



Operator Mixing with the Gradient Flow

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Motivation

- QCD contributions to Charge-Parity violation from BSM sources?
 - Sakharov: CPV (*etc.*) \Rightarrow nonzero neutron electric dipole moment (nEDM)
 - Neutron is a nonperturbative, low-energy system \Rightarrow Lattice methods
- “Integrate out” high-energy modes, leaving local effective (compound) operators ($d_i \geq d_S$)
- In order to take continuum limit, regulator a must be renormalized away.
- Two problems:
 - Operators mix under renormalization:
 - Operator-Product Expansion (OPE): $O_A^R(x) O_B^R(y) \xrightarrow{y \rightarrow x} \sum_{O_j \in \mathcal{A}} c_j(x-y) O_j^R(x) \Rightarrow Z_i \rightarrow z_{ij}$
 - On the lattice, $c_{ij} \propto a^{d_j - d_i}$, so that $\lim_{a \rightarrow 0^+} c_{ij}$ diverges for $d_i \geq d_j$
 - We must disentangle renormalization scale and regulator \Rightarrow new scale



Renormalization Refresher I

- Renormalized parameters = measurable parameters
- Locality of fields causes ultraviolet divergences
 - Think Coulomb potential at zero displacement
 - $\delta E = - \int_{\mathbf{k}} \frac{1}{\hbar\omega} \left| \langle \mathbf{k} | \int_{\mathbf{r}} A_{\mu}(\mathbf{r}) J^{\mu}(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) | 0 \rangle \right|^2 = E_1 + E_2 + E_{12}(\mathbf{r}_1, \mathbf{r}_2)$
 - $E_{12}(\mathbf{r}_1, \mathbf{r}_2) = \frac{q_1 q_2}{\epsilon_0} \int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}}{\mathbf{k}^2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$
 - $E_{12}(\mathbf{r}_1, \mathbf{r}_1 + \epsilon) = \frac{q_1 q_2}{\epsilon_0} \int_{|\mathbf{k}| \leq \Lambda} \frac{1}{\mathbf{k}^2} + \mathcal{O}(\epsilon) = \frac{\Lambda}{2\pi^2} \xrightarrow[\epsilon \rightarrow 0]{\Lambda \rightarrow \infty} \infty$



Renormalization Refresher II

- Type I
 - Consider ϕ^4 theory:

$$\text{---} \circ \text{---} = \text{---} + \text{---} + \text{---} \otimes \text{---} + O(g^4)$$

$$\text{---} = \int_k \frac{1}{k^2 + m^2} \sim \int dk \ k \rightarrow \Lambda^2$$

$$= \int_k \frac{1}{k^2 + m^2} = \frac{2(4\pi)^{-\frac{d}{2}}}{\Gamma(d/2)} \int dk \frac{k^{d-1}}{k^2 + m^2} = -\frac{m^2}{(4\pi)^2} \left\{ \frac{1}{\epsilon} + \log \frac{4\pi}{\gamma' m^2} + 1 \right\}$$



Renormalization Refresher III

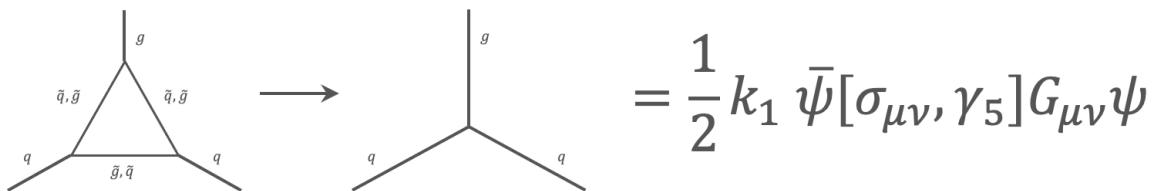
- Type II

- Consider a prototypical OPE for $\delta^2(x)$

$$\begin{aligned}\delta(x)\delta(x+\epsilon) &= \int_{p,q} e^{iq\cdot\epsilon} e^{i(p+q)\cdot x} = \int_{p,q} \left\{ 1 + i\epsilon_\mu q_\mu - \frac{1}{2}\epsilon_{\mu\nu}q_{\mu\nu} + \mathcal{O}(\epsilon^3) \right\} e^{-i(p+q)\cdot x} \\ &= \frac{1}{2} \left\{ 1 + \epsilon_\mu \partial_\mu + \frac{1}{2}\epsilon_{\mu\nu}\partial_{\mu\nu} + \mathcal{O}(\epsilon^3) \right\} \delta^2(x)\end{aligned}$$

- Quark chromoelectric dipole moment:

- Integrate out heavy fields from SUSY model: three vertices merge into one





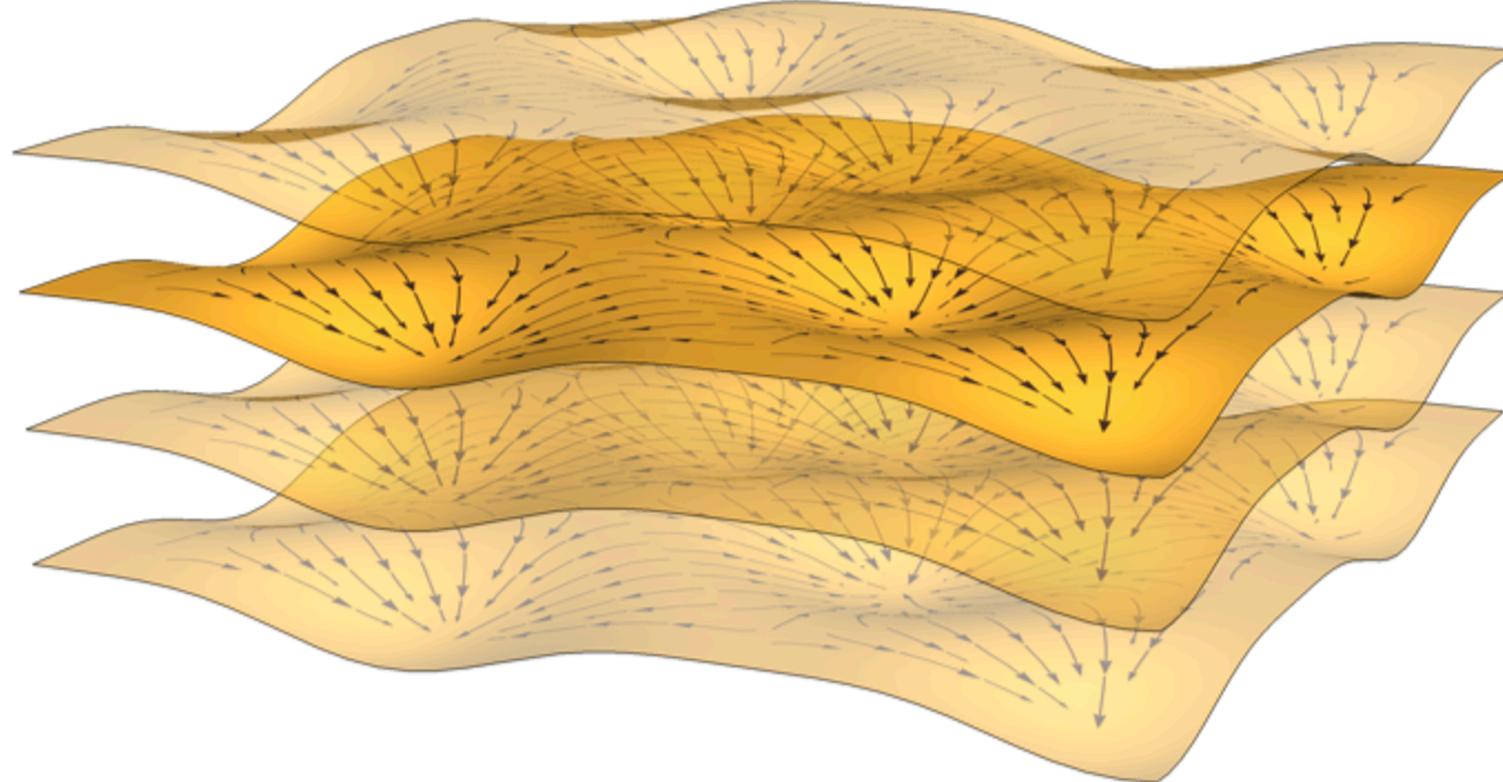
Gradient flow I

- Gauge Fields [1,2]
 - Drive fields toward a local minimum of the action
 - "change in flow time" $\sim (-1)$ "gradient of action"
 - $\partial_t B_\mu = -\frac{\delta S[A]}{\delta A_\mu}|_{A=B} + g.f.$ such that $B|_{\partial\mathcal{M}_t} = A$
- Fermions:
 - Minimally extend flow to fermions in a covariant way
- Nonlinear diffusion equations [1,2,3]:
 - $\partial_t B_\mu(x; t) = D_\nu G_{\nu\mu}(x; t) + \alpha_0 D_\mu \partial_\nu B_\nu(x; t)$
 - $\partial_t \chi(x; t) = D_\mu D_\mu \chi(x; t), \quad \partial_t \bar{\chi}(x; t) = \bar{\chi}(x; t) \bar{D}_\mu \bar{D}_\mu$
 - $B_\mu(x; 0) = A_\mu(x), \quad \chi(x; 0) = \psi(x), \quad \bar{\chi}(x; 0) = \bar{\psi}(x)$



Gradient flow I

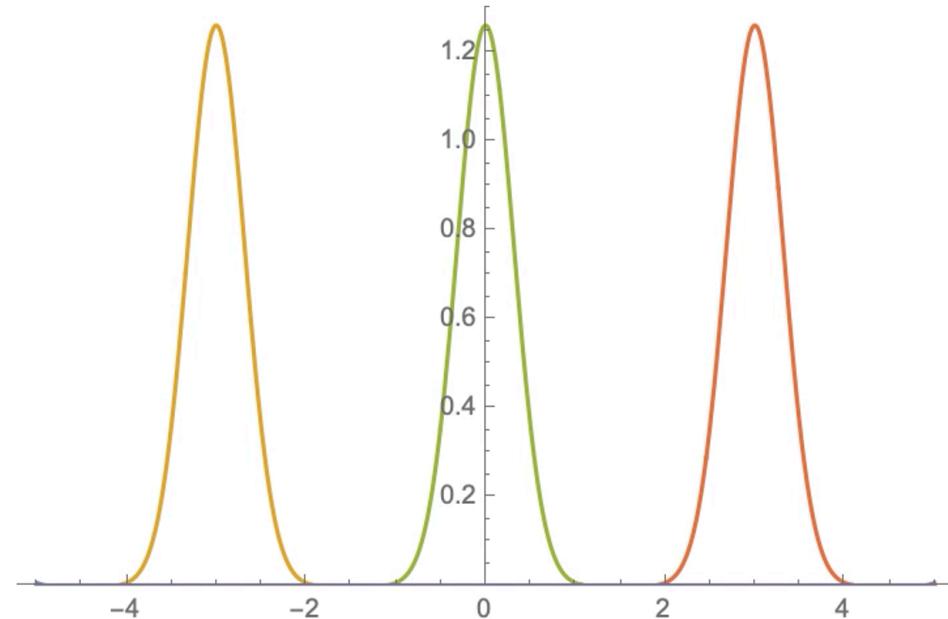
$$\partial_t B_\mu = - \frac{\delta S[A]}{\delta A_\mu} |_{A=B}$$





Gradient flow II

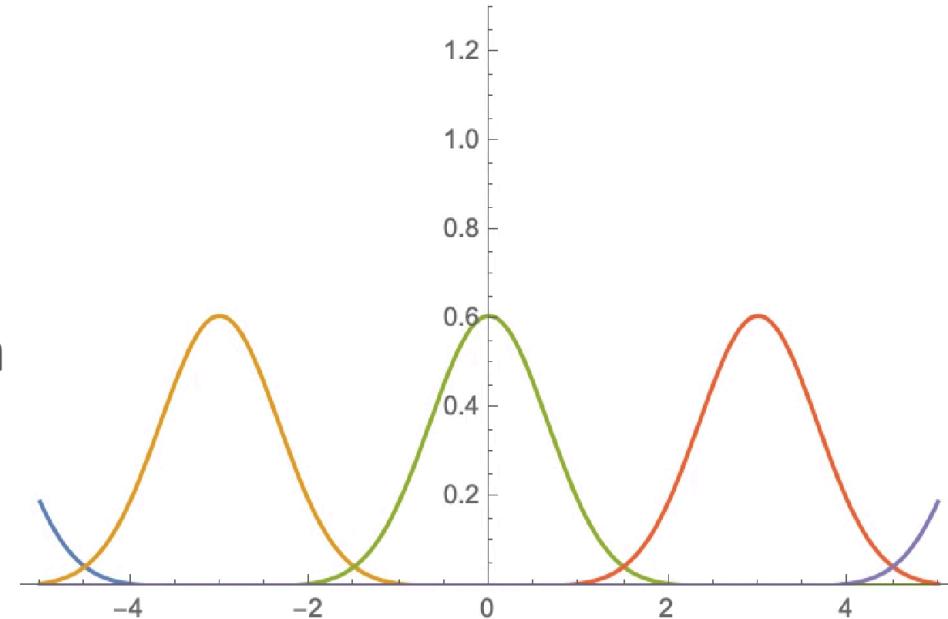
- Look again at δ -distribution
 - $\delta(x) = \lim_{\epsilon \rightarrow 0^+} \delta_\epsilon(x) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/2\epsilon}$
 - $\partial_t f - \partial^2 f = 0, \quad f(x; 0) = \delta(x)$
- Impose PDE as constraint on Lagrangian
 - Lagrange multipliers (fields)
 - New interactions





Gradient flow II

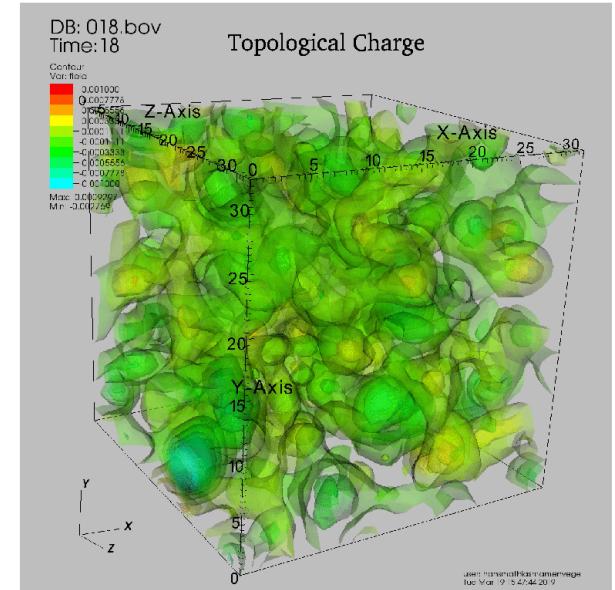
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 - $\partial_t f - \partial^2 f = 0, \quad f(x; 0) = \delta(x)$
- Impose PDE as constraint on Lagrangian
 - Lagrange multipliers (fields)
 - New interactions





Gradient flow III

- Recursive solution [2]:
 - $\partial_t B_\mu = [\delta_{\mu\nu} \partial^2 + (\alpha_0 - 1) \partial_\mu \partial_\nu] B_\nu + R_\mu$
- Leading order solution dampens high-energy modes
 - $\tilde{B}_\mu(p; t) = e^{-p^2 t} \tilde{A}_\mu(p) + \dots \rightarrow 0; p \rightarrow \infty, t > 0$
 - $r_f = x_{rms} = \sqrt{8t}$
- Remainder generates new interactions:
 - $R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu]$
 $+ [B_\nu, [B_\nu, B_\mu]]$





Renormalization with the Gradient Flow

- Renormalization of flowed operators is multiplicative:
 - Given a flowed operator $O_i(\{g_0\}, \{m_0\}, \{\xi_0\})$ with $2n_\chi = n_\chi + n_{\bar{\chi}}$ spinor fields,
$$O_i^R = Z_\chi^{\pm n_\chi} O_i(\{g\}, \{m\}, \{\xi\})$$
- OPE → Short-Flow-Time Expansion (SFTE):
 - $O_i^R(t) \stackrel{t \rightarrow 0^+}{\sim} \sum_{O_j \in \mathcal{A}} c_{ij}(t) O_j^R(0)$
 - Operator-level relation: probe with different choices of external field and different kinematics.



Renormalized Physical Operators

- $O_i^R(t) \xrightarrow{t \rightarrow 0^+} \sum_{O_j \in \mathcal{A}} c_{ij}(t) O_j^R(0); \quad d_{i+1} \geq d_i$
- $c_{ij} = \delta_{ij} + g^2 c_{ij}^{(1)} + \mathcal{O}(g^4); \quad c_{ij}^{(1)} \propto t^{\frac{1}{2}(d_j - d_i)}$
- $$\begin{pmatrix} O_1^R(t) \\ \vdots \\ O_i^R(t) \end{pmatrix} = \begin{pmatrix} c_{1,1} & \cdots & O(t^{n \geq 1}) & \cdots \\ \vdots & \ddots & \vdots & \cdots \\ c_{i,1} & \cdots & c_{i,i} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} O_1^R(0) \\ \vdots \\ O_i^R(0) \end{pmatrix}$$
- We can invert this equation to find the physical operators, perturbatively renormalized to arbitrary order:
 - $\mathbf{O}^R(0) = \lim_{t \rightarrow 0^+} c^{-1}(t) \mathbf{O}^R(t)$

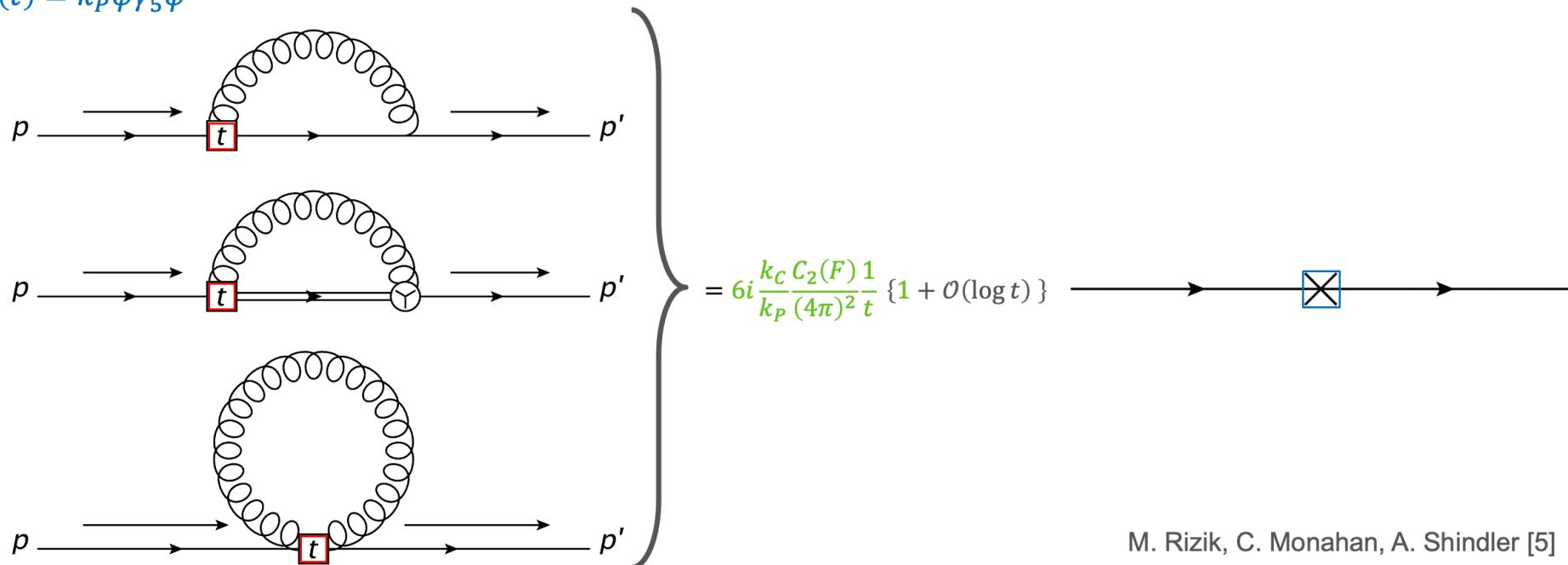


Results I: Power Divergences

$$O_C^R(t) = c_{CP}(t) O_P^R(0) + c_{Cq}(t) O_q^R(0) + \mathcal{O}(\log t)$$

$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi = \frac{1}{2} [\sigma_{\mu\nu}, \gamma_5];$$

$$O_P(t) = k_P \bar{\psi} \gamma_5 \psi$$



M. Rizik, C. Monahan, A. Shindler [5]

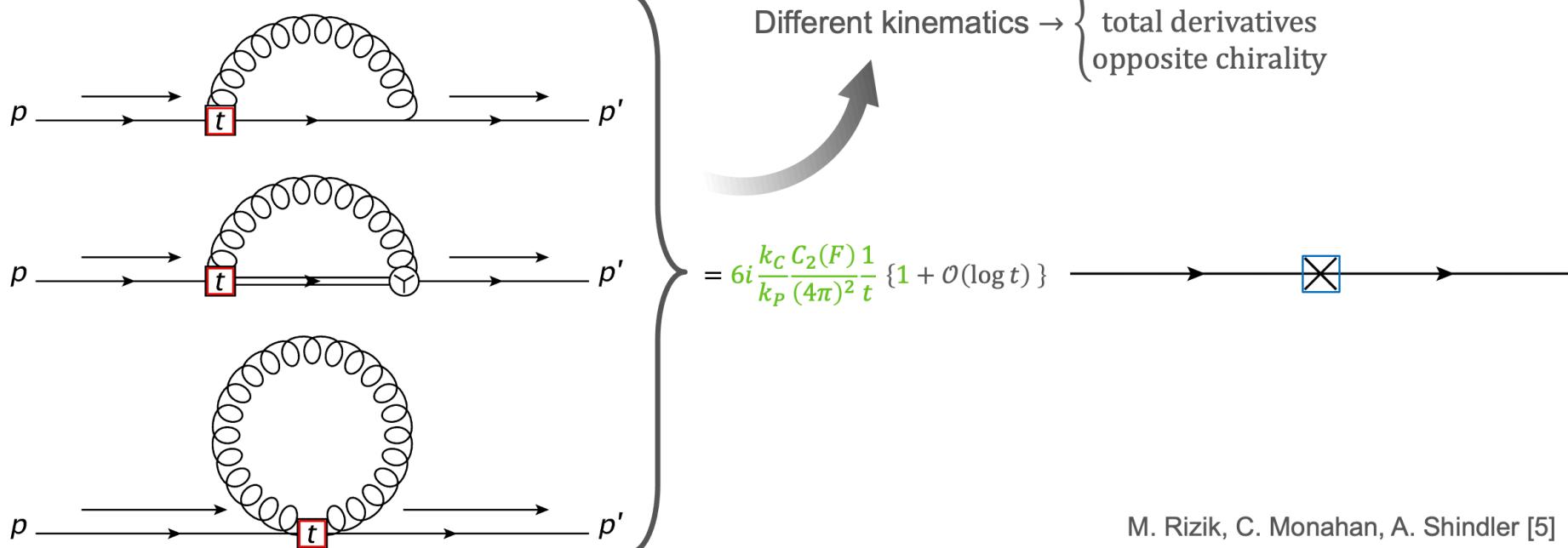


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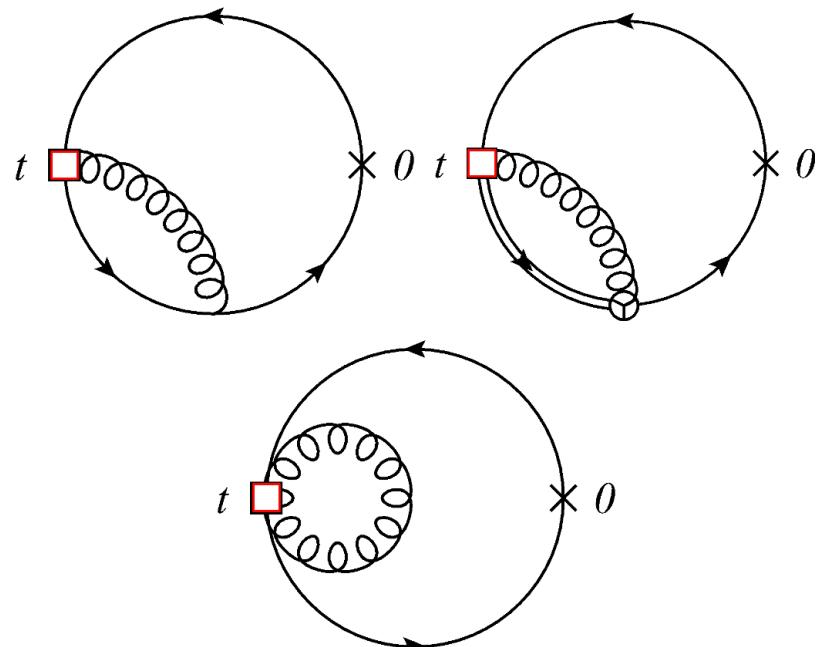


Results I: Power Divergences

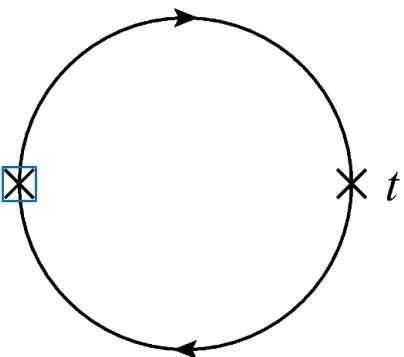
$$O_C^R(t) = c_{CP}(t) O_P^R(0) + c_{Cq}(t) O_q^R(0) + \mathcal{O}(\log t)$$

$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi$$

$$O_P(t) = k_P \bar{\psi} \gamma_5 \psi$$



$$= 6i \frac{k_C}{k_P} \frac{C_2(F)}{(4\pi)^2} \frac{1}{t} \{ 1 + \mathcal{O}(\log t) \} O \otimes t$$



J. Kim, T. Luu, M. Rizik, A. Shindler [6]

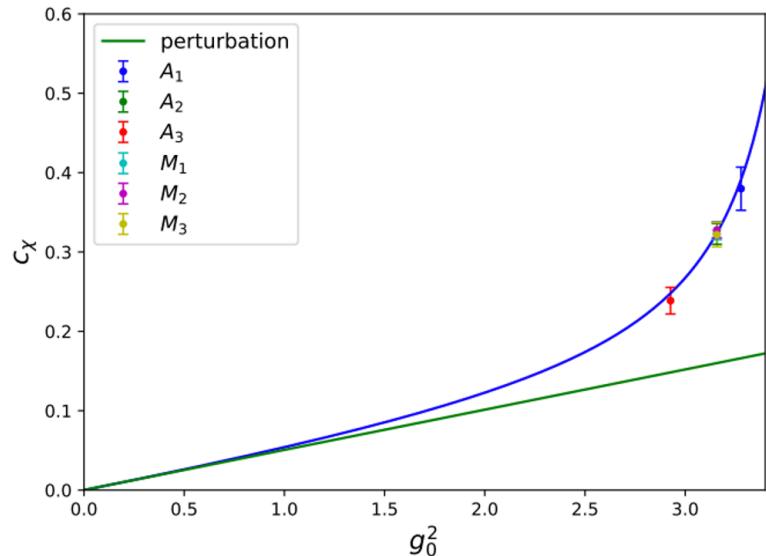


Results I: Power Divergences

$$O_C^R(t) = c_{CP}(t) O_P^R(0) + c_{Cq}(t) O_q^R(0) + \mathcal{O}(\log t)$$

$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi$$

$$O_P(t) = k_P \bar{\psi} \gamma_5 \psi$$



$$c_{CP}^{(1)}(t) = 6i \frac{k_C}{k_P} \frac{C_2(F)}{(4\pi)^2} \frac{1}{t}$$

J. Kim, T. Luu, M. Rizik, A. Shindler [6]

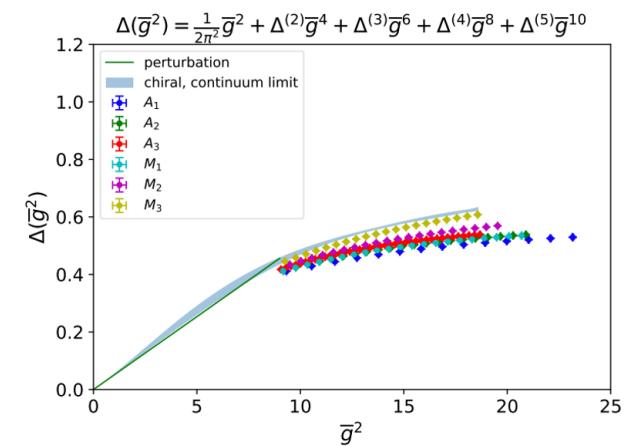
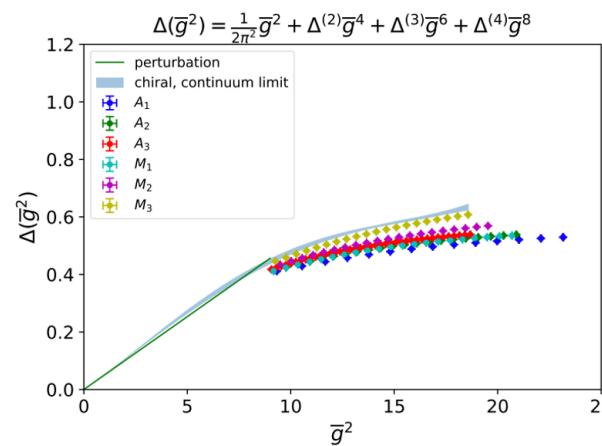
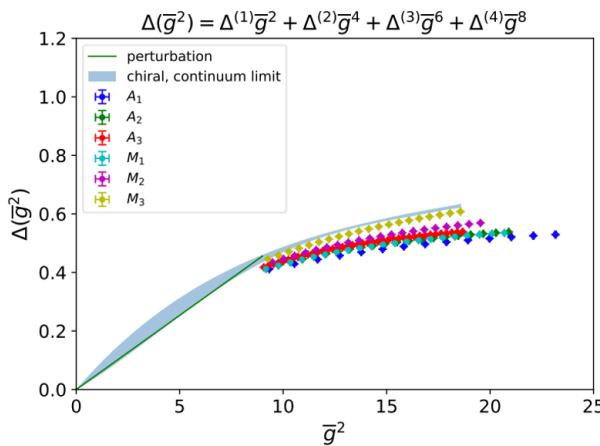


Results I: Power Divergences

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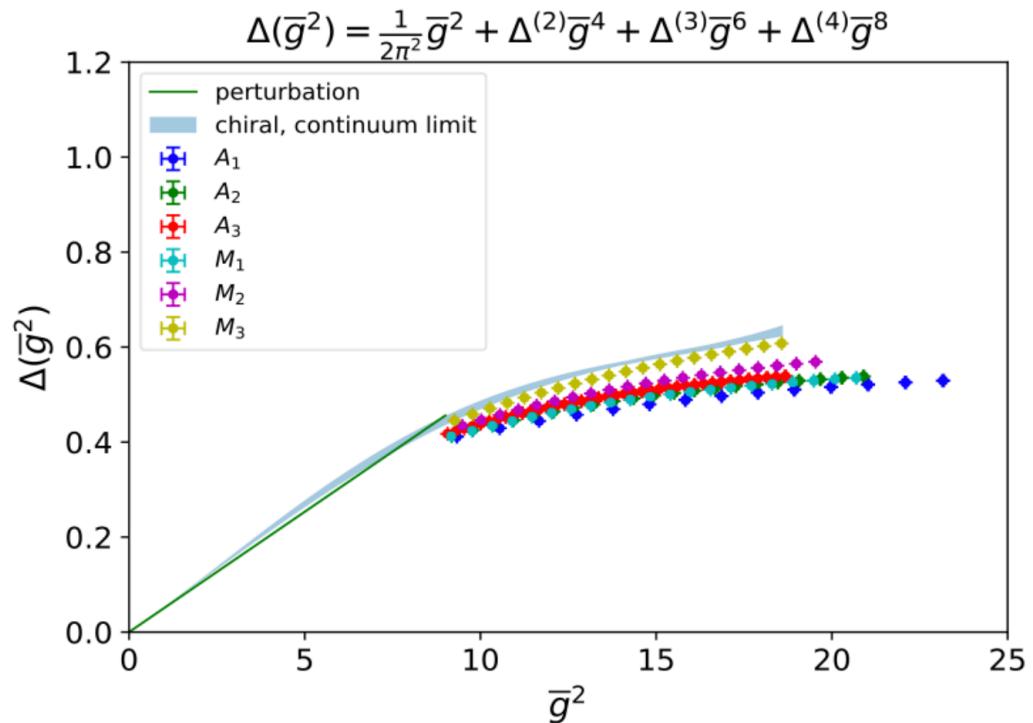


J. Kim, T. Luu, M. Rizik, A. Shindler [6]



Results I: Power Divergences

- $\Delta(t) = t \frac{c_{CP}(t)}{c_{PP}(t)} = t c_{CP}(t) + \mathcal{O}(t \log(t))$
 $t \rightarrow 0$
→ constant
- Δ is RGI:
 - Numerator and denominator scale with $\gamma_\chi + \gamma_P$
 - Scale dependence confined to $\beta(g)$
 - Continuum limit at each flow time



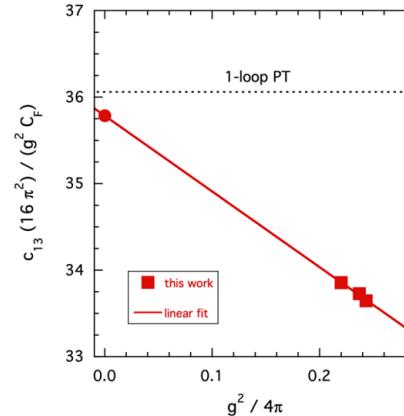
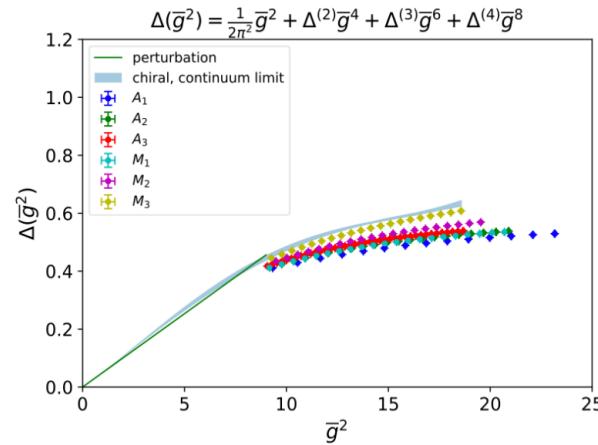
J. Kim, T. Luu, M. Rizik, A. Shindler [6]



Results I: Power Divergences

- $\Delta(t) = t \frac{c_{CP}(t)}{c_{PP}(t)} = t c_{CP}(t) + \mathcal{O}(t \log(t))$
 $t \rightarrow 0 \quad \longrightarrow \text{constant}$

- Δ is RGI:
 - Numerator and denominator scale with $\gamma_\chi + \gamma_P$
 - Scale dependence confined to $\beta(g)$
 - Continuum limit at each flow time
- Compare to previous methods
 - Scalar and CMDM
 - $\sim 10\%$ PT:Lat agreement as $g_0^2 \rightarrow 0$



M. Constantinou, M. Costa, R. Frezzotti, V. Lubicz, G. Martinelli, D. Meloni, H. Panagopoulos , S. Simula [7]



Results I: Power Divergences

$$O_W^R(t) = c_{Wq}(t) O_q^R(0) + \mathcal{O}(\log t)$$

$$O_W(t) = k_W \text{Tr} \left[[G_{\rho\mu}, G_{\rho\nu}] \tilde{G}_{\mu\nu} \right]$$

$$O_q(t) = k_q \text{Tr} [G_{\mu\nu} \tilde{G}_{\mu\nu}]$$

$$\left. \begin{array}{c} \text{Diagram 1: } \alpha a \xrightarrow{p} \text{loop} \xrightarrow{k} \beta b \\ \text{Diagram 2: } \alpha a \xrightarrow{p} \text{loop} \xrightarrow{k} \beta b \\ \text{Diagram 3: } \alpha a \xrightarrow{p} \text{loop} \xrightarrow{k} \beta b \end{array} \right\} = -\frac{9 k_W C_2(F)}{4 k_q (4\pi)^2 t} \{1 + \mathcal{O}(\log t)\} \alpha a \xrightarrow{p} \text{loop} \xrightarrow{\square} \beta b$$

M. Rizik, C. Monahan, A. Shindler [5]

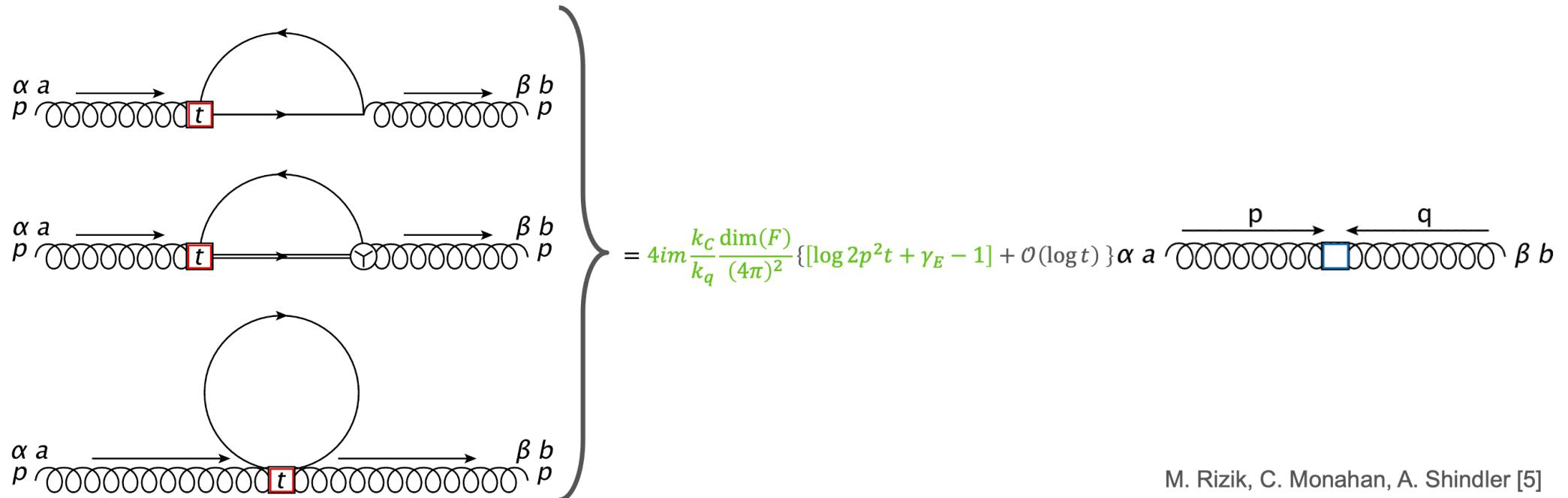


Results II: Logs

$$O_C^R(t) = c_{CP}(t)O_P^R(0) + c_{Cq}(t)O_q^R(0) + \mathcal{O}(\log t)$$

$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi$$

$$O_q(t) = k_q \text{Tr}[G_{\mu\nu} \tilde{G}_{\mu\nu}]$$



M. Rizik, C. Monahan, A. Shindler [5]



Results II: Logs

$$O_i^R(t) = c_{ii}(t) O_i^R(0) + \mathcal{O}(t)$$

$$O_i(t) = k_i \bar{\psi} \Gamma_i \psi; \quad \Gamma_i = 1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu\nu}, \quad i = S, P, V, A, T$$

$$\left. \begin{array}{c} \text{Diagram 1: Two horizontal gluon lines with loops. The top line has a loop at position } t \text{ with a red box. The bottom line has a loop at position } t \text{ with a red box.} \\ \text{Diagram 2: Two horizontal gluon lines with loops. The top line has a loop at position } t \text{ with a red box. The bottom line has a loop at position } t \text{ with a red box.} \\ \text{Diagram 3: Two horizontal gluon lines with loops. The top line has a loop at position } t \text{ with a red box. The bottom line has a loop at position } t \text{ with a red box.} \end{array} \right\} = -\frac{C_2(F)}{(4\pi)^2} \left\{ 4(\log 2\bar{\mu}^2 t + \gamma_E) + B_i^2 (\log 2p^2 t + \gamma_E) - \frac{1}{2} (B_i^2 - 4) \right\} p \rightarrow \square \rightarrow p$$

+ $\frac{1}{3} \frac{C_2(F)}{(4\pi)^2} B_i (2B_i^2 - 5) \frac{\not{p} \Gamma_i^{(0)} \not{p}}{p^2}$

M. Rizik

$$\left. \begin{array}{c} \text{Diagram 4: Four vertical gluon lines with loops. Each line has a loop at position } t \text{ with a red box.} \\ \text{Diagram 5: Four vertical gluon lines with loops. Each line has a loop at position } t \text{ with a red box.} \\ \text{Diagram 6: Four vertical gluon lines with loops. Each line has a loop at position } t \text{ with a red box.} \\ \text{Diagram 7: Four vertical gluon lines with loops. Each line has a loop at position } t \text{ with a red box.} \end{array} \right\} = \frac{C_2(F)}{(4\pi)^2} \left\{ (B_i^2 - 4)(\log 2\bar{\mu}^2 t + \gamma_E) - 6 \log 3 + 20 \log 2 + 2B_i^2 - 2B_i - 2 + \mathcal{O}(t) \right\} o \times \square o$$

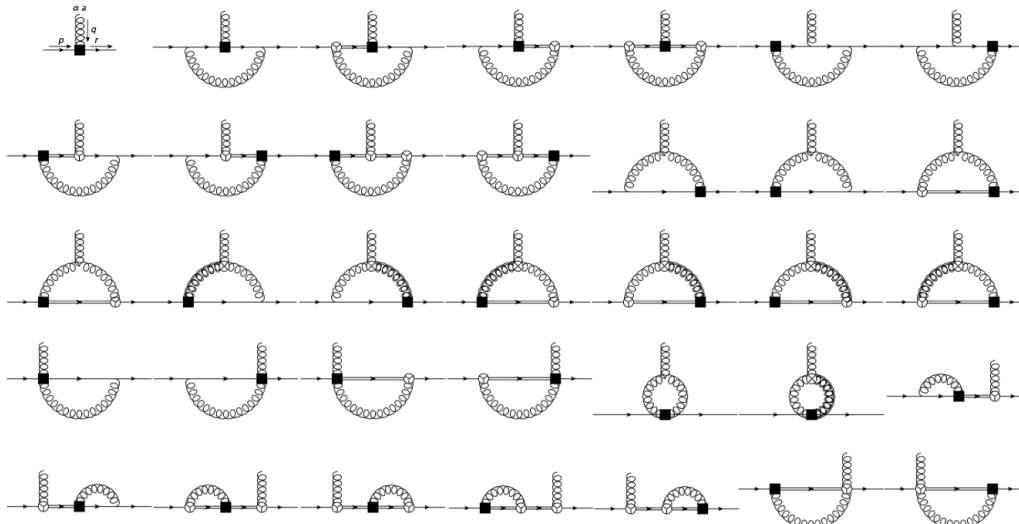
A. Hasenfratz, C. Monahan, M. Rizik, A. Shindler, O. Witzel [8,9,10]



Results II: Logs

$$O_C^R(t) = c_{CP}(t)O_P^R(0) + c_{Cq}(t)O_q^R(0) + \text{c}_{CC}(t)O_C^R(0) + \mathcal{O}(t)$$

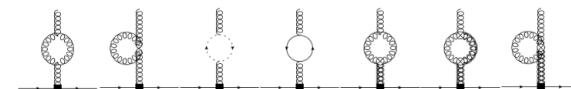
$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi$$



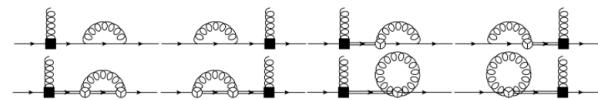
$$\sim c_{CP}(t) \langle A \bar{\psi} O_P(0) \psi \rangle + c_{Cq}(t) \langle A \bar{\psi} O_q(0) \psi \rangle + \{ A \log 2 \lambda^2 t + \text{finite} \} \langle A \bar{\psi} O_C(0) \psi \rangle + \text{E.o.M. operators}$$

E. Mereghetti, C. Monahan, M. Rizik, A. Shindler, P. Stoffer [11,12]

Z_g, Z_ξ



Z_χ

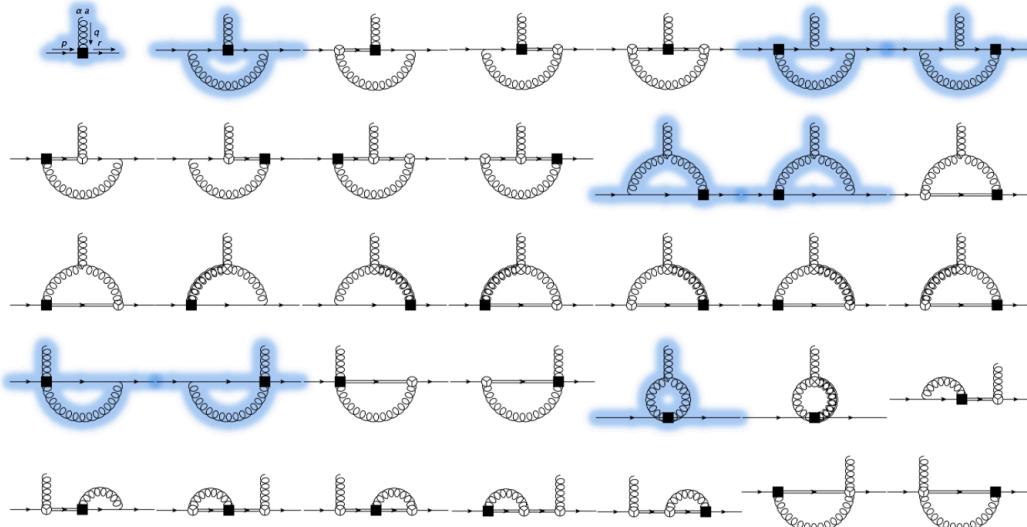




Results II: Logs

$$O_C^R(t) = c_{CP}(t)O_P^R(0) + c_{Cq}(t)O_q^R(0) + c_{CC}(t)O_C^R(0) + \mathcal{O}(t)$$

$$O_C(t) = k_C \bar{\psi} G_{\mu\nu} \tilde{\sigma}_{\mu\nu} \psi$$



$$\sim c_{CP}(t) \langle A \bar{\psi} O_P(0) \psi \rangle + c_{Cq}(t) \langle A \bar{\psi} O_q(0) \psi \rangle + \{ A \log 2 \lambda^2 t + \text{finite} \} \langle A \bar{\psi} O_C(0) \psi \rangle + \text{E.o.M. operators}$$

E. Mereghetti, C. Monahan, M. Rizik, A. Shindler, P. Stoffer [11,12]



Mixing Matrix

$$\begin{pmatrix} O_P^R(0) \\ O_q^R(0) \\ O_{CE}^R(0) \\ \vdots \\ O_W^R(0) \end{pmatrix} = \lim_{t \rightarrow 0^+} \begin{pmatrix} c_{P,P} & 0 & 0 & \cdots & 0 & \cdots \\ c_{q,P} & c_{q,q} & 0 & \cdots & 0 & \cdots \\ c_{CE,P} & c_{CE,q} & c_{CE,CE} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & c_{W,q} & X & \cdots & X & \cdots \end{pmatrix}^{-1} \begin{pmatrix} O_P^R(t) \\ O_q^R(t) \\ O_{CE}^R(t) \\ \vdots \\ O_W^R(t) \end{pmatrix}$$



Mixing Matrix

$$\begin{pmatrix} O_P^R(0) \\ O_q^R(0) \\ O_{CE}^R(0) \\ \vdots \\ O_W^R(0) \\ \vdots \end{pmatrix} = \lim_{t \rightarrow 0^+} \begin{pmatrix} c_{P,P} & 0 & 0 & \cdots & 0 & \cdots \\ c_{q,P} & c_{q,q} & 0 & \cdots & 0 & \cdots \\ c_{CE,P} & c_{CE,q} & c_{CE,CE} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & c_{W,q} & X & \cdots & X & \cdots \end{pmatrix}^{-1} \begin{pmatrix} O_P^R(t) \\ O_q^R(t) \\ O_{CE}^R(t) \\ \vdots \\ O_W^R(t) \\ \vdots \end{pmatrix}$$

There are ~ 15 operators for $d = 5$ [12]
and ~ 30 operators for $d = 6$ [13]



Summary and Outlook

- Perturbation Theory:
 - Mixing of $d = 5$ operators now (preliminarily) determined in both t'Hooft-Veltman (HV) and naïve dimensional regularization (NDR).
 - NDR automatically gives CP-even mixing matrix (CMDM+) up to identity mixing.
 - Weinberg's three-gluon operator needs attention
 - $d = 6$ Mixing
 - N+LO
- Lattice:
 - Smaller coupling → new ensembles (OpenLat)



Thank you

Special Thanks:

- Anna Hasenfratz
- Jangho Kim
- Thomas Luu
- Emanuele Mereghetti
- Christopher Monahan
- Giovanni Pederiva
- Andrea Shindler
- Peter Stoffer
- Jordy de Vries
- Oliver Witzel



Works Cited

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Perturbative Calculation of Wilson coefficients

- Feynman Rules ("pseudo"-Feynman gauge $\xi, \alpha_0 = 1$):
- $\tilde{S}(p; t, s) = e^{-p^2(t+s)} \tilde{S}(p) = \frac{-i p_\mu \gamma_\mu + m}{p^2 + m^2} e^{-p^2(t+s)}$
- $\tilde{D}_{\alpha\beta}^{ab}(q; t, s) = e^{-q^2(t+s)} \tilde{D}_{\alpha\beta}^{ab}(q) = \frac{\delta_{\alpha\beta} \delta^{ab}}{q^2} e^{-q^2(t+s)}$
- Kernel lines: $\int_0^t ds e^{-p^2(t-s)} f(s)$
- New interactions from higher-order terms in flow equations



Perturbation theory I

1. Dimensional regularization:

- Angular integral analytically continued to $(d - 1)$ -dimensions
- $\int_k f(k^2) = \prod_{i=1}^d \left[\int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \right] f(k^2) = \frac{(4\pi)^{2-d/2}}{(4\pi)^2} \frac{2}{\Gamma(d/2)} \int_{-\infty}^{\infty} dk k^{d-1} f(k^2)$
- Requires spherical integrand

2. Feynman/ Schwinger parameters

3. Fubini/ Tonelli theorems

4. Tensor Decomposition



Perturbation theory II

- Flowed propagators: $\frac{1}{p_1^2 p_2^2 \cdots p_n^2} \rightarrow \frac{e^{-p_1^2 t_1} e^{-p_2^2 t_2} \cdots e^{-p_n^2 t_n}}{p_1^2 p_2^2 \cdots p_n^2}$
- Conservation at vertices forces most p_i to be linear combinations of external and loop momenta
- Example:
 - $n = 2, p_1 = k, p_2 = p + k$
 - $\frac{1}{p_1^2 p_2^2} \rightarrow \frac{e^{-k^2 t} e^{-(p+k)^2 t}}{k^2 (p+k)^2} = \int_t^\infty dz e^{-p^2 z} \frac{e^{-k^2(t+z)}}{k^2} e^{-2z p \cdot k}$



Perturbation theory III

- Expand cross-terms in MacLaurin series [4]:
 - $e^{-2z p \cdot k} = \sum_{n=0}^{\infty} \frac{(2z p \cdot k)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} p_{I_n} k_{I_n}$; $p_{I_n} = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_N}$
 - Integrand now of the form $f_n(k^2; \dots) k_{I_m} k_{I_n} p_{I_n}$ (implicit Einstein notation)
- Tensor integral decomposition at each order n is now a combinatorial problem (Isserlis)
 - For any even order of the tensor power $n + m$ of the momentum k , the integral will proportional to a symmetrized sum over $(n + m)$ -fold products of metric tensors (even-dimensional isotropic tensors):
 - $S_{I_{2n}}^{(2n)} = \sum_{i=1}^{(2n-1)!!} \prod_{j=1}^n \delta_{\mu_{\sigma_i(2j-1)} \mu_{\sigma_i(2j)}}$
 - E.g.: $S_{I_4}^{(4)} = \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} + \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3}$