Hadronic contributions to the muon's anomalous magnetic moment from analyticity

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JLab theory seminar — 26/10/2020

Outline

The anomalous magnetic moment of the muon

Aoyama et al., Phys. Rept. (2020)

Hadronic vacuum polarisation

- The $\pi^+\pi^-\pi^0$ channel — Hoferichter, Hoid, BK, JHEP **1908** (2019) 137
- The $\pi^0\gamma$ channel — Hoid, Hoferichter, BK, Eur. Phys. J. (2020)

Hadronic light-by-light scattering

- Dispersive analysis of the π^0 transition form factor
- High-energy asymptotics
- Putting pieces together

Hoferichter, Hoid, BK, Leupold, Schneider, Phys. Rev. Lett. **121** (2018) 112002; JHEP **1810** (2018) 141

Summary / Outlook

The anomalous magnetic moment of the muon

- gyromagnetic ratio: magnetic moment $\leftrightarrow$ spin

$$\vec{\mu} = g \frac{e}{2m} \vec{S}$$

- Dirac theory for spin-1/2 fermions: $g_\mu = 2$
  rad. corr.: $g_\mu = 2(1 + a_\mu)$, a_μ "anomalous magnetic moment"
- one of the most precisely measured quantities in particle physics

$$a_\mu^{\mathrm{exp}} = (116\,592\,089 \pm 63) \times 10^{-11}$$

BNL E821 2006

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- ... with a significant (??) deviation from the Standard Model:

$$a_\mu^{\rm SM} = (116\,591\,810 \pm 43) \times 10^{-11}$$

$\longrightarrow 3.7\sigma$ discrepancy:

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = (279 \pm 76) \times 10^{-11}$$

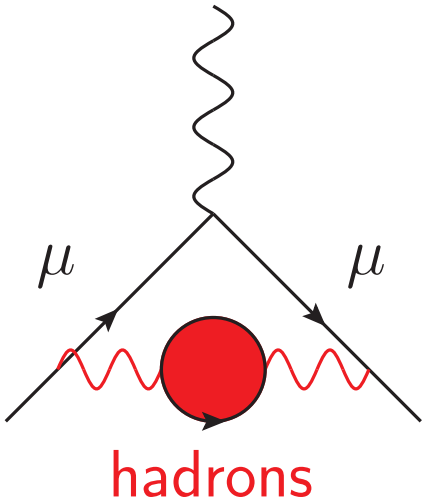
Aoyama et al. 2020

- new Fermilab experiment: reduce experimental error by factor 4

Hadronic contributions to a_μ

Aoyama et al. 2020

	a_μ $[10^{-11}]$	Δa_μ $[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.321	0.023
QED $\mathcal{O}(\alpha^2)$	413 217.626	0.007
QED $\mathcal{O}(\alpha^3)$	30 141.902	0.000
QED $\mathcal{O}(\alpha^4)$	381.004	0.017
QED $\mathcal{O}(\alpha^5)$	5.078	0.006
QED total	116 584 718.931	0.030
electroweak	153.6	1.0
had. VP (LO)	6931.	40.
had. VP (NLO)	–98.3	0.7
had. LbL	92.	19.
total	116 591 810.	43.



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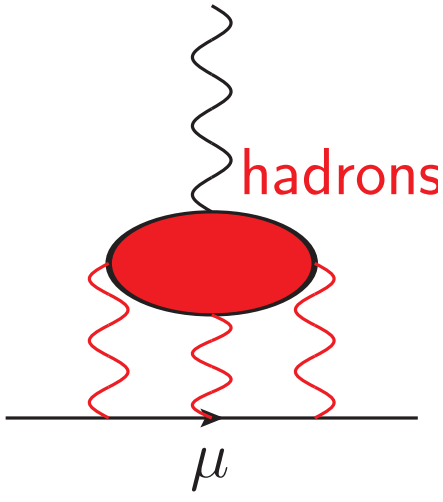
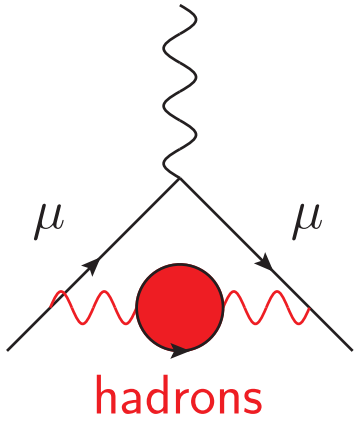


Hadronic vacuum polarization

- how to control hadronic vacuum polarization?
- characteristic scale set by muon mass
  $\longrightarrow$ this is not a perturbative QCD problem!
- dispersion relations to the rescue:
  use the optical theorem!



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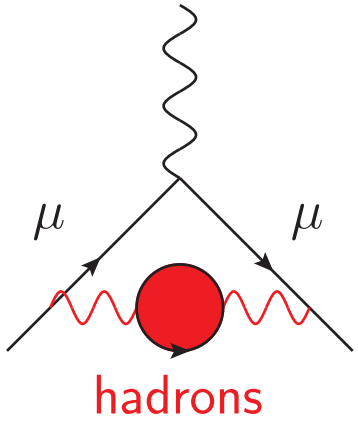
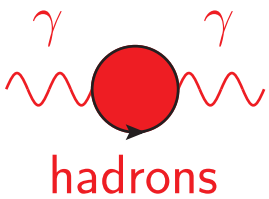
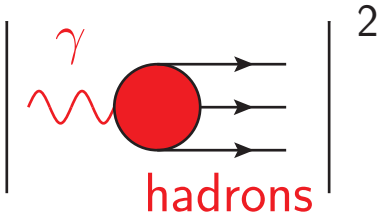


Im 

$\Leftrightarrow$ 

$\propto \sigma_{\mathrm{tot}}(e^+e^- \to \mathrm{hadrons})$

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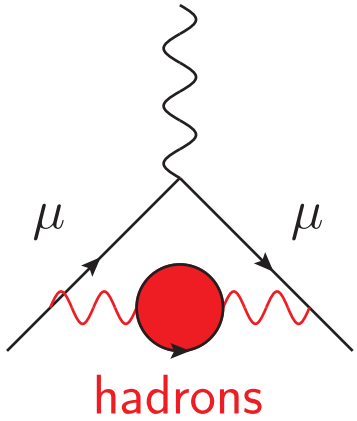

$$a_\mu^{\text{had VP}} \propto \int_{4M_\pi^2}^{\infty} K(s)\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$$

- $K(s)$: kinematical function, for large s: $K(s) \propto 1/s$,
  $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons}) \propto 1/s$

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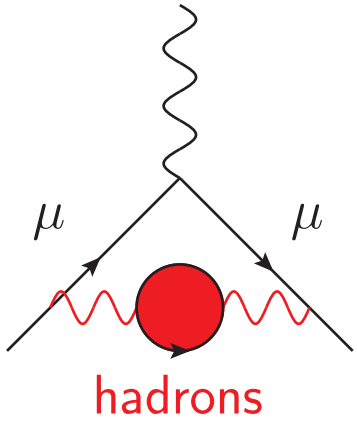

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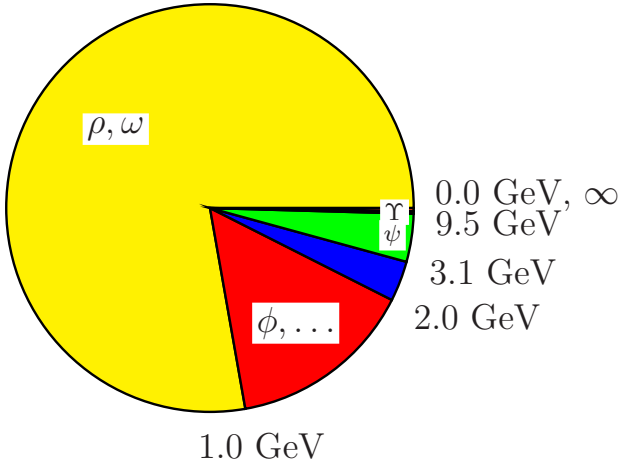
- more than 75% of $a_\mu^{\text{had VP}}$ given by
  energies $s \leq 1\,\text{GeV}^2$ Jegerlehner, Nyffeler 2009
- dominated by $e^+e^- \to \pi^+\pi^-$
  $\longrightarrow$ pion electromagnetic form factor
- well constrained by data KLOE, BABAR...



Motivation: $\pi^+\pi^-\pi^0$ with analyticity constraints

- second largest exclusive channel next to $\pi^+\pi^-$
- large discrepancy between different data integration analyses:

Channel	KNT18	DHMZ17	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^+\pi^-$	503.74 ± 1.96	506.70 ± 2.58	-2.96
$\pi^+\pi^-\pi^0$	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12
K^+K^-	23.00 ± 0.22	23.06 ± 0.41	-0.06
$K^0_S K^0_L$	13.04 ± 0.19	12.82 ± 0.24	0.22
Total	693.3 ± 2.5	693.1 ± 3.4	0.2

in units of 10^{-10} (!) A. Keshavarzi, Mainz 2018

$\longrightarrow$ factor $\sim$10 smaller overall, absolute uncertainty comparable

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Channel	KNT18	FJ17	Difference
Data based channels ($0.318 \leq \sqrt{s} \leq 2$ GeV)			
$\pi^+\pi^-$	501.68 ± 1.71	502.16 ± 2.44	-0.48
$\pi^+\pi^-\pi^0$	47.83 ± 0.89	44.32 ± 1.48	3.51
$\pi^+\pi^-\pi^+\pi^-$	15.17 ± 0.21	14.80 ± 0.36	0.37
$\pi^+\pi^-\pi^0\pi^0$	19.80 ± 0.79	19.69 ± 2.32	0.11
K^+K^-	23.05 ± 0.22	21.99 ± 0.61	1.06
$K^0_S K^0_L$	13.05 ± 0.19	13.10 ± 0.41	-0.05
Total	693.27 ± 2.46	688.07 ± 4.14	5.20

in units of 10^{-10} (!) A. Keshavarzi, Mainz 2018

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- independent cross-check with dispersion-theoretical amplitude:

  analyticity, unitarity, QCD constraints

  analogous to $\pi^+\pi^-$ Colangelo, Hoferichter, Stoffer 2018

Dispersive representation $\gamma^* \to 3\pi$

- $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ amplitude:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

  s,t,u: pion–pion invariant masses, $s+t+u = q^2 + 3M_\pi^2$

- "reconstruction theorem": neglect discontinuities in F-waves...
  $\longrightarrow$ decomposition into single-variable functions

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

- normalisation fixed from Wess–Zumino–Witten anomaly:

$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3}$$

- (s-channel) P-wave projection: $f_1(s,q^2) = \mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2)$
  $\hat{\mathcal{F}}(s,q^2)$: contribution from crossed channels $\mathcal{F}(t/u,q^2)$

Dispersive representation $\gamma^* \to 3\pi$

Unitarity relation for $\mathcal{F}(s, q^2)$:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i\Big\{\underbrace{\mathcal{F}(s,q^2)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s,q^2)}_{\text{left-hand cut}}\Big\} \times \theta(s - 4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

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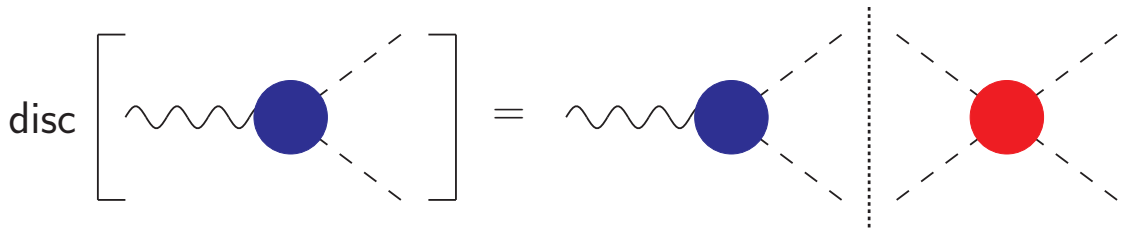
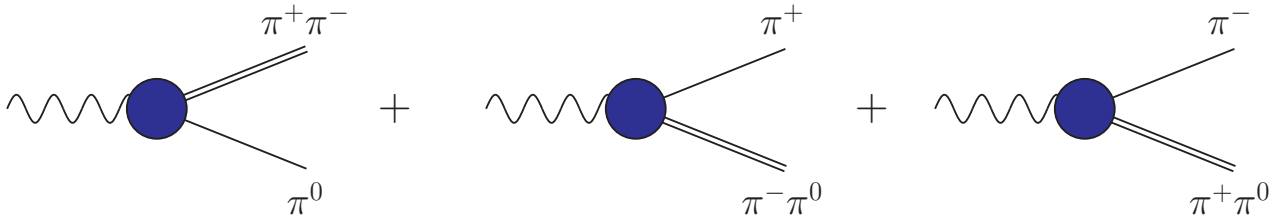


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\,, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$\mathcal{F}(s,t,u;q^2) =$



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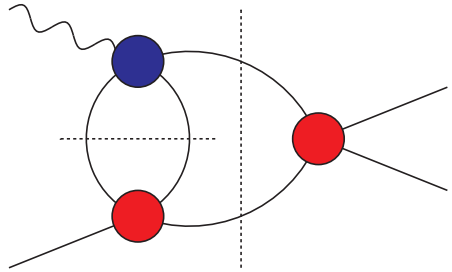
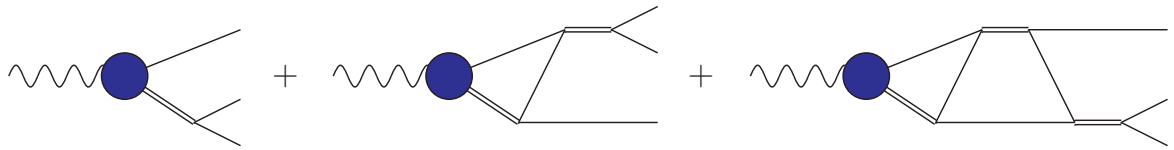


- inhomogeneities $\hat{\mathcal{F}}(s,q^2)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\left\{1 + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

$\mathcal{F}(s,q^2) =$  $+ \ldots$

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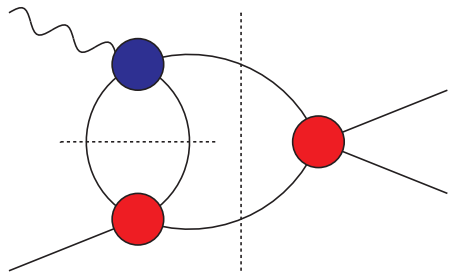

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- crossed-channel scatt. between s-, t-, u-channel (left-hand cuts)

Dispersive representation $\gamma^* \to 3\pi$

- parameterisation of subtraction function $a(q^2)$

  $\longrightarrow$ to be fitted to $e^+e^- \to 3\pi$ cross section data:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{\rm thr}^{\infty} ds' \frac{\mathrm{Im}\,\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

- $\mathcal{A}(q^2)$ includes resonance poles:

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\Gamma_V(q^2)} \qquad V = \omega, \phi, \omega', \omega''$$

c_V real

- conformal polynomial (inelasticities); S-wave cusp eliminated:

$$C_n(q^2) = \sum_{i=1}^{n} c_i \Big(z(q^2)^i - z(0)^i \Big), \qquad z(q^2) = \frac{\sqrt{s_{\rm inel} - s_1} - \sqrt{s_{\rm inel} - q^2}}{\sqrt{s_{\rm inel} - s_1} + \sqrt{s_{\rm inel} - q^2}}$$

- exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\rm thr}}^{\infty} ds' \frac{\mathrm{Im}\,a(s')}{s'}$$

$e^+e^- \to 3\pi$ cross section data sets

experiment	region of $\sqrt{s}$ [GeV]	# data points	normalisation uncertainty
SND 2002	$[0.98, 1.38]$	67	5.0% or 5.4%
SND 2003	$[0.66, 0.97]$	49	3.4% or 4.5%
SND 2015	$[1.05, 1.80]$	31	3.7%
CMD-2 1995	$[0.99, 1.03]$	16	4.6%
CMD-2 1998	$[0.99, 1.03]$	13	2.3%
CMD-2 2004	$[0.76, 0.81]$	13	1.3%
CMD-2 2006	$[0.98, 1.06]$	54	2.5%
DM1 1980	$[0.75, 1.10]$	26	3.2%
ND 1991	$[0.81, 1.39]$	28	10% or 20%
DM2 1992	$[1.34, 1.80]$	10	8.7%
BaBar 2004	$[1.06, 1.80]$	30	all systematics

- normalisation-type systematics assumed 100% correlated
- avoid biased fit for empirical full covariance matrix
  $\longrightarrow$ iterative solution D'Agostini 1994, NNPDF 2010

Fit results

Parameters:

- resonance parameters M_ω, Γ_ω, M_ϕ, Γ_ϕ, c_ω, c_ϕ, $c_{\omega'}$, $c_{\omega''}$
- conformal parameters c_1, c_2, c_3
- energy rescaling $\sqrt{s} \to \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$
  $\longrightarrow$ far less an issue than for $\pi^+\pi^-$

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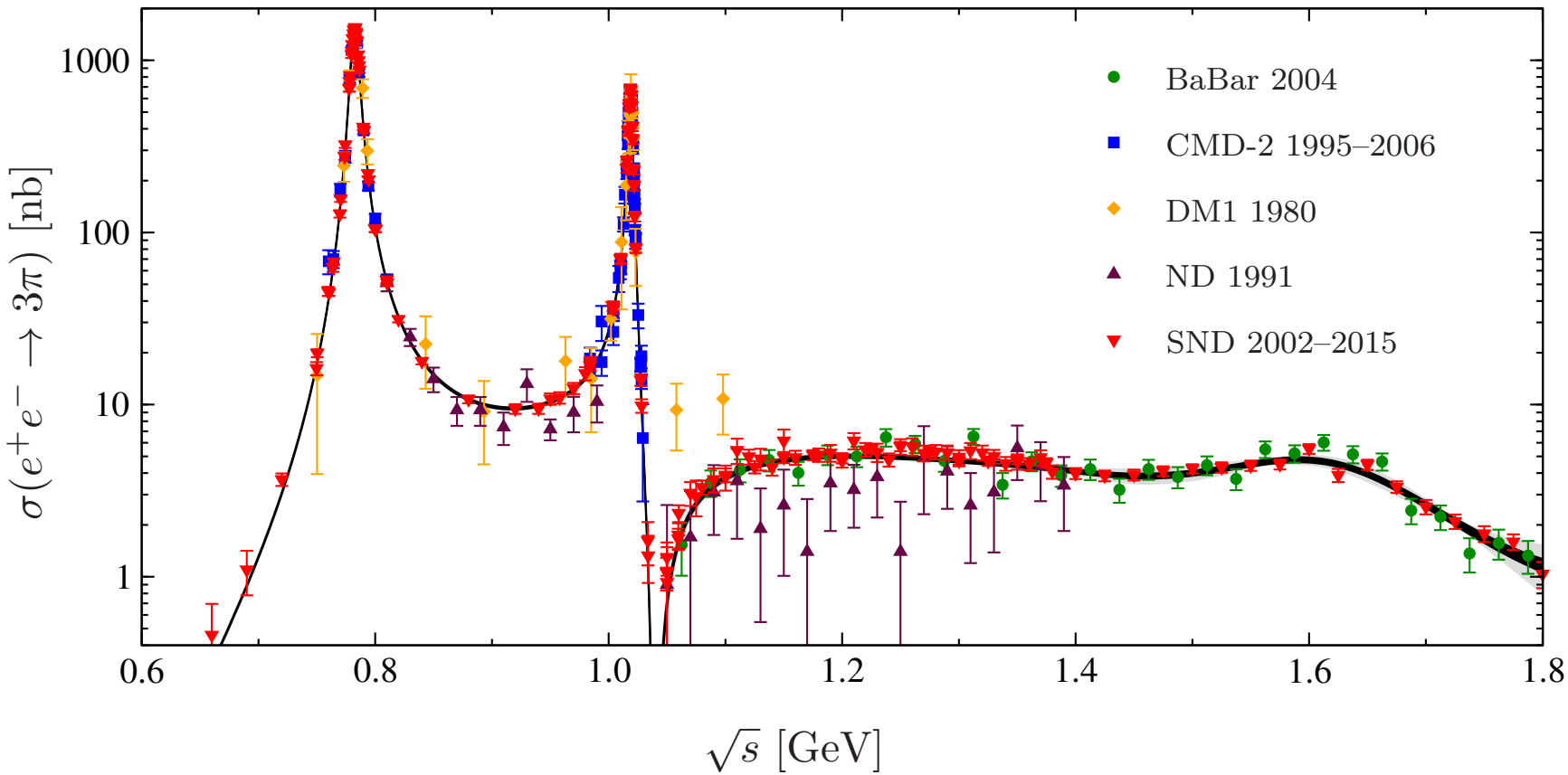
- quality of the combined fit to all data:

	this work		
χ^2/dof	430.8/305	=	1.41

  - ▷ correlations increase χ^2/dof by $\sim$ 0.3
  - ▷ significantly better fits to individual data sets
    $\longrightarrow$ fit errors inflated by scale factor $S = \sqrt{\chi^2/\text{dof}}$

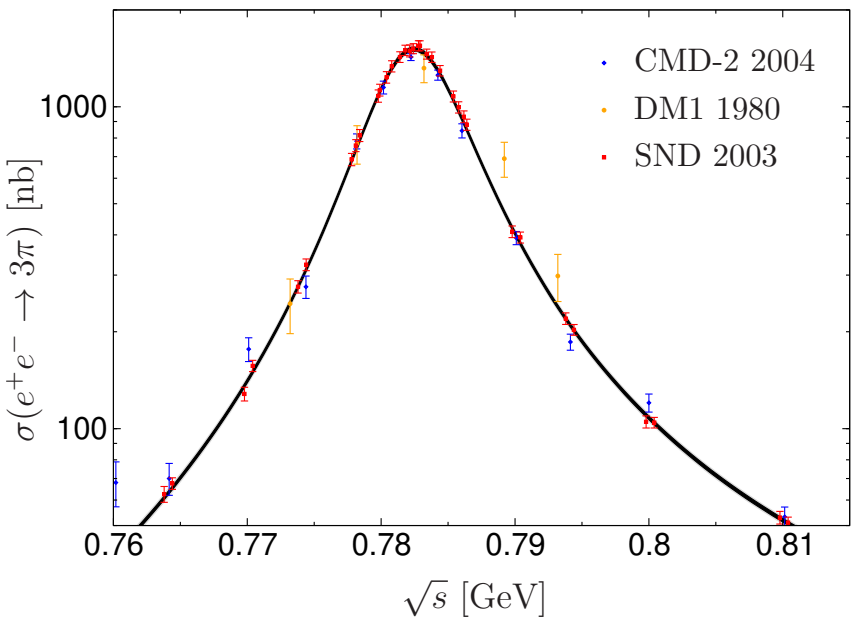
Fit results $e^+e^- \to 3\pi$ data up to 1.8 GeV

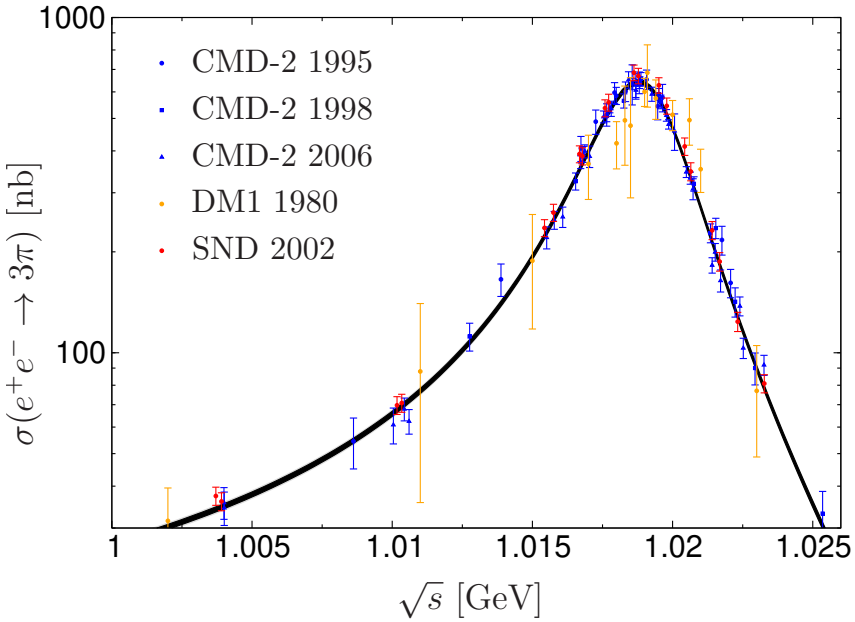


Hoferichter, Hoid, BK 2019

- black / gray bands represent fit and total uncertainties
- vacuum polarisation removed from the cross section

Fit results $e^+e^- \to 3\pi$: ω, ϕ peaks





- VP-subtraction: expect $\Delta M_\omega = -0.13$ MeV, $\Delta M_\phi = -0.26$ MeV

$$M_\omega = 782.63(3)\ \text{MeV} \qquad \Gamma_\omega = 8.71(6)\ \text{MeV}$$

$$M_\phi = 1019.20(2)\ \text{MeV} \qquad \Gamma_\phi = 4.23(4)\ \text{MeV}$$

...vs. PDG:

$$M_\omega = 782.65(12)\ \text{MeV} \qquad \Gamma_\omega = 8.49(8)\ \text{MeV}$$

$$M_\phi = 1019.461(16)\ \text{MeV} \qquad \Gamma_\phi = 4.249(13)\ \text{MeV}$$

$\longrightarrow$ M_ω compatible with PDG [but tension with $\pi\pi$ (!) channel]

Fit results: 3π contribution to HVP

- central result for the 3π contribution to HVP:

$$a_\mu^{3\pi}|_{\leq 1.8\,\text{GeV}} = 462(6)(6) \times 10^{-11} = 462(8) \times 10^{-11}$$

Hoferichter, Hoid, BK 2019

- interpolation errors main discrepancy between different groups:

Davier et al. 2017, 2019	Keshavarzi et al. 2018
$462.0(14.5)$	$477.0(8.9)$

$\longrightarrow$ linear interpolation overestimates narrow resonances

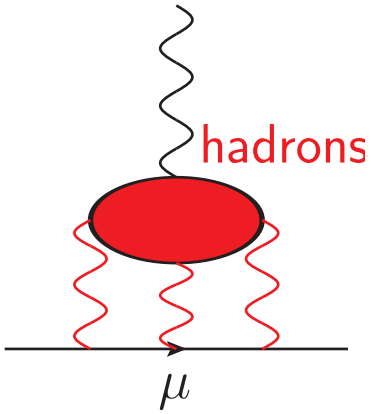
- consistently above values as small as

$$a_\mu^{3\pi}|_{\leq 2.0\,\text{GeV}} = 443(15) \times 10^{-11}$$

Jegerlehner 2017

Hadronic light-by-light scattering

- hadronic light-by-light:
  - ▷ subleading in α_{QED}
  - ▷ large relative uncertainty



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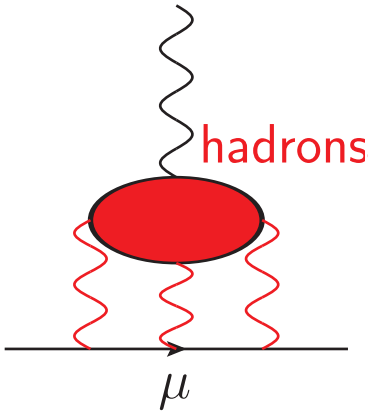
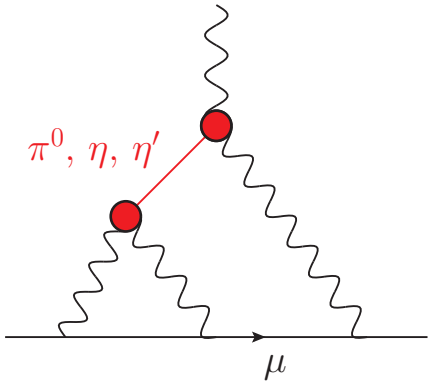
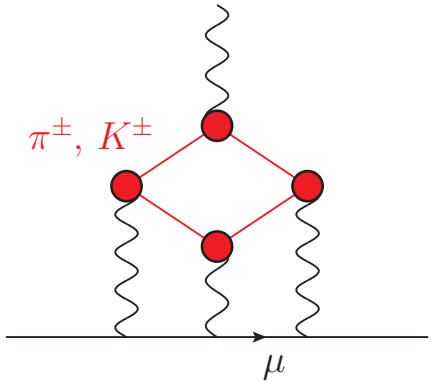


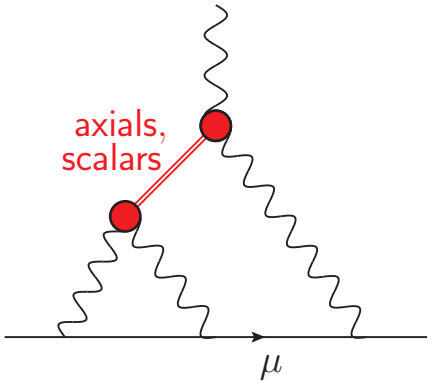
- different contributions calculated or estimated (in 10^{-11}):



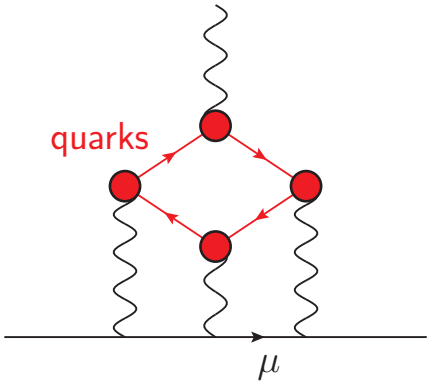
94±4



–24±1



5±7



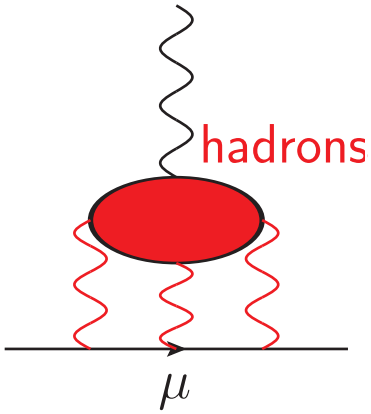
18±10

$\longrightarrow$ increasing systematic control over HLbL

Aoyama et al. 2020

Hadronic light-by-light scattering

- hadronic light-by-light:
  - ▷ subleading in α_{QED}
  - ▷ large relative uncertainty



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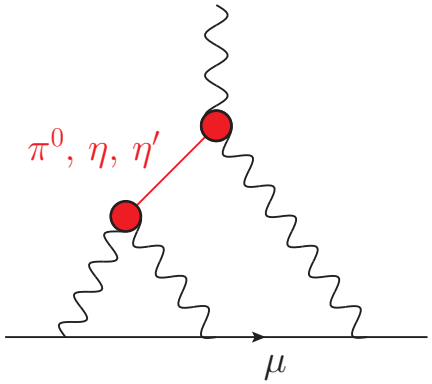

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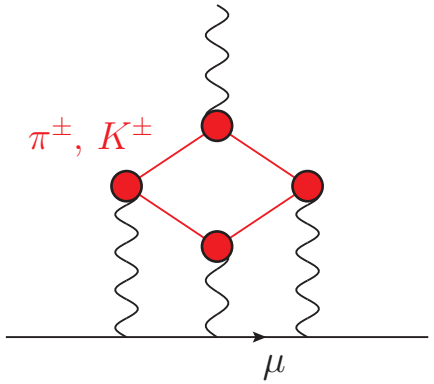

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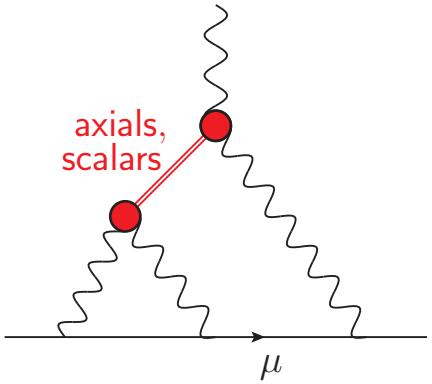

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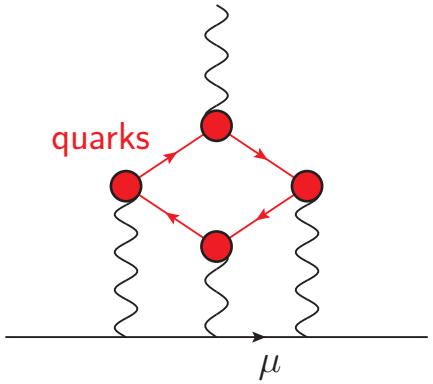


18±10

$\longrightarrow$ increasing systematic control over HLbL

Aoyama et al. 2020

- largest (and unambiguous!) individual contribution: π^0 pole term
  $\longrightarrow$ depends on $\pi^0 \to \gamma^*\gamma^*$ form factor (doubly virtual in general!)

Hadronic light-by-light: the π^0 pole

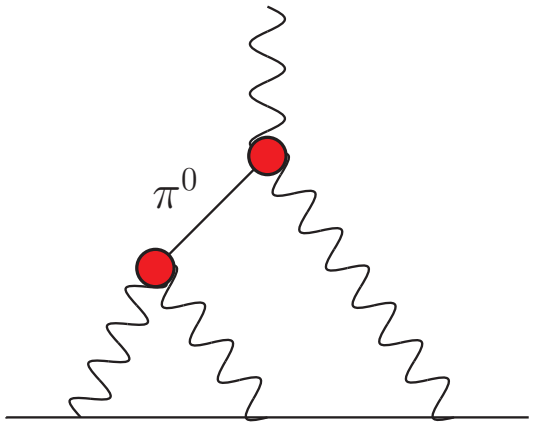
dispersive approach to HLbL: Colangelo, Hoferichter, Procura, Stoffer 2013–

- largest individual HLbL contribution:

  π^0 pole term

  singly / doubly virtual form factors

  $F_{\pi^0\gamma^*\gamma^*}(q^2, 0)$ and $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$



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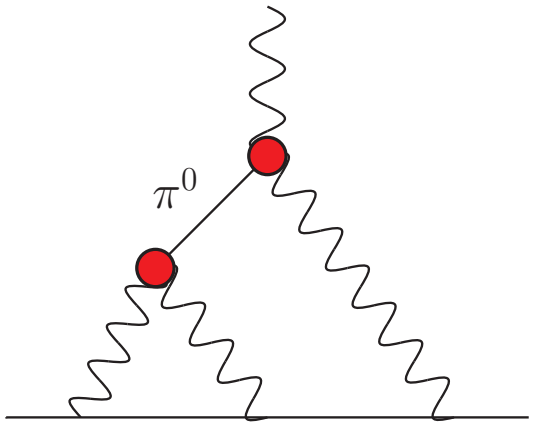

- normalisation fixed by Wess–Zumino–Witten anomaly:

$$F_{\pi^0\gamma^*\gamma^*}(0, 0) = \frac{1}{4\pi^2 F_\pi}$$

$\longrightarrow$ measured at 0.75% (F_π: pion decay constant) PrimEx 2020

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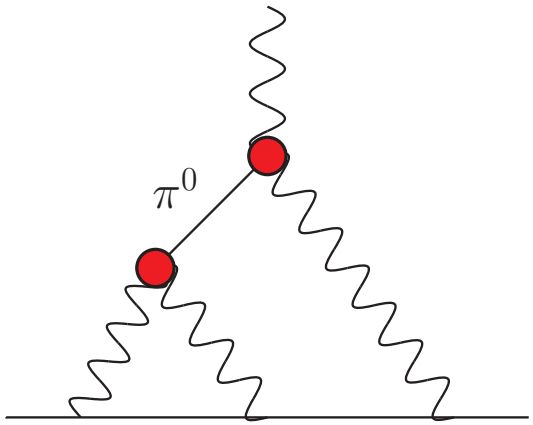

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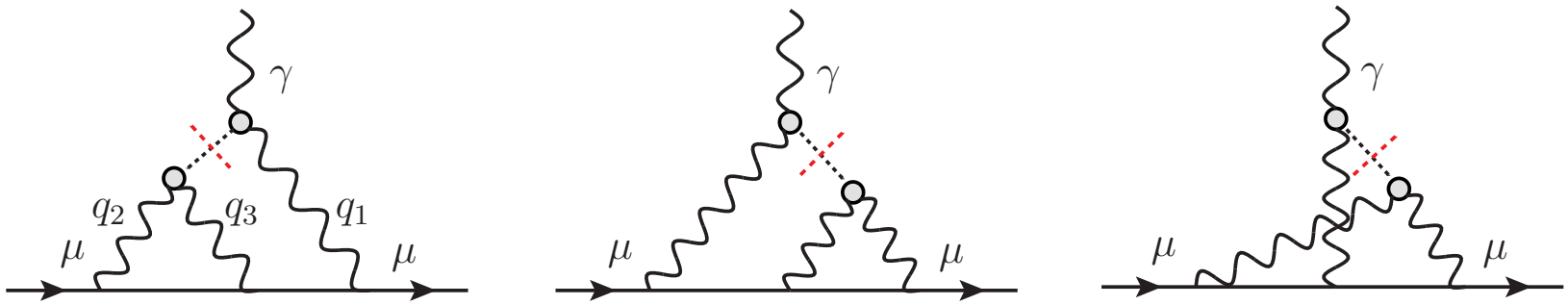
- two-loop integral with constant form factors does not converge

  $\longrightarrow$ no full prediction from e.g. chiral perturbation theory

  $\longrightarrow$ sensible high-energy behaviour required!

Pion-pole contribution to a_μ

- 3-dimensional integral representation: Jegerlehner, Nyffeler 2009

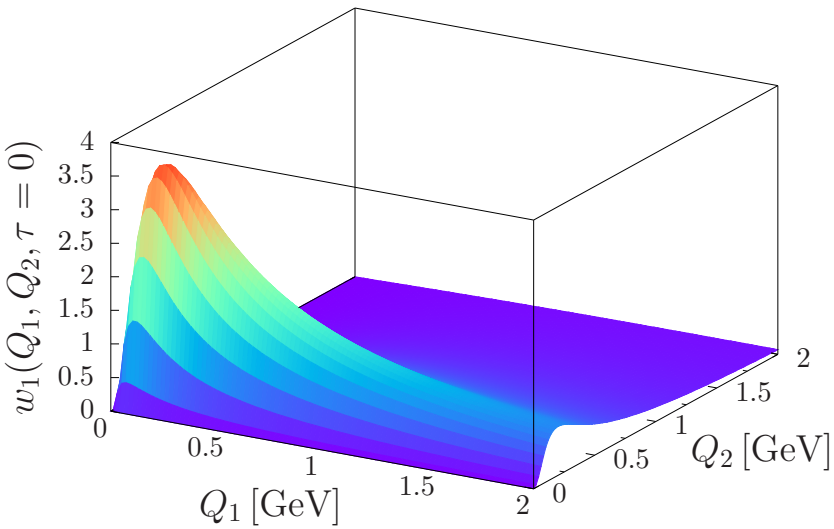


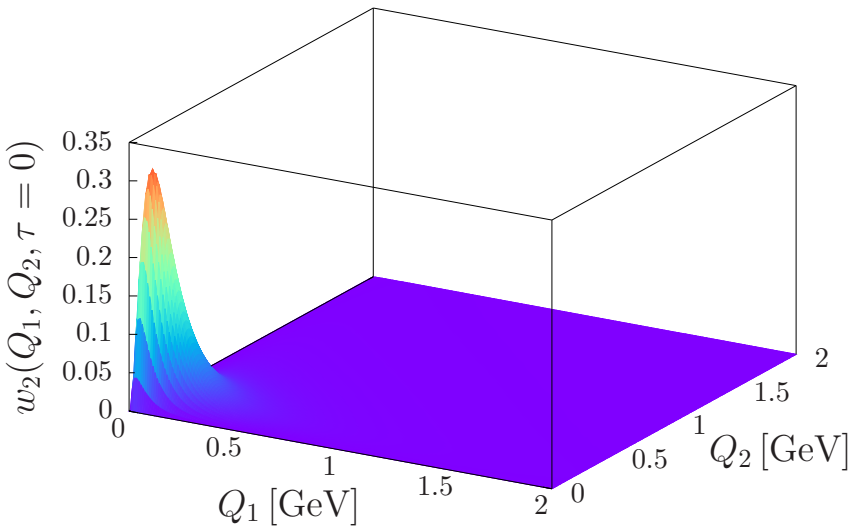
$$a_\mu^{\pi^0\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau$$
$$\times \Big[w_1(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)$$
$$+ w_2(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0)\Big]$$

- $w_{1/2}(Q_1,Q_2,\tau)$: kinematical weight functions, $\tau = \cos\theta$
- $F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$: space-like on-shell π^0 TFF

Pion-pole contribution to a_μ

- weight functions $w_{1/2}(Q_1, Q_2, \tau = 0)$: Nyffeler 2016





- concentrated for $Q_i \leq 0.5$ GeV

  $\longrightarrow$ pion-pole contribution dominantly from low-energy region

  $\longrightarrow$ pion transition form factor can be determined model-independently and with high precision using dispersion relations

Dispersive analysis of $\pi^0 \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$

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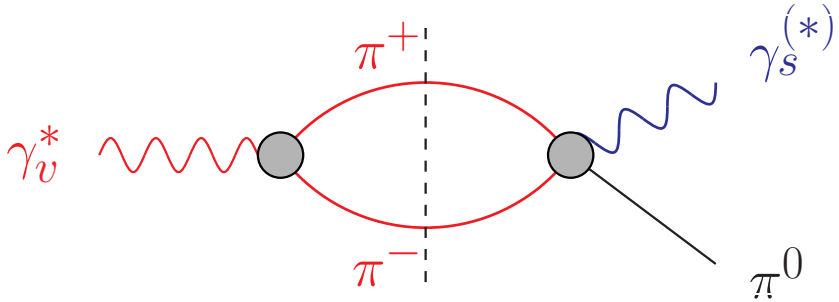
- analyse the leading hadronic intermediate states:

Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor $\times$ $\gamma^* \to 3\pi$

all determined in terms of pion–pion P-wave phase shift

+ Wess–Zumino–Witten anomaly for normalisation

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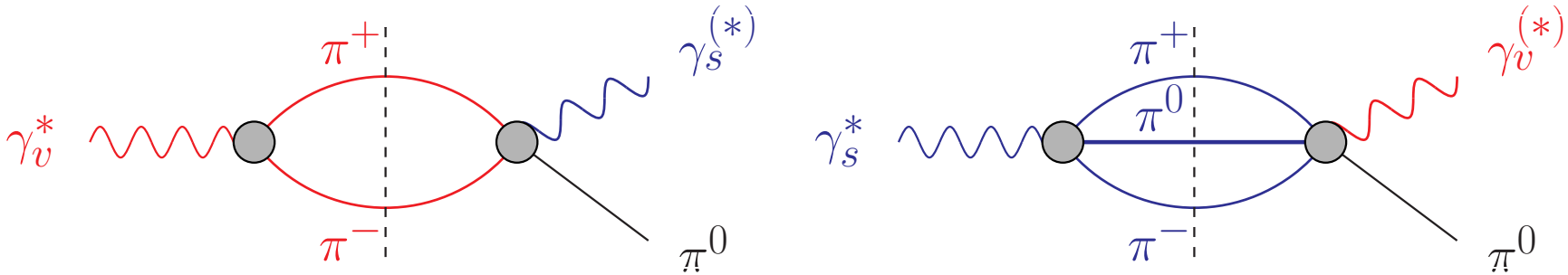

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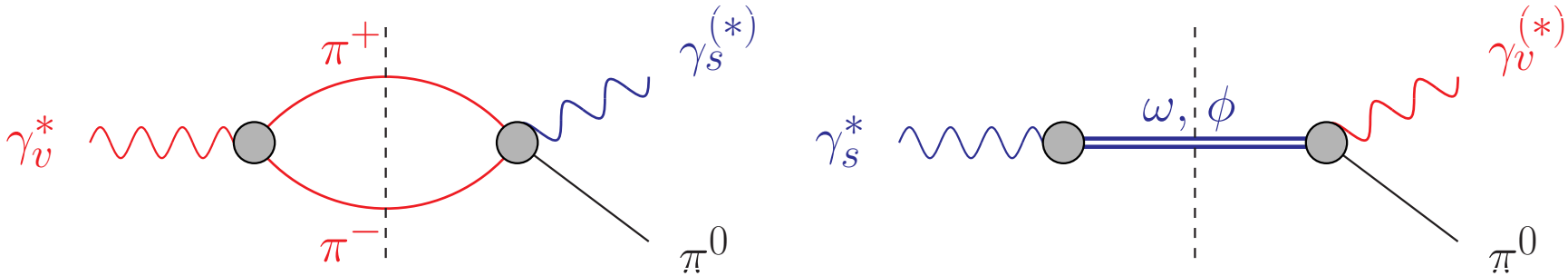

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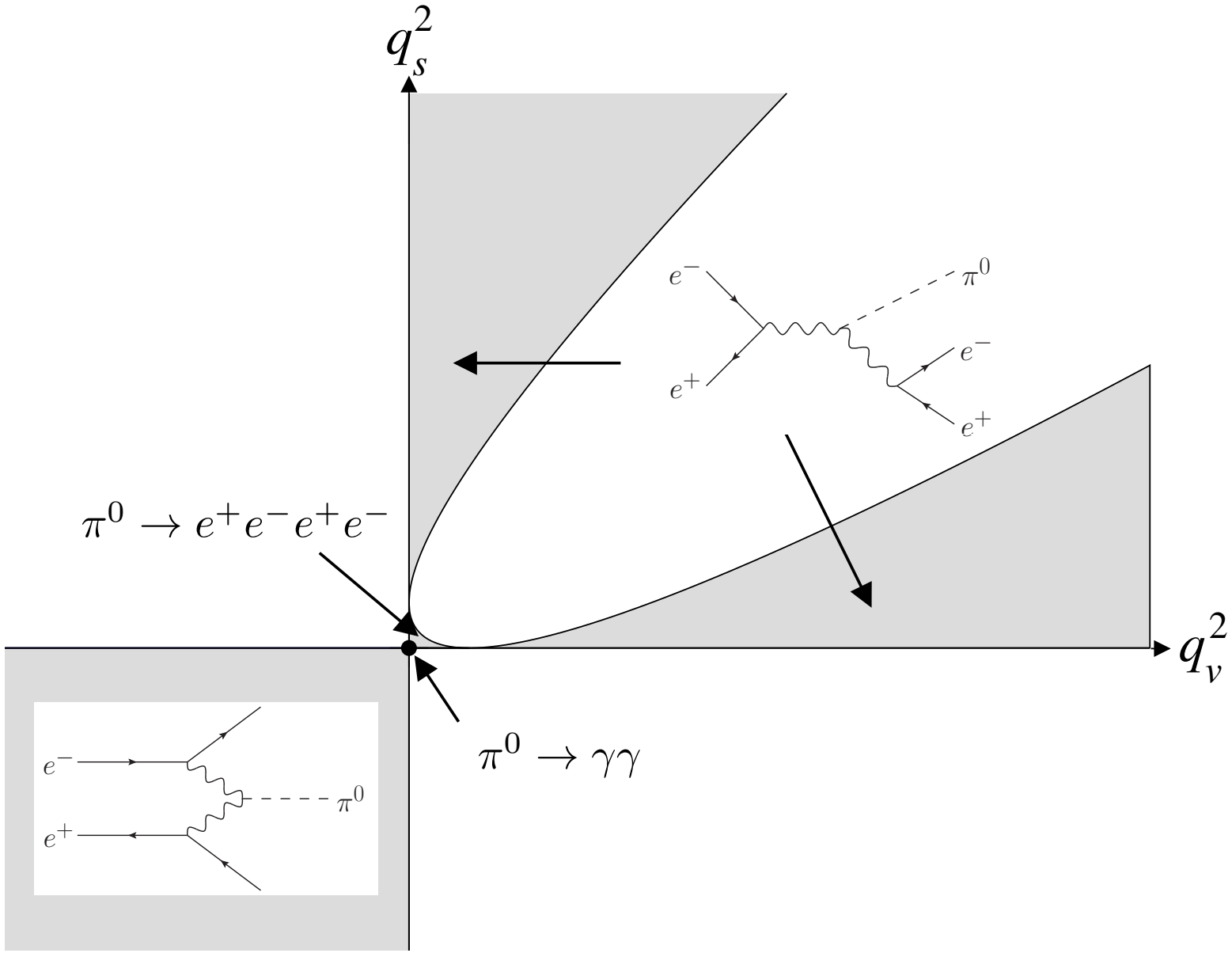
    dominated by narrow resonances ω, ϕ

$\pi^0 \to \gamma^*(q_v^2)\gamma^*(q_s^2)$ transition form factor



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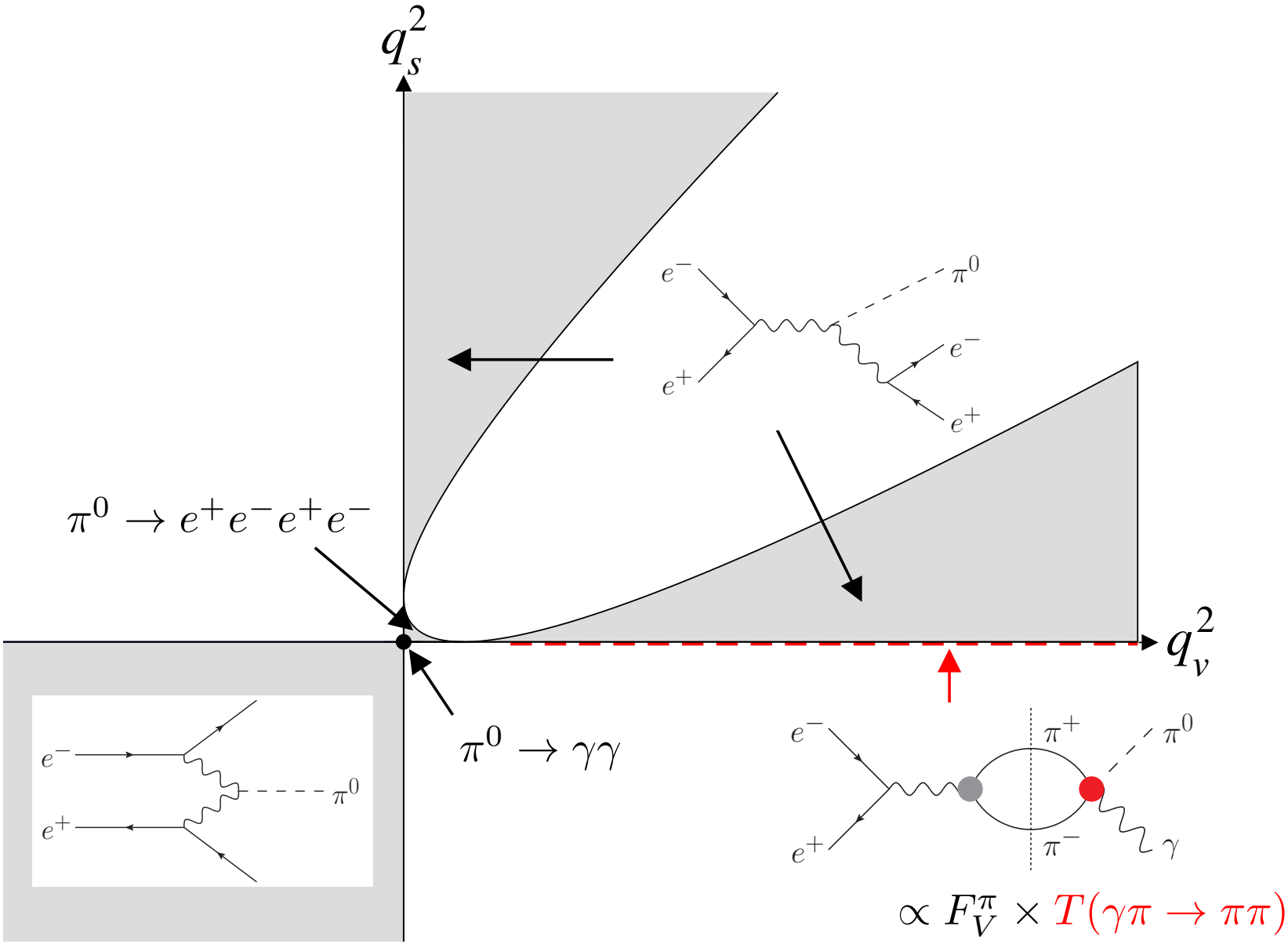

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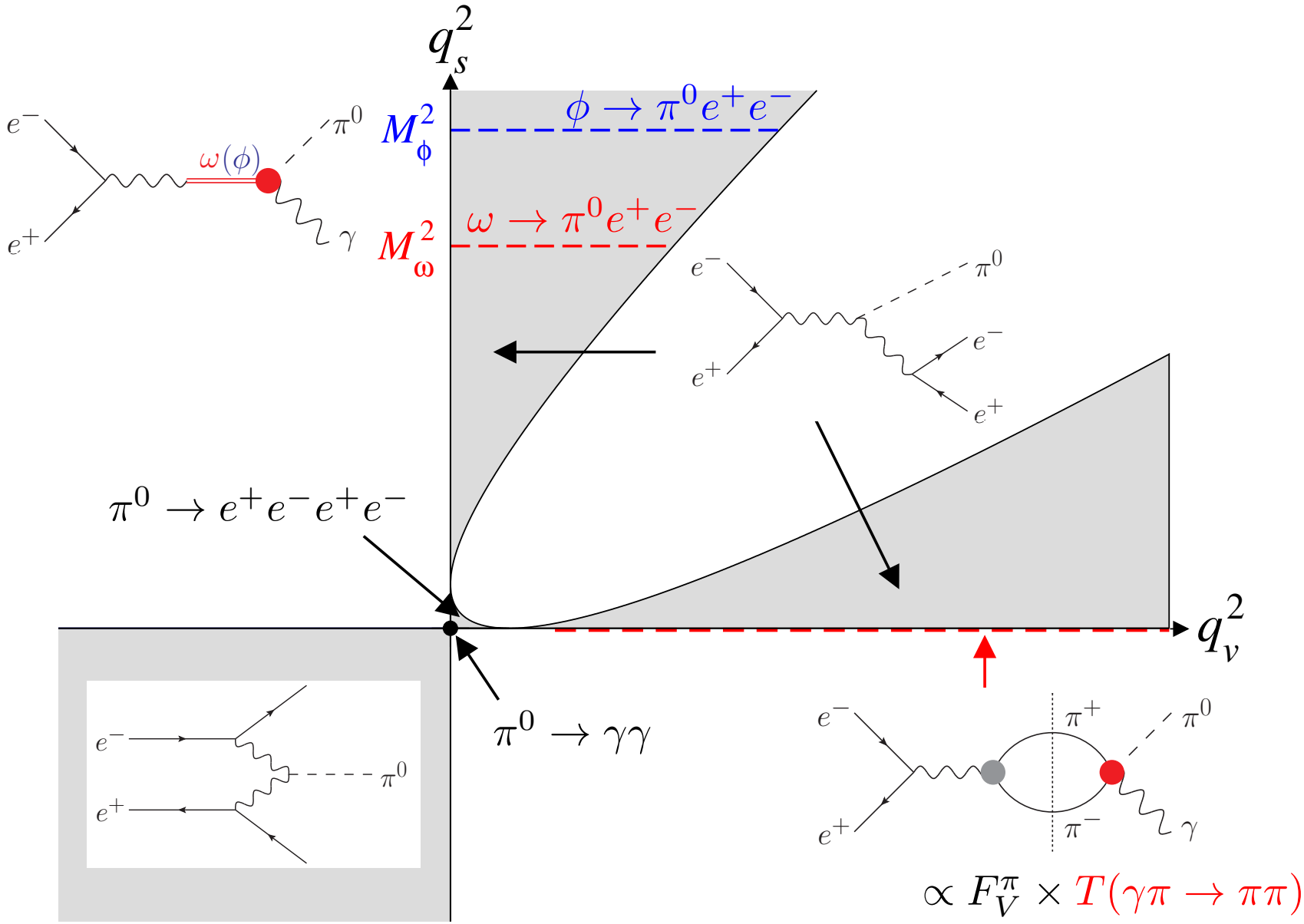
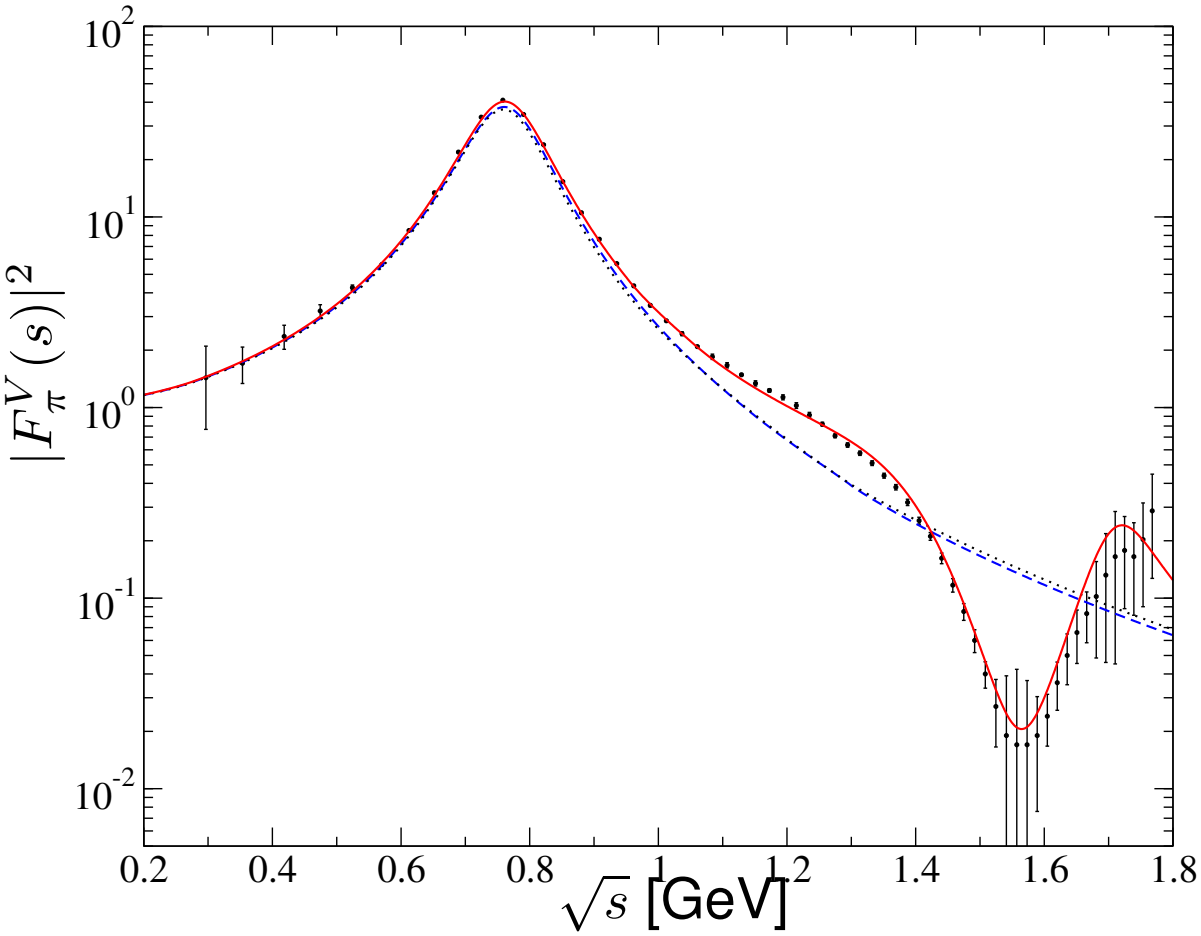


Pion vector form factor vs. Omnès representation

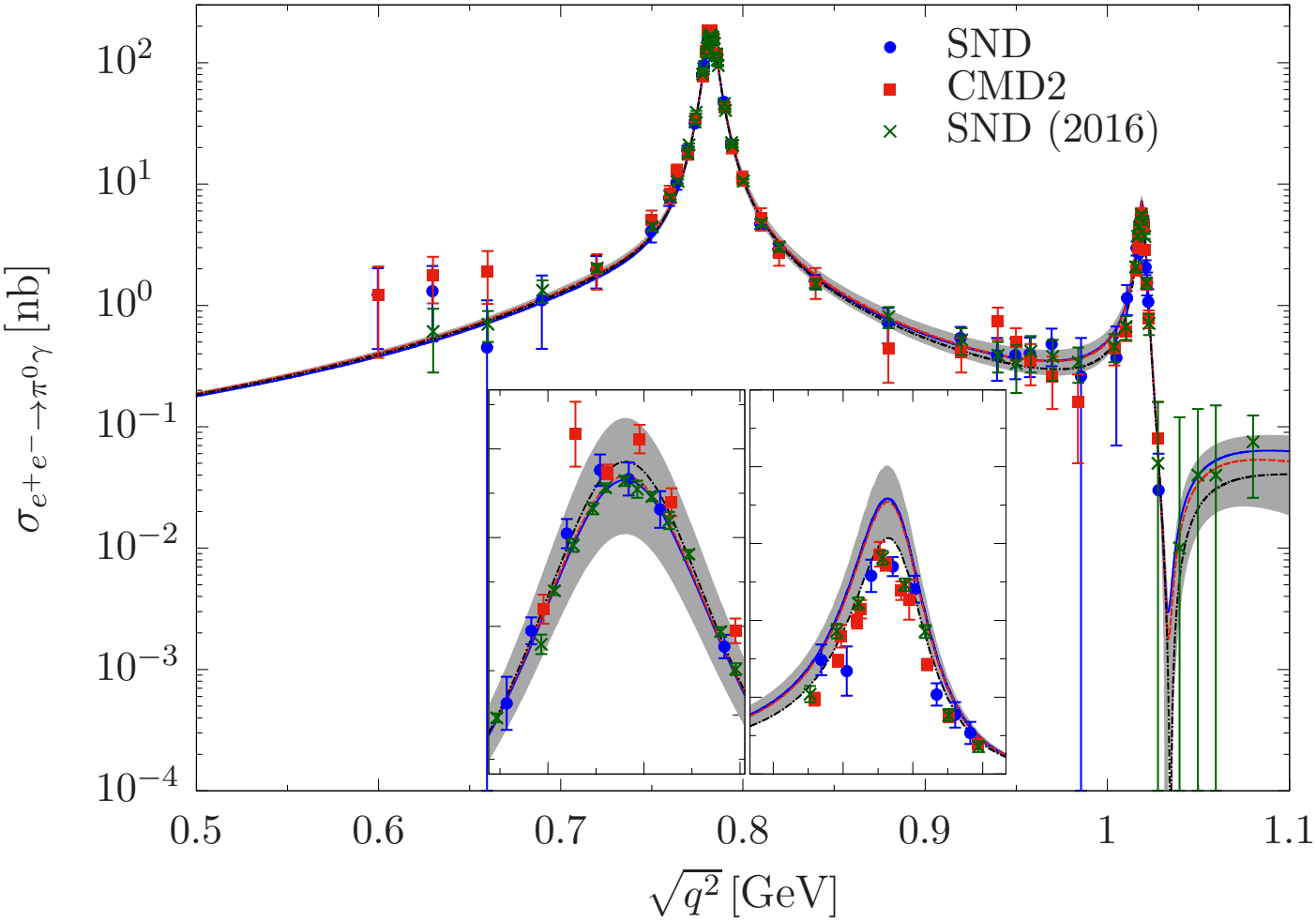
Data on pion form factor in $\tau^- \to \pi^-\pi^0\nu_\tau$ Belle 2008



$\pi\pi$ P-wave phase shift / effective form factor phase incl. ρ', ρ''

Schneider et al. 2012

Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, Hoid, BK, Leupold, Schneider 2018

- "prediction"—no further parameters adjusted
- data very well reproduced

Fit to $\pi^0\gamma$ instead: HVP contribution

- fit disp. representation to $e^+e^- \to \pi^0\gamma$ instead of 3π data
  $\longrightarrow$ excellent consistency, average pole parameters:

	$e^+e^- \to 3\pi, \pi^0\gamma$	PDG
$\bar{M}_\omega$ [MeV]	$782.736(24)$	$782.65(12)$
$\bar{\Gamma}_\omega$ [MeV]	$8.63(5)$	$8.49(8)$
$\bar{M}_\phi$ [MeV]	$1019.457(20)$	$1019.461(16)$
$\bar{\Gamma}_\phi$ [MeV]	$4.22(5)$	$4.249(13)$

- $\pi^0\gamma$ HVP contribution: Hoid, Hoferichter, BK 2020

$$a_\mu^{\pi^0\gamma}|_{\leq 1.35\,\mathrm{GeV}} = 43.8(6)(1) \times 10^{-11} = 43.8(6) \times 10^{-11}$$

- good agreement (except small interpolation errors):

Davier et al. 2019	Keshavarzi et al. 2019
$44.1(1.0)$	$45.8(1.0)$

Towards the doubly-virtual π^0 transition form factor

- so far: subtracted dispersion relation
  - ▷ high precision at low energies
  - ▷ uses $\pi^0 \to \gamma\gamma$ as input for form factor normalisation
  - ▷ asymptotically: $\lim\limits_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}\big(-Q^2, 0\big) = \text{const.}$

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  $\longrightarrow$ switch to unsubtracted representation

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- double-spectral-function representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$

$$\rho(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} \mathrm{Im}\left[F_\pi^{V*}(x) f_1(x,y)\right] + [x \leftrightarrow y]$$

$f_1(s, q^2)$: $\gamma^*(q^2)\pi \to \pi\pi$ P-wave

$\longrightarrow$ doubly-virtual form factor fixed from singly-virtual input!

Asymptotics and pQCD constraints (1)

- asymptotically, π^0 TFF can be expressed via pion wave function:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2F_\pi}{3}\int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2} + \mathcal{O}(Q^{-4})$$

$\phi_\pi(x) = 6x(1-x) + \ldots$ Brodsky, Lepage 1979–1981

- implies

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2) = \frac{2F_\pi}{3Q^2} + \mathcal{O}(Q^{-4})$$

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- change of variables: rewrite this as dispersion relation

$$F^{\mathrm{pQCD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi}\int_0^\infty dx \frac{\mathrm{Im}\, F^{\mathrm{pQCD}}_{\pi^0\gamma^*\gamma^*}(x, q_2^2)}{x - q_1^2}$$

$$\mathrm{Im}\, F^{\mathrm{pQCD}}_{\pi^0\gamma^*\gamma^*}(x, y) = \frac{2\pi F_\pi}{3(x-y)}\,\phi_\pi\left(\frac{x}{x-y}\right)$$

Khodjamirian 1999

Asymptotics and pQCD constraints (2)

- formally, write this as a double-spectral representation:

$$F^{\mathrm{pQCD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_0^\infty dx \int_0^\infty dy \frac{\rho^{\mathrm{pQCD}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

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- decomposition of the full form factor:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_{\mathrm{m}}} dx \int_0^{s_{\mathrm{m}}} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\mathrm{m}}}^\infty dx \int_{s_{\mathrm{m}}}^\infty dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$
$$+ \frac{1}{\pi^2}\int_0^{s_{\mathrm{m}}} dx \int_{s_{\mathrm{m}}}^\infty dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\mathrm{m}}}^\infty dx \int_0^{s_{\mathrm{m}}} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$

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- use dispersive $\rho(x,y)$ at low energies
- doubly-asymptotic: $\rho^{\mathrm{pQCD}}(x,y) \propto x\,y$
  $\longrightarrow$ does not contribute to singly-virtual TFF!
- mixed regions: nothing known, $\rho^{\mathrm{pQCD}}(x,y)$ vanishes there
  $\longrightarrow$ all constraints can be fulfilled setting these to zero

Asymptotics and pQCD constraints (3)

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\rm m}}^{\infty} dx \int_{s_{\rm m}}^{\infty} dy \frac{\rho^{\rm pQCD}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

- pQCD piece alone: $F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \dfrac{2F_\pi}{3Q^2} + \mathcal{O}\big(Q^{-4}\big)$
- dispersive part: $\dfrac{1}{\pi^2}\displaystyle\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x+Q^2)(y+Q^2)} = \mathcal{O}\big(Q^{-4}\big)$

Asymptotics and pQCD constraints (3)

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^{\mathrm{disp}}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty} dx \int_{s_\mathrm{m}}^{\infty} dy \frac{\rho^{\mathrm{pQCD}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

- pQCD piece alone: $F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \dfrac{2F_\pi}{3Q^2} + \mathcal{O}\big(Q^{-4}\big)$
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- all constraints okay if $\rho^{\mathrm{disp}}(x,y)$ fulfils two sum rules:

$$\frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^{\mathrm{disp}}(x,y)}{x\,y} = \frac{1}{4\pi^2 F_\pi} \qquad \text{[anomaly]}$$

$$\frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^{\mathrm{disp}}(x,y)}{x} = 2F_\pi \qquad \text{[Brodsky–Lepage]}$$

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- add effective pole: $\rho^{\mathrm{eff}} = \dfrac{g_{\mathrm{eff}}}{4\pi^2 F_\pi}\pi^2 M_{\mathrm{eff}}^4 \delta\big(x-M_{\mathrm{eff}}^2\big)\delta\big(y-M_{\mathrm{eff}}^2\big)$

find $g_{\mathrm{eff}} \sim 10\%$ (small), $M_{\mathrm{eff}} \sim 1.5\ldots 2.0\,\mathrm{GeV}$ (reasonable)

Uncertainties in the dispersive approach

Normalisation

- uncertainty on $\pi^0 \to \gamma\gamma$ $\pm 1.5\%$ PrimEx 2020

  $\longrightarrow$ implemented via effective pole coupling $g_{\rm eff}$

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- different $\pi\pi$ phase shift inputs:
  - ▷ Bern vs. Madrid Colangelo et al. 2011, García-Martín et al. 2011
  - ▷ effective form factor phase (incl. ρ', ρ'') Schneider et al. 2012
- cutoff in Khuri–Treiman integrals $1.8 \ldots 2.5\,\text{GeV}$

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Brodsky–Lepage limit uncertainty

- allow for $^{+20\%}_{-10\%}$, 3σ band around data BaBar 2009, Belle 2012

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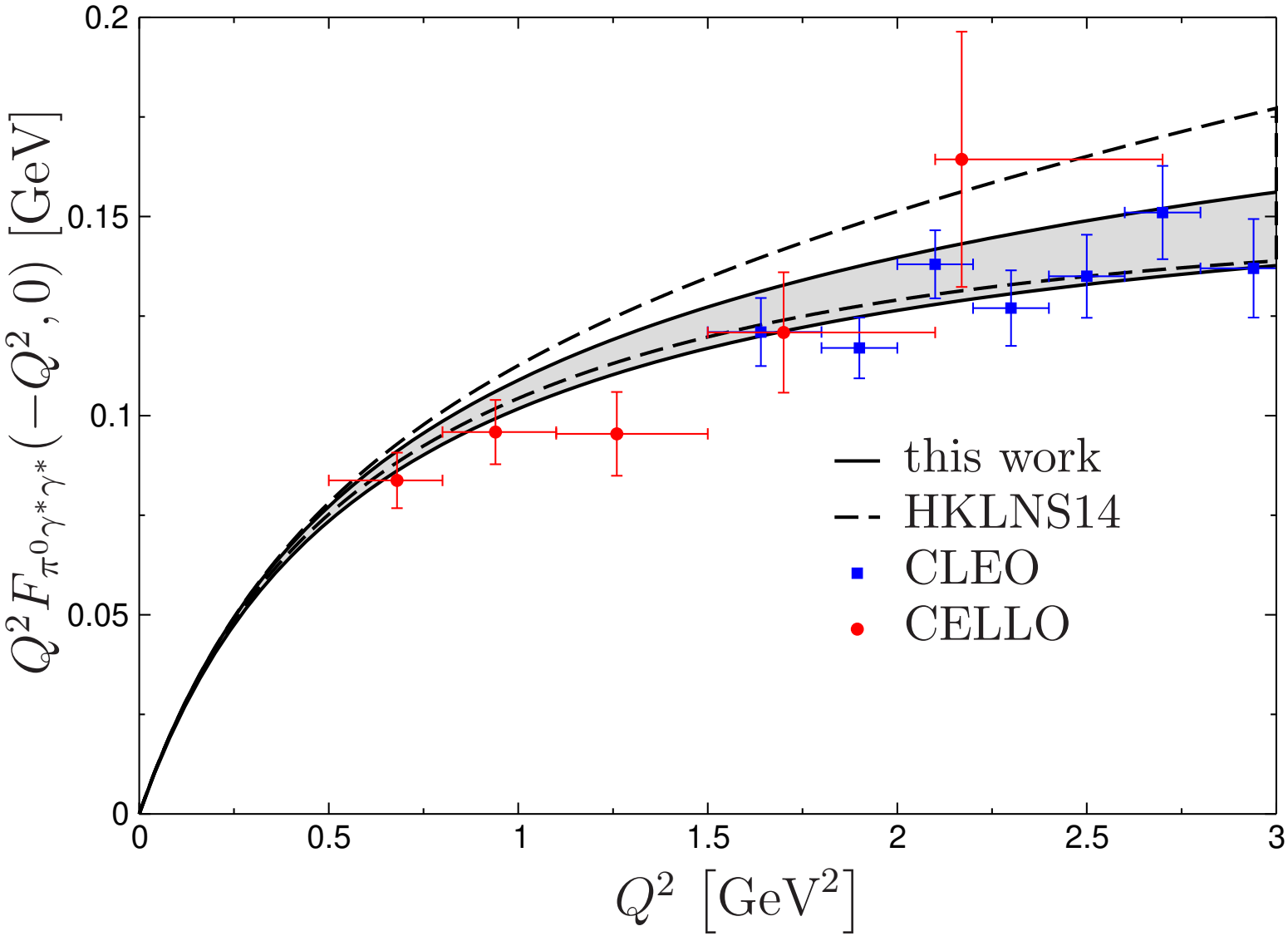
Onset of pQCD asymptotics

- vary $s_m = 1.7(3)\text{GeV}^2$

Results: singly-virtual



Hoferichter, Hoid, BK, Leupold, Schneider 2018

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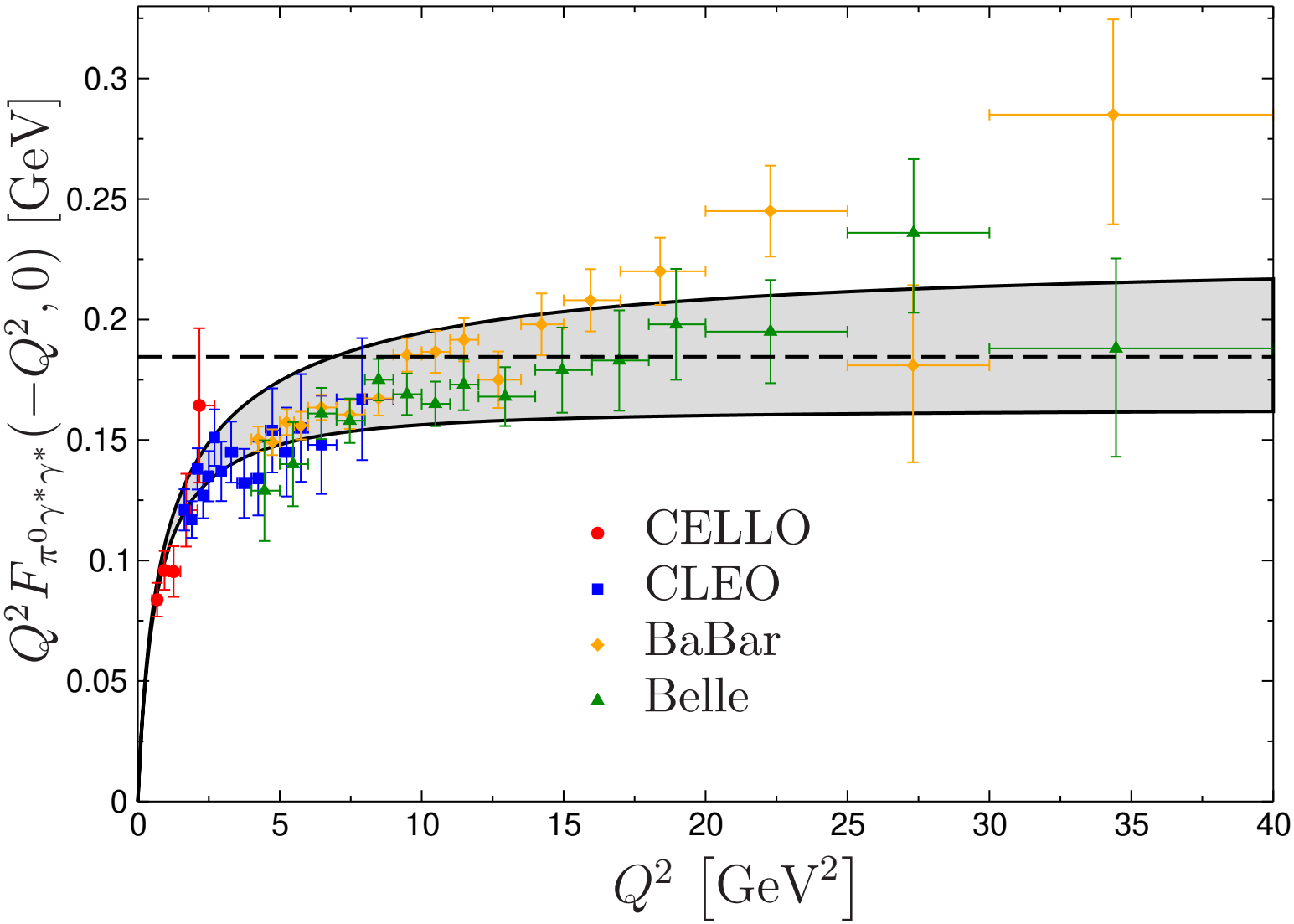
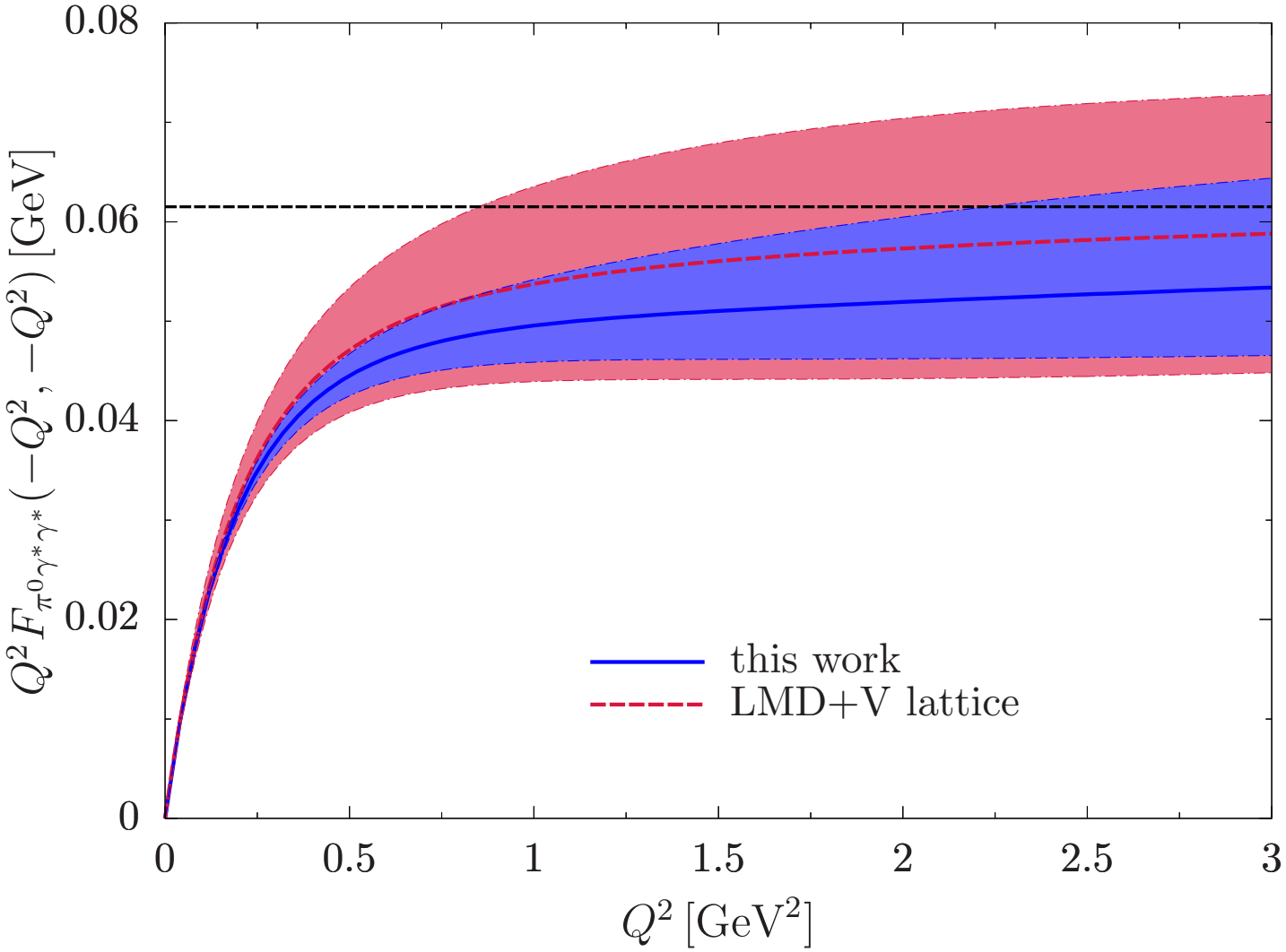


Hoferichter, Hoid, BK, Leupold, Schneider 2018

Results: doubly-virtual (diagonal)

in comparison to Gérardin, Meyer, Nyffeler 2016



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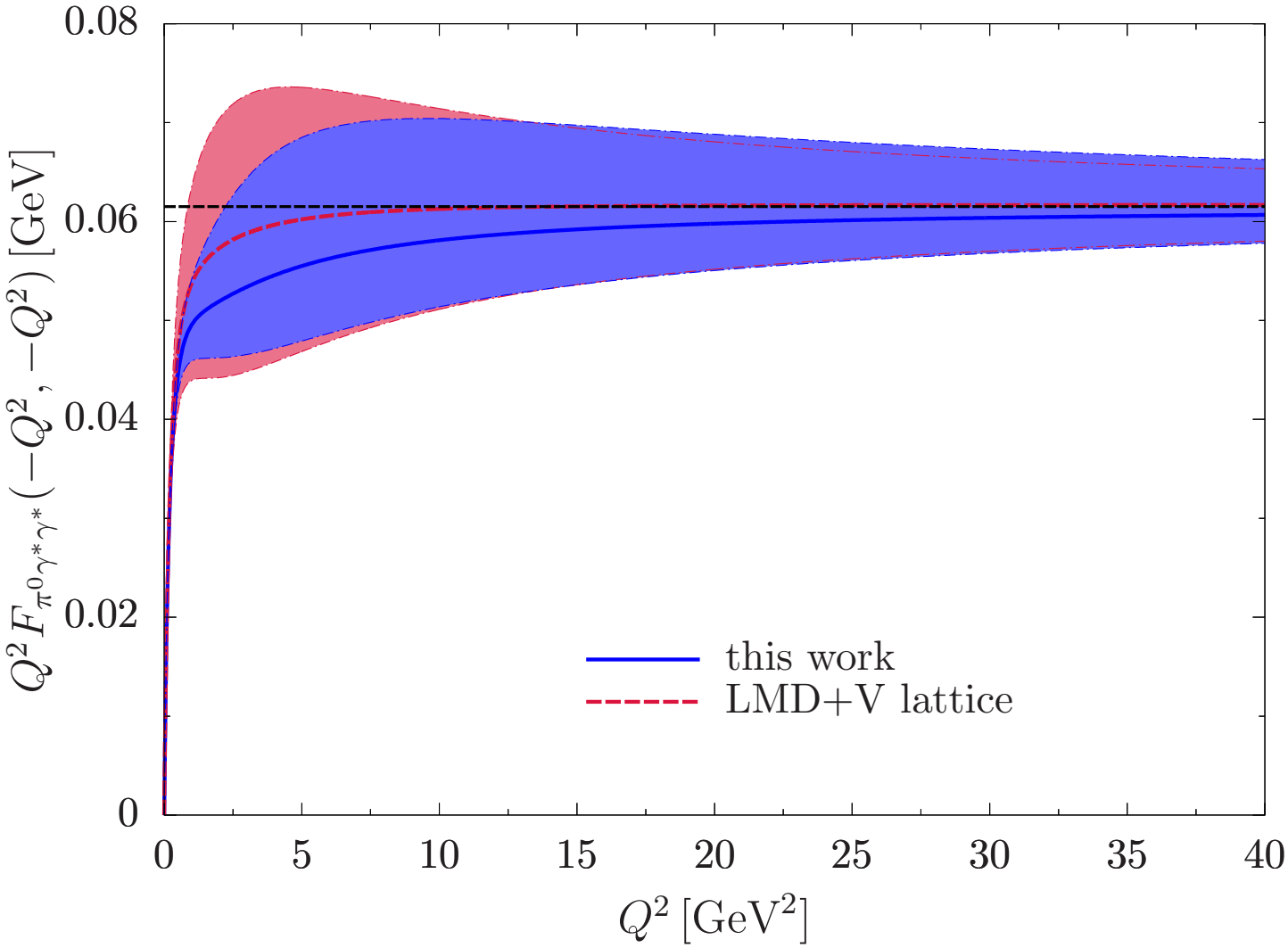
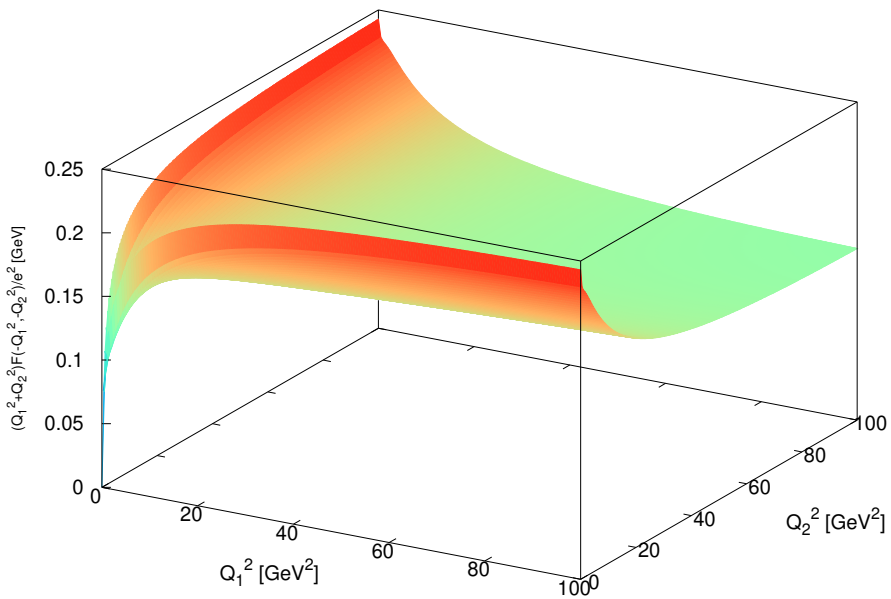


Hoferichter, Hoid, BK, Leupold, Schneider 2018

Comparison dispersive vs. LMD+V-lattice

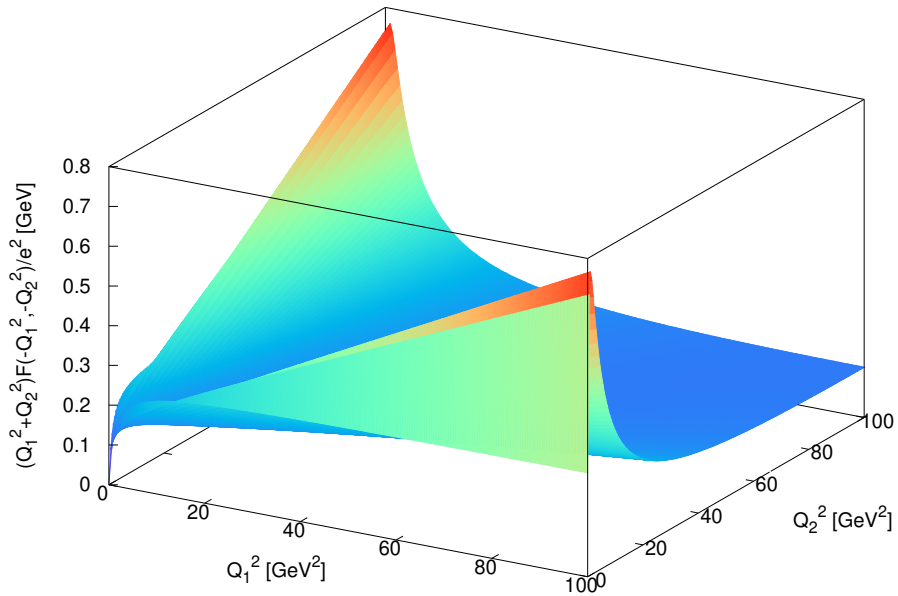
- plot $\left(Q_1^2+Q_2^2\right)F_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2,-Q_2^2\right)$:

dispersive



Hoferichter, Hoid, BK, Leupold, Schneider 2018

LMD+V fit to lattice



Gérardin, Meyer, Nyffeler 2016

Results: $(g-2)_\mu$ from π^0 pole

Final result for the π^0 pole contribution $[10^{-11}]$

63.0 ± 0.9	chiral anomaly / $\pi^0 \to \gamma\gamma$
± 1.1	dispersive input
$^{+\,2.2}_{-\,1.4}$	Brodsky–Lepage
± 0.6	onset of pQCD contribution s_m
$= 63.0\ ^{+\,2.7}_{-\,2.1}$	

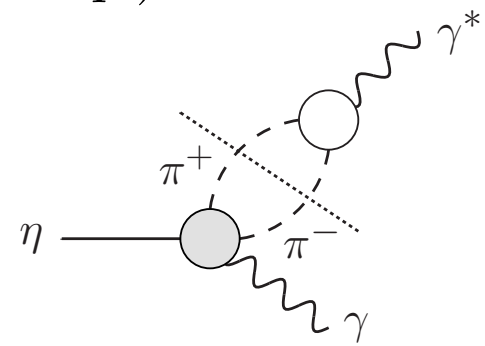
- first model-independent, data-driven determination with all physical low- and high-energy constraints implemented
- dominant uncertainties are all accessible in real-photon or singly-virtual experiments

Towards η, η': transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013, BK, Plenter 2015

$$F_{\eta\gamma^*\gamma}(q^2,0) = F_{\eta\gamma\gamma} + \frac{q^2}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds\, \frac{q_\pi^3(s)[F_\pi^V(s)]^* f_1^{\eta\to\pi\pi\gamma}(s)}{s^{3/2}(s-q^2)}$$

$$+ \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0)\ [\longrightarrow \text{VMD}]$$



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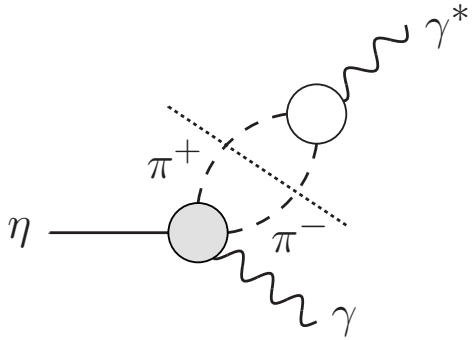
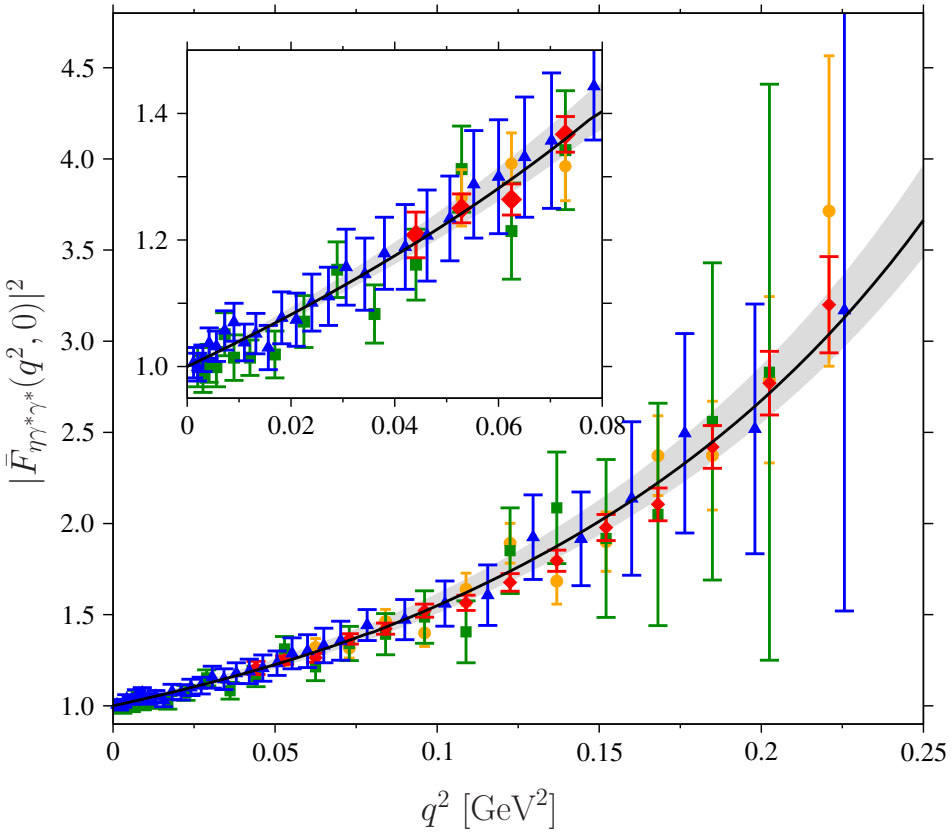


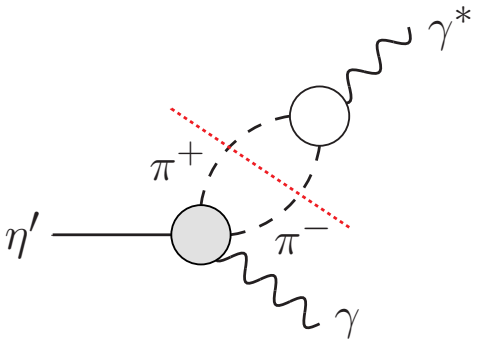


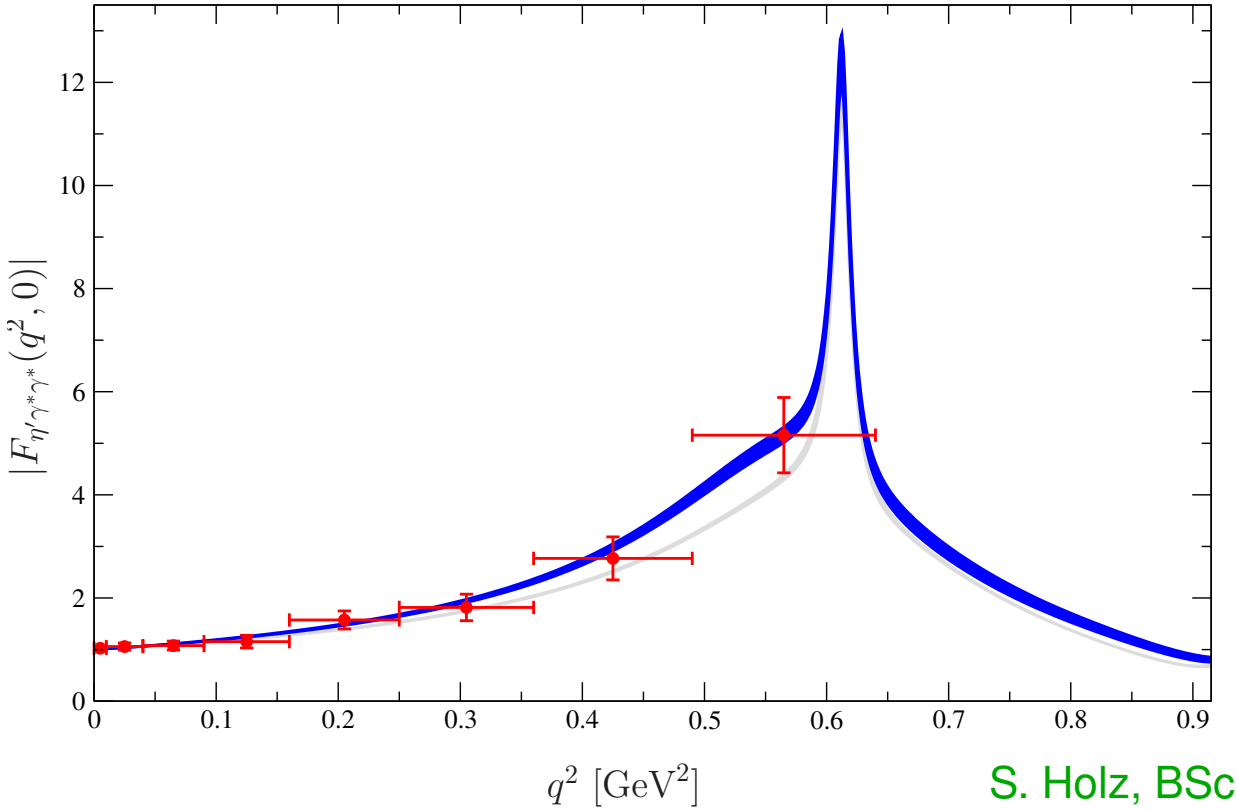
$\longrightarrow$ huge statistical advantage of using hadronic input for $\eta \to \pi^+\pi^-\gamma$ over direct measurement of $\eta \to \ell^+\ell^-\gamma$ (rate suppressed by α^2_{QED})

data: NA60 2009, 2016
A2 2014, 2017

Prediction for η' transition form factor

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$



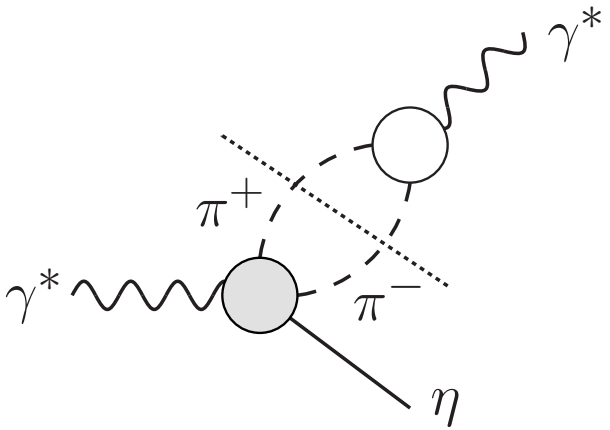


S. Holz, BSc thesis 2016

data: BESIII 2015

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

- idea (again): beat α^2_{QED} suppression of $e^+e^- \to \eta e^+e^-$ by measuring $e^+e^- \to \eta\pi^+\pi^-$ instead



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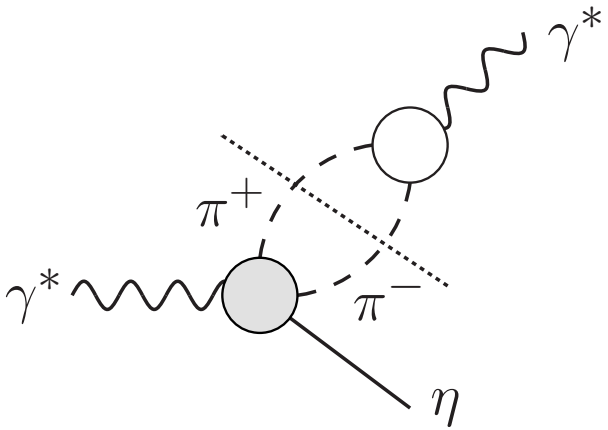


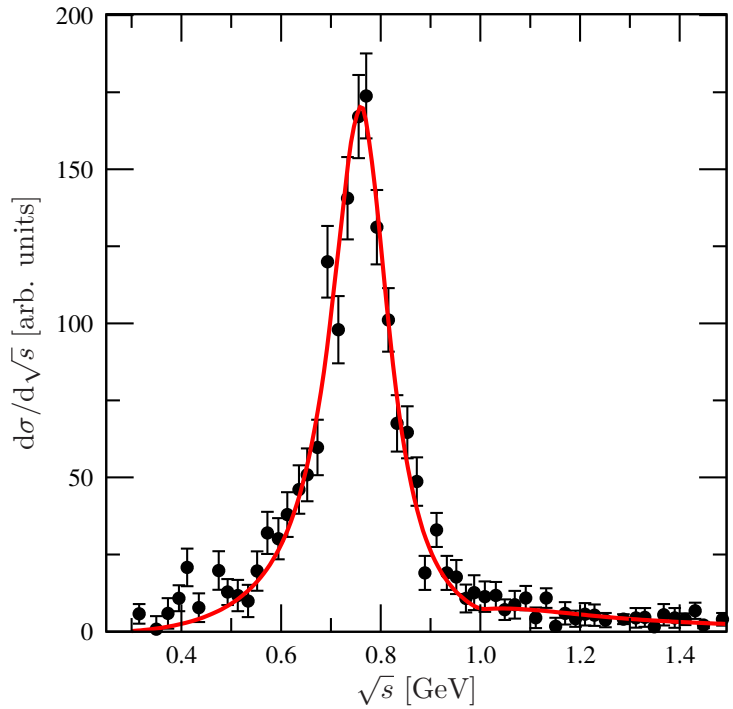
- test factorisation hypothesis in $e^+e^- \to \eta\pi^+\pi^-$:

$$f_1^{\eta\pi\pi\gamma^*}(s,k^2) \stackrel{!?}{=} f_1^{\eta\pi\pi\gamma}(s) \times F^{\eta\gamma\gamma^*}(k^2)$$

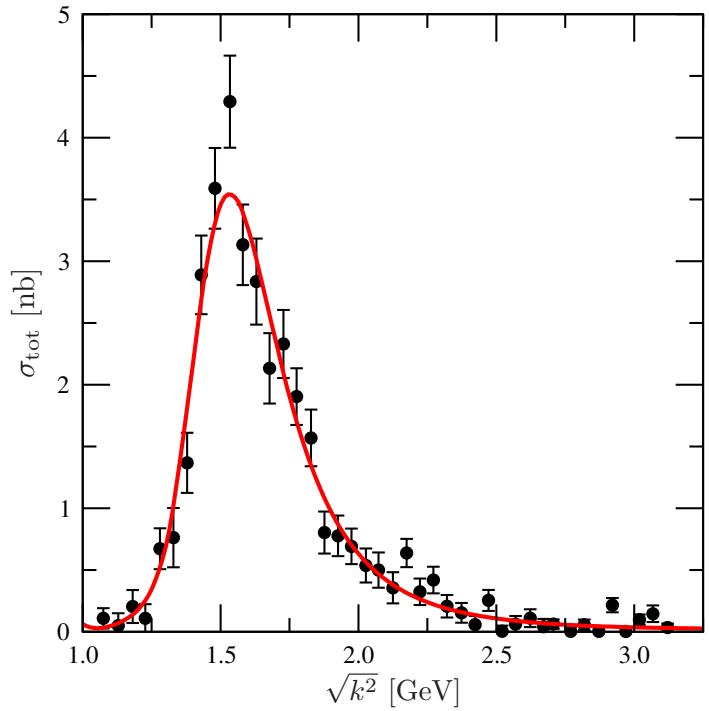
  - ▷ allow same form for $f_1^{\eta\pi\pi\gamma}(s)$ as in $\eta \to \pi^+\pi^-\gamma$
  - ▷ fit subtractions to $\pi^+\pi^-$ distribution in $e^+e^- \to \eta\pi^+\pi^-$
    $\longrightarrow$ are they compatible to the ones in $\eta \to \pi^+\pi^-\gamma$?
  - ▷ parametrise $F^{\eta\gamma\gamma^*}(k^2)$ by sum of Breit–Wigners (ρ, ρ')

Xiao et al. (preliminary)

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$



$$\frac{d\sigma}{d\sqrt{s}}$$



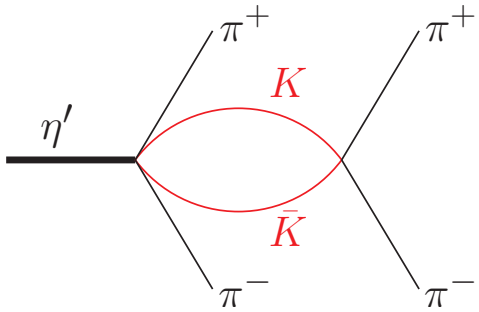
$$\sigma_{\rm tot}(k^2)$$

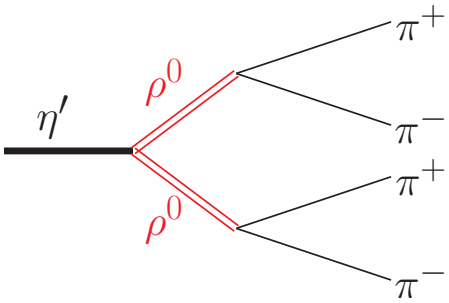
Xiao et al. (preliminary); data: BaBar 2007

- $d\sigma/d\sqrt{s}$ integrated over 1 GeV $\leq \sqrt{k^2} \leq$ 4.5 GeV
- factorisation seems to work only if curvature ($\simeq a_2$) retained
- more differential/binned data highly desirable!

How to go *doubly* virtual? — $\eta' \to \pi^+\pi^-\pi^+\pi^-$

- prediction of $\eta' \to 4\pi$ branching ratios based on ChPT + VMD:





$\longrightarrow \quad \mathcal{B}(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (10 \pm 3) \times 10^{-5}$ Guo, BK, Wirzba 2012

exp: $\mathcal{B}(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (8.5 \pm 0.7 \pm 0.6) \times 10^{-5}$ BESIII 2014

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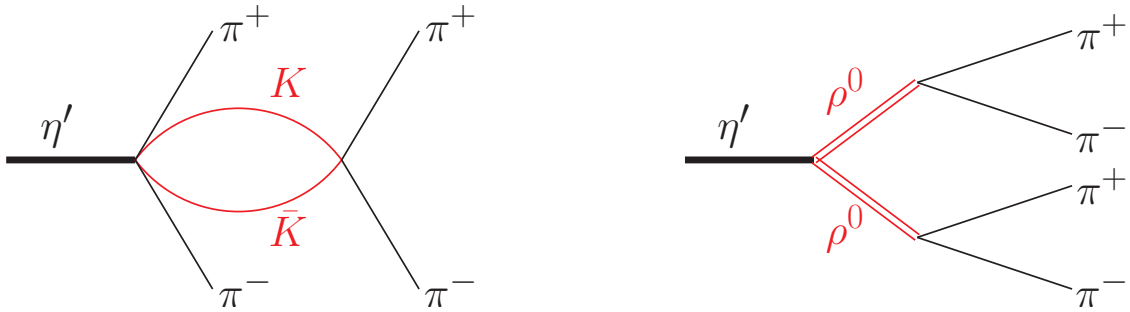

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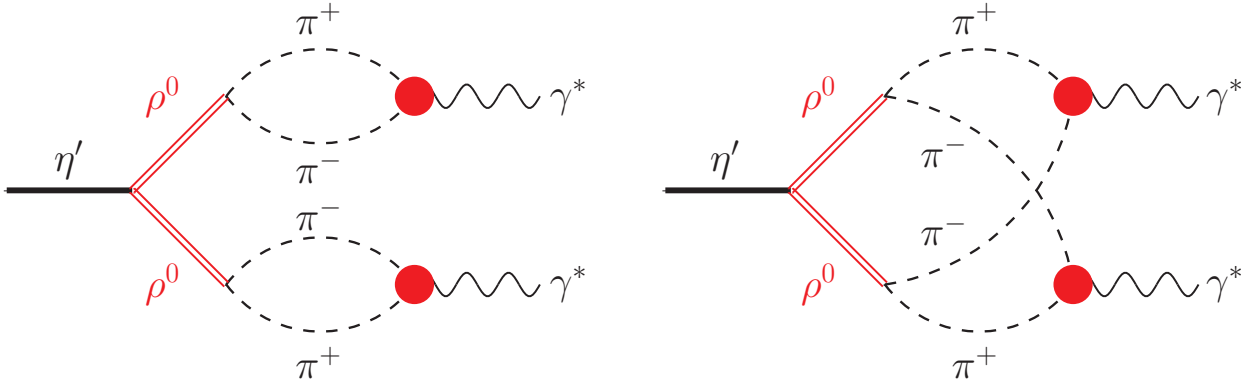
- start analysis of *doubly* virtual η' transition form factor from here?



factorising non-factorising

- aim: build double-spectral function for $\eta^{(\prime)}$ transition form factor

Summary / Outlook

Dispersive analyses of π^0, $\eta^{(\prime)}$ transition form factors:

- high-precision data on

  $e^+e^- \to \pi^+\pi^-\pi^0$ var. / $\eta \to \pi^+\pi^-\gamma$ KLOE / $\eta' \to \pi^+\pi^-\gamma$ BESIII

  allow for high-precision dispersive predictions of π^0, $\eta^{(\prime)} \to \gamma^*\gamma$

Merge with high-energy constraints on π^0:

- double-spectral representation, dispersive (low) + pQCD (high)
- addition of one effective pole allows to fulfil constraints:

  anomaly Brodsky–Lepage limit pQCD limit

- π^0 pole $(g-2)_\mu^{\pi^0} = 63.0^{+2.7}_{-2.1} \times 10^{-11}$

  uncertainties: $\pi^0 \to \gamma\gamma$ PrimEx-II

  dispersive uncertainties BES III

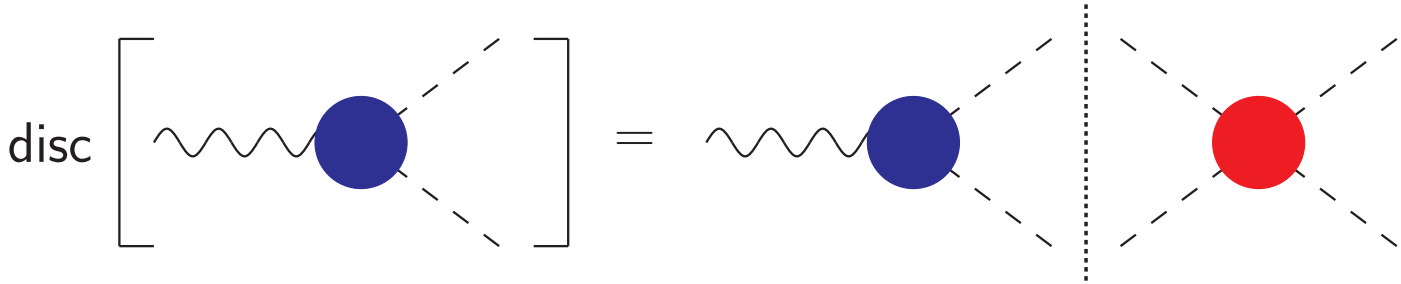
  BL limit (BaBar vs. Belle) Belle II

In progress:

- $\pi^0 \to e^+e^-$ & similar program for η / η' B.-L. Hoid, S. Holz et al.

Spares

Charged pion form factor



$$\frac{1}{2i}\,\text{disc}\,F_\pi^V(s) = \text{Im}\,F_\pi^V(s) \;=\; F_\pi^V(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_\pi^V(s)$ is just $\delta_1^1(s)$ Watson 1954

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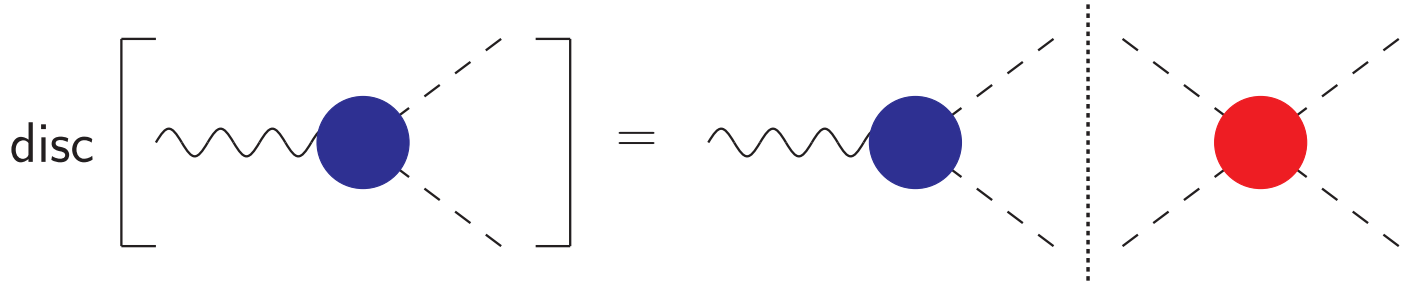

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- solution:

$$F_\pi^V(s) = P(s)\Omega(s)\;, \quad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

$P(s)$ polynomial, $\Omega(s)$ Omnès function Omnès 1958

  ▷ $\pi\pi$ phase shifts from Roy equations

  Ananthanarayan et al. 2001, García-Martín et al. 2011

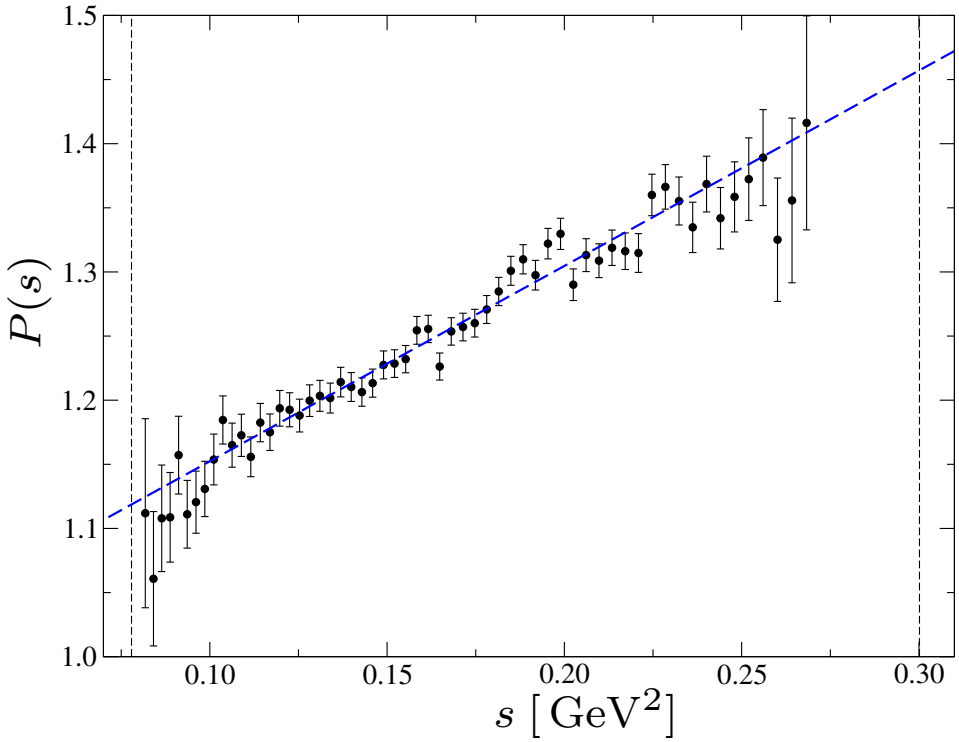
  ▷ $P(0) = 1$ from symmetries (gauge invariance)

- below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2}s)\Omega(s)$

  slope due to inelastic resonances ρ', $\rho''\ldots$ Hanhart 2012

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

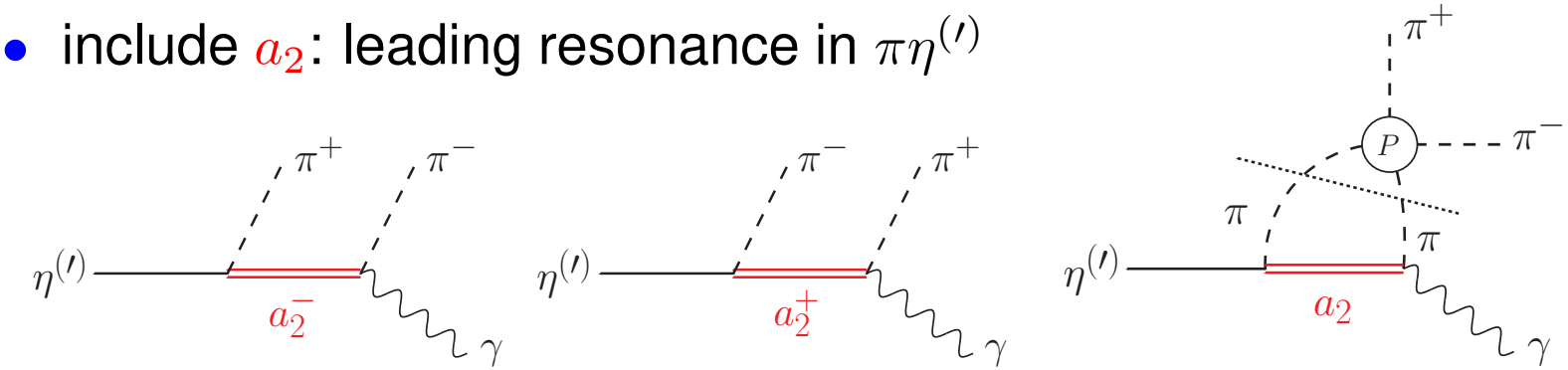
- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{F}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s) \times \Omega(s),\ \ P(s) = 1 + \alpha^{(\prime)} s,\ \ s = M^2_{\pi\pi}$
- divide data by pion form factor $\longrightarrow P(s)$ Stollenwerk et al. 2012

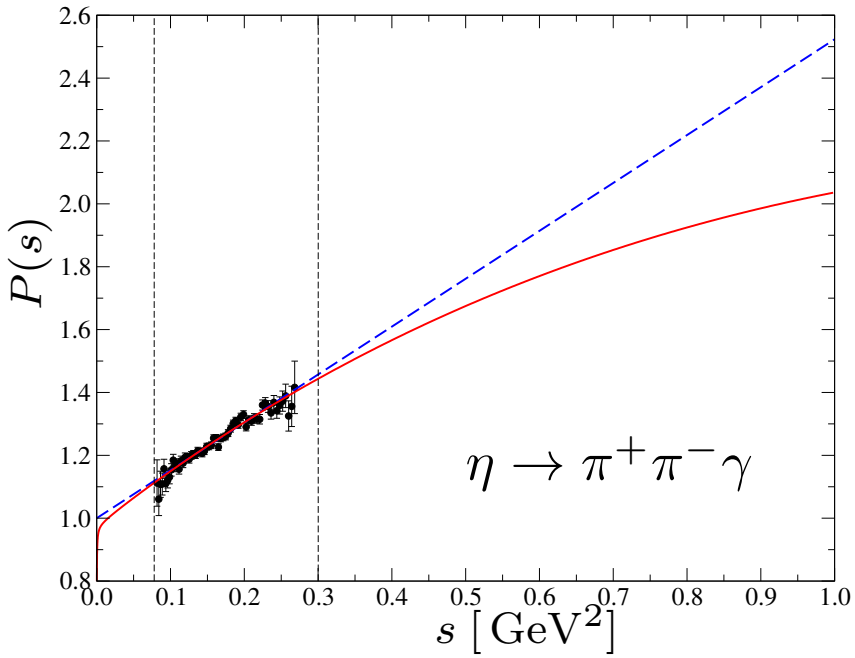


$\longrightarrow$ exp.: $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)\,\text{GeV}^{-2}$ cf. KLOE 2013

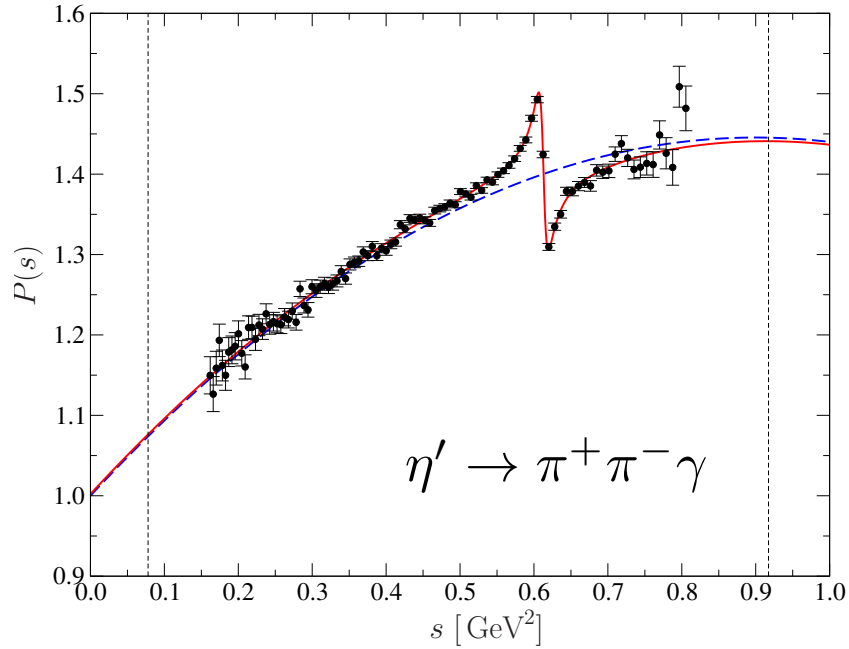
$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$





KLOE 2013; BK, Plenter 2015



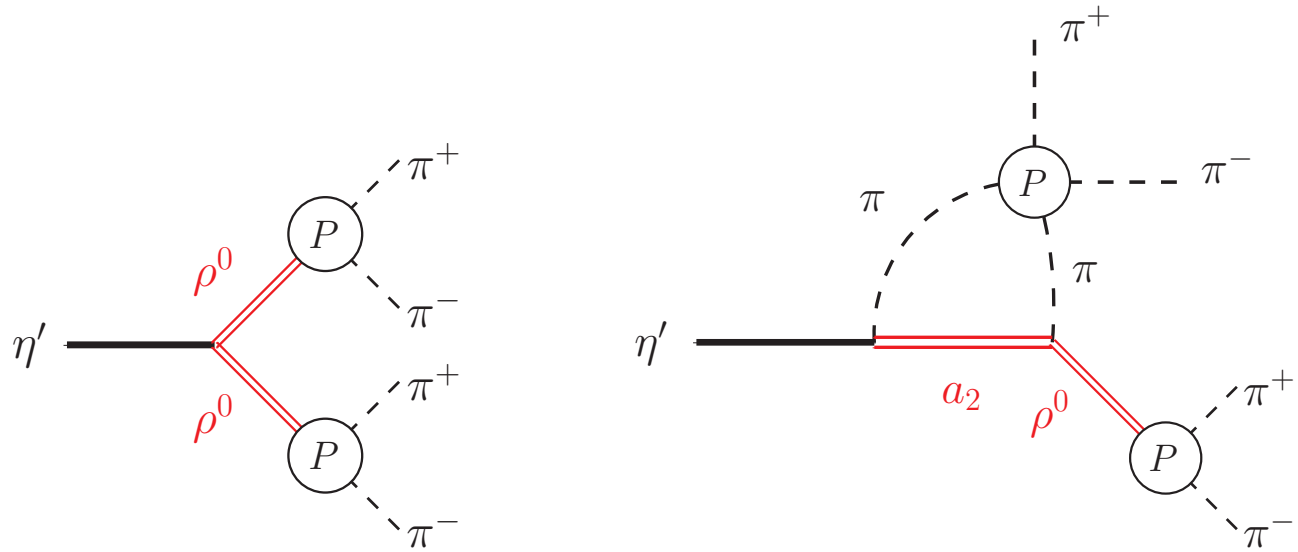
BESIII 2017; Hanhart et al. 2017

- induces curvature in $P(s)$
- curvature, plus ρ–ω mixing

How to go *doubly* virtual? — $\eta' \to \pi^+\pi^-\pi^+\pi^-$

- work in progress: S. Holz, J. Plenter, MSc theses

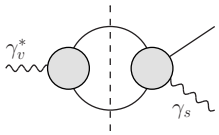
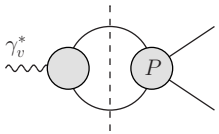
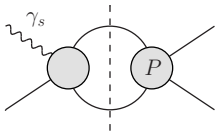
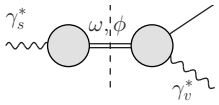
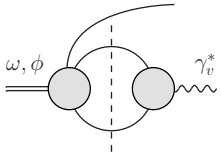
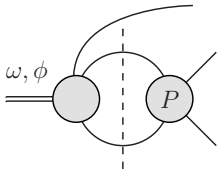
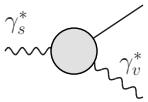
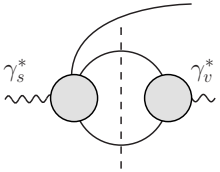
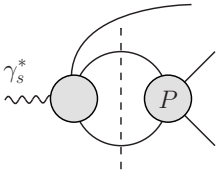
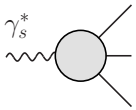
  implement a_2 left-hand cut for $\eta' \to \pi^+\pi^-\pi^+\pi^-$, too!



- pairwise rescattering only (no Khuri–Treiman for four-body!)
- a_2-exchange provides natural factorisation-breaking
- aim: build double-spectral function for $\eta^{(\prime)}$ transition form factor based on $\eta^{(\prime)} \to \pi^+\pi^-\pi^+\pi^-$ amplitude
- next step: consistency with $\eta' \to \pi^+\pi^-\gamma$ data

Summary: processes and unitarity relations for $\pi^0 \to \gamma^*\gamma^*$

Colangelo, Hoferichter, BK, Procura, Stoffer 2014

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi \to \pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{ds\,dt}(\omega, \phi \to 3\pi)$
			$\sigma(e^+e^- \to \pi^0\gamma)$
		$\sigma(e^+e^- \to 3\pi)$	$\sigma(\gamma\pi \to \pi\pi)$ $\frac{d^2\Gamma}{ds\,dt}(\omega, \phi \to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^- \to 3\pi)$

$\gamma\pi \to \pi\pi$

$\omega \to 3\pi$, $\phi \to 3\pi$

$\gamma^* \to 3\pi$

common theme:
resum $\pi\pi$ rescattering