Three-body scattering in the presence of bound states

Sebastian M. Dawid

Thomas Jefferson National Accelerator Facility
"Cake" Seminar
Thursday, September 13th, 2021
Previous JLab seminars on three-body physics

Monday, October 15, 2018, 1:00 p.m. Room L102
Maxim Mai (GWU)
3 hadrons in (in-)finite volume

Monday, October 29, 2018, 1:00 p.m. Room L102
Andrew Jackura (Indiana U.)
Phenomenology of 3→3 Scattering

Monday, February 11, 2019, 1:00 p.m. Room L102
Peng Guo (California Bakerfield University)
Multiple particles interaction in finite volume

Wednesday, March 13, 2019, 1:00 p.m. Room L102
Raúl Briceño (JLab/ODU)
The three-body problem

Monday, December 14th, 2020, 1:00 p.m.
Fernando Romero López (University of Valencia)
Two- and three-particle scattering amplitudes from lattice QCD

Monday, March 15th, 2021, 1:00 p.m.
Maxwell Hansen (University of Edinburgh)
Three-pion scattering from lattice QCD
Efimov physics - picture

Short range, near-resonant interaction (barely supports two-body bound state)

Two bodies

Induced long-range interaction
Discrete scale symmetry
Borromean binding

\[ E_n \propto - \left( e^{-\frac{2\pi}{s_0}} \right)^n \]
Three-body processes and hadronic spectrum

- Exotic resonances decay to three-particle final states
  - \( X(3872), N^*(1440), a_1(1260), a_1(1420), ... \)

- Interpretations
  - molecules,
  - hybrids,
  - diquark-antidiquark,
  - kinematical effects (triangle singularity)

- Goal — to build three-body scattering formalism
  - derive properties of hadrons from the (lattice) QCD
  - having convenient three-body framework for phenomenology

Light-meson spectroscopy with COMPASS
Ketzer, Grube, Ryabchikov, Prog. Part. Nucl. Phys. 113 (2020) 103755

not only a question of computational cost. With decreasing pion mass also the kinematical thresholds for three- and four-hadron channels decrease. In particular highly excited states couple strongly to such multi-hadron final states. However, the current method is not applicable to these channels. A complete finite-volume formalism for three or even more particles would therefore be a major breakthrough for the calculation of masses and decay modes of hadron resonances. Such a formalism is already under development (see Ref. [33] and references therein).
Exotic states

Roper $N^*(1440)$  

X(3872)

Pion-nucleon scattering in the Roper channel from lattice QCD, Lang, Leskovec, Padmanath, Prelovsek, Phys. Rev. D 95 (2017) 1, 014510

Combining experimental $N\pi$ phase shifts with elastic approximation and the Lüscher formalism suggests in the spectrum an additional energy level near the Roper mass $m_R = 1.43$ GeV for our lattice. We do not observe any such additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$.

- About 7 MeV from $D\bar{D}\pi$
- About 0.4 MeV from $D^*\bar{D}$
Principles of the S–matrix theory:

- **Unitarity** (probability conservation)
- **Analyticity** (causality)
- **Crossing symmetry** (particles ↔ antiparticles)
- **Poincaré symmetry** (frame independence)
- **Internal symmetries** (charge, isospin, G–parity)

Analyticity on the first Riemann sheet,
Bound–states & resonances correspond to poles,
Branch cuts correspond to open channels.
Elastic unitarity in the two-body scattering

\[ 2 \text{Im} \begin{array}{c} \includegraphics[width=0.5\textwidth]{diagram.png} \end{array} = \begin{array}{c} \includegraphics[width=0.5\textwidth]{diagram.png} \end{array} \]

- Unitarity relation: \( SS^\dagger = 1 \) implies \( \frac{1}{2i} (M^\dagger - M) = \frac{1}{2i} M M^\dagger \) for \( S = 1 + iM \)

- After partial wave projection

\[ \text{Im} M_\ell(s) = \rho(s) |M_\ell(s)|^2 \theta(s - 4m^2) \quad \text{where} \quad \rho(s) \propto \sqrt{1 - \frac{4m^2}{s}} \]

- In consequence

\[ M^{-1}_\ell(s) = K^{-1}_\ell(s) - i\rho(s) \]

- Phase shift

\[ K^{-1}_\ell(s) = \frac{q^*}{8\pi E^*} \cot \delta_l(E^*) \]
Two-body Lüscher formalism – general idea

- One-dimensional NR problem in infinite volume (IV)
  \[ \psi(x) \propto \cos(k|x| + \delta(k)) \]

- One-dimensional NR problem in finite volume (FV)

  - Functions
    \[ \cos\left(\frac{kL}{2} + \delta(k)\right) = \cos\left(\frac{kL}{2} + \delta(k)\right) \]
  
  - Derivatives
    \[ -\sin\left(\frac{kL}{2} + \delta(k)\right) = \sin\left(\frac{kL}{2} + \delta(k)\right) \]

  Periodic boundary conditions

Quantization condition
\[ \frac{kL}{2} + \delta(k) = n\pi \]
Two-body quantization condition

- Relation between IV phase shift and FV spectrum

\[ \det \left[ \mathcal{M}(s) + F^{-1}(s, P, L) \right] = 0 \]

Matrices in the angular momentum space (truncation)

Complicated but known function

An a0 resonance in strongly coupled \( \pi\eta, \bar{K}K \) scattering from lattice QCD
Briceño, Dudek, Young, Rev. Mod. Phys. 90 (2018) 2, 025001

Resonance in coupled \( \pi\omega, \pi\phi \) scattering from lattice QCD
Woss et al. (HadSpec), Phys. Rev. D 100 (2019) 5, 054506
The canonical workflow

- Finite volume spectrum → Quantization Condition → Three-body K-matrix
- K-matrix + two-body subprocesses → integral equations → 3-body amplitudes
- Final amplitudes analytically continued to the unphysical Riemann sheets

Two main frameworks

- Relativistic Effective Field Theory
  - generic scalar EFT,
  - summation of 2PI, 3PI diagrams,
    (references — next few slides)
  - First results — three pions at I=3

- Unitarity-based framework
  - parametrization based on the S matrix unitarity,
    (references — next few slides)
  - First results — three pions at I=3
Three-body S-matrix unitarity in one slide

Phenomenology of relativistic $3 \rightarrow 3$ reaction amplitudes within the isobar approximation

Pair/Isobar and Spectator choice

Partial-wave projection

Three-body unitarity relation

Amputation of two-body interactions
The B-matrix approach

- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation

Three-body amplitude

$$A_{\ell m_\ell; \ell' m_\ell} (\sigma', s, \sigma)$$

- pair–spectator,
- partial waves,
- symmetrization,

One Particle Exchange

Short Range Interactions

$$A = M_2 B M_2 + M_2 \int B \tau A$$

$$\tilde{A} = B + \int B M_2 \tau \tilde{A}$$

$$M_2 = K + K i \rho M_2$$
The unitarity-based approach

✧ Formalism

Relativistic three-body theory with applications to \( \pi-N \) scattering
Aaron, Amado, Young, Phys. Rev. 174, 2022 (1968)

Three-body unitarity in finite volume

Three-body unitarity with Isobars revisited

Three-body spectrum in a finite volume: the role of cubic symmetry

Phenomenology of relativistic \( 3 \rightarrow 3 \) reaction amplitudes within the isobar approximation

Three-body scattering: ladders and resonances
Mikhasenko et al. (JPAC), JHEP 08 (2019) 1, 080

Bound states in the B-matrix formalism for the three-body scattering
Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009

✧ Lattice spectrum

and applications

Finite-Volume Spectrum of \( \pi^+\pi^- \) and \( \pi^0\pi^0\pi^0 \) Systems

Three-body unitarity vs finite-volume \( \pi^0\pi^0\pi^0 \) spectrum from lattice QCD

Dalitz plots and lineshape of \( a_1(1260) \) from relativistic three-body unitary approach

Three-pion spectrum in the I=3 channel from lattice QCD

Finite volume energy spectrum of the K-K-K system

Three-body interactions from the finite-volume spectrum
Brett et al., Phys. Rev. D 104 (2021) 1, 014501

Three-body dynamics of the \( a_1(1260) \) resonance from lattice QCD
Mai et al. (GWQCD), arxiv:2107.03973 (2021)

older papers from 60's by
Fleming, Holman, Grisaru, Blankenbecler, Cook, Lee, Hwa, Greben, Kok, etc.
REFT three-body formalism

\[
C_L(E, \hat{P}) = \sum \text{diagrams} + \sum \text{diagrams} + \sum \text{diagrams} + \cdots
\]

\[
\text{det} \left[ \mathcal{K}_{\text{df},3}(s) + F_3(s, P, L)^{-1} \right] = 0
\]

- Infinite volume integral equations:

\[
\mathcal{M}_3^{(u,u)} = \mathcal{D}^{(u,u)} + \mathcal{M}_{\text{df},3}^{(u,u)}
\]
REFT approach

Formalism

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude

Threshold expansion of the three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 93 (2016) 9, 096006

Relating the FV spectrum and the 2-and-3-particle $S$ matrix for relativistic systems of identical scalar particles
Briceño, Hansen, Sharpe, Phys. Rev. D 95 (2017) 7, 074510

Three-particle systems with resonant subprocesses in a finite volume

Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism

Generalizing the relativistic quantization condition to include all three-pion isospin channels
Hansen, Romero-López, Sharpe, JHEP 07 (2020) 047

Relativistic three-particle quantization condition for nondegenerate scalars
Blanton, Sharpe, Phys. Rev. D 103 (2021) 5, 054503

Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ and related systems
Blanton, Sharpe, Phys. Rev. D 104 (2021) 3, 034509

Lattice spectrum and applications

Numerical study of the relativistic three-body quantization condition in the isotropic approximation
Briceño, Hansen, Sharpe, Phys. Rev. D 98 (2018) 1, 014506

Implementing the three-particle quantization condition including higher partial waves
Blanton, Romero-López, Sharpe, JHEP 03 (2019) 106

$I=3$ Three-Pion Scattering Amplitude from Lattice QCD

Interactions of two and three mesons including higher partial waves from lattice QCD
Blanton et al., arxiv:2106.05690 (2021)

Energy dependent $\pi^+\pi^-\pi^+$ scattering amplitude from lattice QCD
Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021) 012001
Both formalisms equivalent in the FV and IV form

\[ \int \left[ \delta_{p'k'} + \frac{1}{3} \int_{k'} U_{p'k'} K_{df,k'k} \right] R_{kp} = \int_{k'} \int_k U_{p'k'} K_{df,k'k} U_{kp} \]

\[ \text{det} \left[ \overline{K}^{-1}_{2,L} + \tilde{F} + \tilde{G} - (2\omega L^3)^{-1} R^{(u,u)} (2\omega L^3)^{-1} \right] = 0 \]

In practice, differences in the regularization schemes exist in the literature.
Short summary

- Three-body physics is not only interesting, but also important for studies of resonances,
  - Lattice QCD,
  - Phenomenology,
- The two-body Lüscher philosophy is well-developed, and guides the three-body research,
- There are two equivalent relativistic formalisms,
  - Unitarity-based,
  - EFT,
- $R$-matrix or $K_{df}$ are obtained from lattice or modeled; they are related to the physical amplitudes via integral equations,
- Amplitudes must have good analytic properties,
Solving the EFT three-body ladder equation

- **Ladder approximation,** $B = G + (R=0)$

- **Numerical solution of the three-body EFT equations**


- **Similar studies**

  - **weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$**
    Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

  - **decay** $a_1(1260) \rightarrow \rho^0\pi^- \rightarrow \pi^-\pi^+\pi^-$
Nonrelativistic studies

The Three-Boson System with Short-Range Interactions

◆ NREFT approach

\[ \mathcal{L} = \psi^\dagger (i\partial_0 + \frac{\nabla^2}{2m})\psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}} (T^\dagger \psi \psi + \text{h.c.}) + \hbar T^\dagger T \psi^\dagger \psi + \ldots \]

◆ The three-body equation

\begin{align*}
T & = \quad + \quad + \quad + \\
\end{align*}

Three-body spectrum in a finite volume: the role of cubic symmetry

\begin{align*}
\text{Three-body scattering in the presence of bound states}
\end{align*}
**Bound-state system from REFT QC**

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

- **Start with the quantization condition**

\[
\det \left[ \mathcal{K}_{df,3}(s) + F_3(s, P, L)^{-1} \right] = 0
\]

- **S-wave, P=0, and K_{df}=0**

- **Geometric function**

\[
L^3 F_3^s \equiv \frac{\tilde{F}_s^s}{3} - \tilde{F}_s^s \frac{1}{1/\tilde{K}_2^s + \tilde{F}_s^s + \tilde{G}_s^s}
\]

- **Energy levels**

- **Noninteracting 3-particle states,**
- **Noninteracting dimer-particle states,**
- **Interacting 3-particle, dimer-particle, and trimer states**

- **2-body quantization condition**
Bound state system from REFT QC

Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states
Romero-López et al., JHEP 10 (2019) 007

Graphs showing the relationship between $\frac{k}{m}$ cot $\delta_0$ and $(k/m)^2$ for different values of $a$: $a=2$ and $a=16$. The graphs compare linear fits, quartic global fits, and the energy $E = 3m$. The graphs indicate the behavior of the system under different conditions, with $a=2$ and $a=16$ showing distinct patterns in the scatterplots.
Numerical procedure

- Discretization of the integral equation → N linear equations (Matrix equation)
- Regulation of the bound-state pole via $\epsilon$-prescription

\[ A_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} A_2(s; N, \epsilon) \]

Systematics

- **Unitarity test:** \[ \text{Im} A_2(s) = \rho_2(s) |A_2(s)|^2 \quad \rightarrow \quad \Delta \rho_2 = 100 \times \left| \frac{\text{Im} A_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right| \]

- **Convergence test:** \[ \Delta_N A_2 = 2 \times \left| \frac{A_2(s; N + 1, \epsilon) - A_2(s; N, \epsilon)}{A_2(s; N + 1, \epsilon) + A_2(s; N, \epsilon)} \right| \]

Methods

- **“Brute force”**
- **Explicit pole removal**
- **Spline-based quadratures**
Brute force method

Three-body scattering in the presence of bound states

\[ D_s^{(u,u)}(p, k) = -M_2(E_{2,p}^*)G_s(p, k, \epsilon)M_2(E_{2,k}^*) - \int_0^{k_{\text{max}}} \frac{k'^2}{(2\pi)^2\omega_{k'}} G_s(p, k', \epsilon) D_s^{(u,u)}(k', k), \]

\[
\begin{align*}
\text{Discretization:} & \quad d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot M \cdot d(N, \epsilon) \\
\text{Matrix inversion:} & \quad d(N, \epsilon) = -\left[ \mathbb{1} + P \cdot G(\epsilon) \cdot M \right]^{-1} \cdot G(\epsilon) \\
\text{Limits:} & \quad d^{(u,u)}(p, k) = \lim_{\epsilon \to 0} \lim_{N \to \infty} d(N, \epsilon) \\
\text{Amputation:} & \quad D^{(u,u)}(p, k) \equiv M_2(p)d^{(u,u)}(p, k)M_2(k) \\
\text{Double ordered limit \rightarrow single limit:} & \quad \epsilon \propto \eta/N
\end{align*}
\]
Numerical procedures – continuation

Interpolation

\[ d_{S}^{(u,u)}(p,k) = -G_{S}(p,k) - \int_{0}^{\infty} \frac{dk'k'^{2}}{(2\pi)^{2}\omega_{k'}}G_{S}(p,k')M_{2}(k')d_{S}^{(u,u)}(k',k) \]

1. Plug the solution
2. Plug the BS momentum
3. Integrate (sum over discrete momenta)

Limits

Poisson summation formula \(\rightarrow\) error

\[ \sigma(\epsilon, N) \approx |G_{S} \rho_{2} d^{(u,u)}| e^{-\eta} \]

\[ \eta = 2\pi \epsilon N \times \text{(energy factors)} \]
Splines-based method

\[ d_{S}^{(u,u)}(p, k) = -G_{S}(p, k) - \int_{0}^{\infty} \frac{dk'k'^2}{(2\pi)^2 \omega_{k'}} G_{S}(p, k') M_{2}(k') d_{S}^{(u,u)}(k', k) \]

\[ d_{S}^{(u,u)}(\sigma_k, \sigma_p) = -G_{S}(\sigma_k, \sigma_p) - \sum_{n=0}^{N_s} \omega_{n}(\sigma_k, s) d_{S}^{(u,u)}(\sigma_q, s) \]

\[ d_{S}^{(u,u)}(\sigma_k, \sigma_p) \approx \sum_{n=0}^{N_s} S_{n}(\sigma_k) d_{S}^{(u,u)}(\sigma_q, \sigma_p) \]

Possible improvement

\[ \int_{0}^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{\pi} G_{S}(\sigma_k, \sigma_q) \frac{\lambda^{1/2}(s, \sigma_q, m^2)}{16\pi s} M_{2}(\sigma_q) S_{i}(\sigma_q) = \sum_{n=0}^{N_s} \omega_{n}(\sigma_k) S_{i}(\sigma_{q,n}) \]
Extrapolations of the results

- Expansion in $1/N$
- Epsilon regulator fixed to $\epsilon \propto \eta/N$
Example results, $M^2 = 3m^2$, i.e. $a=2$

\[ \Delta \rho_2(q/m) \cot \delta \]
Example results, $M^2 = 3.89m^2$, i.e. $a=6$

\[ \Delta \rho_2 (\rho/m) \cot \delta \]

\[ \rho_2 A_2 \]

\[ \Delta \rho_2 \]

\[ (E/m)^2 \]
Example result, three-body scattering length

This work, $\eta = 15$
Romero-López et al. ○
NREFT $\Lambda = 0.75m$

Romero-Lopez et al., JHEP 10 (2019) 007
Example result, $2 \rightarrow 3$ amplitude

We are not limited below the three-body threshold,
Below the threshold — circular cut

- Analytic structure of the integration kernel
- OPE develops the circular cut

- Real Particle Exchange cut

\[ \sigma_{\pm} = \frac{1}{2}(s - \sigma_{p'} + 3m^2) \pm \frac{1}{2\sqrt{\sigma_{p'}}}\sqrt{\sigma_{p'} - 4m^2} \lambda^{1/2}(s, \sigma_{p'}, m^2) \]

- Crossing the real axis

\[ \sigma_{c1} = \frac{(m^2 - s)(m^2 - \sigma_{p'} + s)}{(m^2 - \sigma_{p'} - s)} \quad \sigma_{c2} = \frac{(m^2 - s)(2m^2 - \sigma_{p'})}{\sigma_{p'}} \]
Circular cut in motion picture

\[ \epsilon = 0.3 \]

\[ \text{Im} \sigma_p, 0.0 \]

\[ \text{Re} \sigma_p \]

\[ \text{Re OPE} \]

\[ \text{Im OPE} \]

Three-body scattering in the presence of bound states

work in progress
Interpolation and bound–state poles

- Interpolation requires contour deformation
- We need to solve for complex isobar energies lying along the contour
- Three–body poles lie close to the threshold
  - no circular cut
  - no deformation needed
Bound–state–particle scattering

- Model study—formation of the three–body bound states
- Analytic properties of the B–matrix formalism

Dawid, Szczepaniak, Phys. Rev. D 103 (2021) 1, 014009

Generalization of the B-matrix formalism

- Amplitude constrained by unitarity in the physical interval

![Diagram showing the B-matrix formalism]

\[
\int_q \equiv \int \frac{d\Omega_q}{4\pi} \int_0^{q_{\text{max}}} \frac{dq}{2\pi^2 \omega_q} = \int \frac{d\Omega_q}{4\pi} \int_{\sigma_{\text{min}}}^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} \tau(s, \sigma_q)
\]

- One can not continue below two-body threshold

- Generalization needed to include coupled channels

\[
A_{22} = B_{22} + \int_k B_{22} \rho_2 A_{22} + \int_q B_{22,q} A_{22,q},
\]

\[
A_{23,p} = B_{23,p} F_p + \int_k B_{23} \rho_2 A_{23,p} + \int_q B_{23,q} A_{33,q},
\]

\[
A_{32,p'} = F_{p'} B_{32,p'} + \int_k F_{p'} B_{32} \rho_2 A_{32,qp'} + \int_q F_{p'} B_{32,q} A_{32,q},
\]

\[
A_{33,p'} = F_{p'} B_{33,p'} F_p + \int_k F_{p'} B_{33} \rho_2 A_{33,p} + \int_q F_{p'} B_{33,p} A_{33,q}.
\]
Generalization of the B–matrix formalism

- Satisfies unitarity above the three–particle threshold
- New kernels include multi–channel couplings, for example

\[ \tilde{A}_{33} = \mathcal{H}_{33} + \int \mathcal{H}_{33} \mathcal{M}_2 \tilde{A}_{33} \]

- Formal solutions can be found

\[ \tilde{A}_{33} = \frac{1}{1 - \mathcal{H}_{33} \mathcal{M}_2} \mathcal{H}_{33} \]
\[ \tilde{A}_{32} = \frac{1}{1 - \mathcal{H}_{33} \mathcal{M}_2} \mathcal{H}_{32} \]
\[ \tilde{A}_{23} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{23} \]
\[ \tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22} i \rho_2} \mathcal{H}_{22} \]
Generalization of the B–matrix formalism

Approximation: all multi–particle interactions are constant and real (couplings \( g_{ij} \)),

\[
\begin{align*}
  a_{33}(s) &= \frac{g_{33}}{1 - g_{22}i\rho_2 - g_{33}\mathcal{I}(s)}, \\
  a_{22}(s) &= \frac{g_{22}}{1 - g_{22}i\rho_2 - g_{33}\mathcal{I}(s)}. 
\end{align*}
\]

Solutions do not satisfy unitarity below the three–body threshold

Spurious singularities start arbitrarily close to the two–body threshold

Kernel suffering from non–physical left–hand cuts:

\[
\mathcal{I}(s) = \int_{\sigma_{\text{min}}}^{(\sqrt{s}-m)^2} \frac{\sigma_q}{2\pi} \tau(s, \sigma_q) \mathcal{M}_2(\sigma_q)
\]
Analyticity of the B-matrix equations

- Dispersion procedure ensures analyticity

Kernel free from those problems:

\[ \mathcal{I}_d(s) = \frac{(s - s_s)^2}{\pi} \int_\infty ds' \frac{\text{Im} \, \mathcal{I}(s')}{(3m)^2 (s' - s - i\epsilon)(s' - s_s - i\epsilon)^2} \]

- Spurious singularities pushed to non-physical Riemann sheets and three-body physics influences the amplitude,
- General dispersion procedure for the three-body unitarity formalism is needed

\[ b_0 = -\frac{g_{33}}{1 - g_{33} \mathcal{I}_d(s_{th,2})} \]
Including OPE

How does this generalized B-matrix approach compare with EFT?

- **B-matrix formal solution:**
  \[
  \tilde{A}_{22} = \frac{1}{1 - \mathcal{H}_{22}i\rho_2} \mathcal{H}_{22}
  \]

  where the \( \mathcal{H} \) matrix is

  \[
  \mathcal{H}_{22} = B_{22} + B_{23} M_2 \left( \frac{1}{1 - B_{33} M} B_{32} \right)
  \]

  \[
  \mathcal{H}_{22} = g_B^2 \frac{1}{1 + G_S \mathcal{M}_2} \mathcal{M}_2
  \]

- **REFT ladder formal solution:**
  \[
  d = \frac{1}{1 - \mathcal{K}i\rho_2} \mathcal{K}
  \]

  where the \( \mathcal{K} \) matrix is

  \[
  \mathcal{K} = g^2 \frac{1}{1 + G_S \text{P.V.}[\mathcal{M}_2]} (-G_S)
  \]

Those two are matrices defined in different momentum spaces!
N/D approximation

Non-perturbative approximation of the ladder solution

\[ A_2 = \frac{N(s)}{D(s)} \]

- **N** - contains left hand cuts,
- **D** - right hand cuts

\[ N(s) = \frac{1}{\pi} \int_{-\infty}^{s_{L2}} ds' \frac{\text{Im} A_2(s')}{s' - s - i\epsilon} D(s') , \]

\[ D(s) = 1 - \frac{s}{\pi} \int_{(M+m)^2}^{\infty} ds' \frac{\rho_2(s')}{s'(s' - s - i\epsilon)} N(s') \]

\[ D(s) = 1 + \frac{s}{\pi^2} \int_{-\infty}^{s_{L2}} ds' \ g(s, s') \ \text{Im} A_2(s') \ D(s') \]
Conclusions

- Systematic procedure for solving the integral equations
- Agreement with previous studies
- Generalization of the B-matrix formalism
- Study of the analytic properties

Future prospects

- Continuing below the two-body threshold and to complex energies
- Understanding the Efimov aspects of the model
- Controlling the regularization scheme/ cutoff dependence
- Formulation of the B-matrix satisfying analyticity
- Generalization to arbitrary spins
THANK YOU

I AM BOUND TO PLEASE THEE WITH MY ANSWERS
For three-body experts – black to move

chessskill.blogspot.com/2020/06/three-pawns-problem.html