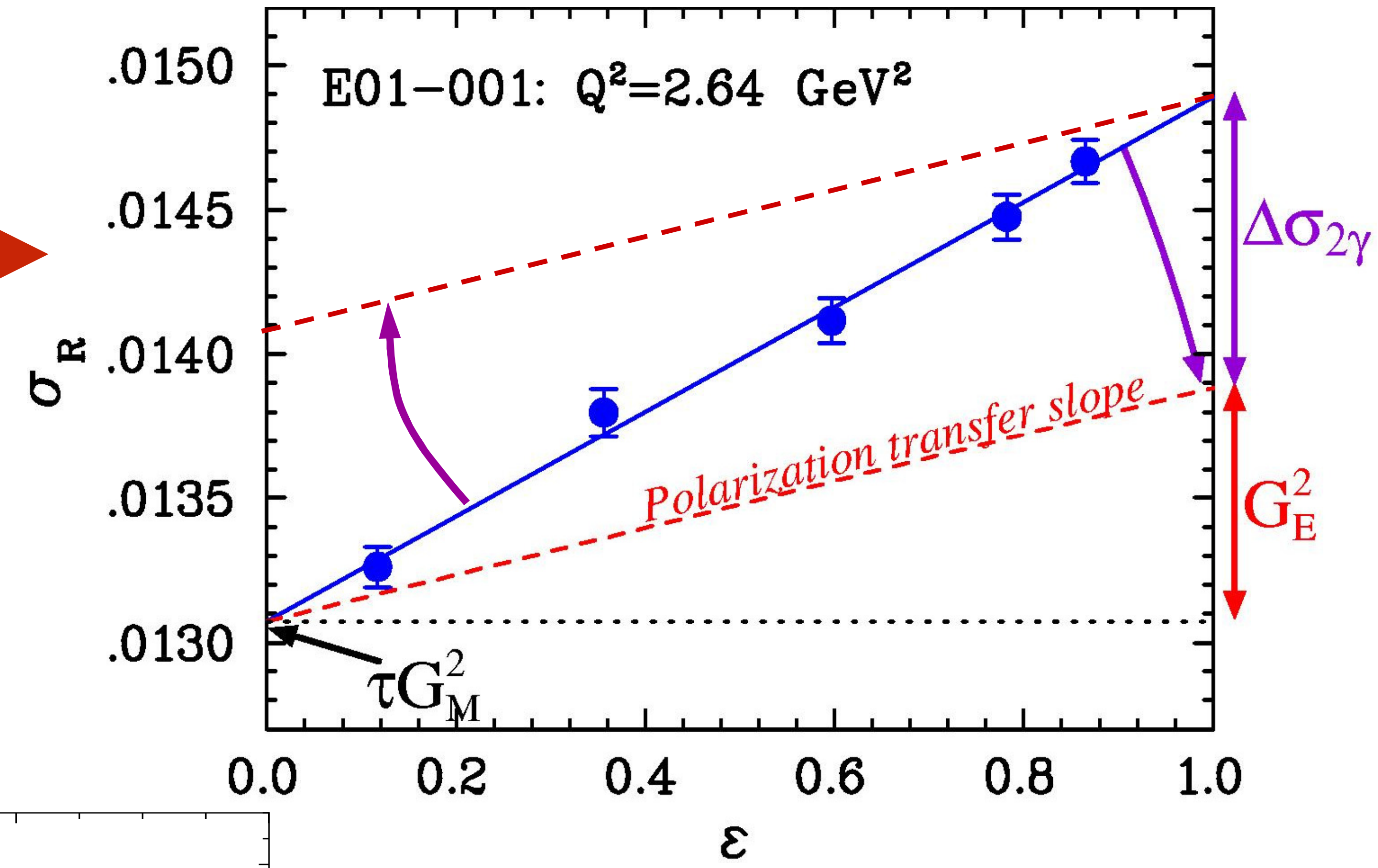


Black dots from direct LT separations
(interpolate to a common Q^2)

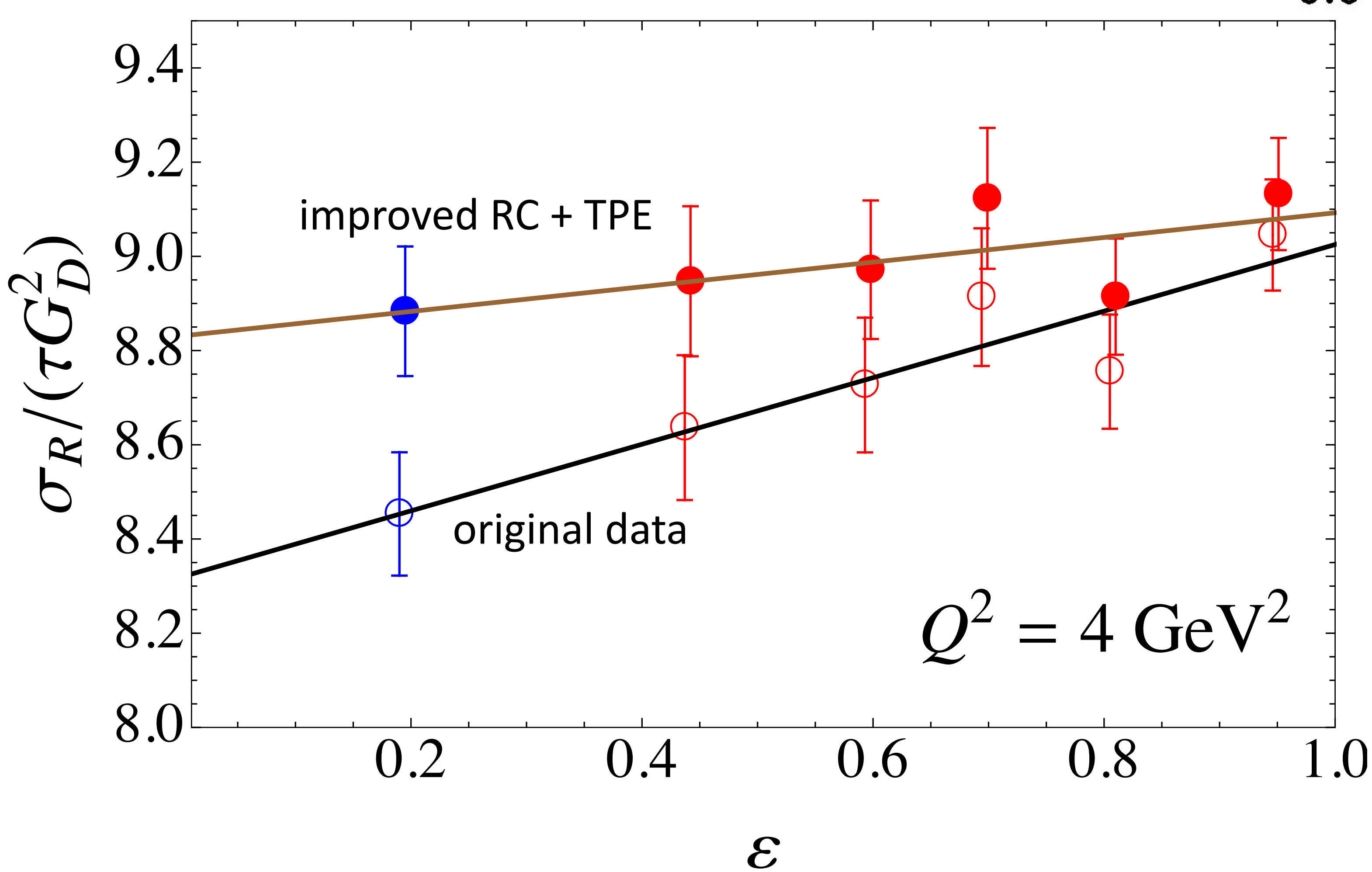
Is this meaningful?

Radiative corrections and TPE

Conceptual picture (no TPE)



$$\frac{1}{\tau}\sigma_{\rm red} = G_M^2 + \frac{\varepsilon}{\tau}G_E^2$$



- Example from Andivahis data
- Uses improved RC + our TPE applied to data
- No evidence of non-linearity in ε

GMp12 experiment

New data, updated RC: $\langle\Delta_{2\gamma}\rangle$ = 4.2 ± 2.0% (Q^2>6 GeV^2)
Using conventional RC: $\langle\Delta_{2\gamma}\rangle$ = 6.6 ± 2.1% (Q^2>6 GeV^2)

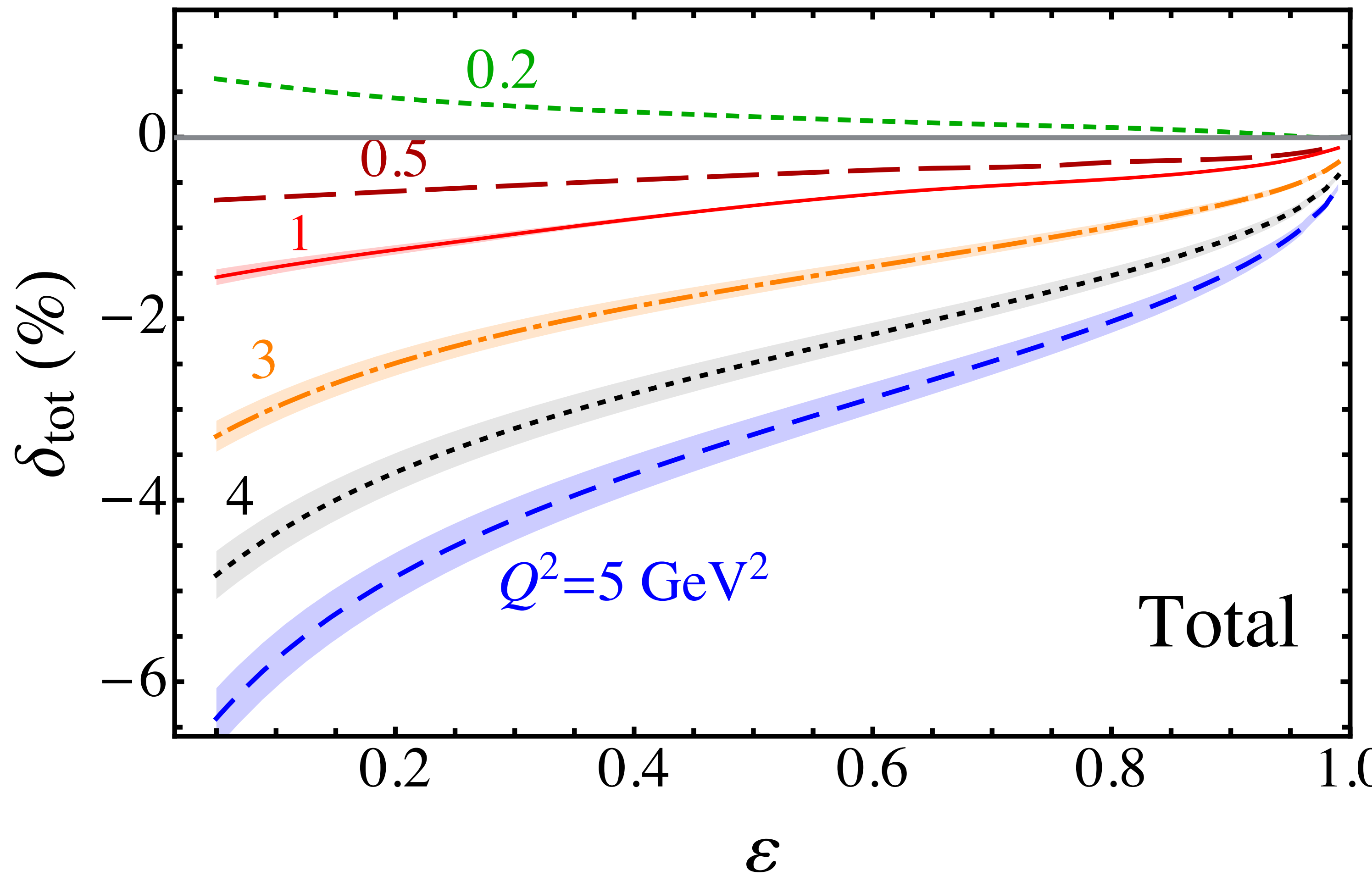
New RC resolves roughly one-third of the discrepancy

Even after this, discrepancy between continues to Q^2 > 15 GeV^2

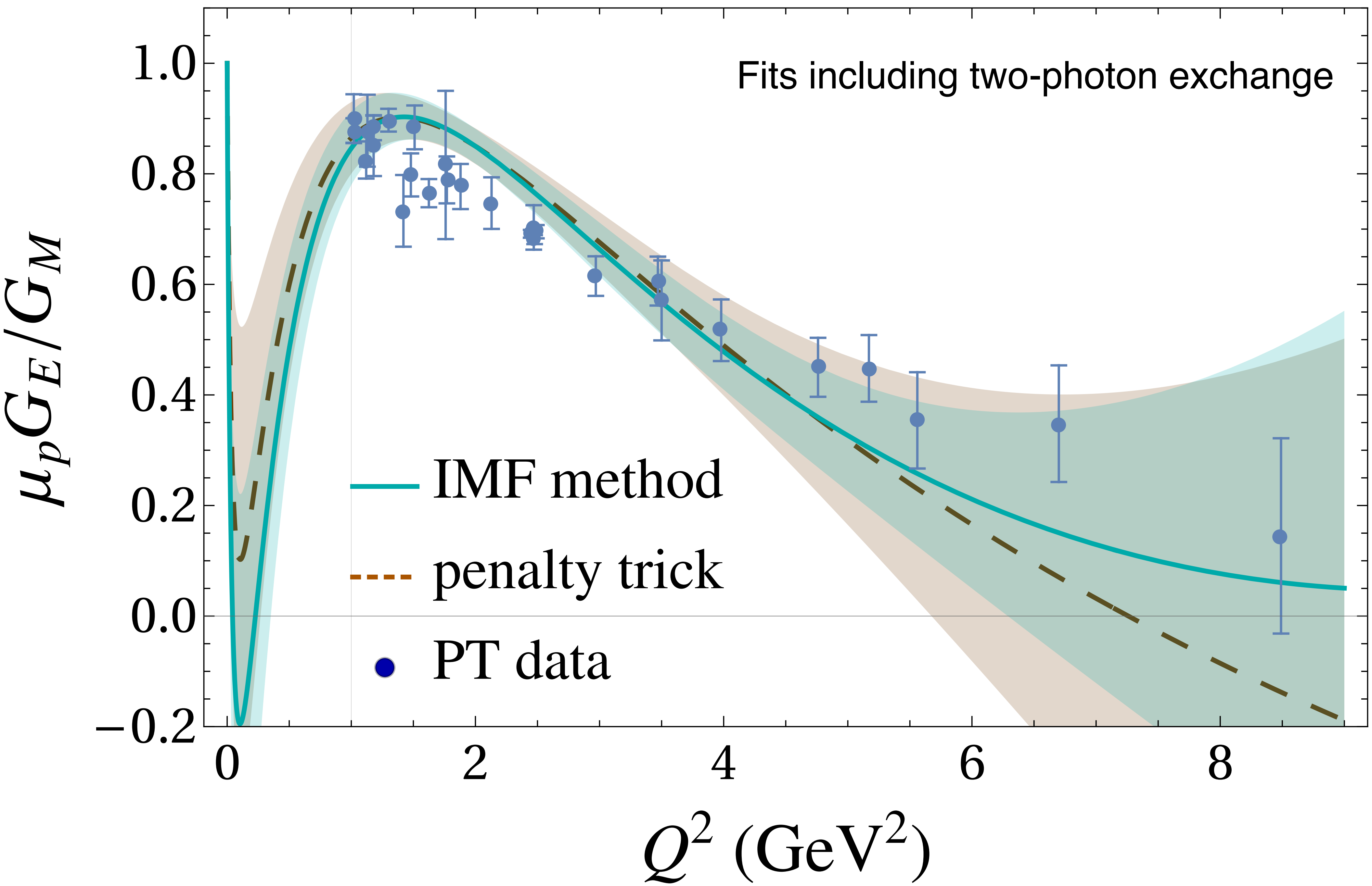
TPE corrections to cross section (CLAS resonances)

- Linear over mid-range of ε values, but curves towards endpoints
- Presence of TPE allows for smaller contribution of G_E^2

Ahmed, Blunden & Melnitchouk, PRC**102**, 045205 (2020)



IMF fit including TPE as a RC



Suggests TPE can fully resolve the proton form factor discrepancy

Summary

- Model-dependent but statistically unbiased fitting leads to minor improvements to global fits to cross section data
- Penalty trick seems to consistently provide similar fits. Why bother with the more involved method?
- Lesson from D'Agostini bias: naive treatment of normalization uncertainties can lead to "repulsion" of the fit from the data while keeping the chi-square the same.
- No *a priori* criteria to determine when such misbehaviour is likely to occur.
- Plan to reanalyze penalty trick fits to global data sets including data at low Q^2 (over 1,500 data points) to determine possible impact on proton radius discrepancy.