Developments in subleading power factorization and resummation

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Theory Seminar
Jefferson Laboratory
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Leading-logarithmic threshold resummation of the Drell-Yan process at next-to-leading power

Martin Beneke, Alessandro Broggio, Mathias Garny, Sebastian Jaskiewicz, Robert Szafron, Leonardo Vernazza and Jian Wang

JHEP, 2019(3):43 arXiv:1809.10631

Threshold factorization of the Drell-Yan process at next-to-leading power

Martin Beneke, Alessandro Broggio, Sebastian Jaskiewicz and Leonardo Vernazza

JHEP, 2020(7):78 arXiv:1912.01585

NLP two loop soft functions for the Drell-Yan process at threshold

Alessandro Broggio, Sebastian Jaskiewicz, Leonardo Vernazza

JHEP, 2021(10):061 arXiv:2107.07353 + *in preparation*

Next-to-leading power endpoint factorization and resummation for off-diagonal "gluon" thrust

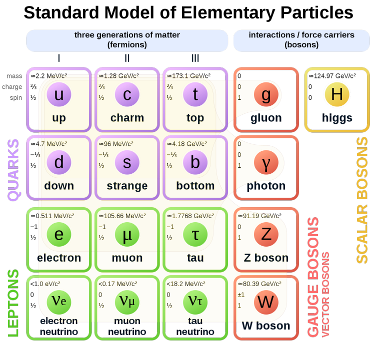
Martin Beneke, Mathias Garny, Sebastian Jaskiewicz, Julian Strohm, Robert Szafron, Leonardo Vernazza and Jian Wang

JHEP, 2022(07):144 arXiv:2205.04479

Developing our understanding and striving for precise predictions

Standard Model

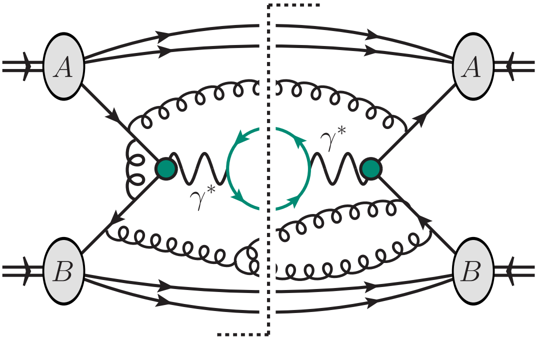
However, we know it is not the complete picture yet



Motivations and focus: High perturbative precision

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Recently calculated to N^{3}LO
[C. Duhr, F.Dulat, B. Mistlberger, 2001.07717]



Motivations and focus: beyond fixed-order

$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)\,] + X(p_X)$$

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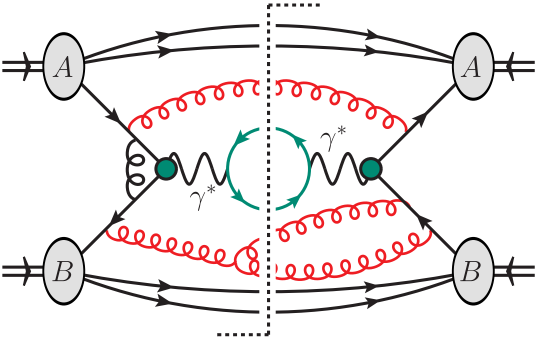
Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \to 1$$

Define power counting parameter λ:

$$\lambda = \sqrt{1-z}$$



Schematic form for production cross-sections near threshold, $z \to 1$:

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[\; c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(\;\; c_{nm} \left[\frac{\ln^m(1-z)}{1-z}\right]_+ + \quad\; d_{nm} \ln^m(1-z) \quad\;\; \right) + \ldots \right]$$

Motivations and focus: The Drell-Yan process

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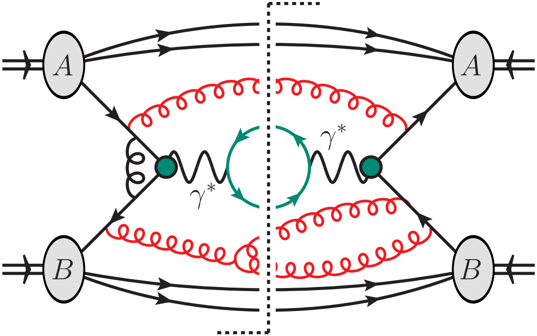

The general idea is to factorise the physics appearing at different scales:

$$\sigma \sim H \otimes J \otimes \ldots \otimes J \otimes S$$

and solve RG equations for each object to sum the large logarithms → SCET

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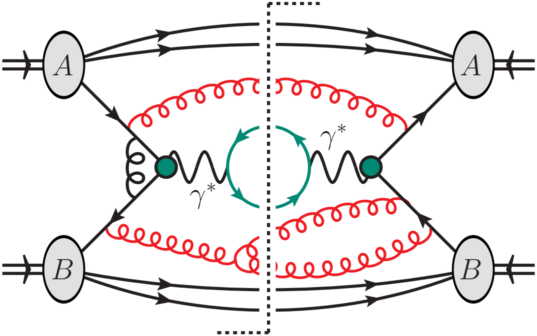

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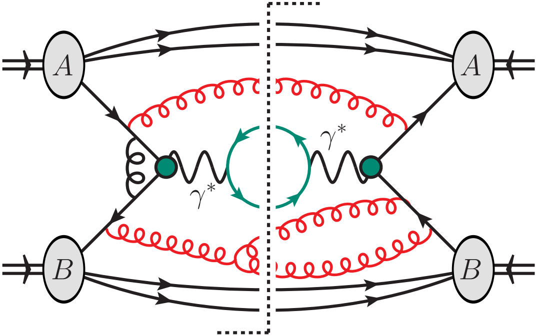

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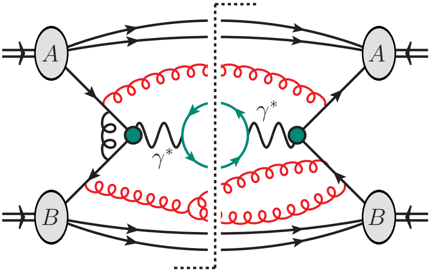
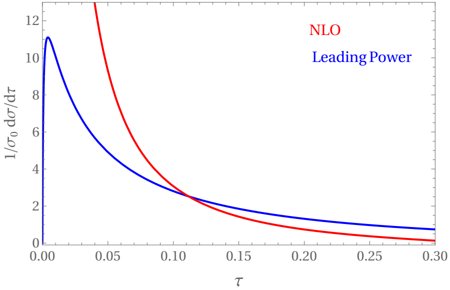




DY threshold $\tau = (1 - z)$

$$\frac{d\sigma}{dQ^2} = f_{a/A} \otimes f_{b/B} \otimes |C^{A0}|^2 \times S_{\rm LP}$$

Event shapes, thrust $\tau = 1 - T$

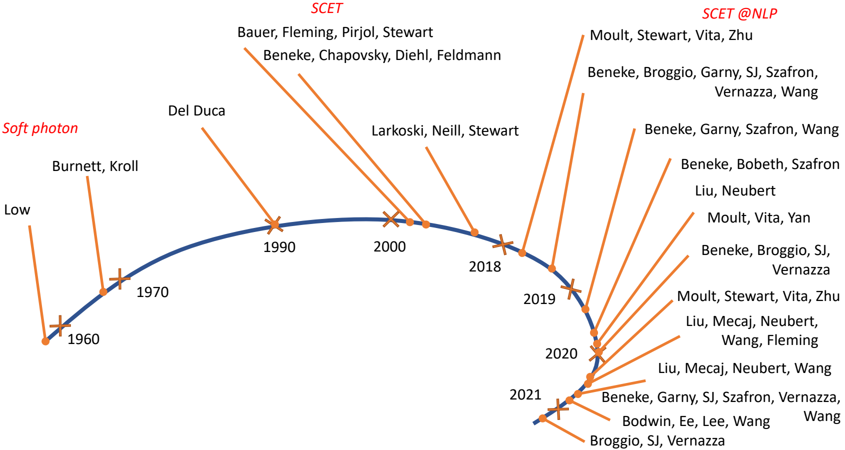
$$\frac{d\sigma}{d\tau} = |C^{A0}|^2 \times J_c^{(q)} \otimes J_{\bar{c}}^{(\bar{q})} \otimes S_{\rm LP}$$

LP logarithms, $\alpha_s^n \left[\frac{\ln^m \tau}{\tau}\right]_+$, $m \leq 2n - 1$ are well studied: LL, NLL, NNLL, N3LL

[T. Becher, M. Neubert, A. Xu, 0710.0680] [T. Becher, M. Schwartz, 0803.0342]

Towards SCET at NLP

Tremendous progress in understanding of the NLP terms has been made in the last few years.

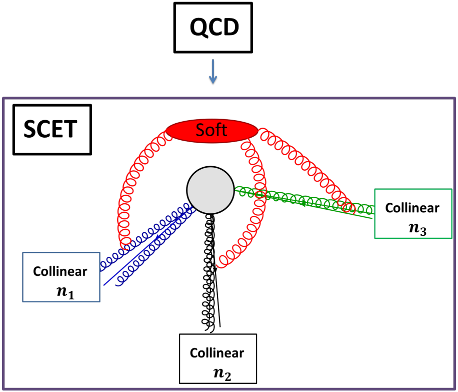


Diagrammatic approach: Laenen, Magnea, White; Anastasiou, Penin; Ravindran

Introduction to Soft Collinear Effective Field Theory (SCET)

SCET

Soft collinear effective theory is contained within QCD. It is an EFT which describes energetic particles.



- Process specific description, with collinear sectors formed by energetic particles. These can only interact with each other, and not between different sectors.
- Interactions between sectors are mediated by the soft degrees of freedom.
- Every interaction is well defined in terms of power counting - this allows for systematic expansion.

SCET introduction: Modes

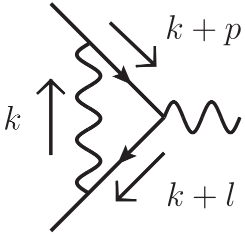
Convenient set-up utilises light-cone vectors

$$n_-^\mu = (1,0,0,1) \qquad\qquad n_+^\mu = (1,0,0,-1)$$

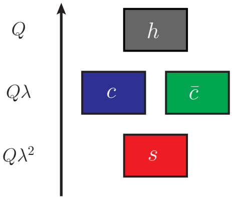
satisfying $n_-^2 = n_+^2 = 0$ and $n_- \cdot n_+ = 2$. In terms of which a momentum is decomposed as

$$p^\mu \;=\; (n_+p)\,\frac{n_-^\mu}{2} + (n_-p)\,\frac{n_+^\mu}{2} + p_\perp^\mu \;=\; (n_+p, n_-p, p_\perp)$$

Using the expansion-by-regions method we find the relevant modes for the process.
[M. Beneke, V. A. Smirnov, hep-ph/9711391]



$$\begin{aligned}
\lambda &= \sqrt{(1-z)} \to 0 \\
k^\mu &\sim Q(1,1,1) \\
{\color{blue}k^\mu} &\sim Q(1,\lambda^2,\lambda) \\
{\color{green}k^\mu} &\sim Q(\lambda^2,1,\lambda) \\
{\color{red}k^\mu} &\sim Q(\lambda^2,\lambda^2,\lambda^2)
\end{aligned}$$



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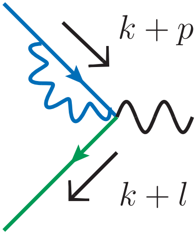

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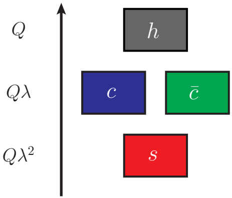

SCET introduction: Notation and Modes

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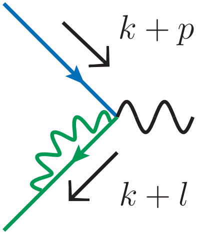

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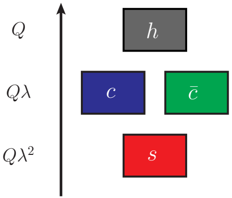

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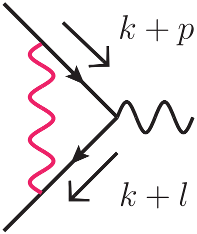

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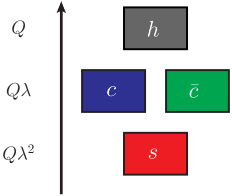


SCET introduction: Lagrangian

In this talk we employ position-space SCET [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358]

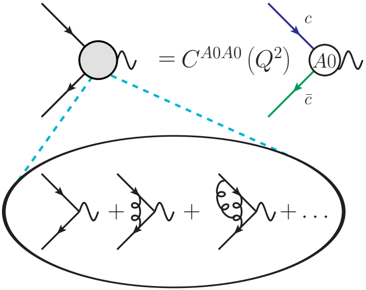
$$\psi(x) \to \underbrace{\psi_1(x) + \cdots + \psi_N(x)}_{N \text{ collinear fermion fields}} + q(x) \qquad \mathcal{L}_{\mathrm{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\mathrm{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter $\lambda = \sqrt{1-z}$:

$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\mathrm{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots \qquad \text{E.g.} \quad \mathcal{L}_c^{(1)} = \bar{\chi}_c\, i x_\perp^\mu \left[i n_- \partial \mathcal{B}_\mu^+ \right] \frac{\not{n}_+}{2} \chi_c$$

We keep the *collinear*, *anti-collinear*, and *soft* degrees of freedom. The *hard* modes are integrated out:

$$\bar{\psi}\gamma^\mu \psi = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k}\; C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, \ldots, t_{i_{n_i}})$$



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Separate collinear sectors interact only through soft gluon interactions. Focusing on the LP term:

$$\mathcal{L}_c^{(0)} = \bar{\xi}_c \left(i n_- D_c + g\, n_- A_s(x_-) + i \not{D}_{\perp c} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi_c + \mathcal{L}_{c,\text{YM}}^{(0)}$$

with $i n_- D_c = i n_- \partial + g\, n_- A_c(x)$, $x_-^\mu = (n_+ x) n_-^\mu / 2$.

The decoupling transformation, $\xi_c \to Y_+ \xi_c^{(0)}$ and $A_c^\mu \to Y_+ A_c^{(0)\mu} Y_+^\dagger$, separates the soft and collinear sectors at LP [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

$$\bar{\xi}_c \left(i n_- D_c + g_s n_- A_s \right) \frac{\not{n}_+}{2} \xi_c = \bar{\xi}_c^{(0)} \, i n_- D_c^{(0)} \, \frac{\not{n}_+}{2} \xi_c^{(0)}$$

where

$$Y_\pm(x) = \mathbf{P} \exp \left[i g_s \int_{-\infty}^{0} ds \, n_\mp A_s(x + s n_\mp) \right]$$

SCET introduction: Lagrangian

The structure of the SCET Lagrangian beyond LP is more intricate

[M. Beneke, Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}\left(x_\perp^\mu n_-^\nu \, W_c \, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger\right) \frac{\not{n}_+}{2} \xi + \mathcal{L}^{(1)}_{\mathrm{YM}} + \left(\bar{q} \, W_c^\dagger \, i \not{D}_\perp \xi + \mathrm{h.c.}\right)$$

$$\begin{aligned}\mathcal{L}_\xi^{(2)} = {} & \frac{1}{2}\bar{\xi}\left((n_- x)\, n_+^\mu n_-^\nu \, W_c \, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu W_c \left[D_{\mathrm{s}}^\rho, gF^{\mathrm{s}}_{\mu\nu}\right] W_c^\dagger\right) \frac{\not{n}_+}{2} \xi \\ & + \frac{1}{2}\bar{\xi}\left(i\not{D}_{\perp c} \frac{1}{in_+ D_c} x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger \frac{1}{in_+ D_c} i\not{D}_{\perp c}\right) \frac{\not{n}_+}{2} \xi\end{aligned}$$

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$$+\frac{1}{2}\,\bar{\xi}\left(i\not{D}_{\perp c}\,\frac{1}{in_+D_c}\, x_\perp^\mu \gamma_\perp^\nu\, W_c\, g F^{\mathrm{s}}_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu\, W_c\, g F^{\mathrm{s}}_{\mu\nu} W_c^\dagger\,\frac{1}{in_+D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi$$

- Importantly, there are no purely collinear interactions at subleading powers. In each vertex there is at least one soft field.
- Coordinate space arguments appear in the Lagrangian due to multipole expansion of the soft modes:
  $\phi_s(x)\phi_c(x) = (\phi_s(x_-) + \underbrace{x_\perp \cdot \partial_\perp \phi_s(x_-)}_{\mathcal{O}(\lambda)} + \ldots)\phi_c(x)$
- For Feynman rules see [M. Beneke, M. Garny, R. Szafron, J. Wang, 1808.04742]

SCET introduction: N-jet operator basis

Generic N-jet operator has the form:
[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

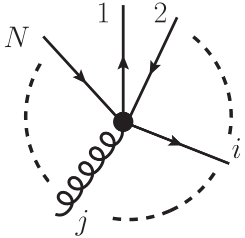
$$J = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k} \; C(\{t_{i_k}\}) \prod_{i=1}^{N} \; J_i(t_{i_1}, t_{i_2}, ..., t_{i_{n_i}})$$

where the Js are constructed by multiplying collinear gauge invariant building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad\qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

by acting on these with derivatives $i\partial^\mu_{\perp i} \sim \lambda$, and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

Generic leading power N-jet operator:



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We adopt the notation: $J_i^{An}, J_i^{Bn}, J_i^{Cn}, J_i^{Tn}$ where:

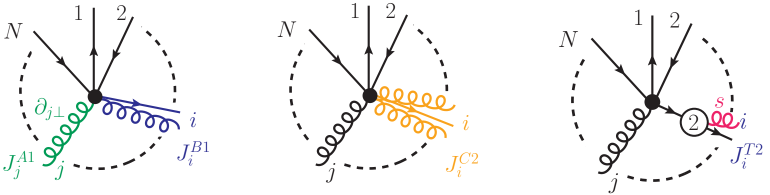
- A, B, C... refers to number of fields in a given collinear direction
- n is the power of λ suppression (relative to $A0$) in a given sector.

For example, up to $\mathcal{O}(\lambda^2)$ we can construct $J_i^{A1}, J_i^{B1}, J_i^{C2}, J_i^{T2}$ respectively:

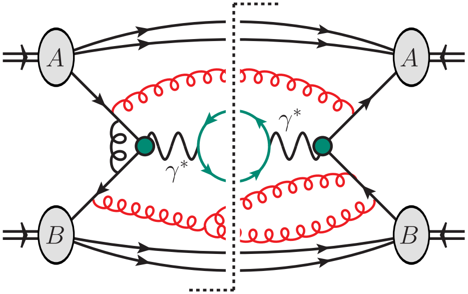
$$i\partial^{\mu}_{\perp i}\chi_i, \quad \chi_i(t_{i_1})\mathcal{A}^{\mu}_{i\perp}(t_{i_2}), \quad \chi_i(t_{i_1})\mathcal{A}^{\nu}_{i\perp}(t_{i_2})\mathcal{A}^{\mu}_{i\perp}(t_{i_3}), \quad i\int d^4z\, \mathbf{T}\left[\chi_i(t_{i_1})\mathcal{L}^{(2)}(z)\right]$$

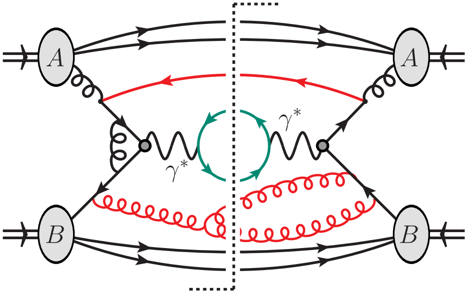
And $\mathcal{O}(\lambda^2)$ N-jet operators are

$J_1^{A0} \dots J_i^{B1} J_j^{A1} \dots J_N^{A0}, \qquad J_1^{A0} \dots J_i^{C2} \dots J_N^{A0}, \qquad J_1^{A0} \dots J_i^{T2} \dots J_N^{A0}, \dots$



Factorization of the Drell-Yan process

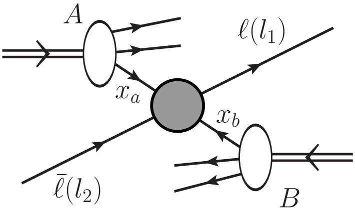




Drell-Yan Process at Threshold: Leading Power

The Drell-Yan Process

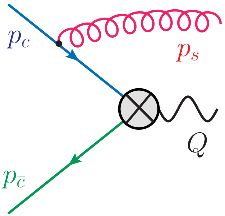
$$A(p_A) + B(p_B) \to \gamma^*(Q^2)[\to \ell(l_1)\bar{\ell}(l_2)\,] + X(p_X)$$

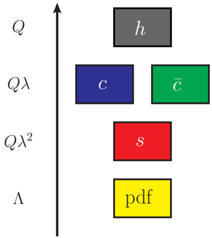


$$z = \frac{Q^2}{\hat{s}} \to 1 \qquad \lambda = \sqrt{(1-z)}$$

$$\begin{aligned} p_c &= (n_+p_c, n_-p_c, p_{c\perp}) \sim Q(1,\lambda^2,\lambda) \\ p_{\bar{c}} &= (n_+p_{\bar{c}}, n_-p_{\bar{c}}, p_{\bar{c}\perp}) \sim Q(\lambda^2,1,\lambda) \\ p_s &= (n_+p_s, n_-p_s, p_{s\perp}) \sim Q(\lambda^2,\lambda^2,\lambda^2) \end{aligned}$$

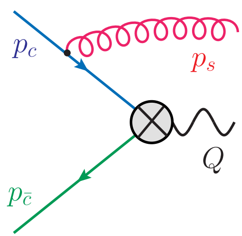
$$Q^2\lambda^2 = Q^2(1-z) \gg \Lambda^2_{\rm QCD} \qquad p_{c-\rm PDF} \sim (Q, \Lambda^2/Q, \Lambda)$$





The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t}\,)\, J^{A0}_\mu(t,\bar{t}\,)$$



$$J^{A0}_\mu(t,\bar{t}\,) = \bar{\chi}_{\bar{c}}(\bar{t}n_-)\,\gamma_{\perp\mu}\,\chi_c(tn_+)$$

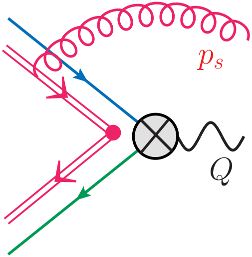
The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t}\,)\, J_\mu^{A0}(t,\bar{t}\,)$$

Use decoupled fields $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$.

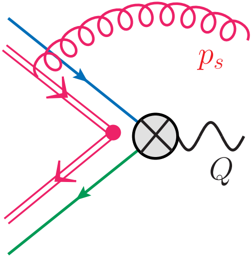
Leading power current becomes

$$J_\mu^{A0}(t,\bar{t}\,) = \bar{\chi}_{\bar{c}}^{(0)}(\bar{t}n_-)Y_-^\dagger(0)\gamma_{\perp\mu}Y_+(0)\chi_c^{(0)}(tn_+)$$



The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, J^{A0}_\mu(t,\bar{t})$$



Use decoupled fields $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$.

Leading power current becomes

$$J^{A0}_\mu(t,\bar{t}) = \bar{\chi}^{(0)}_{\bar{c}}(\bar{t}n_-)Y_-^\dagger(0)\gamma_{\perp\mu}Y_+(0)\chi_c^{(0)}(tn_+)$$

Consider the matrix element:

$$\langle X|\bar{\psi}\gamma^\mu\psi(0)|A(p_A)B(p_B)\rangle \quad = \quad \int \frac{d(n_+p_a)}{2\pi}\frac{d(n_-p_b)}{2\pi}C^{A0}(n_+p_a,-n_-p_b)$$

$$\times\langle X^{\text{PDF}}_{\bar{c}}|\hat{\bar{\chi}}^{\text{PDF}}_{\bar{c}}(n_-p_b)|B(p_B)\rangle\gamma_\perp^\mu\langle X_c^{\text{PDF}}|\hat{\chi}_c^{\text{PDF}}(n_+p_a)|A(p_A)\rangle$$

$$\times\langle X_s|\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\right]|0\rangle$$

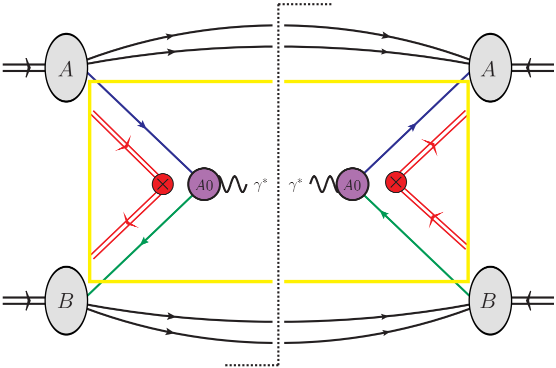
The states factorize: $\langle X| = \langle X^{\text{PDF}}_{\bar{c}}|\,\langle X^{\text{PDF}}_c|\,\langle X_s|$. The threshold collinear mode does not appear. Only the PDF collinear mode with scaling

$$p_{c-\text{PDF}} \sim (Q, \Lambda^2_{\text{QCD}}/Q, \Lambda_{\text{QCD}})$$

The Drell-Yan Proccess: Leading Power Cross-Section

The leading power factorisation formula has the following form:

$$\frac{d\sigma_{\mathrm{DY}}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\boxed{\hat{\sigma}_{ab}^{\,\mathrm{LP}}} + \hat{\sigma}_{ab,dyn}^{\,\mathrm{NLP}} + \hat{\sigma}_{ab,kin}^{\,\mathrm{NLP}} + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$



$$\hat{\sigma}^{\,\mathrm{LP}}(z) = Q \, |C^{A0}(Q^2)|^2 \, S_{\mathrm{DY}}(\Omega)$$

[G. Sterman 1987] [S. Catani, L. Trentadue 1989] [G. P. Korchemsky G. Marchesini, 1993]

[S. Moch, A. Vogt, hep-ph/0508265] [T. Becher, M. Neubert, G. Xu, 0710.0680]

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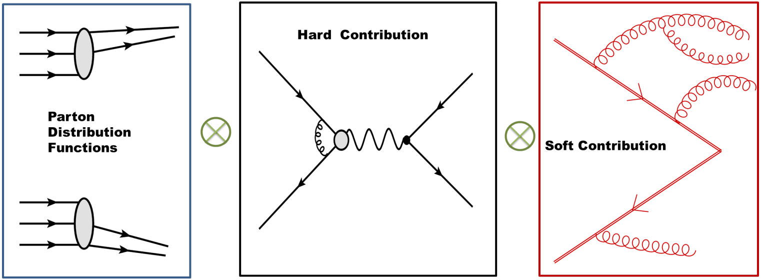

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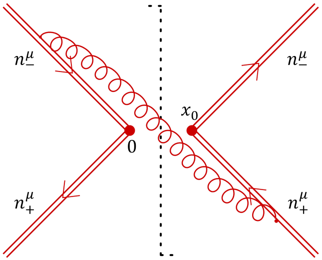
The Drell-Yan Proccess: Leading Power Cross-Section

The leading power factorisation formula has the following form:

$$\frac{d\sigma_{\rm DY}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}^{\rm LP}_{ab} + \hat{\sigma}^{\rm NLP}_{ab,dyn} + \hat{\sigma}^{\rm NLP}_{ab,kin} + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{1}{N_c}\, {\rm Tr}\, \langle 0 | \bar{\mathbf{T}}(Y_+^\dagger(x^0) Y_-(x^0))\, \mathbf{T}(Y_-^\dagger(0) Y_+(0)) | 0 \rangle$$



Note, inserting complete set of states, and performing $\int dx_0$

$$\delta(\Omega - 2E_X)$$

Drell-Yan Process at Threshold: Next-to-Leading Power

Factorisation formula at NLP

First let us compare leading power and *next*-to-leading power cross-sections schematically:

$$\frac{d\sigma_{\rm DY}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b \, f_{a/A}(x_a) f_{b/B}(x_b) \left(\, \hat{\sigma}^{\rm LP}_{ab} + \hat{\sigma}^{\rm NLP}_{ab,dyn} + \hat{\sigma}^{\rm NLP}_{ab,kin} + \ldots \right) + \mathcal{O}\left(\Lambda/Q\right)$$

We have discussed the LP piece

$$\hat{\sigma}^{\,\rm LP}(z) = Q \, |C^{A0}(Q^2)|^2 \; S_{\rm DY}(\Omega)$$

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We have discussed the LP piece

$$\hat{\sigma}^{\,\rm LP}(z) = Q\, |C^{A0}(Q^2)|^2\; S_{\rm DY}(\Omega)$$

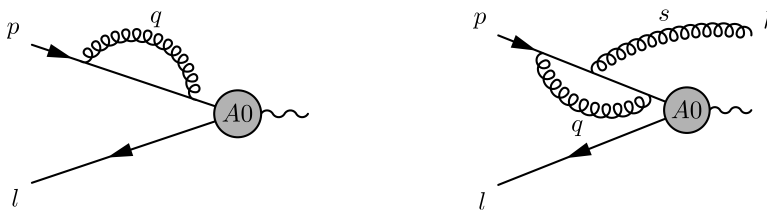
and as will be shown the NLP is given by

$$\hat{\sigma}^{\,\rm NLP}(z) = \sum_{\rm terms} [C \otimes J \otimes \bar{J}\,]^2 \otimes S$$

- C is the hard Wilson matching coefficient
- S is the generalised soft function
- Js are the collinear functions

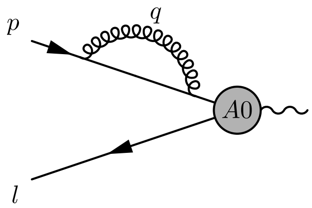
Let us now motivate the emergence of this structure at next-to-leading power.

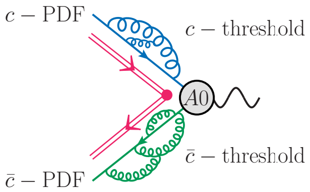
Absence of collinear functions at LP



The purely threshold-collinear loops are scaleless and vanish.

Absence of collinear functions at LP





The purely threshold-collinear loops are scaleless and vanish.

At leading power, we can apply the decoupling transformation $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$ which removed the soft-collinear interactions from the Lagrangian.

[C. Bauer, D. Pirjol, and I. Stewart, hep-ph/0109045]

The threshold-collinear modes can be identified with c-PDF modes, $\chi_c \to \chi_c^{\mathrm{PDF}}$

$$\chi_c(tn_+) = \int du\, \tilde{J}(t,u)\, \chi_c^{\mathrm{PDF}}(un_+) \qquad \tilde{J}(t,u) = \delta(t-u)$$

Collinear functions at NLP

Beyond LP, the decoupling transformation does not remove soft-collinear interactions in the Lagrangian.

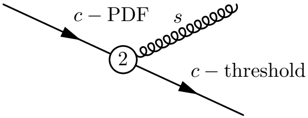
Consider an example of subleading SCET Lagrangian:

[M. Beneke and Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2}\bar{\chi}_c^{(0)} z_\perp^\mu z_\perp^\rho \left[i\partial_\rho i n_- \partial \mathcal{B}_\mu^+(z_-)\right] \frac{\not{n}_+}{2}\chi_c^{(0)} \qquad \text{where} \qquad \mathcal{B}_\pm^\mu = Y_\pm^\dagger \left[iD_s^\mu Y_\pm\right]$$

The subleading power Lagrangian terms enter the basis through time-ordered product operators

$$\left(J_{A0,2\xi}^{T2}(t)\right)^\mu = i\int d^4z\, \mathbf{T}\left[J_{A0}^\mu(t)\, \mathcal{L}_{2\xi}^{(2)}(z)\right]$$



Collinear functions at NLP

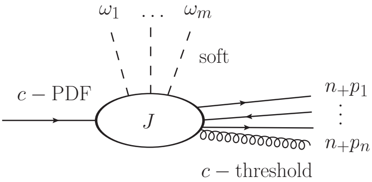
PDF collinear modes can be radiated into the final state:
$p_c \sim Q(1, \lambda^2, \lambda)$ and $p_{c-\mathrm{PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$

$$i \int d^4z \, \mathbf{T}\Big[\chi_c(tn_+) \; \mathcal{L}^{(2)}(z)\Big] = 2\pi \sum_i \int du \int \frac{d(n_+z)}{2} \, \tilde{J}_i\left(t, u; \frac{n_+z}{2}\right) \chi_c^{\mathrm{PDF}}(un_+) \, \mathfrak{s}_i(z_-)$$

$$\mathfrak{s}_i(z_-) \in \left\{ \frac{i\partial_\perp^\mu}{in_-\partial} \mathcal{B}^+_{\mu_\perp}(z_-) \, , \; \frac{i\partial_{[\mu_\perp}}{in_-\partial} \mathcal{B}^+_{\nu_\perp]}(z_-) \, , \; \frac{1}{(in_-\partial)} \left[\mathcal{B}^+_{\mu_\perp}(z_-), \mathcal{B}^+_{\nu_\perp}(z_-)\right], \ldots \right\}$$

[M. Beneke, A. Broggio, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]



Collinear functions at NLP

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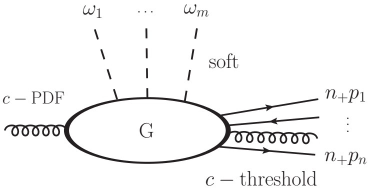
$p_c \sim Q(1, \lambda^2, \lambda)$ and $p_{c-\mathrm{PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$

$$i \int d^d z \, \mathbf{T} \left[\xi_c(tn_+) \mathcal{L}^{(1)}_{\xi q}(z) \right] = 2\pi \int du \int dz_- \, \tilde{G}_{\xi q}\left(t, u; z_-\right) \mathcal{A}^{\mathrm{PDF}}_{c\perp}(un_+) \, \mathfrak{s}_{\xi q}(z_-)$$

Only one soft structure present at NLP

$$\mathfrak{s}_{\xi q}(z_-) = \frac{g_s}{in_- \partial_z} q^+(z_-)$$



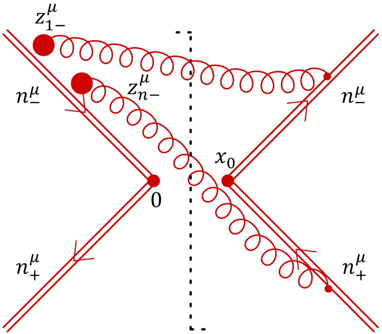
Generalised soft functions

The necessary presence of the time-ordered products at NLP gives rise to the concept of generalized soft functions with explicit gauge fields. Schematically we have

$$S(\Omega, \{\omega\}) = \int \frac{dx^0}{4\pi} e^{ix^0\Omega/2} \left(\prod_{j=1}^{n} \int \frac{d(n_+ z_j)}{4\pi} \, e^{-i\omega_j (n_+ z_j)/2} \right)$$
$$\times \operatorname{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \times \mathcal{L}_s^{(n)}(z_{1-}) \times ... \times \mathcal{L}_s^{(n)}(z_{n-}) \right] |0\rangle$$

$\mathcal{L}_s^{(n)}(z_{j-})$ contains $\mathcal{B}^+_{\perp\nu}(z_{j-})$ fields, not only Wilson lines.

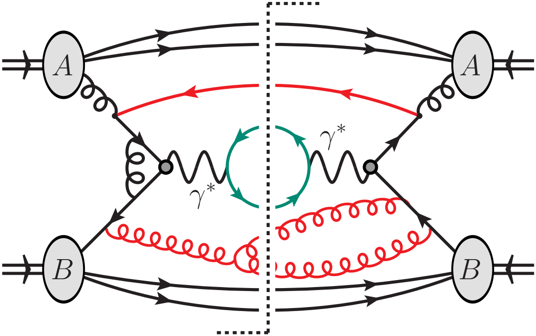
[M. Beneke , F. Campanario, T. Mannel, B.D. Pecjak, hep/0411395]



Factorisation formula at NLP

The off-diagonal DY process

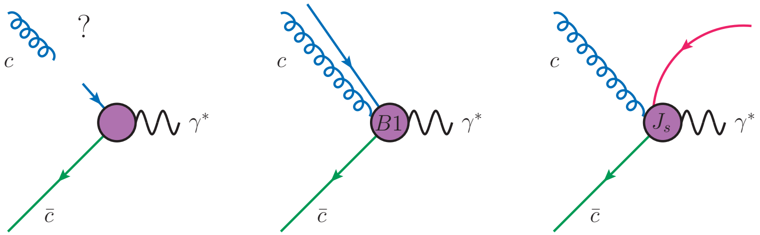
Now we consider initial state gluon from a proton A, which must be converted into a collinear quark via an emission of a soft antiquark.



In the EFT picture

This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\rho\psi(0) = \sum_{m_1,m_2} \int \{dt_k\}\,\{d\bar{t}_{\bar{k}}\}\,\widetilde{C}^{\,m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)\,J_s(0)\,J_\rho^{\,m_1,m_2}\left(\{t_k\},\,\{\bar{t}_{\bar{k}}\}\right)$$



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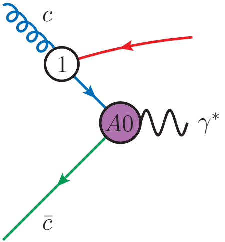

Use an insertion of a subleading power Lagrangian term with a soft quark, starting at $\mathcal{O}(\lambda)$

$$\left(J^{T2}_{A0,\xi q}(t)\right)^\mu = i\int d^dz\, \mathbf{T}\Big[\chi_c\,(tn_+)\;\mathcal{L}^{(1)}_{\xi q}(z)\Big] \qquad\qquad \mathcal{L}^{(1)}_{\xi q} = \bar{q}_+ \not{\mathcal{A}}_{c\perp}\chi_c + \text{h.c.}$$

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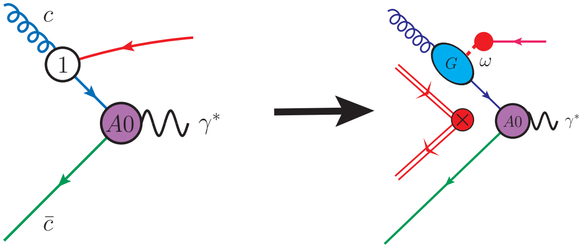
This is inherently a subleading power channel.

$$\bar{\psi}\gamma_\mu\psi = \int dt\, d\bar{t}\, \widetilde{C}^{A0}(t,\bar{t})\, \underbrace{\bar{\chi}_{\bar{c}}(\bar{t}n_-)\,\gamma_{\perp\mu}\,\chi_c(tn_+)}_{J^{A0}_\mu(t,\bar{t})}$$



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NLP factorization formula for Drell-Yan: $g\bar{q}$

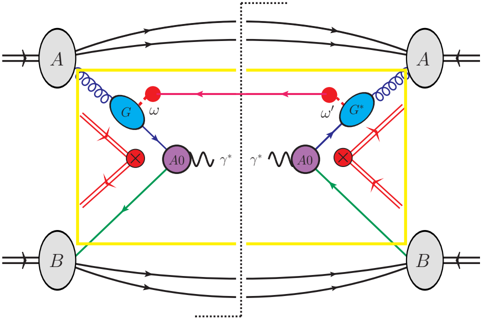
The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \qquad \Delta_{g\bar{q},\mathrm{NLP}}(z) = \boxed{\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z)}$$

where

[**SJ**, PhD thesis][A. Broggio, **SJ**, L. Vernazza, to appear]

$$\boxed{\Delta^{dyn}_{g\bar{q},\mathrm{NLP}}(z)} = 8H(Q^2) \int d\omega\, d\omega'\, G^*_{\xi q}(x_a n_+ p_A; \omega')\, G_{\xi q}(x_a n_+ p_A; \omega)\, S(\Omega, \omega, \omega')$$



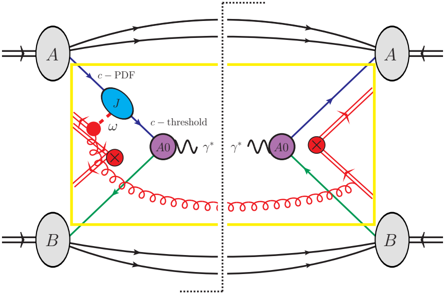
Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\Delta^{dyn}_{\rm NLP}(z) = -2\, Q\, \left[\left(\frac{\not{n}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{n}_+}{4} \right) \gamma^{\rho}_{\perp} \right]_{\beta\gamma} \int d(n_+p)\; C^{\,A0}\,(n_+p, x_b n_-p_B)$$

$$\times\, C^{*A0}\,(\,x_a\, n_+p_A,\, x_b n_-p_B) \sum_{i=1}^{4} \int \{d\omega_j\}\; J_i\,(n_+p, x_a\, n_+p_A; \{\omega_j\})\, S_i(\Omega; \{\omega_j\}) + \text{h.c.}$$



For comparison, LP result is:

$$\Delta_{\,\rm LP}(z) = |C(Q^2)|^2\; Q\; S_{\rm DY}(Q(1-z))$$

Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

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$$\times\, C^{*A0}\,(\,x_a\,n_+p_A,\,x_b n_- p_B)\sum_{i=1}^{4}\int\{d\omega_j\}\;J_i\,(n_+p, x_a\,n_+p_A;\{\omega_j\})\,S_i(\Omega;\{\omega_j\}) + \text{h.c.}$$

where the *generalised* soft functions have the structure:

$$\widetilde{S}_{\,i}\,(x;\{\omega_j\}) = \int\left\{\frac{dz_{j-}}{2\pi}\right\}\,e^{-i\omega_j z_{j-}}\,\times\,\frac{1}{N_c}\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left(\left[Y_+^\dagger\,Y_-\right](x)\right)\mathbf{T}\left(\left[Y_-^\dagger\,Y_+\right](0)\,\mathfrak{s}_i\,(\,\{z_{j-}\})\right)|0\rangle$$

with

$$\mathfrak{s}_i(\{z_{j-}\})\;\;\in\;\left\{\frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_{1-}),\,\frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_{1-}),\left[in_-\partial\mathcal{B}^+_{\mu_\perp}(z_{1-})\right]\right],\right.$$

$$\left.\frac{1}{(in_-\partial)}\mathcal{B}^+_{\mu_\perp}(z_{1-})\mathcal{B}^+_{\nu_\perp}(z_{2-}),\,\frac{1}{(in_-\partial)^2}q_{+\sigma}(z_{1-})\bar{q}_{+\lambda}(z_{2-})\right\}$$

Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\Delta^{dyn}_{\mathrm{NLP}}(z) = -2\,Q\left[\left(\frac{\not{n}_-}{4}\right)\gamma_{\perp\rho}\left(\frac{\not{n}_+}{4}\right)\gamma^{\rho}_{\perp}\right]_{\beta\gamma}\int d(n_+p)\;C^{A0}\,(n_+p, x_b n_- p_B)$$

$$\times\, C^{*A0}\,(x_a\,n_+p_A,\, x_b n_- p_B)\sum_{i=1}^{4}\int\{d\omega_j\}\;J_i\,(n_+p, x_a\,n_+p_A;\{\omega_j\})\,S_i(\Omega;\{\omega_j\}) + \text{h.c.}$$

where the *generalised* soft functions have the structure:

$$\widetilde{S}_i\,(x;\{\omega_j\}) = \int\left\{\frac{dz_{j-}}{2\pi}\right\}\,e^{-i\omega_j z_{j-}}\times\frac{1}{N_c}\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left(\left[Y_+^\dagger\,Y_-\right](x)\right)\mathbf{T}\left(\left[Y_-^\dagger\,Y_+\right](0)\,\mathfrak{s}_i\,(\{z_{j-}\})\right)|0\rangle$$

with

$$\mathfrak{s}_i(\{z_{j-}\})\;\in\;\left\{\frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_{1-}),\,\frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_{1-}),\left[in_-\partial\mathcal{B}^+_{\mu_\perp}(z_{1-})\right]\right],\right.$$

$$\left.\frac{1}{(in_-\partial)}\mathcal{B}^+_{\mu_\perp}(z_{1-})\mathcal{B}^+_{\nu_\perp}(z_{2-}),\,\frac{1}{(in_-\partial)^2}q_{+\sigma}(z_{1-})\bar{q}_{+\lambda}(z_{2-})\right\}$$

At NNLO accuracy there are three contributions:

- Collinear: 1-loop collinear and NLO soft functions, $\Delta^{dyn(2)}_{\mathrm{NLP-coll}}$.
- Hard: 1-loop hard and NLO soft functions, $\Delta^{dyn(2)}_{\mathrm{NLP-hard}}$.
- Soft: NNLO soft functions, $\Delta^{dyn(2)}_{\mathrm{NLP-soft}}$.

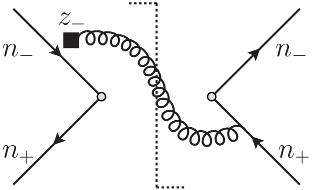
Only one of the soft building blocks starts with a single gluon emission.

Soft function at NLO

The only generalised soft function with non-vanishing contributions at $\mathcal{O}(\alpha_s)$ is

$$S_1(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi}\, e^{ix^0\Omega/2 - i\omega(n_+z)/2}$$

$$\times \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right]|0\rangle$$



$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\,\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\,\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\; e^{iz_-k}$$

$\to \delta(n_-k-\omega)$

Soft function at NLO

The only generalised soft function with non-vanishing contributions at $\mathcal{O}(\alpha_s)$ is

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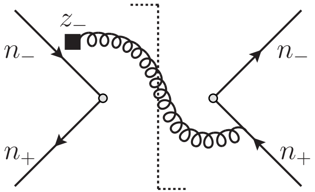
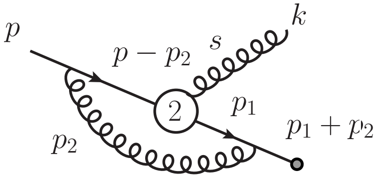


$$S_1^{(1)}(\Omega,\omega) = \frac{\alpha_s C_F}{2\pi}\frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]}\frac{1}{\omega^{1+\epsilon}}\frac{1}{(\Omega-\omega)^\epsilon}\,\theta(\omega)\theta(\Omega-\omega)\ .$$

After $\int d\omega$ the NLO cross section $\Delta_{\mathrm{NLP}}^{dyn(1)}$ is correctly reproduced.

Collinear functions at one-loop



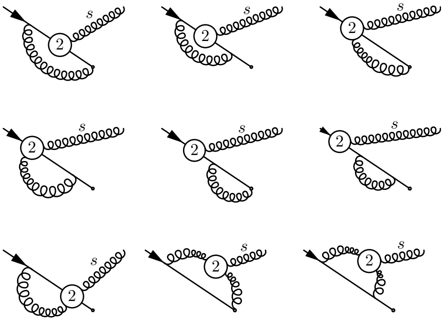
Use NLP Feynman rules to calculate these new objects up to $\mathcal{O}(\alpha_s)$. [M. Beneke, M. Garny, R. Szafron, J. Wang, 1808.04742]
Some non-standard features present!

$$X^\mu = \partial^\mu \left[(2\pi)^d \delta^{(d)} \left(\sum p_{\text{in}} - \sum p_{\text{out}}\right)\right]$$

$$\mathcal{O}(\lambda^2): \qquad i g_s \mathbf{T}^A A^{\rho\nu}(k,p,p') \frac{\not{n}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu})$$

$$A^{\rho\nu}(k,p,p') \equiv \frac{1}{2}\left[(n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left(\frac{\not{p}'_\perp}{n_+ p'} \gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p}\right)\right]$$

Collinear functions at one-loop



$$J_1\left(n_+q, n_+p;\, \omega\right) = -\frac{1}{n_+p}\delta(n_+q - n_+p) + 2\,\frac{\partial}{\partial n_+q}\delta(n_+q - n_+p)$$
$$+\frac{\alpha_s}{4\pi}\,\frac{1}{(n_+p)}\left(\frac{n_+p\,\omega}{\mu^2}\right)^{-\epsilon}\frac{e^{\epsilon\gamma_E}\,\Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]}$$
$$\times\left(C_F\left(-\frac{4}{\epsilon}+3+8\epsilon+\epsilon^2\right) - C_A\left(-5+8\epsilon+\epsilon^2\right)\right)\delta(n_+q - n_+p)$$

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]

Validation at Fixed Order: $\Delta^{dyn\,(2)}_{\mathrm{NLP-coll}}$ piece

Focus on one piece of the factorization formula

$$\Delta^{dyn\,(2)}_{\mathrm{NLP-coll}}(z) \quad = \quad 4Q \int d\omega \, J^{(1)}_{1,1}(x_a\, n_+p_A;\omega)\, S^{(1)}_1(\Omega;\omega)\,.$$

The factorization formula is valid for unrenormalized objects. Performing the convolution in d - dimensions reproduces fixed NNLO result:

$$\Delta^{(2)}_{\mathrm{NLP-coll}} = \frac{\alpha_s^2}{(4\pi)^2}\Bigg(C_A C_F\left(\frac{20}{\epsilon} - 60\log(1-z) + 8 + \mathcal{O}(\epsilon^1)\right)$$
$$+C_F^2\left(\frac{-16}{\epsilon^2} - \frac{20}{\epsilon} + \frac{48}{\epsilon}\log(1-z) + 60\log(1-z) - 72\log^2(1-z) + \mathcal{O}(\epsilon^1)\right)\Bigg)$$

after we set the scale to hard. In agreement with equation (4.22) of [D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza, C. White, 1503.05156]

We have also computed and checked the remaining contributions

- Hard: 1-loop hard and NLO soft functions, $\Delta^{dyn(2)}_{\mathrm{NLP-hard}}$.
  [M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]
- Soft: NNLO soft functions, $\Delta^{dyn(2)}_{\mathrm{NLP-soft}}$ $\rightarrow$ focus on next.
  [A. Broggio, **SJ**, L. Vernazza, 2107.07353]

NLP factorization formula for Drell-Yan

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)}\frac{\hat{\sigma}(z)}{z} \qquad \Delta_{\rm NLP}(z) = \Delta^{dyn}_{\rm NLP}(z) + \Delta^{kin}_{\rm NLP}(z)$$

where [M. Beneke, A.Broggio, **SJ**, L. Vernazza, 1912.01585]

$$\begin{aligned}\Delta^{dyn}_{\rm NLP}(z) \quad = \quad & -\frac{2}{(1-\epsilon)}\, Q \left[\left(\frac{\not{n}_-}{4}\right)\gamma_{\perp\rho}\left(\frac{\not{n}_+}{4}\right)\gamma^{\rho}_{\perp}\right]_{\beta\gamma} \\ & \times \int d(n_+p)\, C^{A0,A0}(n_+p, x_b n_- p_B)\, C^{*A0A0}(x_a n_+ p_A, x_b n_- p_B) \\ & \times \sum_{i=1}^{4}\int \{d\omega_j\}\;\; J_{i,\gamma\beta}(n_+p, x_a n_+ p_A; \{\omega_j\})\;\; S_i(\Omega;\{\omega_j\}) \;+\text{h.c.}\end{aligned}$$

We now focus on $\mathcal{O}(\alpha_s^2)$ calculation of:

$$S_i(\Omega;\{\omega_j\}) = \int \frac{dx^0}{4\pi}\, e^{i\Omega\, x^0/2} \int \left\{\frac{dz_{j-}}{2\pi}\right\} e^{-i\omega_j z_{j-}}\, S_i(x_0;\{z_{j-}\})$$

Generalised Soft Functions

We make use of the soft building blocks

$$\mathcal{B}^{\mu}_{\pm} = Y^{\dagger}_{\pm}\,[i\,D^{\mu}_{s}\,Y_{\pm}]\,, \quad q^{\pm} = Y^{\dagger}_{\pm}\,q_{s}$$

The relevant soft functions are

$$S_1(x^0; z_-) = \frac{1}{N_c}\,\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{i\partial^{\nu}_{\perp}}{in_-\partial}\mathcal{B}^{+}_{\nu_\perp}(z_-)\right)|0\rangle$$

$$S_3(x^0; z_-) = \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]\frac{1}{(in_-\partial)^2}\left[\mathcal{B}^{+\,\mu_\perp}(z_-),\left[in_-\partial\mathcal{B}^{+}_{\mu_\perp}(z_-)\right]\right]\right)|0\rangle$$

$$S^{AB}_{4;\mu\nu,bf}(x^0; z_{1-}, z_{2-}) = \frac{1}{N_c}\,\mathrm{Tr}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba}$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\mathcal{B}^{+A}_{\mu_\perp}(z_{1-})\,\mathcal{B}^{+B}_{\nu_\perp}(z_{2-})\right)|0\rangle$$

$$S_{5;bfgh,\sigma\lambda}(x^0; z_{1-}, z_{2-}) = \frac{1}{N_c}\,\langle 0|\bar{\mathbf{T}}\left[Y^{\dagger}_{+}(x^0)Y_{-}(x^0)\right]_{ba}$$
$$\times\,\mathbf{T}\left(\left[Y^{\dagger}_{-}(0)Y_{+}(0)\right]_{af}\frac{g_s^2}{(in_-\partial_{z_1})(in_-\partial_{z_2})}\,q_{+\sigma g}(z_{1-})\,\bar{q}_{+\lambda h}(z_{2-})\right)|0\rangle$$

Not functions of Wilson lines only!

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\;\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\,\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\;e^{iz_-k}$$

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\ e^{iz_-k}$$

Double emission:

$$\langle g^{K_1}(k_1)g^{K_2}(k_2)|\mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\ \frac{i\partial_\perp^\mu}{in_-\partial}\mathcal{B}^+_{\mu_\perp}(z_-)\right]|0\rangle \quad =$$

$$g_s^2\mathbf{T}^{K_2}\,\mathbf{T}^{K_1}\,\frac{1}{(n_-k_1)}\frac{n_-^{\eta_2}}{(n_-k_2)}\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-k_1}$$

$$+\dots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\left(-\frac{(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2})\,n_-^{\eta_1}}{(n_-k_1)} + \frac{(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1})\ n_-^{\eta_2}}{(n_-k_2)}\right.$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\ \left(k_{1\perp}^2 + k_{1\perp}\cdot k_{2\perp}\right)$$

$$\left.-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp} + k_{2\perp}^2\right)\Big]\right)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\ e^{iz_-(k_1+k_2)}$$

$$+\dots$$

$\rightarrow$ $\delta(\omega - n_-k_1)$ constraint in the integrand.

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k)|\mathbf{T}\left(Y_-^\dagger(0)Y_+(0)\;\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle = \mathbf{T}^A\frac{g_s}{(n_-k)}\left[k_\perp^\eta - \frac{k_\perp^2}{(n_-k)}n_-^\eta\right]\epsilon^*_\eta(k)\;e^{iz_-k}$$

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$$g_s^2\mathbf{T}^{K_2}\,\mathbf{T}^{K_1}\,\frac{1}{(n_-k_1)}\,\frac{n_-^{\eta_2}}{(n_-k_2)}\,\left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_-k_1)}n_-^{\eta_1}\right]\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\;e^{iz_-k_1}$$

$$+\ldots$$

$$+g_s^2\,if^{K_1K_2K}\mathbf{T}^K\frac{1}{n_-(k_1+k_2)}\Bigg(-\frac{\left(k_{1\perp}^{\eta_2}+k_{2\perp}^{\eta_2}\right)n_-^{\eta_1}}{(n_-k_1)} + \frac{\left(k_{1\perp}^{\eta_1}+k_{2\perp}^{\eta_1}\right)\,n_-^{\eta_2}}{(n_-k_2)}$$

$$-\frac{n_-^{\eta_1}n_-^{\eta_2}}{n_-(k_1+k_2)(n_-k_1)(n_-k_2)}\Big[(n_-k_1)\;\left(k_{1\perp}^2+k_{1\perp}\cdot k_{2\perp}\right)$$

$$-(n_-k_2)\left(k_{2\perp}\cdot k_{1\perp}+k_{2\perp}^2\right)\Big]\Bigg)\epsilon^*_{\eta_1}(k_1)\epsilon^*_{\eta_2}(k_2)\;e^{iz_-(k_1+k_2)}$$

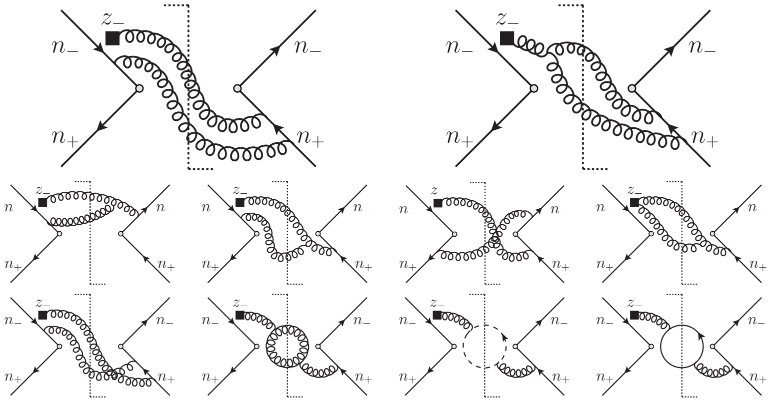
$$+\ldots$$

$\rightarrow \quad \delta(\omega - n_-k_1 - n_-k_2)$ constraint in the integrand.

Some sample diagrams

For the S_1 soft function:

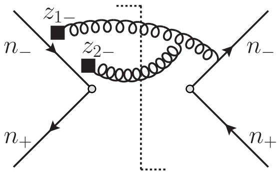
$$S_1(x^0; z_-) \;=\; \frac{1}{N_c}\,\mathrm{Tr}\langle 0|\bar{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right]\mathbf{T}\left(\left[Y_-^\dagger(0)Y_+(0)\right]\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\nu_\perp}(z_-)\right)|0\rangle$$

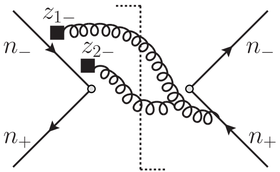


...

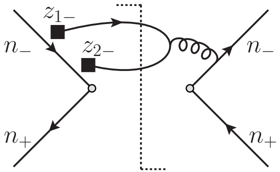
Some sample diagrams

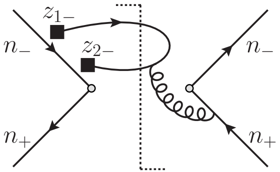
For the S_4 soft function:





For the S_5 soft function:



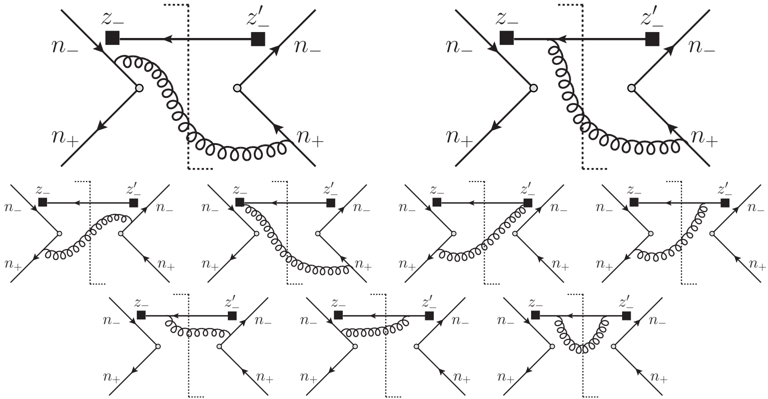


Some sample two-loop diagrams: double real

$$\mathcal{S}(\Omega,\omega,\omega') = \sum_{s,\lambda} \int \frac{d^dk_1}{(2\pi)^{d-1}}\delta^+(k_1^2) \int \frac{d^dk_2}{(2\pi)^{d-1}}\delta^+(k_2^2)\delta(\omega - n_-k_1 - n_-k_2)\delta(\omega' - n_-k_1)$$

$$\times \frac{1}{C_F C_A}\langle 0|\bar{\mathbf{T}}\Big(\frac{g_s}{in_-\partial_{z'}}\, \bar{q}_{+\sigma k}(z'_-)\ \mathbf{T}^D\ \{Y_+^\dagger(0)Y_-(0)\}\Big)|X_s\rangle$$

$$\times \frac{\not{n}_{-\sigma\beta}}{4}\langle X_s|\mathbf{T}\Big(\{Y_-^\dagger(0)Y_+(0)\}\ \mathbf{T}^D\ \frac{g_s}{in_-\partial_z}\ q_{+\beta}(z_-)\Big)|0\rangle\delta(\Omega - 2k_1^0 - 2k_2^0)$$



The calculation

Methods developed for calculations of two-loop soft functions at *leading power.*

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglia, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = n_-(k_1 + k_2),$$

$$P_4 = k_1^2\ , \quad P_5 = k_2^2\ , \quad P_6 = \big(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\big)\ , \quad P_7 = \big(\omega - n_- k_1\big)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i} \left[\frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+} \right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

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- The reduction is implemented in `LiteRed`
- 9 topologies are needed to reduce the soft functions
- We have 8 Master Integrals (MIs)
  - 5 MIs implementing the $\delta(\omega - n_- k_1)$ constraint: $\hat{I}_1$ – $\hat{I}_5$
  - 2 MIs with $\delta(\omega - n_- k_1 - n_- k_2)$: $\hat{I}_6$ and $\hat{I}_7$
  - 1 MI with $\delta(\omega_1 - n_- k_1)\, \delta(\omega_2 - n_- k_2)$: $\hat{I}_8$

The calculation

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$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = k_1^2\,, \quad P_4 = k_2^2\,, \quad P_5 = \big(\omega - n_- k_1\big)$$

$$P_6 = \big(\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2\big)\,, \quad P_7 = \big(\omega' - n_- k_1 - n_- k_2\big)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^{7} \frac{1}{P_i^{\alpha_i}}$$

- The reduction is implemented in `LiteRed`
- 4 + 1 topologies are needed to reduce the soft functions
- We have 6 + 2 Master Integrals (MIs)
  - 4 MIs implementing the $\delta(\omega - n_- k_1)\delta(\omega' - n_- k_1 - n_- k_2)$ and $\omega \leftrightarrow \omega'$ constraint: $\hat{I}_1$ – $\hat{I}_4$
  - 1 MI with $\delta(\omega - n_- k_1)\delta(\omega - \omega')$: $\hat{I}_5$
  - 1 MI with $\delta(\omega - n_- k_1 - n_- k_2)\,\delta(\omega - \omega')$: $\hat{I}_6$

Reduced expressions

Double real

$$
\begin{aligned}
&S_{g\bar{q}}^{(2)2r0v}(\Omega,\omega,\omega') \\
&\quad = \frac{\alpha_s^2}{(4\pi)^2} T_F \Bigg[C_F \left(\frac{(4-\epsilon)(1-\epsilon)(1-2\epsilon)}{\epsilon^2\omega^2(\Omega-\omega)}\,\hat{I}_6 + \frac{4(2-3\epsilon)(1-3\epsilon)}{\epsilon^2\omega(\Omega-\omega)^2}\,\hat{I}_5 \right) \delta(\omega-\omega') \\
&\qquad + (C_A - 2C_F) \Bigg(\frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\,\omega\omega'(\Omega-\omega)(\omega-\omega')}\,\hat{I}_3 - \frac{(1-2\epsilon)(\omega+\omega')}{\epsilon\,\omega\omega'(\Omega-\omega')(\omega-\omega')}\,\hat{I}_1 \\
&\qquad + \frac{(\Omega-\omega)(\omega+\omega')}{2\omega\omega'}\hat{I}_4 + \frac{(\Omega-\omega')(\omega+\omega')}{2\omega\omega'}\hat{I}_2 \Bigg) \Bigg]
\end{aligned}
$$

where for example

$$
\hat{I}_1(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(0,0,1,1,1,1,1), \quad \hat{I}_2(\Omega,\omega,\omega') \equiv \hat{I}_{\mathcal{A}}(1,1,1,1,1,1,1)
$$

and family $\mathcal{A}$ propagators are

$$
\begin{aligned}
&P_1 = (k_1+k_2)^2, \quad P_2 = n_+k_2, \quad P_3 = k_1^2, \quad P_4 = k_2^2, \\
&P_5 = \big(\Omega - n_-k_1 - n_-k_2 - n_+k_1 - n_+k_2\big), \quad P_6 = \big(\omega - n_-k_1\big), \\
&P_7 = \big(\omega' - n_-k_1 - n_-k_2\big)
\end{aligned}
$$

The integrals are calculated using the differential equations method.

DE method for MIs

Convenient to change to dimensionless variable $\omega \to r\,\Omega$

$$I_1'(r) = \frac{1}{\Omega^2}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_1(\Omega, r), \qquad I_2'(r) = \frac{1}{\Omega}\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_2(\Omega, r),$$

$$I_3'(r) = \left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_3(\Omega, r), \qquad I_4'(r) = \Omega\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_4(\Omega, r),$$

$$I_5'(r) = \Omega^2\left(\frac{\Omega}{\mu}\right)^{4\epsilon}\hat{I}_5(\Omega, r)$$

System of DEs can be put into canonical form [J. Henn, 1304.1806]

$$\frac{d\vec{I}(r)}{dr} = \epsilon\, A(r)\cdot\vec{I}(r)$$

$$I_1'(r) = \frac{2(1-r)^2}{2-9\epsilon+9\epsilon^2}I_1(r),$$

$$I_3'(r) = \frac{1}{\epsilon^2}I_3(r),$$

$$I_4'(r) = -\frac{1}{\epsilon^2(1-r)}I_4(r),$$

$$I_5'(r) = -\frac{1+r}{2\epsilon^2(1-r)r}I_4(r) + \frac{1}{\epsilon^2 r}I_5(r)$$

$$A(r) = \begin{bmatrix} -\frac{1}{r}+\frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

Virtual real result

This contribution is given by [A. Broggio, **SJ**, L. Vernazza, to appear]

$$
\begin{aligned}
S^{(2)1r1v}_{g\bar{q}}(\Omega,\omega,\omega') &= \frac{\alpha_s^2\, T_F}{(4\pi)^2}(2C_F - C_A)\frac{e^{2\epsilon\gamma_E}\,\Gamma[1+\epsilon]}{\epsilon\,\Gamma[1-\epsilon]} \\
&\cdot \mathrm{Re}\Bigg\{\left[\frac{\omega'-\omega}{\omega(\omega')^2}\,{}_2F_1\left(1,1+\epsilon,1-\epsilon,-\frac{\omega}{\omega'}\right) - \frac{1}{\omega\omega'}\right]\left(\frac{\mu^4}{\omega\omega'(\Omega-\omega')^2}\right)^{\epsilon}\theta(\omega) \\
&\quad + \frac{2(\omega+\omega')}{\omega\omega'(\omega'-\omega)}\left(\frac{\mu^4}{(\omega'-\omega)^2(\Omega-\omega')^2}\right)^{\epsilon}\frac{\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]}\theta(\omega'-\omega)\Bigg\}\theta(\omega')\theta(\Omega-\omega')
\end{aligned}
$$

Integrating over ω,ω'

$$
\begin{aligned}
\Delta^{(2)}_{g\bar{q}}(z)|_{\mathrm{NLP,s,1r1v}} &= -\left(\frac{\alpha_s}{4\pi}\right)^2 T_F\,(C_A - 2C_F)\left(\frac{\mu^2}{\Omega^2}\right)^{2\epsilon} \\
&\times \frac{2\mathrm{Re}[e^{-i\epsilon\pi}]e^{2\epsilon\gamma_E}\,\Gamma[1-2\epsilon]\Gamma[1-\epsilon]^2\Gamma[1+\epsilon]^2}{\epsilon^3\Gamma[1-4\epsilon]}
\end{aligned}
$$

Results at NNLO

The S_1 soft function is the only one with C_F^2 contributions. Combine our results for two-loop soft with tree-level collinear according to factorisation theorem

$$\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) = 4\,Q\,\Omega\,H^{(0)}(Q^2)\,\int dr\,J^{(0)}_{1,1}\,(x_a(n_+p_A);\omega)\,S^{(2)2r0v}_{1,C_F^2}\,(\Omega,r)$$

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F^2}}(z) \quad &= \quad \frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\bigg(\frac{32}{\epsilon^3} - \frac{128}{\epsilon^2}\ln(1-z) + \frac{256}{\epsilon}\ln^2(1-z) - \frac{112\pi^2}{3\epsilon} \\ &\quad + \frac{32}{3}\left(-32\ln^3(1-z) + 14\pi^2\ln(1-z) - 62\zeta(3)\right) + \mathcal{O}(\epsilon)\bigg)\end{aligned}$$

The S_1 1r1v gives leading logarithms proportional to $C_F C_A$, which are cancelled exactly by the leading poles of 2r0v contribution to S_1

$$\begin{aligned}\Delta^{dyn\,(2)2r0v}_{\mathrm{NLP-soft},S_{1,C_F C_A}}(z) \quad &= \quad \frac{\alpha_s^2}{(4\pi)^2}\,C_F C_A\bigg(\frac{8}{\epsilon^3} - \frac{4}{3\epsilon^2}(24\ln(1-z) - 11) \\ &\quad - \frac{16}{9\epsilon}\left(-36\ln^2(1-z) + 33\ln(1-z) + 6\pi^2 - 16\right) \\ &\quad - \frac{256}{3}\ln^3(1-z) + \frac{352}{3}\ln^2(1-z) + \frac{128}{3}\pi^2\ln(1-z) \\ &\quad - \frac{1024}{9}\ln(1-z) - \frac{616\zeta(3)}{3} - \frac{154\pi^2}{9} + \frac{1484}{27} + \mathcal{O}(\epsilon)\bigg)\end{aligned}$$

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

Results at N^{3}LO

Interestingly, we can partially validate our results at N^{3}LO by comparing to [N. Bahjat-Abbas, J. Sinninghe Damsté, L. Vernazza, C. White, 1807.09246]

$$\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad 4Q \int d\omega \, J^{(1)}_{1,1}(x_a\, n_+p_A;\omega)\, S^{(2)}_{1,C_F^2}(\Omega;\omega)$$

Using one-loop collinear function from [M.Beneke, A.Broggio, **SJ**, L.Vernazza, 1912.01585]

$$S^{(2)2r0v}_{1,C_F^2}(\Omega,\omega) = 8\,\frac{\alpha_s^2}{(4\pi)^2}\,C_F^2\,\left(\frac{\omega\,(\Omega-\omega)^3}{\mu^4}\right)^{-\epsilon}\,\frac{1}{\omega}\,\frac{1}{\epsilon^2}\,\frac{e^{2\epsilon\gamma_E}\,\Gamma[1-\epsilon]}{\Gamma[1-3\epsilon]}\theta(\Omega-\omega)\theta(\omega)$$

We find a closed d-dimensional result, expanding

$$\begin{aligned}
\Delta^{dyn\,(3)}_{\mathrm{NLP-coll},\,C_F^3}(z) \quad = \quad & \frac{\alpha_s^3}{(4\pi)^3}C_F^3\Bigg(-\frac{64}{\epsilon^4}+\frac{80(4\ln(1-z)-1)}{\epsilon^3}+\frac{16}{\epsilon^2}\Big(-50\ln^2(1-z) \\
& +25\ln(1-z)+7\pi^2-6\Big)+\frac{1}{\epsilon}\Big(\frac{4000}{3}\ln^3(1-z)-1000\ln^2(1-z) \\
& -560\pi^2\ln(1-z)+480\ln(1-z)+2624\zeta(3)+140\pi^2-128\Big) \\
& -\frac{5000}{3}\ln^4(1-z)+\frac{5000}{3}\ln^3(1-z)+1400\pi^2\ln^2(1-z) \\
& -1200\ln^2(1-z)-700\pi^2\ln(1-z)+640\ln(1-z) \\
& +\zeta(3)(3280-13120\ln(1-z))+\frac{62\pi^4}{5}+168\pi^2-192\Bigg)
\end{aligned}$$

Problems at the Endpoint

Focus on one piece of the factorization formula

$$\int d\omega \, J_1^{(1)}(x_a n_+ p_A; \omega) \, S_1^{(1)}(\Omega, \omega)$$

with

$$J_1^{(1)}(x_a \, n_+ p_A; \, \omega) = \frac{\alpha_s}{4\pi} \frac{1}{(x_a \, n_+ p_A)} \left(\frac{(x_a \, n_+ p_A) \, \omega}{\mu^2} \right)^{-\epsilon} \frac{e^{\epsilon \, \gamma_E} \, \Gamma[1+\epsilon]\Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon)\Gamma[2-2\epsilon]} \times \left(C_F \left(-\frac{4}{\epsilon} + 3 + 8\epsilon + \epsilon^2 \right) - C_A \left(-5 + 8\epsilon + \epsilon^2 \right) \right)$$

$$S_1(\Omega, \omega) = \frac{\alpha \, C_F}{2\pi} \frac{\mu^{2\epsilon} \, e^{\epsilon \gamma_E}}{\Gamma[1-\epsilon]} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^{\epsilon}} \theta(\omega)\theta(\Omega - \omega) \, + \, \mathcal{O}(\alpha^2)$$

As we have seen before, performing the $d\omega$ convolution integral in d-dimensions, and only *after* expanding in ϵ gives the correct results.

Problems at the Endpoint

$$\int_0^\Omega d\omega \, \underbrace{\left(n_+p\,\omega\right)^{-\epsilon}}_{\text{collinear piece}} \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega-\omega)^\epsilon}}_{\text{soft piece}}$$

For resummation, we treat the two objects independently, and expand in ϵ prior to performing the final convolution. However, there is a problem! At two loops:

$$J_1^{(1)}(x_a n_+ p_A; \omega) \sim \alpha_s \log(\omega)$$

and

$$S_1\,(\Omega, \omega) \sim \alpha_s\,\delta(\omega) + \mathcal{O}(\alpha^2)$$

Hence, first expanding in ϵ and performing convolution after yields

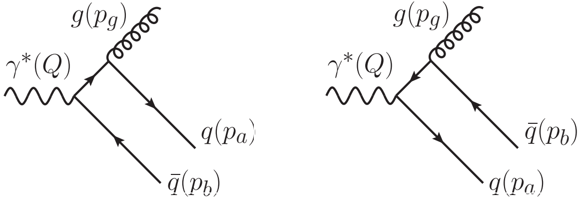
$$\begin{aligned}\Delta^{dyn\,(2)}_{\mathrm{NLP-coll}}(z) \quad &= \quad \frac{\alpha_s^2}{(4\pi)^2}\Bigg(C_F^2\left(-\frac{32}{\epsilon^2}-\frac{8}{\epsilon}\left[5-8\ln(1-z)-4\int d\omega\,\delta(\omega)\ln\left(\frac{\omega}{Q}\right)\right]\right)\\ &\quad + C_A C_F\,\frac{40}{\epsilon} + \mathcal{O}(\epsilon^0)\Bigg)\end{aligned}$$

A couple of issues arise. The convolution $d\omega$ integral is now divergent at the endpoint. This prohibits the application of standard RG methods.

Off-diagonal thrust:
Standard factorization

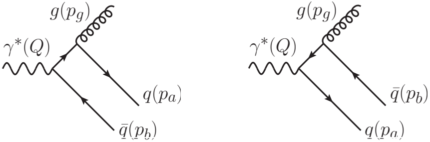
Thrust power corrections: off-diagonal channel

$$e^+e^- \to \gamma^* \to [g]_c + [q\bar{q}]_{\bar{c}}$$



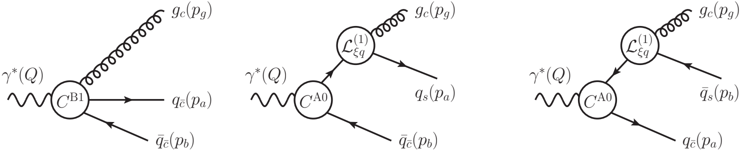
Thrust power corrections: off-diagonal channel

$$e^+e^- \to \gamma^* \to [g]_c + [q\bar{q}]_{\bar{c}}$$



$$\begin{aligned}\bar{\psi}\gamma_\perp^\mu\psi(0) \quad &= \quad \int dt d\bar{t}\, \widetilde{C}^{\mathrm{A0}}(t,\bar{t})\, \bar{\chi}_c(tn_+)\gamma_\perp^\mu \chi_{\bar{c}}(\bar{t}n_-) + (c \leftrightarrow \bar{c}) \\ &\quad + \sum_{i=1,2} \int dt d\bar{t}_1 d\bar{t}_2\, \widetilde{C}_i^{\mathrm{B1}}(t,\bar{t}_1,\bar{t}_2)\, \bar{\chi}_{\bar{c}}(\bar{t}_1 n_-)\Gamma_i^{\mu\nu}\mathcal{A}_{c\perp\nu}(tn_+)\chi_{\bar{c}}(\bar{t}_2 n_-) + \ldots\end{aligned}$$

$$\mathcal{L}_{\xi q}(x) = \bar{q}_s(x_-)\not{\mathcal{A}}_{c\perp}(x)\chi_c(x) + \mathrm{h.c.} \qquad \Gamma_1^{\mu\nu} = \frac{\not{n}_-}{2}\gamma_\perp^\nu\gamma_\perp^\mu\,, \Gamma_2^{\mu\nu} = \frac{\not{n}_-}{2}\gamma_\perp^\mu\gamma_\perp^\nu$$



Bare factorization theorem: Endpoint divergence in A-type

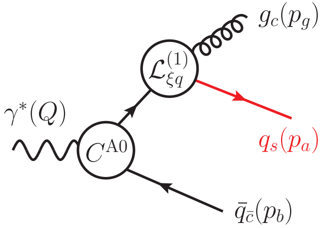
$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R^2 dM_L^2}\Big|_{\mathrm{A-type}} \sim \int d\omega d\omega' \, |C^{\mathrm{A0}}|^2 \, \mathcal{J}_{\bar{c}}^{(\bar{q})} \, \mathcal{J}_c(\omega,\omega') \, S_{\mathrm{NLP}}(\omega,\omega')$$

Consider the asymptotic behaviour

$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R^2 dM_L^2}\Big|_{\mathrm{A-type}} \sim \int_{M_R^2/Q}^{\infty} d\omega \, \frac{1}{\omega^{1+\epsilon}} + \dots$$

Endpoint divergence from the soft quark or soft antiquark becoming too energetic, $\omega \to \infty$.

This signals breaking of standard factorization.



Bare factorization theorem: Endpoint divergence in B-type

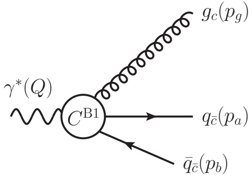
$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R^2 dM_L^2}|_{\mathrm{B-type}} \sim \int_0^1 dr dr'\, C^{\mathrm{B1}*}(r')C^{\mathrm{B1}}(r) \times \mathcal{J}_{\bar{c}}^{q\bar{q}}(r,r') \otimes \mathcal{J}_c^{(g)} \otimes S^{(g)}$$

Consider the asymptotic behaviour

$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R^2 dM_L^2}|_{\mathrm{B-type}} \sim \int_0^1 dr \left[\frac{1}{r^{1+\epsilon}} + \frac{1}{(1-r)^{1+\epsilon}}\right]$$

Endpoint divergence originating from the quark (or antiquark) becoming soft, $r \to 0$ ($r \to 1$) limit.

So we have breaking of standard factorization in the B-type contribution as well.



No issue at fixed orders

We can check the lowest fixed order results:

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dM_R^2\,dM_L^2}\Big|_{\substack{\text{A–type,}\\ \text{tree}}} = \frac{\alpha_s}{4\pi}\frac{2C_F}{Q^2}\frac{f(\epsilon)}{e^{-\epsilon\gamma_E}\Gamma(1-\epsilon)} \times\left[\frac{1}{\epsilon}\delta^+(M_L^2)\left(\frac{(M_R^2)^2}{Q^2\mu^2}\right)^{-\epsilon} - \delta^+(M_R^2)\frac{1}{1-\epsilon}\left(\frac{(M_L^2)^2}{Q^2\mu^2}\right)^{-\epsilon}\right]$$

The pole comes from logarithmic divergence in $\int_{l_+}^{\infty} d\omega/\omega$

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dM_L^2\,dM_R^2}\Big|_{\substack{\text{B–type,}\\ \text{tree}}} = \frac{\alpha_s}{4\pi}\frac{4C_F}{Q^2}f(\epsilon)\left\{-\frac{1}{\epsilon}+\frac{\epsilon}{(1-\epsilon)^2}\right\}\delta^+(M_L^2)\left(\frac{M_R^2}{\mu^2 e^{\gamma_E}}\right)^{-\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}$$

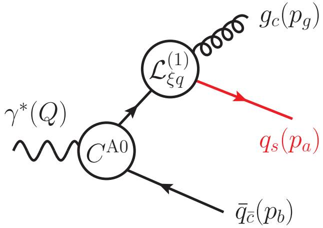
Here the pole is due to logarithmic divergence as $r \to 0$.

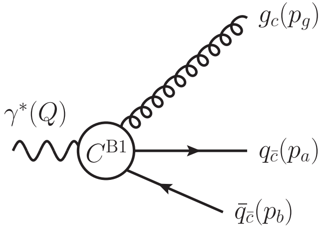
Summing we find

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dM_L^2\,dM_R^2} = \frac{\alpha_s C_F}{\pi}\frac{1}{Q^2}\left\{\delta^+(M_L^2)\left[\ln\frac{Q^2}{M_R^2}-1\right]-\delta^+(M_R^2)\right\}$$

Integrand level observations

$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R dM_L} = \int d\omega d\omega' \left|C^{A0}\right|^2 \times \mathcal{J}_{\bar{c}}^{(\bar{q})} \times \mathcal{J}_c\left(\omega,\omega'\right) \otimes S_{\mathrm{NLP}}\left(\omega,\omega'\right)$$
$$+ \int dr dr'\, C^{B1}(r) C^{B1}(r')^* \otimes \mathcal{J}_{\bar{c}}^{q\bar{q}}\left(r,r'\right) \times \mathcal{J}_c^{(g)} \times S^{(g)}$$





We see that $S_{\mathrm{NLP}} \to S^{(g)}$ and $\mathcal{J}_{\bar{c}}^{(\bar{q})} \to \mathcal{J}_{\bar{c}}^{q\bar{q}}$. The *integrands* of the two terms should become identical.

Integrand level observations

$$\frac{1}{\sigma_0}\frac{d\sigma}{dM_R dM_L} = \int d\omega d\omega' \left|C^{A0}\right|^2 \times \mathcal{J}_{\bar{c}}^{(\bar{q})} \times \mathcal{J}_c\left(\omega,\omega'\right) \otimes S_{\mathrm{NLP}}\left(\omega,\omega'\right)$$
$$+ \int dr dr'\, C^{B1}(r) C^{B1}(r')^* \otimes \mathcal{J}_{\bar{c}}^{q\bar{q}}\left(r,r'\right) \times \mathcal{J}_c^{(g)} \times S^{(g)}$$

We see this explicitly at first order!
A-type terms becomes:

$$\frac{\alpha_s}{4\pi}\frac{2C_F}{Q^2}\delta^+(M_L^2)\frac{\delta(\omega-\omega')}{\omega\omega'}\frac{f(\epsilon)}{e^{-\epsilon\gamma_E}\Gamma(1-\epsilon)}\omega\left(\frac{M_R^2\omega}{Q\mu^2}\right)^{-\epsilon}$$

and the B-type:

$$\frac{\alpha_s}{4\pi}\frac{2C_F}{Q^2}\delta^+(M_L^2)\frac{\delta(r-r')}{rr'}\frac{f(\epsilon)}{\Gamma(1-\epsilon)}r\left(\frac{M_R^2 r}{\mu^2 e^{\gamma_E}}\right)^{-\epsilon}$$

Identical if we identify $r=\omega/Q$, $r'=\omega'/Q$

Factorization theorem rearrangement

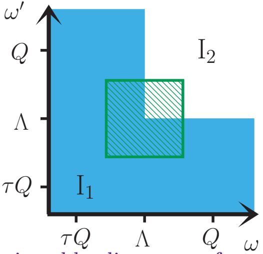
We add the following scaleless integral

$$0 = \frac{2C_F}{Q} f(\epsilon)\, |C^{\mathrm{A0}}(Q^2)|^2 \widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R)\widetilde{\mathcal{J}}_{c}^{(g)}(s_L) \times \int_0^\infty d\omega d\omega' \, \frac{D^{\mathrm{B1}}(\omega Q)}{\omega} \frac{D^{\mathrm{B1}*}(\omega' Q)}{\omega'} \left[\!\left[\widetilde{S}_{\mathrm{NLP}}(s_R, s_L, \omega, \omega') \right]\!\right]$$

where D^{B1} are universal objects describing the splitting of a collinear quark into a collinear gluon and a soft-collinear quark. The same object appears in the endpoint factorization theorem for $H \to gg$ [Z. L. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

We split it by introducing a parameter Λ: $0 = I_1 + I_2$.

Then subtract I_1 from the B-type term and I_2 from the A-type term.



Renormalized factorization theorem

A-type

$$\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}\Big|_{\mathrm{A-type}} = \frac{2C_F}{Q} f(\epsilon)\,|C^{\mathrm{A0}}(Q^2)|^2\,\widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R)\int_0^\infty d\omega d\omega'$$

$$\times\Big\{\widetilde{\mathcal{J}}_c(s_L,\omega,\omega')\,\widetilde{S}_{\mathrm{NLP}}(s_R,s_L,\omega,\omega')$$

$$-\,\theta(\omega-\Lambda)\theta(\omega'-\Lambda)\,[\![\widetilde{\mathcal{J}}_c(s_L,\omega,\omega')]\!][\![\widetilde{S}_{\mathrm{NLP}}(s_R,s_L,\omega,\omega')]\!]$$

$$+\,\widetilde{\widetilde{\mathcal{J}}}_c(s_L,\omega,\omega')\,\widetilde{\widetilde{S}}_{\mathrm{NLP}}(s_R,s_L,\omega,\omega')\Big\}$$

B-type

$$\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}\Big|_{\substack{\mathrm{B-type}\\ i=i'=1}} = \frac{2C_F}{Q^2} f(\epsilon)\,\widetilde{\mathcal{J}}_c^{(g)}(s_L)\,\widetilde{S}^{(g)}(s_R,s_L)\int_0^\infty dr dr'$$

$$\times\Big[\theta(1-r)\theta(1-r')\,C_1^{\mathrm{B1}*}(Q^2,r')C_1^{\mathrm{B1}}(Q^2,r)\,\widetilde{\mathcal{J}}_{\bar{c}}^{q\bar{q}(8)}(s_R,r,r')$$

$$-\big[1-\theta(r-\Lambda/Q)\theta(r'-\Lambda/Q)\big]$$

$$\times\,[\![C_1^{\mathrm{B1}*}(Q^2,r')]\!]_0\,[\![C_1^{\mathrm{B1}}(Q^2,r)]\!]_0\,[\![\widetilde{\mathcal{J}}_{\bar{c}}^{q\bar{q}(8)}(s_R,r,r')]\!]_0\Big]$$

Resummed result for NLP-LL gluon thrust

Putting everything together

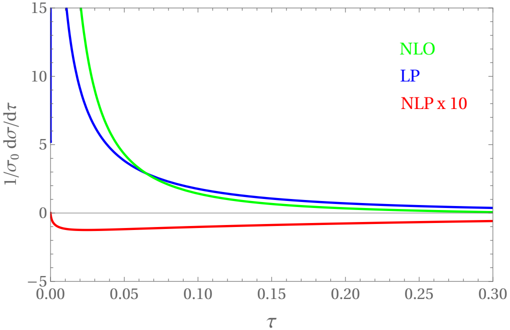
$$\frac{1}{\sigma_0}\frac{\widetilde{d\sigma}}{ds_R ds_L}|_{\mathrm{LL}} = 2\cdot\frac{2C_F}{Qs_R}\frac{\alpha_s(\mu_c)}{4\pi}\;\exp\left[4C_F S\left(\mu_h,\mu_{\bar{c}}\right)+4C_A S\left(\mu_s,\mu_c\right)\right]$$
$$\times\left(\frac{Q^2}{\mu_h^2}\right)^{-2C_F A(\mu_h,\mu_{\bar{c}})}\left(\frac{1}{s_L s_R e^{2\gamma_E}\mu_s^2}\right)^{-2C_A A(\mu_s,\mu_c)}$$
$$\times\int_\sigma^Q\frac{d\omega}{\omega}\;\exp\left[4\left(C_F-C_A\right)S\left(\mu_{s\Lambda},\mu_{h\Lambda}\right)\right]\left(\frac{\omega}{s_R e^{\gamma_E}\mu_{s\Lambda}^2}\right)^{-2(C_F-C_A)A(\mu_{s\Lambda},\mu_{h\Lambda})}$$
$$\times\;(s_R e^{\gamma_E}Q)^{2C_F A(\mu_{h\Lambda},\mu_{\bar{c}})+2C_A A(\mu_c,\mu_{h\Lambda})}$$

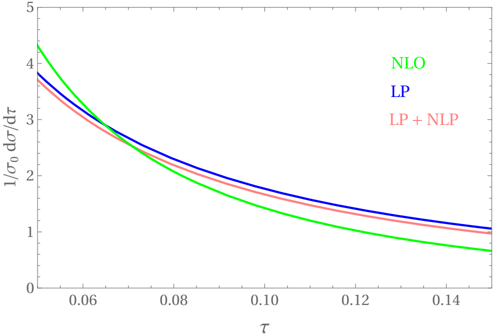
And in the double logarithmic limit, the result of previous works is recovered

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}|_{\mathrm{DL}} = \frac{C_F}{C_F-C_A}\frac{1}{\ln(1/\tau)}\,e^{-\frac{\alpha_s C_A}{\pi}\ln^2\tau}\left\{1-e^{-\frac{\alpha_s}{\pi}(C_F-C_A)\ln^2\tau}\right\}$$

[I. Moult, I. Stewart, G. Vita, H. X. Zhu, 1910.14038] [M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 2008.04943]

Numerics


15
10
5
0
−5
0.00
0.05
0.10
0.15
0.20
0.25
0.30
τ
$1/\sigma_0\, d\sigma/d\tau$
NLO
LP
NLP x 10


5
4
3
2
1
0
0.06
0.08
0.10
0.12
0.14
τ
$1/\sigma_0\, d\sigma/d\tau$
NLO
LP
LP + NLP

Summary

- SCET is designed to systematically include power corrections.
- There has been significant progress in understanding subleading power factorization theorems in the last years.
- The structure of the factorisation formula is more involved than the leading power counter part. New functions appear!
- We have verified the factorisation theorem at fixed orders by explicitly computing all the objects.
- We have uncovered issues preventing application of standard RG methods.
- Succeeded in refactorizing and resumming to LL accuracy the off diagonal "gluon" thrust.
- Interesting conceptual challenges ahead. Important to understand from the point of view of gauge theories, as well as for delivering precise theoretical predictions.

Thank you!

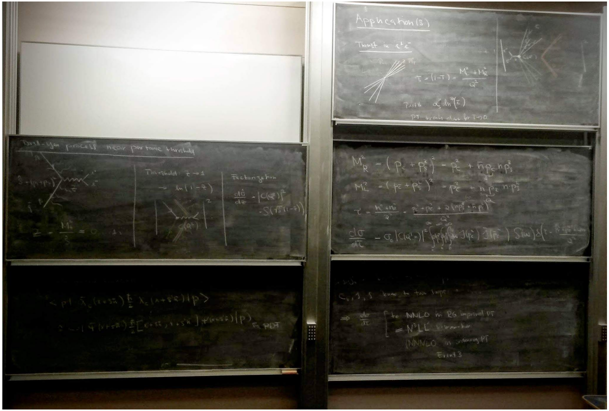
Auxiliary slides

Resources

TASI2018 Lecture notes by T. Cohen arXiv:1903.03622

Higgs Centre School of Theoretical Physics 2016 Lecture series by T. Becher based on arXiv:1403.03622 with videos:

https://media.ed.ac.uk/media/The-method-of-regions



Thank you