Enabling lattice calculation of TMDs via factorization

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Iain Stewart (MIT)
Markus Ebert (Max Planck)

Support:

Based on:
2004.14831
2201.08401
2205.12369
Soon, we’ll have even higher precision experimental data about the proton’s full 3D internal structure…

It’s crucial to develop a corresponding first-principles understanding!

Figure: CERN
Roadmap for today’s lecture

1. Background
2. Factorization
3. Implications

Figure: I. Stewart
Parton Distribution Functions: 1D momentum distribution of quarks and gluons inside the proton

- Universal factors in processes
- SLAC-MIT experiment (1969): deep inelastic scattering

Figure: PDG
Transverse momentum dependent PDFs: full 3D picture

- Key factor in SIDIS, Drell-Yan, W/Z production, Higgs, ...
- Challenge: important non-perturbative contributions even at perturbative scales
TMDs in cross-sections

Drell-Yan process:

\[ TMDs \text{ in cross-sections} \]

Virtual corrections

Collins-Soper (CS) scale:
\[ \zeta = 2(xP^+e^{-\gamma_E})^2 \]

Tree-level

\[
\frac{d\sigma_{\text{DY}}}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{i, j} H_{ij}(Q, \mu) \\
\times \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T \cdot \vec{b}_T} f_{i/h_1}(x_1, \vec{b}_T, \mu, \zeta_1) f_{j/h_2}(x_2, \vec{b}_T, \mu, \zeta_2)
\]

Collins, Foundations of Perturbative QCD.
Possible to relate TMDs at different scales $(\mu, \zeta)$ & $(\mu_0, \zeta_0)$:

$$f_q(x, \vec{b}_T, \mu, \zeta) = \exp \left[ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma^q_{\mu}(\mu', \zeta_0) \right] \exp \left[ \frac{1}{2} \gamma^q_\xi(\mu, b_T) \ln \frac{\zeta}{\zeta_0} \right] f_q(x, \vec{b}_T, \mu_0, \zeta_0)$$

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\mu, \sqrt{\zeta} \sim Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lattice</td>
<td>$\mu, \sqrt{\zeta} \sim 1$ GeV</td>
</tr>
</tbody>
</table>

Data used in Scimemi & Vladimirov global fit

Projected EIC data

Split TMD into two pieces:

\[ f_i(x, b_T, \mu, \zeta) = f_i^P[x, b^*(b_T), \mu, \zeta] \ f^{NP}(x, b_T, \zeta) \]

**Perturbative piece:**
- Expand in \( \alpha_s(b_T^{-1}) \) about collinear PDF
- Known to three loops! Ebert, Mistlberger, Vita (JHEP 2020).
  Luo, Yang, Zhu, Zhu (JHEP 2021).

**Non-perturbative piece:**
- Construct model, fit to data e.g., JAM collaboration
Example: 11-parameter model

**TMD-PDF**

\[ f_{NP}(x, b) = \exp \left( -\frac{\lambda_1 (1 - x) + \lambda_2 x + x(1 - x) \lambda_5}{\sqrt{1 + \lambda_3 x^4 b^2}} b^2 \right) \]

**TMD-FF**

\[ D_{NP}(x, b) = \exp \left( -\frac{\eta_1 z + \eta_2 (1 - z) b^2}{\sqrt{1 + \eta_3 (b/z)^2} z^2} \right) \left( 1 + \frac{\eta_4 b^2}{z^2} \right) \]

**CS kernel**

\[ \gamma_{\xi}^q(\mu, b) = \gamma_{\xi}^{q, \text{pert}}(\mu, b^*) - \frac{1}{2} c_0 b b^* \]

\[ b^*(b) = \frac{b}{\sqrt{1 + b^2 / B_{NP}^2}} \]

- Describes data well at wide range of energy scales
- **But** large uncertainties from experiment at high b
- Uncertainties from choice of functional form are *not* known

Scimemi & Vladimirov (JHEP 2019).
Example: global fits of the CS kernel

Figure: MAP collaboration, Bacchetta et al. (2022).

- Large non-perturbative contributions to TMDs
- At low $b_T$, good fits; agree by construction
- Larger uncertainty in non-perturbative region
Lattice QCD in a nutshell

General premise:

- Discretize QFT to regulate divergences
- Only known systematically improvable numerical approach

Want correlation functions:

\[ \langle \mathcal{O} \rangle = \frac{\int [dU] e^{-S[U]} \mathcal{O}}{\int [dU] e^{-S[U]}} \]

Generate representative set of gauge configurations using Monte Carlo

Figures: Argonne, D. Leinweber (QCD vacuum).
Recipe for TMDs on the lattice

1. TMD in cross-sections
2. Project to equal-time slice & discretize
3. Lattice-calculable TMD
4. Run simulations
5. Raw lattice TMD data
6. Limit of zero lattice spacing, $\infty$ size
7. Continuum extrapolation of lattice TMD

Factorization formula
CS kernel from the lattice

\[ \mathcal{D}(b, \text{2GeV}) \]

- Dots = lattice data
- Lines = global fits

Martinez & Vladimirov (2022).
Why lattice QCD?

Complementary to experiment & phenomenology:

- Good to check that QFT and experiment match
- Easier to access CS kernel, spin and flavor dependence than in experiment
- Can improve global fit errors with lattice data
- Calculations beyond $b_T > 1$ GeV$^{-1}$
TMDs from field theory
Many schemes to define TMDs…

Modern Collins

\[ \tilde{f}_{i/p}(x, b_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{uv}(\mu, \zeta, \epsilon) \lim_{y_B \to -\infty} \frac{\tilde{f}^0(u)}{\tilde{f}_{i/p}(x, b_T, \epsilon, y_B, xP^+)} \]

Chiu, Jain, Neill, Rothstein

\[ \tilde{f}_{i/p}(x, b_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z^i_{uv}(\mu, \zeta, \epsilon) \frac{\tilde{f}^0(u)}{\tilde{f}_{i/p}(x, b_T, \epsilon, \eta, xP^+) \sqrt{S^0_{CJNR}(b_T, \epsilon, \eta)}} \]

Echevarria, Idilbi, Scimemi

\[ \tilde{f}_{i/p}(x, b_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z^i_{uv}(\mu, \zeta, \epsilon) \frac{\tilde{f}^0(u)}{\tilde{f}_{i/p}(x, b_T, \epsilon, \delta^+/xP^+)} \]

Becher & Neubert

\[ \lim_{\epsilon \to 0} \left[ \tilde{f}_{i/p}^{BN}(x_1, b_T, \epsilon, \alpha, x_a P^+ A) \tilde{f}^0(u)_{BN}(x_2, b_T, \epsilon, \alpha, x_b P^- B) \right] \]

Ji, Ma, Yuan

\[ \tilde{f}_{i/p}(x_a, b_T, \mu, x_a \zeta_a; \rho) = \lim_{\epsilon \to 0} Z^i_{uv}(\mu, \rho, \epsilon) \frac{\tilde{f}^0(u)}{\tilde{f}_{i/p}(x_a, b_T, \epsilon, \nu, xP^+)} \]

\[ + O(\nu^+, \nu^-) \]

\[ \sqrt{S^0_{v\bar{v}}(b_T, \epsilon, \rho)} \]

Etc!

First goal: sort this out.
Definition of TMDs in QFT

\[ f = \lim \ Z_{\text{UV}} \left( B^{\Gamma}_{q_i/H} \right) \frac{\sqrt{S^R}}{\text{lightcone, renormalization}} \]

Scheme-dependent renormalization

Beam function

Soft factor
TMD matrix elements

Beam function:

- Analog of parton in QCD: quark field attached to lightcone Wilson line

Soft factor:

- Soft & collinear particle interactions: approximated by gauge links
- Gauge invariance: need closed paths
New: unified notation

Can describe lattice & continuum off-lightcone schemes using the same generic beam function & soft factor

\[
f = \lim_{Z_{UV}} \frac{B^{[\Gamma]}_{q_i/H}(b, P, \epsilon, \eta\nu, \delta)}{\sqrt{S^R(b, \epsilon, \eta\nu, \bar{\eta}\bar{\nu})}}
\]

Quasi-TMD
Collins TMD
JMY scheme
Etc.

Each scheme is characterized by a distinct set of arguments & limits

Ebert, Schindler, Stewart, and Zhao (2022).
Meaning of the correlators

**Beam**

\[ \text{Beam} = \langle P | \bar{q}_i \frac{\Gamma}{2} W^F_{\Sigma} (b, \eta \nu, \delta) q_i | P \rangle \]

**Soft**

\[ \text{Soft} = \frac{1}{d^R} \langle 0 | \text{Tr} [S^R_\Delta (b, \eta \nu, \bar{\eta} \bar{\nu})] | 0 \rangle \]

- \( b^\mu, \eta \nu^\mu, \delta^\mu \): parametrize Wilson lines
- **Length** \( \eta \): finite (lattice) or infinite (physical TMD)
- \( \delta^\mu = (0,0,0,\tilde{b}^Z) \) for quasi
  \[ = (0,0,0,0) \] for MHENS

Ebert, Schindler, Stewart, and Zhao (2022).
Now, neat & tidy tables of schemes

<table>
<thead>
<tr>
<th>TMD</th>
<th>Collins scheme</th>
<th>Quasi-TMDs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam</td>
<td>[ \Omega_{i/h} \left[ b, P, \epsilon, -\infty n_B(y_B), b^{-} n_b \right] \xrightarrow{FT} B_{i/h} ]</td>
<td>[ \Omega_{i/h} \left( \tilde{b}, \tilde{P}, a, \tilde{\eta} \tilde{z}, \tilde{b}^{\tilde{z}} \right) \xrightarrow{FT} \tilde{B}_{i/h} ]</td>
</tr>
<tr>
<td>Soft</td>
<td>[ S_{\kappa i}^{} \left[ b_{\perp}, \epsilon, -\infty n_A(2y_n), -\infty n_B(2y_B) \right] ]</td>
<td>[ S_{\kappa i}^{} \left[ b_{\perp}, a, -\tilde{\eta} \frac{n_A(2y_n)}{</td>
</tr>
<tr>
<td>( b^\mu )</td>
<td>( (0, b^{-}, b_{\perp}) )</td>
<td>( (0, b^{\tilde{x}}, b^{\tilde{y}}, \tilde{b}^{\tilde{z}}) )</td>
</tr>
<tr>
<td>( \nu^\mu )</td>
<td>( (-e^{2y_B}, 1, 0_{\perp}) )</td>
<td>( (0, 0, 0, -1) )</td>
</tr>
<tr>
<td>( \delta^\mu )</td>
<td>( (0, b^{-}, 0_{\perp}) )</td>
<td>( (0, 0, 0, \tilde{b}^{\tilde{z}}) )</td>
</tr>
<tr>
<td>( P^\mu )</td>
<td>[ \frac{m_h}{\sqrt{2}} (e^{y_P}, e^{-y_P}, 0_{\perp}) ]</td>
<td>( m_h (\cosh y_{\bar{P}}, 0, 0, \sinh y_{\bar{P}}) )</td>
</tr>
</tbody>
</table>

TMDs from the lattice?

Naïve lattice QCD:

1. Make TMD Wilson lines finite
2. Rotate to Euclidean space
3. Discretize path integral
4. Run Monte Carlo simulations
5. Extrapolate results back to continuum

Problem: Wilson lines are on the lightcone

Real Minkowski time variable $\rightarrow$ complex Euclidean action
Trick: Project the desired **physical** Wilson line onto an equal-time slice

(Nontrivial! More later.)

- **Lattice TMD** is numerically tractable
- Want **physical TMD** & “lattice TMD” to be same in IR
- At worst, differ in UV & related by perturbative matching
Things are progressing rapidly…

2013

**First lattice TMD (MHENS scheme) proposed**
Musch, Hägler, Engelhardt, Negele, and Schäfer

2014

**New lattice TMD (quasi-TMD) proposed, 1-loop studies**
Xiangdong Ji

**Lattice calculations of MHENS beam functions**
MHENS and collaborators

2018

**Quasi-TMD theory put on firmer footing**
Ebert, Stewart, and Zhao

2019

**Proposal for lattice calculation of quasi-soft function**
Ji, Liu, and Liu

2022

**First lattice results for CS kernel & quasi-soft function**
MIT, LPC, ETMC, and Regensburg lattice groups

**Factorization connecting quasi & physical TMDs**
Ebert, *Schindler*, Stewart, and Zhao
TMD factorization

Drell-Yan cross-section

\[ d\sigma = H \int f \otimes f \]

\[ q_T \ll Q \]

Formal definition

\[ f = Z_{UV} \frac{B}{\sqrt{S}} \]

SCET/QCD

\[ q_T \ll Q \]

On the lattice

\[ f = C \times \tilde{f}_{lattice} \]

LaMET

\[ P^z \gg \Lambda_{QCD} \]

Proof of factorization
Connecting physical & lattice TMDs

Lattice schemes

Quasi

MHENS

Large Rapidity

Continuum schemes

Continuum limits

Matching relation

New!

Collins

Ebert, Schindler, Stewart, and Zhao (2022).
Two main lattice approaches

MHENS scheme

- Pioneered lattice TMDs
- Focused on $x$-moments
- Renormalization, soft function, factorization not fully known

Quasi-TMDs

- Newer; fewer results for proton
- Focused on full TMD
- Renormalization, soft function have been proposed

Ji (2014).
Physical TMD schemes in this talk

<table>
<thead>
<tr>
<th></th>
<th>Collins</th>
<th>Large Rapidity (LR)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Limits</strong></td>
<td>( \lim_{\epsilon \to 0} \lim_{y_B \to -\infty} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}} )</td>
<td>( \lim_{-y_B \gg 1} \lim_{\epsilon \to 0} Z_{UV}^R \frac{\Omega_{i/h}}{\sqrt{S^R}} )</td>
</tr>
<tr>
<td><strong>Beam</strong></td>
<td>( \Omega_{q/h}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b] )</td>
<td>( \Omega_{q/h}^{[\gamma^+]}[b, P, \epsilon, -\infty n_B(y_B), b^- n_b] )</td>
</tr>
<tr>
<td><strong>Soft</strong></td>
<td>( S^R[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)] )</td>
<td>( S^R[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)] )</td>
</tr>
</tbody>
</table>

- Closely related to lattice TMDs
- Regularize by taking off lightcone (characterize by rapidity \( y_B \))
- Only differ by an order of limits
Definition of the schemes

Lattice

Quasi

LR

Collins

Continuum
Factorization derivation steps

Step 1: same at large rapidity $P^z \gg \Lambda_{\text{QCD}}$
- Expand & relate their variables
- Take Wilson line length $|\eta| \to \infty$

Step 2: need a matching coefficient
- Different UV renormalizations
- Nontrivial relationship

Focus on beams: quasi-soft function is chosen to reproduce the Collins soft function
### Step 1: Quasi to Large Rapidity

Compare Lorentz invariants formed from beam function arguments $b^\mu, P^\mu, \delta^\mu, \eta \nu^\mu$

Use boosts to show quasi = LR as $|\eta| \to \infty$ & $P^z >> \Lambda_{QCD}$

<table>
<thead>
<tr>
<th></th>
<th>Quasi</th>
<th>LR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^2$</td>
<td>$-b_T^2 - (\tilde{b}^z)^2$</td>
<td>$-b_T^2$</td>
</tr>
<tr>
<td>$(\eta \nu)^2$</td>
<td>$-\tilde{\eta}^2$</td>
<td>$-2\eta^2 e^{2y_B}$</td>
</tr>
<tr>
<td>$P \cdot b$</td>
<td>$-m_h \tilde{b}^z \sinh y_P$</td>
<td>$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$</td>
</tr>
<tr>
<td>$\frac{b \cdot (\eta \nu)}{\sqrt{(</td>
<td>\eta \nu</td>
<td>^2 b^2)}}$</td>
</tr>
<tr>
<td>$\frac{P \cdot (\eta \nu)}{\sqrt{P^2</td>
<td>\eta \nu</td>
<td>^2}}$</td>
</tr>
<tr>
<td>$\frac{\delta^2}{b^2}$</td>
<td>$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{b \cdot \delta}{b^2}$</td>
<td>$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{P \cdot \delta}{P \cdot b}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\delta \cdot (\eta \nu)}{b \cdot (\eta \nu)}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$P^2$</td>
<td>$m_h^2$</td>
<td>$m_h^2$</td>
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</tbody>
</table>
Matching up Lorentz invariants implies:

\[ \sinh(\tilde{y}_P) \text{sgn}(\eta) = \sinh(y_P - y_B) \text{sgn}(\eta) \]

Need \( y_P - y_B = y_{\tilde{P}} \)
Quasi to LR: same at Large Rapidity

Previous slide:
\[ y_B = y_\bar{p} - y_P \]

Need:
\[-m_h \tilde{b}_z \sinh y_\bar{p} = \frac{m_h}{\sqrt{2}} b^- e^{y_P} \]

Finite \( P \cdot b \) & \( y_P \rightarrow \) finite \( b^- \)

For quasi, \( y_\bar{p} \rightarrow \infty \), so \( \tilde{b}^z \rightarrow 0 \)

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<tr>
<td>( b^2 )</td>
<td>(-b_T^2 - (\tilde{b}^z)^2)</td>
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</tr>
<tr>
<td>( (m_t)^2 )</td>
<td>(-\tilde{m}^2)</td>
<td>(-2m_T^2 e^{2y_B})</td>
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<tr>
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<td>0</td>
</tr>
<tr>
<td>( b \cdot \delta )</td>
<td>(\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2})</td>
<td>0</td>
</tr>
<tr>
<td>( P \cdot \delta )</td>
<td>(\frac{\delta}{P \cdot b})</td>
<td>1</td>
</tr>
<tr>
<td>( P \cdot (\eta v) )</td>
<td>(\frac{(\eta v)^2}{P^2 \eta v^2})</td>
<td>1</td>
</tr>
<tr>
<td>( \delta \cdot (\eta v) )</td>
<td>(\frac{\delta}{b \cdot (\eta v)})</td>
<td>1</td>
</tr>
<tr>
<td>( P^2 )</td>
<td>(m_h^2)</td>
<td>(m_h^2)</td>
</tr>
</tbody>
</table>
Quasi to LR: same at Large Rapidity

Need $\tilde{\eta} = \sqrt{2} e^{y_B} \eta$

In $y_\tilde{P} \to -\infty$ limit, $b_T \gg \tilde{b}_T$
Factorization derivation steps

Step 1: Same at large rapidity

Step 2: need a matching coefficient
- Different UV renormalizations
- Nontrivial relationship
### Step 2: Large Rapidity to Collins

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<tbody>
<tr>
<td><strong>Limits</strong></td>
<td>$\lim_{\epsilon \to 0} \lim_{y_B \to -\infty} Z^R_{UV} \frac{\Omega_i/h}{\sqrt{S^R}}$</td>
<td>$\lim_{-y_B &gt; 1} \lim_{\epsilon \to 0} Z^R_{UV} \frac{\Omega_i/h}{\sqrt{S^R}}$</td>
</tr>
<tr>
<td><strong>Beam</strong></td>
<td>$\Omega_{q/h}^{[r^+]}[b, P, \epsilon, -\infty n_B(y_B), b^{-} n_b]$</td>
<td>$\Omega_{q/h}^{[r^+]}[b, P, \epsilon, -\infty n_B(y_B), b^{-} n_b]$</td>
</tr>
<tr>
<td><strong>Soft</strong></td>
<td>$S^R[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$</td>
<td>$S^R[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)]$</td>
</tr>
</tbody>
</table>

- Order of UV limits **cannot** affect IR physics
- But if non-commuting $\rightarrow$ perturbative matching coefficient
- Non-commutativity can arise from divergences

Intuition for rapidity divergences, which arise from factorization:

$$
\int_{q_T}^Q \frac{dk}{k} = \lim_{\tau \to 0} \left[ \int_{0}^{Q} \frac{dk}{k} R_c(k, \tau) + \int_{q_T}^{\infty} \frac{dk}{k} R_s(k, \tau) \right] = \ln \frac{Q}{q_T}
$$

**full**  \quad  **collinear**  \quad  **soft**
Rapidity divergences and matching

Can see even at one loop. Contains terms like:

\[
I = \int \frac{d^d k}{(2\pi)^d} \frac{p^+ - k^+}{k^2(p - k)^2} \left( e^{ik_T \cdot \vec{b_T}} - 1 \right) \left( \frac{1}{n_B \cdot k + i\delta} + \frac{1}{n_B \cdot k - i\delta} \right)
\]

**Collins:** directly carry out the integration

**LR:** integrate over \( k^0 \) & \( k^z \), get a log, then expand in \( p'_z \gg k_T \) before integrating over \( k_T \):

\[
I = \frac{i}{4} \int \frac{d^{d-2} k_T}{(2\pi)^{d-1}} \left( e^{ik_T \cdot \vec{b_T}} - 1 \right) \left( \frac{2}{k_T^2} + \frac{1}{p'_z^2} \right) \left( \frac{\ln \left( \frac{k_T^2 + 2p'_z \sqrt{k_T^2 + p'_z^2 + 2p'_z^2}}{k_T^2} \right)}{\sqrt{1 + k_T^2/p'_z^2}} - \frac{4}{k_T^2} \right)
\]

Yield different values:

**Collins**

\[
\frac{i}{(4\pi)^2} \left[ \frac{1}{e^2} - \frac{1}{2} \ln^2 \left( \frac{4p'_z b_T}{b_0} \right) + \ln \left( \frac{4p'_z b_T}{b_0} \right)^2 + \ln \left( \frac{b_T^2 - \mu^2}{b_0^2} \right) - 2 \right]
\]

**LR**

\[
\frac{i}{(4\pi)^2} \left[ \frac{1}{e^2} + \frac{1}{e} \left( 2 + \ln \left( \frac{\mu^2}{4p'_z^2} \right) \right) - \frac{1}{2} \ln^2 \left( \frac{b_T^2 - \mu^2}{b_0^2} \right) + \ln \left( \frac{b_T^2 - \mu^2}{b_0^2} \right)^2 \left( 2 + \ln \left( \frac{\mu^2}{4p'_z^2} \right) \right) - \frac{\pi^2}{12} \right]
\]

\[f_{LR} = C_i(\tilde{P}^z, \mu) f_{Collins}\]

Schindler, Stewart, and Zhao (2022).
Factorization derivation steps

- **Step 1:** Same at large rapidity
- **Step 2:** Pick up a matching coefficient
- **Step 3:** Combine to get full factorization
Lattice-to-physical factorization

Quasi-TMD (lattice)  Matching  RGE for $\zeta$  Collins TMD (continuum)

$$\tilde{f}_{i/H}^{[s]} (x, \vec{b}_T, \mu, \tilde{\zeta}, x\vec{P}^z) = C_i (x\vec{P}^z, \mu) \exp \left[ \frac{1}{2} \gamma^i_\zeta (\mu, b_T) \ln \frac{\tilde{\zeta}}{\zeta} \right] f_{i/H}^{[s]} (x, \vec{b}_T, \mu, \zeta)$$

$$\tilde{\zeta} = (2x\vec{P}^z)^2 e^{2(\gamma_B - \gamma_n)}$$

Power corrections

$$\times \left\{ 1 + \mathcal{O} \left[ \frac{1}{(x\vec{P}^z b_T)^2}, \frac{\Lambda_{QCD}^2}{(x\vec{P}^z)^2} \right] \right\}$$

Note that this formula connects physical continuum TMDs to the renormalized continuum limit of lattice calculations.

Ebert, Schindler, Stewart, and Zhao (2022).
What is the matching coefficient?

\[ \tilde{f}_{i/H}^{[s]} (x, \vec{b}_T, \mu, \zeta, x\vec{P}^z) = C_i (x\vec{P}^z, \mu) \exp \left[ \frac{\gamma^i}{2} \ln \frac{\zeta}{\tilde{\zeta}} \right] f_{i/H}^{[s]} (x, \vec{b}_T, \mu, \zeta) \]

Convenient properties:

- Independent of spin
- No quark-gluon or flavor mixing
- Known at one-loop & logarithmic terms
Recall: only rapidity divergences contribute! For the gluon:

Casimir scaling for quarks and gluons

\[ C_i(\mu, x\bar{P}^z) = 1 + \frac{\alpha_s C_R}{4\pi} \left[ -\ln^2 \frac{(2xP^z)^2}{\mu^2} + 2 \ln \left(\frac{(2xP^z)^2}{\mu^2}\right) - 4 + \frac{\pi^2}{6} \right] + O(\alpha_s^2) \]

Schindler, Stewart, and Zhao, 2205.12369.

Note that Casimir scaling only holds if one chooses \( F^{+\rho} \) rather than \( F^{0\rho} \) or \( F^{3\rho} \) (Zhang et al., 2209.05443)
N^nLL terms

RG evolution:

\[
\frac{d}{d \ln(2x\bar{P}^z)} \ln C_q(x\bar{P}^z, \mu) = \gamma_C^q (2x\bar{P}^z, \mu)
\]

Turn the crank, get matching coefficient:

\[
C_i(x\bar{P}^z, \mu) = C_i[\alpha_s(\mu)] \exp \left[ \int_{\alpha_s(\mu)}^{\alpha_s(2x\bar{P}^z)} \frac{d\alpha}{\beta[\alpha]} \int_{\alpha}^{\alpha_s(\mu)} \frac{d\alpha'}{\beta[\alpha']} (2\Gamma_{\text{cusp}}^i[\alpha'] + \gamma_C^i[\alpha]) \right]
\]

→ N^nLL straightforward to compute from higher-order anomalous dimensions.

Example, NLL:

\[
C_q(x\bar{P}^z, \mu)^{NLL} = -2K_{\Gamma}^q (2x\bar{P}^z, \mu) - K_{\gamma}^q (2x\bar{P}^z, \mu)
\]

\[
K_{\Gamma}^q(\mu_0, \mu) = -\frac{\Gamma_0^q}{4\beta_0} \left\{ \frac{4\pi}{\alpha_s(\mu_0)} \left( 1 - \frac{1}{r} - \ln r \right) + \left( \frac{\Gamma_1^q}{\Gamma_0^q} - \beta_1 \beta_0 \right) (1 - r + \ln r) + \frac{\beta_1}{2\beta_0} \ln^2 r \right\} \quad K_{\gamma}^q(\mu_0, \mu) = -\frac{\gamma_{\text{C0}}^q}{2\beta_0} \ln r
\]

Ebert, Schindler, Stewart, and Zhao (2022).
Spin independence

Beam = \left\langle P \left| \bar{q}_i \, \frac{\Gamma}{2} \, W^F (b, \eta \nu, \delta) q_i \right| P \right\rangle

\Gamma \in \{ \bar{\nu}, \bar{\nu} \gamma_5, i \sigma^\alpha \gamma_5 \}

Spin structure example:

\[ f_{q/h_S}^{[\vec{q}_T]} (x, q_T) = f_1 (x, q_T) - \frac{\epsilon_{\rho \sigma} q_\perp S^\sigma}{M} f_{1T} (x, q_T) \]

### Quark Polarization

<table>
<thead>
<tr>
<th>Nucleon Polarization</th>
<th>Un-Polarized (U)</th>
<th>Longitudinally Polarized (L)</th>
<th>Transversely Polarized (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U \quad f_1 = \bullet</td>
<td>Unpolarized</td>
<td>\quad \quad g_1 = \rightarrow \rightarrow</td>
<td>h_{1L} = \rightarrow \rightarrow</td>
</tr>
<tr>
<td>L \quad \epsilon_{\rho \sigma} q_\perp S^\sigma</td>
<td>Boer-Mulders</td>
<td>\quad \quad h_{1L} = \rightarrow \rightarrow</td>
<td></td>
</tr>
<tr>
<td>T \quad f_{1T} = \bullet</td>
<td>Sivers</td>
<td>\quad \quad g_{1T} = \rightarrow \rightarrow</td>
<td>h_{1T} = \rightarrow \rightarrow</td>
</tr>
</tbody>
</table>

Ebert, Schindler, Stewart, and Zhao (2020).
No flavor or quark-gluon mixing

The diagrams above are the same for quasi, LR, and Collins:

- Can see directly from factorization derivation
- So, only two coefficients $C_q$ & $C_g$
- Can do gluon TMDs!

Ebert, Schindler, Stewart, and Zhao (2022).
Status of the lattice
Lattice targets

\[ d\sigma = H \int f \otimes f \]

\[ f = Z_{UV} \frac{B}{\sqrt{S}} \]

\[ f = C \times \tilde{f}_{lattice} \]

Full TMDs (hadrons, flavors, spins)
Beam functions (& ratios)
Soft factor (indirect)
CS kernel (\(\zeta\) evolution)
From the factorization formula:

\[
\gamma^q_\xi(\mu, b_T) = \frac{1}{\ln P_1^Z/P_2^Z} \ln \frac{C^{TMD}(\mu, xP_2^Z) \int db^z e^{ib^z xP_1^Z} \tilde{Z}_q \tilde{Z}_{uv}^q \tilde{B}_q(b^z, b_T, a, L, P_1^Z)}{C^{TMD}(\mu, xP_1^Z) \int db^z e^{ib^z xP_2^Z} \tilde{Z}_q \tilde{Z}_{uv}^q \tilde{B}_q(b^z, b_T, a, L, P_2^Z)}
\]

Dependent on few parameters compared to RHS!

- No soft function needed
- Can set up $\tilde{Z}_{uv}^q$ to remove power law divergences in numerator and denominator

Ebert, Stewart, and Zhao (PRD 2019).
CS kernel lattice results

Recently, first lattice results!

However, large systematic uncertainties.

Shanahan, Wagman, Zhao (PRD 2021).
- Soft function also runs into lightcone Wilson staple issues
- Can express soft as ratio of meson form factor with convolution of two meson wavefunctions
Ratios of different TMD spins, flavors, or hadrons can be calculated directly from lattice beam functions:

$$\lim_{\tilde{\eta} \to \infty} \frac{f[\tilde{\Gamma}_1]}{f[\tilde{\Gamma}_2]}_{q_i/h} = \lim_{\tilde{\eta} \to \infty} \frac{\tilde{B}[\tilde{\Gamma}_1]}{\tilde{B}[\tilde{\Gamma}_2]}_{q_i/h'}$$

This follows from the quasi-to-Collins factorization formulas:

$$C_i \exp \left[ \frac{1}{2} \gamma^i_\zeta \ln \frac{\zeta}{\tilde{\zeta}} \right] f[\Gamma]_{q_i/H} = \tilde{f}[\Gamma]_{q_i/H} = \lim Z_{UV} \frac{\tilde{B}[\Gamma]}{\sqrt{S^R}}_{q_i/H}$$

Lattice-to-continuum TMD factorization

Factorization of a lattice TMD into matrix elements
First lattice studies!

- Suggested ratio method
- Focus on $x$-integrated TMDs, so renormalization is less of a problem

Caveat:

- So far, no matching corrections
- Procedure to carry out simulations with matching, $x$-dependence, soft functions not yet known
MHENS lattice results: Sivers sign change

- Sivers has different signs in DY & SIDIS
- Can verify on the lattice using ratios at various $b_T$ values

Transversely Polarized (T)

\[ h_1^\perp = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Boer-Mulders

- Pion u-quark Boer-Mulders shift in SIDIS

\[ m_N \frac{\tilde{h}_1^\perp}{\tilde{f}_1} \]

MHENS lattice results: Boer-Mulders shift

MHENS (PRD 2016).
For the case $P \cdot b = 0$ (focus of all studies so far) MHENS and quasi have an equivalent renormalization, soft function, etc.

$$\int dx \ f_{q_i/h}^{[\Gamma]}(x, \vec{b}_T, \mu, \tilde{\zeta}, z \vec{P}_z, \tilde{\eta}) = f_{q_i/h}^{[\Gamma]}(b^z = 0, \vec{b}_T, \mu, \vec{P}_z, y_n - y_B, \tilde{\eta})$$

For the case $P \cdot b \neq 0$:

- Non-trivial cusp angles $\gamma$, even as $\eta \to \infty$
- $b^z$-dependent Wilson length
- Implies renormalization, soft are $b^z$-dependent and won’t cancel out in ratios at finite $\eta$

Ebert, Schindler, Stewart, and Zhao (JHEP 2022).
### Status of the lattice

<table>
<thead>
<tr>
<th>Category</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>CS kernel</td>
<td>⏳</td>
</tr>
<tr>
<td>Spin-dependent TMD ratios</td>
<td>⏳</td>
</tr>
<tr>
<td>3D structure ratios</td>
<td>⏳</td>
</tr>
<tr>
<td>Flavor ratios</td>
<td>⏳</td>
</tr>
<tr>
<td>Normalized TMD</td>
<td>❌</td>
</tr>
<tr>
<td>Proton-pion TMD ratios</td>
<td>❌</td>
</tr>
<tr>
<td>Gluon TMDs</td>
<td>❌</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>
Conclusion
Implications of factorization

Quasi-to-Collins matching coefficient: quite convenient...

- No spin dependence
- No quark-gluon or flavor mixing (simpler to get gluon TMDs!)
- NLO & $N^n$LL results: generalized Casimir scaling
- Same as LR-to-Collins coefficients, so can compute as the rapidity-divergent diagrams in different orders of limits

Implies validity of taking quasi-TMD ratios...

- $P^z$ ratios for CS kernel
- Beam hadron, flavor, spin ratios for full TMD ratios
Our contributions

1. New unified TMD notation

2. New scheme (LR)

3. Lattice-to-physical TMD factorization: convenient!

Quasi-TMDs have a straightforward, rigorous connection to physical TMDs