

Observables for scattering on targets with any spin

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Physics Seminar at Jefferson Lab

March 20, 2023

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Current matrix elements for composite particles with arbitrary spin

- Covariant decomposition of matrix element in independent non-perturbative objects

e.g. one-photon-exch.: $\langle p', s' | j^\mu | p, s \rangle = \bar{u}_{(p', s')} \Gamma_{(p', p)}^\mu u_{(p, s)} \xrightarrow{\text{spin } 1/2} \bar{u}_{(p', s')} \left[F_{1(t^2)} \gamma^\mu - F_{2(t^2)} \frac{i}{2m} \sigma^{\mu\nu} q_\nu \right] u_{(p, s)}$

Spin- j fields embedded in objects with $> 2j + 1$ components

- Polarization four vector (ε) for spin 1 $\rightarrow p_\mu \varepsilon^\mu(p, s) = 0$
- Rarita Schwinger for spin 3/2 $\rightarrow \gamma^\mu \psi_\mu(p, s) = 0$
- Need for constraints, subsidiary conditions

Use $(2j + 1)$ -component spinors

- Via $SL(2, C)$ fundamental rep tensor products [Zwanziger 60s, Polyzou '18]
- Weinberg's construction [64-65] (not yet applied in this context)

Advantages of Weinberg's construction

- **Systematic approach**, e.g., for any spin $j \rightarrow \langle p', s' | j^\mu | p, s \rangle = \bar{u}_{(p', s')} \Gamma_{(p', p)}^\mu u_{(p, s)}$
- Use only exact degrees of freedom (**chiral reps**), no need for constraints
- Straightforward “multipole” **physical interpretation**
- For parity conserving interactions a **generalized Dirac algebra** is obtained
- “Basic” construction and implementation. From $\text{su}(2) \rightarrow \text{su}(N)$ algebra
- Easy to switch between forms of dynamics (**instant form, light front**)
- No kinematic singularities (**improved analyticity properties of operators**)

Outline

- Review of Weinberg's formalism
- Building upon Weinberg's formalism: Bilinear Calculus
- On-Shell (Gordon) Identities
- Simplification (I): Algorithm for construction of t-tensors
- Simplification (II): Algebra of t-tensors
- Examples: Spin $1/2$, Spin 1
- Different forms of dynamics: Canonical (Instant Form), Light Front

Weinberg's "Feynman rules for Any Spin" [1964]

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n, \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n, \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent $\mathfrak{su}(2)$ subalgebras \rightarrow irreps (j_A, j_B)

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m), \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n, \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n, \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin- $j \rightarrow (2j + 1 \text{ components})$

- Right-handed $(j, 0): \mathbb{K}_m \rightarrow -i\mathbb{J}_m$

- Left-handed $(0, j): \mathbb{K}_m \rightarrow +i\mathbb{J}_m$

[Wigner(1939)]

Causal chiral fields (massive, left- right-handed)

- Lorentz invariant S-matrix using a Hamiltonian density built up from causal fields

$$U_{[\Lambda,a]} \psi_{\sigma}(x) U_{[\Lambda,a]}^{-1} = \sum_{\sigma'} \left(D_{[\Lambda^{-1}]}^{(j)} \right)_{\sigma\sigma'} \psi_{\sigma'}(\Lambda x + a)$$

- No EoM for chiral fields (only obey KG eq.)
- Spinors appearing in the fields (**not invariants**, depend on choice boost)

$$D_{[L(p)]}^{(j)} = e^{-\hat{p} \cdot \vec{J}^{(j)} \theta}$$

Canonical

→

$$\bar{D}_{[L(p)]}^{(j)} = e^{+\hat{p} \cdot \vec{J}^{(j)} \theta}$$

Propagator of chiral fields

- Numerator (invariant)
$$\Pi_{\sigma\sigma'}^{(j)}(\vec{p}, \omega) = m^{2j} D_{\sigma\sigma'}^{(j)}[L(\vec{p})] \left(D_{\sigma'\sigma''}^{(j)}[L(\vec{p})] \right)^\dagger = m^{2j} \left(e^{-2\hat{p} \cdot \vec{J}^{(j)} \theta} \right)_{\sigma\sigma'}$$
$$\bar{\Pi}_{\sigma\sigma'}^{(j)}(\vec{p}, \omega) = m^{2j} \bar{D}_{\sigma\sigma'}^{(j)}[L(\vec{p})] \left(\bar{D}_{\sigma'\sigma''}^{(j)}[L(\vec{p})] \right)^\dagger = m^{2j} \left(e^{2\hat{p} \cdot \vec{J}^{(j)} \theta} \right)_{\sigma\sigma'}$$

- Introduction of the **t-tensors**
$$\Pi_{\sigma\sigma'}^{(j)}(\vec{p}, \omega) = t_{\sigma\sigma'}^{\mu_1 \mu_2 \dots \mu_{2j}} p_{\mu_1} p_{\mu_2} \dots p_{\mu_{2j}}$$
$$\bar{\Pi}_{\sigma\sigma'}^{(j)}(\vec{p}, \omega) = \bar{t}_{\sigma\sigma'}^{\mu_1 \mu_2 \dots \mu_{2j}} p_{\mu_1} p_{\mu_2} \dots p_{\mu_{2j}}$$

- These can also be used to write the **boosts/spinors**
$$D_{[L(p)]}^{(j)} = t^{\mu_1 \mu_2 \dots \mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \dots \tilde{p}_{\mu_{2j}}$$
$$\bar{D}_{[L(p)]}^{(j)} = \bar{t}^{\mu_1 \mu_2 \dots \mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \dots \tilde{p}_{\mu_{2j}}$$

Instant form dynamics
(Canonical)

$$\tilde{p}_C^\mu = \sqrt{\frac{1}{2m(m+p^0)}} (p^0 + m, \vec{p})$$

\tilde{p}^μ not four-vectors

Properties of the t -tensors

- Each $t^{\mu_1 \dots \mu_{2j}}$ is a $2j$ -rank tensor

- **Symmetric** and (covariantly) **traceless**

$$g_{\mu_k \mu_l} t^{\mu_1 \dots \mu_k \dots \mu_l \dots \mu_{2j}} = 0$$

- Transform **covariantly**

$$\left(D_{[\Lambda]}^{(j)} \right)_{\sigma\delta} t^{\mu_1 \dots \mu_{2j}}_{\delta\delta'} \left(D_{[\Lambda]}^{(j)\dagger} \right)_{\delta'\sigma'} = \Lambda_{\nu_1}^{\mu_1} \dots \Lambda_{\nu_{2j}}^{\mu_{2j}} t^{\nu_1 \dots \nu_{2j}}_{\sigma\sigma'}$$

- Right chiral (t) and left chiral (\bar{t})
are related by charge conjugation

$$\bar{t}^{\mu_1 \mu_2 \dots \mu_{2j}}_{\sigma\sigma'} = (\pm) t^{\mu'_1 \mu'_2 \dots \mu'_{2j}}_{\sigma\sigma'}$$

(+ for even (− for odd) spacelike indices)

Introduction: Review of Weinberg's formalism

Bi-Spinors (direct sum representation $(j, 0) \oplus (0, j)$)

- For Parity conserving interactions the direct sum of both chiral representations is used (like the spin 1/2 case)
- Boosts and bispinor (Weyl rep.)

$$u_{(p,s)}^{(j)} = \mathcal{D}_{[L_p]}^{(j)} \overset{\circ}{u}_s^{(j)} = \begin{pmatrix} D_{[L_p]}^{(j)} & 0 \\ 0 & \bar{D}_{[L_p]}^{(j)} \end{pmatrix} \overset{\circ}{u}_s^{(j)} = \begin{pmatrix} \Pi_{(\vec{p})}^{(j)} & 0 \\ 0 & \bar{\Pi}_{(\vec{p})}^{(j)} \end{pmatrix} \overset{\circ}{u}_s^{(j)}$$
$$\overset{\circ}{u}_s^{(j)} = \begin{pmatrix} \overset{\circ}{\phi}_s^{(j)} \\ \overset{\circ}{\phi}_s^{(j)} \end{pmatrix}, \quad \overset{\circ}{\phi}_s^{(j)} = m^j \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \quad (1 \text{ in the } s\text{-th position})$$

- Adjoint bispinor ($\mathcal{D}_{[\Lambda]}^\dagger = \beta \mathcal{D}_{[\Lambda]}^{-1} \beta$)

$$\bar{u}_{(p,s)}^{(j)} = u_{(p,s)}^{(j) \dagger} \beta = \overset{\circ}{u}_s^{(j) \dagger} \mathcal{D}_{[L_p]}^{(j) \dagger} \beta = \overset{\circ}{u}_s^{(j) \dagger} \begin{pmatrix} 0 & \Pi_{(\vec{p})}^{(j)} \\ \bar{\Pi}_{(\vec{p})}^{(j)} & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 0 & \mathbf{1}^{(j)} \\ \mathbf{1}^{(j)} & 0 \end{pmatrix}$$

Dirac Eq. & gamma matrices

- The bispinor satisfy the Dirac eq.

$$(\gamma^{\mu_1 \cdots \mu_{2j}} p_{\mu_1} \cdots p_{\mu_{2j}} - m^{2j}) u_{(p,s)}^{(j)} = 0$$

$$(\gamma^{\mu_1 \cdots \mu_{2j}} p_{\mu_1} \cdots p_{\mu_{2j}} - m^{2j}) u_{(p,s)}^{(j)} = 0$$

- The gamma matrices appear from

$$\begin{pmatrix} D_{[Lp]}^{(j)} & 0 \\ 0 & \bar{D}_{[Lp]}^{(j)} \end{pmatrix} \begin{pmatrix} 0 & D_{[Lp]}^{(j)} \\ \bar{D}_{[Lp]}^{(j)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\Pi_{(\vec{p})}^{(j)})^2 \\ (\bar{\Pi}_{(\vec{p})}^{(j)})^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Pi_{(\vec{p})}^{(j)} \\ \bar{\Pi}_{(\vec{p})}^{(j)} & 0 \end{pmatrix} = \gamma^{\mu_1 \cdots \mu_{2j}} p_{\mu_1} \cdots p_{\mu_{2j}}$$

$$\gamma^{\mu_1 \cdots \mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1 \cdots \mu_{2j}} \\ \bar{t}^{\mu_1 \cdots \mu_{2j}} & 0 \end{pmatrix} ; \quad \beta = \gamma^{0 \cdots 0} ; \quad \gamma_5 = \begin{pmatrix} -\mathbf{1}^{(j)} & 0 \\ 0 & \mathbf{1}^{(j)} \end{pmatrix}$$

Generalized Bilinears

- The bispinor satisfy the Dirac eq.

$$\bar{u}_{(p_f, s_f)}^{(j)} \Gamma u_{(p_i, s_i)}^{(j)} = \frac{1}{(m_f m_i)^{2j}} \overset{\circ}{u}_{s_f}^{(j)\dagger} \left(\begin{array}{cc} 0 & t^{\beta_1 \dots} \tilde{p}_{\beta_1 \dots}^f \\ \bar{t}^{\beta_1 \dots} (\tilde{p}_{\beta_1 \dots}^f)^* & 0 \end{array} \right) \Gamma \left(\begin{array}{cc} t^{\alpha_1 \dots} \tilde{p}_{\alpha_1 \dots}^i & 0 \\ 0 & \bar{t}^{\alpha_1 \dots} (\tilde{p}_{\alpha_1 \dots}^i)^* \end{array} \right) \overset{\circ}{u}_{s_i}^{(j)}$$

- The Dirac basis

$$\Gamma = \mathbf{1}, \quad \gamma^{\mu_1 \dots \mu_{2j}}, \quad [\gamma^{\mu_1 \dots \mu_{2j}}, \gamma^{\nu_1 \dots \nu_{2j}}], \quad \{\gamma^{\mu_1 \dots \mu_{2j}}, \gamma^{\nu_1 \dots \nu_{2j}}\}, \quad \gamma_5 \gamma^{\mu_1 \dots \mu_{2j}}, \quad \gamma_5$$

Examples

- Basis for spin-1/2

$$\mathbf{1}(1), \quad \gamma^\mu(4), \quad [\gamma^\mu, \gamma^\nu](6), \quad (\{\gamma^\mu, \gamma^\nu\} - 2g^{\mu\nu})(0), \quad \gamma^\mu \gamma_5(4), \quad \gamma_5(1)$$

- Basis for spin-1

$$\mathbf{1}(1), \quad \gamma^{\mu\nu}(9), \quad [\gamma^{\mu_1 \mu_2}, \gamma^{\mu_3 \mu_4}](6), \quad (\{\gamma^{\mu_1 \mu_2}, \gamma^{\mu_3 \mu_4}\} - \text{traces})(10), \quad \gamma^{\mu\nu} \gamma_5(9), \quad \gamma_5(1)$$

- **Amplitudes** and **Covariant density matrices** can be constructed

Generalized Gordon Identities

- Dirac Equation $(\gamma^{\mu_1 \dots \mu_{2j}} p_{\mu_1} \dots p_{\mu_{2j}} - m^{2j}) u_p^s = 0$ leads to On-Shell (**Gordon**) identities

$$u_{p'}^{s'}(\Gamma) u_p^s = \frac{1}{2\bar{m}^{2j}} u_{p'}^{s'} \left(\left\{ \not{P}^{(j)}, \Gamma \right\} + \frac{1}{2} \left[\not{A}^{(j)}, \Gamma \right] \right) u_p^s$$

$$0 = u_{p'}^{s'} \left(\frac{1}{2} \left\{ \not{A}^{(j)}, \Gamma \right\} + \left[\not{P}^{(j)}, \Gamma \right] \right) u_p^s$$

$$\bar{m}^{2j} = \frac{1}{2} (m'^{2j} + m^{2j})$$

$$P_{\mu_1 \dots \mu_{2j}} = \frac{\bar{m}^{2j}}{2} \left(\frac{p'_{\mu_1} \dots p'_{\mu_{2j}}}{m'^{2j}} + \frac{p_{\mu_1} \dots p_{\mu_{2j}}}{m^{2j}} \right)$$

$$\Delta_{\mu_1 \dots \mu_{2j}} = \bar{m}^{2j} \left(\frac{p'_{\mu_1} \dots p'_{\mu_{2j}}}{m'^{2j}} - \frac{p_{\mu_1} \dots p_{\mu_{2j}}}{m^{2j}} \right)$$

$$P_{(p', p)}^{\mu_1 \dots \mu_{2j}} = P_{(p, p')}^{\mu_1 \dots \mu_{2j}}$$

$$\rightarrow \Delta^{\mu_1 \dots \mu_{2j}}(p', p) = -\Delta^{\mu_1 \dots \mu_{2j}}(p, p')$$

$$P^{\mu_1 \dots \mu_{2j}} \Delta_{\mu_1 \dots \mu_{2j}} = 0$$

- **Gordon identity** separates general bilinears into convection and magnetization currents. **Useful to reduce independent Dirac structures**

Simplifications: Algorithm for construction of t -tensors

Insightful construction for the t -tensors

- The 0-th degree polynomial in the J 's is always $t^{0\dots 0} = \mathbf{1}$

- The linear polynomials
are the Rotation Group Generators $t^{0\dots i\dots 0} = \frac{2}{2j} J_i = \frac{1}{j} J_i$

- From pairwise symmetrizations of the rotation generators

$$\begin{aligned} t^{0\dots m\dots 0\dots n\dots 0} = t^{mn0\dots 0} &= \frac{1}{\frac{(2j)!}{2!(2j-2)!}} \left(\{J_m, J_n\} - \frac{1}{3} \delta_{mn} \sum_{r=1}^3 \{J_r, J_r\} \right) + \frac{1}{3} t^{0\dots 0} \delta_{mn} \\ &= \frac{j}{(2j-1)} \left(\{t^{m0\dots 0}, t^{n0\dots 0}\} - \frac{1}{j} \delta_{mn} t^{0\dots 0} \right) \end{aligned}$$

Simplifications: Algorithm for construction of t-tensors

- Continues for higher orders
 - Matrices have more and more off-diagonal elements

$$t^{lmn0\dots0} = t^{0\dots0l0\dots0m0\dots0n0\dots0} = \frac{j}{(2j-2)} \frac{1}{3} \left(\{t^{l0\dots0}, t^{mn0\dots0}\} + \{t^{m0\dots0}, t^{nl0\dots0}\} + \{t^{n0\dots0}, t^{lm0\dots0}\} \right. \\ \left. - \frac{2}{j} \{ \delta_{lm} t^{n0\dots0} + \delta_{ln} t^{m0\dots0} + \delta_{mn} t^{l0\dots0} \} \right)$$

- Construction stops after j steps (Cayley-Hamilton) $(J-s)(J-s-1)\dots(J+s) = 0$
- The t-tensors contain a **basis for the su(N) algebra** $(N=2j+1)$
- A basis to decompose operators with physical interpretation for each term
(**multipole expansion** \rightarrow mono-, di-, quadrupole, ...)

$$\hat{O} = Tr [O] \mathbf{1} + Tr [OJ_i] J_i + Tr [OJ_{ij}] J_{ij} + \dots = \langle O \rangle \mathbf{1} + O_i J_i + O_{ij} J_{ij} + \dots$$

Reduction for Monomials of $t^{\mu_1 \dots \mu_{2j}}$ with Odd Powers

- Monomials always appear in bilinear calculus with an alternating “barring” pattern

$$t^{\mu_1 \dots \mu_{2j}} \bar{t}^{\nu_1 \dots \nu_{2j}} t^{\rho_1 \dots \rho_{2j}} = \frac{1}{(2j)^2} \mathcal{S}_{\text{cyclic}} \left\{ \nu_1 \dots \nu_{2j} \right\} \mathcal{S}_{\text{cyclic}} \left\{ \rho_1 \dots \rho_{2j} \right\} \left(\bar{\mathcal{C}}^{\mu_1 \nu_1 \rho_1 \beta_1} \bar{\mathcal{C}}^{\mu_2 \nu_2 \rho_2 \beta_2} \dots \bar{\mathcal{C}}^{\mu_{2j} \nu_{2j} \rho_{2j} \beta_{2j}} \right) t_{\beta_1 \dots \beta_{2j}}$$

- The coefficients

$$\bar{\mathcal{C}}^{\mu\rho\alpha\beta} = g^{\mu\rho} g^{\alpha\beta} - g^{\mu\alpha} g^{\rho\beta} + g^{\mu\beta} g^{\rho\alpha} + i\epsilon^{\mu\rho\alpha\beta}$$

$$\mathcal{C}^{\mu\rho\alpha\beta} = g^{\mu\rho} g^{\alpha\beta} - g^{\mu\alpha} g^{\rho\beta} + g^{\mu\beta} g^{\rho\alpha} - i\epsilon^{\mu\rho\alpha\beta}$$

$$\mathcal{C}^{\mu\rho\alpha\beta} = (\bar{\mathcal{C}}^{\mu\rho\alpha\beta})^*$$

Similar to: $\text{Tr} \{ \gamma^\mu \gamma^\rho \gamma^\alpha \gamma^\beta \} = 4 (g^{\mu\rho} g^{\alpha\beta} - g^{\mu\alpha} g^{\rho\beta} + g^{\mu\beta} g^{\rho\alpha})$

Reduction for Monomials of $t^{\mu_1 \dots \mu_{2j}}$ with Even Powers

- Monomials always appear in bilinear calculus with an alternating “barring” pattern

$$t^{\mu_1 \dots \mu_{2j}} \bar{t}^{\nu_1 \dots \nu_{2j}} (t^{\rho_1 \dots \rho_{2j}} \eta_{\rho_1} \dots \eta_{\rho_{2j}}) = \frac{1}{(2j)} \mathcal{S}_{\text{cyclic}} \left\{ \nu_1 \dots \nu_{2j} \right\} \left(\bar{\mathcal{C}}^{\mu_1 \nu_1 \rho_1 \beta_1} \eta_{\rho_1} \bar{\mathcal{C}}^{\mu_2 \nu_2 \rho_2 \beta_2} \eta_{\rho_2} \dots \bar{\mathcal{C}}^{\mu_{2j} \nu_{2j} \rho_{2j} \beta_{2j}} \eta_{\rho_{2j}} \right) t_{\beta_1 \dots \beta_{2j}}$$

- The condition $t^{\rho_1 \dots \rho_{2j}} \eta_{\rho_1} \dots \eta_{\rho_{2j}} = \mathbf{1}_{(2j+1) \times (2j+1)}$ defines η_ρ in Lorentz coordinates $t^{0 \dots 0} = \mathbf{1}$, thus $\eta^\mu \rightarrow \eta_C^\mu = (1, 0, 0, 0)$

- General result

$$t^{\mu_1 \dots \mu_{2j}} \bar{t}^{\nu_1 \dots \nu_{2j}} = \frac{1}{(2j)} \mathcal{S}_{\text{cyclic}} \left\{ \nu_1 \dots \nu_{2j} \right\} \left(\bar{\mathcal{C}}_{(\eta)}^{\mu_1 \nu_1 \beta_1} \bar{\mathcal{C}}_{(\eta)}^{\mu_2 \nu_2 \beta_2} \dots \bar{\mathcal{C}}_{(\eta)}^{\mu_{2j} \nu_{2j} \beta_{2j}} \right) t_{\beta_1 \dots \beta_{2j}}$$

$$\bar{\mathcal{C}}_{(\eta)}^{\mu\rho\alpha} = g^{\mu\rho} \eta^\alpha - g^{\mu\alpha} \eta^\rho + g^{\rho\alpha} \eta^\mu + i\epsilon^{\mu\rho\alpha\beta} \eta_\beta$$

$$\mathcal{C}_{(\eta)}^{\mu\rho\alpha} = g^{\mu\rho} \eta^\alpha - g^{\mu\alpha} \eta^\rho + g^{\rho\alpha} \eta^\mu - i\epsilon^{\mu\rho\alpha\beta} \eta_\beta$$

Spin 1/2 Example: t^μ tensors

0-th order terms in $J_i^{(1/2)}$: $t^0 = \mathbf{1}$

Linear terms in $J_i^{(1/2)}$: $t^i = \frac{1}{1/2} J_i^{(1/2)} = \sigma_i$

(Pauli matrices)

$$J_1^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2^{(1/2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J_3^{(1/2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1/2)}$

$$(J^{(1/2)} - \frac{1}{2}\mathbf{1})(J^{(1/2)} + \frac{1}{2}\mathbf{1}) = 0 \implies (J^{(1/2)})^2 = c_0\mathbf{1} + c_2 J^{(1/2)}$$

Right Chiral Rep

- $t^0 = \mathbf{1}$, $t^i = \sigma_i$
- t^μ Transform Covariantly: $D_{[\Lambda]}^{(1/2)} t^\mu D_{[\Lambda]}^{(1/2)\dagger} = \Lambda_\rho{}^\mu t^\rho$
- Propagator (Lorentz invariant): $\Pi^{(1/2)}(p) = t^\mu p_\mu = \begin{pmatrix} E - p_z & -(p_x - ip_y) \\ -(p_x + ip_y) & E + p_z \end{pmatrix}$
 $p_\mu = (E_p, \vec{p})$
- Boost/spinors (Canonical): $D_{\text{IF}}^{(1/2)} = t^\mu \tilde{p}_\mu^{\text{C}} = \frac{1}{\sqrt{2m(m+p_0)}} \begin{pmatrix} m + p^- & -p_\ell \\ -p_r & m + p^+ \end{pmatrix}$
 $\tilde{p}_\text{C}^\mu = \sqrt{\frac{m}{2(m+p^0)}}(p^0 + m, \vec{p})$

Similarly for the **Left Chiral Rep**, **only change is**: $J_i^{(1/2)} \rightarrow \bar{J}^\mu = (1, -\vec{J}^{(1/2)})$

Spin 1/2 Bilinears

Final evaluations recover the results of [Lorcé(2017)]

- Scalar $\bar{u}_{(p_f, s_f)}^{(1/2)} u_{(p_i, s_i)}^{(1/2)} = \tilde{N} \phi_{s_f}^\dagger [4P^2 \mathbf{1}_2 + 4mP_\lambda (\sigma^\lambda + \bar{\sigma}^\lambda) - \frac{i}{2} (\sigma_\lambda + \bar{\sigma}_\lambda) \varepsilon^{\lambda\beta\alpha\rho} \Delta_\beta P_\alpha (\sigma_\rho - \bar{\sigma}_\rho)] \phi_{s_i}$
 $= \tilde{N} \phi_{s_f}^\dagger [4(P^2 + mP^0) + 2i\varepsilon^{0\beta\alpha\rho} \Delta_\beta P_\alpha \sigma_\rho] \phi_{s_i}$

- Pseudoscalar

$$\bar{u}_{(p_f, s_f)}^{(1/2)} \gamma_5 u_{(p_i, s_i)}^{(1/2)} = \tilde{N} \phi_{s_f}^\dagger \left[m\Delta_\lambda (\sigma^\lambda - \bar{\sigma}^\lambda) + (P_\mu (\sigma^\mu + \bar{\sigma}^\mu)) (\Delta_\nu (\sigma^\nu - \bar{\sigma}^\nu)) - (\Delta_\mu (\sigma^\mu + \bar{\sigma}^\mu)) (P_\nu (\sigma^\nu - \bar{\sigma}^\nu)) \right] \phi_{s_i}$$
$$= \tilde{N} \phi_{s_f}^\dagger \left[2\Delta^0 \vec{P} \cdot \vec{\sigma} - 2(P^0 + m) \vec{\Delta} \cdot \vec{\sigma} \right] \phi_{s_i}$$

$$\tilde{N} = \tilde{N}_f \tilde{N}_i = \frac{1}{2m} \left[(p^0 + m)^2 - \left(\frac{1}{2} \Delta \right)^2 \right]^{-1}$$

Bilinears

- Vector

$$\begin{aligned}
 \bar{u}_{(p_f, s_f)}^{(j)} \gamma^\mu u_{(p_i, s_i)}^{(j)} &= \tilde{N} \phi_{s_f} + \left[\frac{1}{2} \Delta^2 (\sigma^\mu + \bar{\sigma}^\mu) - \frac{1}{2} \Delta_\lambda (\sigma^\lambda + \bar{\sigma}^\lambda) \Delta^\mu \right. \\
 &\quad \left. + 4m P^\mu \mathbf{1}_2 + 2P_\lambda (\sigma^\lambda + \bar{\sigma}^\lambda) P^\mu \right. \\
 &\quad \left. + i\varepsilon^{\mu\beta\alpha\rho} \Delta_\beta (m (\sigma_\alpha + \bar{\sigma}_\alpha) + P_\alpha) (\sigma_\rho - \bar{\sigma}_\rho) \right] \phi_{s_i} \\
 &= \tilde{N} \phi_{s_f}^\dagger \left[(4(P^0 + m) P^\mu + \Delta^2 g^{0\mu} - \Delta^0 \Delta^\mu) \mathbf{1}_2 \right. \\
 &\quad \left. + 2i\varepsilon^{0\mu\beta\rho} \Delta_\beta \sigma_\rho + i\varepsilon^{\mu\beta\alpha\rho} \Delta_\beta P_\alpha (\sigma_\rho + \bar{\sigma}_\rho) \right] \phi_{s_i}
 \end{aligned}$$

- Pseudovector

$$\begin{aligned}
 \bar{u}_{(p_f, s_f)}^{(1/2)} \gamma^\mu \gamma_5 u_{(p_i, s_i)}^{(1/2)} &= \tilde{N} \phi_{s_f}^\dagger \left[-(4P^\mu P_\alpha - \Delta^\mu \Delta_\alpha) (\sigma^\alpha - \bar{\sigma}^\alpha) \right. \\
 &\quad \left. + \left(P^2 - \frac{1}{4} \Delta^2 \right) (\sigma^\mu - \bar{\sigma}^\mu) - i\varepsilon^{\mu\alpha\beta\rho} \Delta_\alpha P_\beta (\sigma_\rho + \bar{\sigma}_\rho) \right] \phi_{s_i}
 \end{aligned}$$

Spin 1 Example: $t^{\mu\nu}$ tensors

0-th order terms in $J_i^{(1)}$: $t^{00} = \mathbf{1}$

Linear terms in $J_i^{(1)}$: $t^{0i} = J_i^{(1)}$

$$t^{01} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^{02} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1)}$: $t^{ij} = \{J_i^{(1)}, J_j^{(1)}\} - \mathbf{1}\delta_{ij}$

$$t^{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^{22} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$t^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad t^{23} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Cubic terms in $J_i^{(1)}$: $(J^{(1)} - 1)(J^{(1)})(J^{(1)} + 1) = 0 \implies (J^{(1)})^3 = c_0\mathbf{1} + c_2J^{(1)} + c_3(J^{(1)})^2$

Spin 1 Example: Spinors

Right Chiral Rep

- $t^{00} = \mathbf{1}$, $t^{0i} = t^{i0} = J_i^{(1)}$, $t^{ij} = \{J_1^{(1)}, J_1^{(1)}\} - \mathbf{1}\delta_{ij}$

- $t^{\mu\nu}$ Transform covariantly $D_{[\Lambda]}^{(1)} t^{\mu\nu} D_{[\Lambda]}^{(1)\dagger} = \Lambda_{\rho}^{\mu} \Lambda_{\sigma}^{\nu} t^{\rho\sigma}$

- Propagator ($p_{\mu} = (E_p, \vec{p})$): $\Pi^{(1)}(p) = t^{\mu\nu} p_{\mu} p_{\nu} = \begin{pmatrix} (p^{-})^2 & -\sqrt{2}p_{\ell}p^{-} & p_{\ell}^2 \\ \sqrt{2}p_r p^{+} & p^{+}p^{-} + p_{\text{T}}^2 & \sqrt{2}p_{\ell}p^{-} \\ p_r^2 & \sqrt{2}p_r p^{-} & (p^{+})^2 \end{pmatrix}$

- Boost/spinors ($t^{\mu\nu} \tilde{p}_{\mu} \tilde{p}_{\nu}$)

Canonical: $D_{\text{IF}}^{(1)} = \frac{1}{2m(m+p_0)} \begin{pmatrix} (m+p^{-})^2 & -\sqrt{2}p_{\ell}(m+p^{-}) & p_{\ell}^2 \\ -\sqrt{2}p_r(m+p^{-}) & 2(m^2 + mp_0 + p_{\text{T}}^2) & -\sqrt{2}p_{\ell}(m+p^{+}) \\ p_r^2 & -\sqrt{2}p_r(m+p^{+}) & (m+p^{+})^2 \end{pmatrix}$

$$\tilde{p}_{\text{C}}^{\mu} = \sqrt{\frac{m}{2(m+p^0)}}(p^0 + m, \vec{p})$$

Similarly for the Left Chiral Rep, **only change is:** $J_i^{(1)} \rightarrow \bar{J}^{\mu} = (1, -\vec{J}^{(1)})$

Spin 1 Example: EM Current

Using spinor representation: $\langle p', s' | j^\mu(0) | p, s \rangle = \overset{\circ}{\phi}_{s'}^{(1)} \Gamma_{(p', p)}^\mu \overset{\circ}{\phi}_s^{(1)}$

$$m^2 \Gamma_{(p', p)}^\mu = 2P^\mu \left[P^2 \mathbf{1} G_C(Q^2) - \Delta^\rho \Delta^\sigma \left(t_{\rho\sigma} - \frac{1}{3} g_{\rho\sigma} \mathbf{1} \right) G_Q(Q^2) \right]$$

$$P = \frac{1}{2}(p' + p)$$

$$\Delta = p' - p \quad (\Delta^2 = -Q^2)$$

$$n_t^\nu = (1, 0, 0, 0)$$

$$-i\epsilon^{\mu\rho\sigma\lambda} \left[\Delta_\rho P_\sigma \left(t_{\lambda\nu} - \frac{1}{3} g_{\lambda\nu} \mathbf{1} \right) n_t^\nu G_M(Q^2) \right]$$

Using polarization vectors: $\langle p', s' | j^\mu(0) | p, s \rangle = \varepsilon_{s'}^{*\alpha}(p') \Gamma_{\alpha\beta}^\mu(P, \Delta) \varepsilon_s^\beta(p)$

[Wang & Lorcé (2022)]

$$\Gamma^{\mu\alpha\beta} = 2P^\mu \left(\Pi^{\alpha\beta} G_C(Q^2) - \frac{\Delta^\rho \Delta^\sigma (\Sigma_{\rho\sigma})^{\alpha\beta}}{2m^2} \frac{P^2}{m^2} G_Q(Q^2) \right)$$

$$-i\epsilon^{\mu\rho\sigma\lambda} \left(\frac{\Delta_\rho P_\sigma (\Sigma_\lambda)^{\alpha\beta}}{\sqrt{P^2}} G_M(Q^2) \right)$$

Similar structure

Canonical Space-Time Parameterization

Parameterizations (Foliations) of space-time \rightarrow Specify equal time surfaces

Canonical or Instant time: $x^0 = t$

[Wigner(1939)]

- Defined by rotationless boosts from rest: $\overset{\circ}{p}^\mu = (m, 0, 0, 0)$

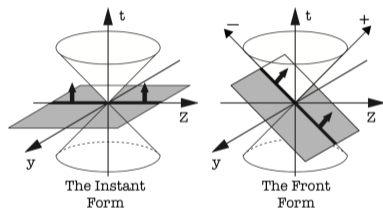
to final momentum: $p^\mu = (E_p, \vec{p}) = (\sqrt{m^2 + \vec{p}^2}, \vec{p})$

$$\Lambda^{\text{IF}} = \exp(i\vec{\mathbb{K}} \cdot \vec{\phi}) = \exp(i\phi \hat{\mathbb{K}} \cdot \hat{\phi})$$

- Then, $p^\mu = (E, \vec{p}) = (\Lambda^{\text{IF}})^\mu{}_\nu \overset{\circ}{p}^\nu$

implies, $\cosh(\phi) = \frac{E}{m}$, $\hat{\phi}_j \sinh(\phi) = \frac{p_j}{m}$

Leading to the well known result: $(\Lambda^{\text{IF}})^\mu{}_\nu = \begin{pmatrix} \frac{E}{m} & & & \\ \frac{\vec{p}}{m} & & & \\ \frac{\vec{p}}{m} & \delta_{ij} + \frac{p_i p_j}{(E+m)m} & & \\ & & & \end{pmatrix}$



Light-Front Space-Time Parameterization

Light Front time: $x^+ = t + z$

$$p^+ = E_p + p_z, \quad p^- = E_p - p_z$$

- Defined by a longitudinal boost followed by a transverse boost

$$\Lambda_{\text{def.}}^{\text{LF}} = \exp [i\vec{\mathbb{G}} \cdot \vec{v}_T] \cdot \exp [i\mathbb{K}_3\eta]$$

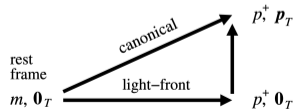
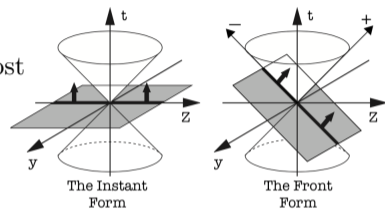
- LF Boost Generators (light front along z -axis),

$$\mathbb{G}_1 = \mathbb{G}_x = \mathbb{K}_x - \mathbb{J}_y, \quad \mathbb{G}_2 = \mathbb{G}_y = \mathbb{K}_y + \mathbb{J}_x, \quad \mathbb{K}_3 = \mathbb{K}_z$$

- Comparing the action of both boosts on the same rest momentum one finds the LF boost parameters

$$e^\eta = \frac{p^+}{m}, \quad \vec{v}_T = \frac{\vec{p}_T}{p^+} \rightarrow \Lambda^{\text{LF}} = \exp \left[i \frac{\eta}{p^+ - m} \vec{p}_T \cdot \vec{\mathbb{G}} + i\eta\mathbb{K}_3 \right]$$

Dirac(1949)



The **boosts/spinors** for the most used forms of dynamics

- In general

$$D_{[L(p)]}^{(j)} = t^{\mu_1 \mu_2 \dots \mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \dots \tilde{p}_{\mu_{2j}}$$

$$\bar{D}_{[L(p)]}^{(j)} = \bar{t}^{\mu_1 \mu_2 \dots \mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \dots \tilde{p}_{\mu_{2j}}$$

Instant form dynamics
(Canonical)

$$\tilde{p}_C^\mu = \sqrt{\frac{1}{2m(m+p^0)}} (p^0 + m, \vec{p})$$

Light-Front dynamics
(Light Cone time)

$$\tilde{p}_{LF}^\mu = \sqrt{\frac{1}{4mp^+}} (p^+ + m, p_\ell, ip_\ell, p^+ - m)$$

\tilde{p}^μ not four-vectors, but same for any spin. Left/right related by complex conjugation.
(Helicity spinor also recovered with specific parameters)

Summary

- Weinberg's construction allows for an efficient and manifestly covariant calculation of currents for any spin
- Central (and multifaceted) role for the covariant t -tensors
- Simple algorithm. Only need to know the matrices for the Generators of rotations in the representation of interest.
- t -tensors has multipole structure "baked in". Physical interpretation (esp. for higher spin)
- Universality of the method for any spin.
Intuition on spin 1/2 can be carried over to higher spin
- No need to work with explicit representations of spinors (Dirac matrices)
Everything reduces to Lorentz covariant t -matrix algebra ($\mathcal{C}^{\mu\nu\rho\lambda}$, just numbers)

Many applications and extensions possible

- Local operators parameterizations: Generalized Form Factors
(two independent four-vectors)
- Bilocal operators parameterizations
(more than two independent four-vectors)
- Transition matrix elements
- Use for EFT's

Thanks!

Questions?