NLP resummed rapidity distributions near threshold for Drell-Yan and diphoton production

Robin van Bijleveld, Eric Laenen, Leonardo Vernazzad and G. W

, arXiv: 2308.00230

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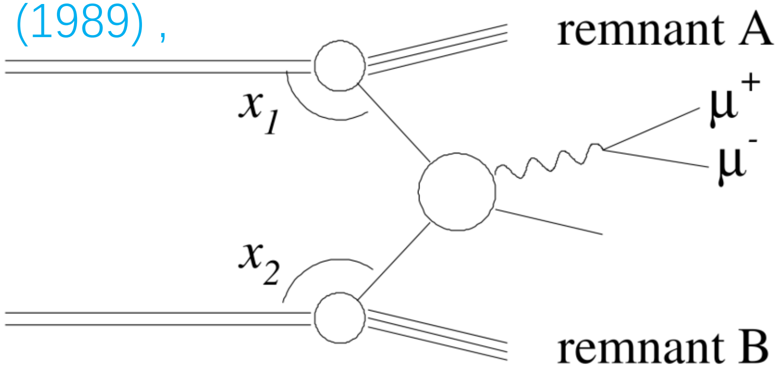
Catalog

- Factorization of the hadronic cross section
- Calculation of the partonic cross section
- Threshold resummation
- Fixed order calculation
  - Drell-Yan process
  - Diphoton production
- Threshold resummation at LP and NLP LL for Drell-Yan and Diphoton
- Summary and outlook

factorization

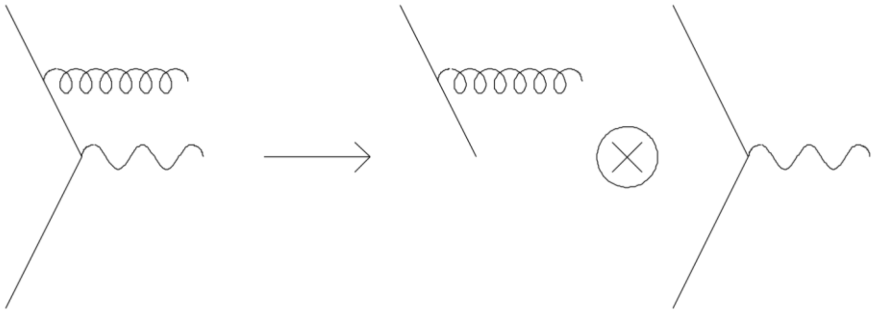
- Cross sections at Hadron colliders:

John C. Collins, *et al.* (1989) , hep-ph/0409313



$$d\sigma = \sum_{ij}\int dx_1\, dx_2\, f_i(x_1,\mu_f^2)\, f_j(x_2,\mu_f^2)\, d\hat{\sigma}_{ij}(\hat{s},\mu_f^2) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

- Mass factorization:



B. Pötter. (1997)

$$\frac{1}{(p_1-k_1)^2} = \frac{1}{-2|p_1||k_1|(1-\cos\theta)}$$

$$d\hat{\sigma}_{ij}(\hat{s}) = \int d\,z_1 dz_2 \Gamma(z_1) d\hat{\sigma}_{ij}^{R}(z_1 z_2 \hat{s})\}\Gamma(z_2)$$

$$f^R(\eta) := \int_0^1 d\,z dx\,\Gamma(x) f(z)\delta(\eta - xz) = \int_\eta^1 \frac{dz}{z} f\left(\frac{\eta}{z}\right)\Gamma(z)$$

$$d\hat{\sigma}_{ij}^{R(1)} = d\hat{\sigma}_{ij}^{(1)} - \int d\,z_1 \Gamma_{ik}^{(1)} d\hat{\sigma}_{kj}^{(0)} - \int d\,z_2 \Gamma_{ik}^{(1)} d\hat{\sigma}_{kj}^{(0)}$$

$$A = A^{(0)} + \frac{\alpha_s}{4\pi}A^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 A^{(2)} + \cdots$$

Partonic cross section

Cross sections

- Total cross sections:

Anti-unitarity relation

$$\hat{\sigma} \sim \left[\prod d^d p_f \delta^+(p_f^2 - m_f^2)\right]\delta^{(d)}\left(\sum p_i - \sum p_f\right)\left[\prod d^d p_l\right]|\mathcal{M}|^2$$

- Differential cross sections:

$$d\hat{\sigma} \sim \left[\prod d^d p_f \delta^+(p_f^2 - m_f^2)\right]\delta^{(d)}\left(\sum p_i - \sum p_f\right)\left[\prod d^d p_l\right]|\mathcal{M}|^2 F(p_i, p_f, \cdots)$$

  - Distributions: $M_{f_1 f_2}\,,\ p_T\,,\ p_{f_i,T}\,,\ y_{f_i}\,,\ \Delta\phi\,,\ \tau\ \cdots$
  - Polarized distributions.

$\delta -$ functions

$\theta -$ functions

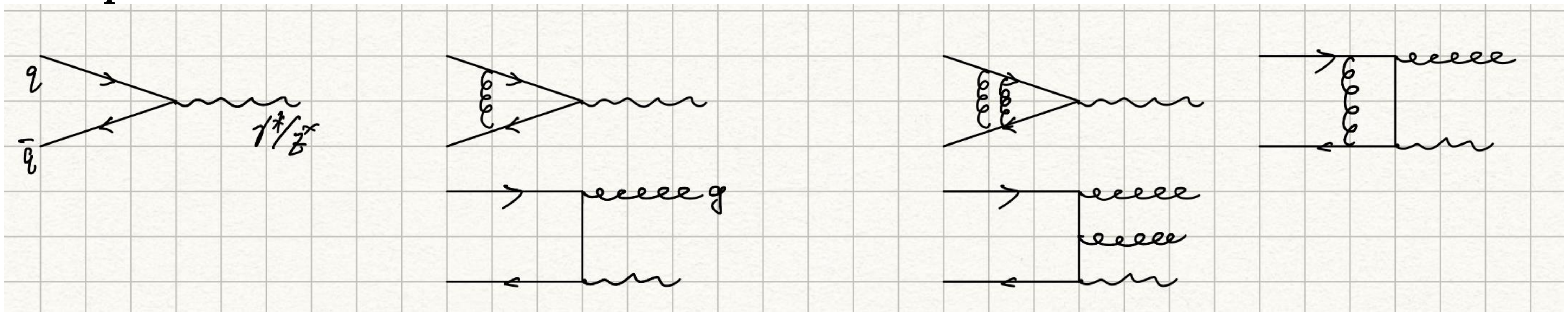
$\cdots$

Squared amplitudes and integrals

Loop- and phase-space integrals

Partonic cross section

- Amplitudes



LO + NLO + NNLO + …

$$A = A^{(0)} + \frac{\alpha_s}{4\pi}A^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 A^{(2)} + \cdots$$

$$\frac{1}{(p-k_1-k_2)^2-m^2} = \frac{1}{p^2-m^2-2p\cdot(k_1+k_2)+(k_1+k_2)^2}$$

$$\xrightarrow{k_i\to 0,\ p^2=m^2} \frac{1}{-2p\cdot(k_1+k_2)} - \frac{(k_1+k_2)^2}{[-2p\cdot(k_1+k_2)]^2} + \cdots$$

$$\xrightarrow{k_i\to 0,\ p^2\neq m^2} \frac{1}{p^2-m^2} - \frac{-2p\cdot(k_1+k_2)}{(p^2-m^2)^2} + \cdots$$

Threshold resummation

- Differential distribution over a kinematic variable ξ

$$\frac{d\hat{\sigma}}{d\xi} = \sigma_0 \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \underbrace{\left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)} \delta(\xi)} + \underset{\big\downarrow}{c_{nm}^{(0)} \log^m \xi} + \mathcal{O}(\xi) \right]$$

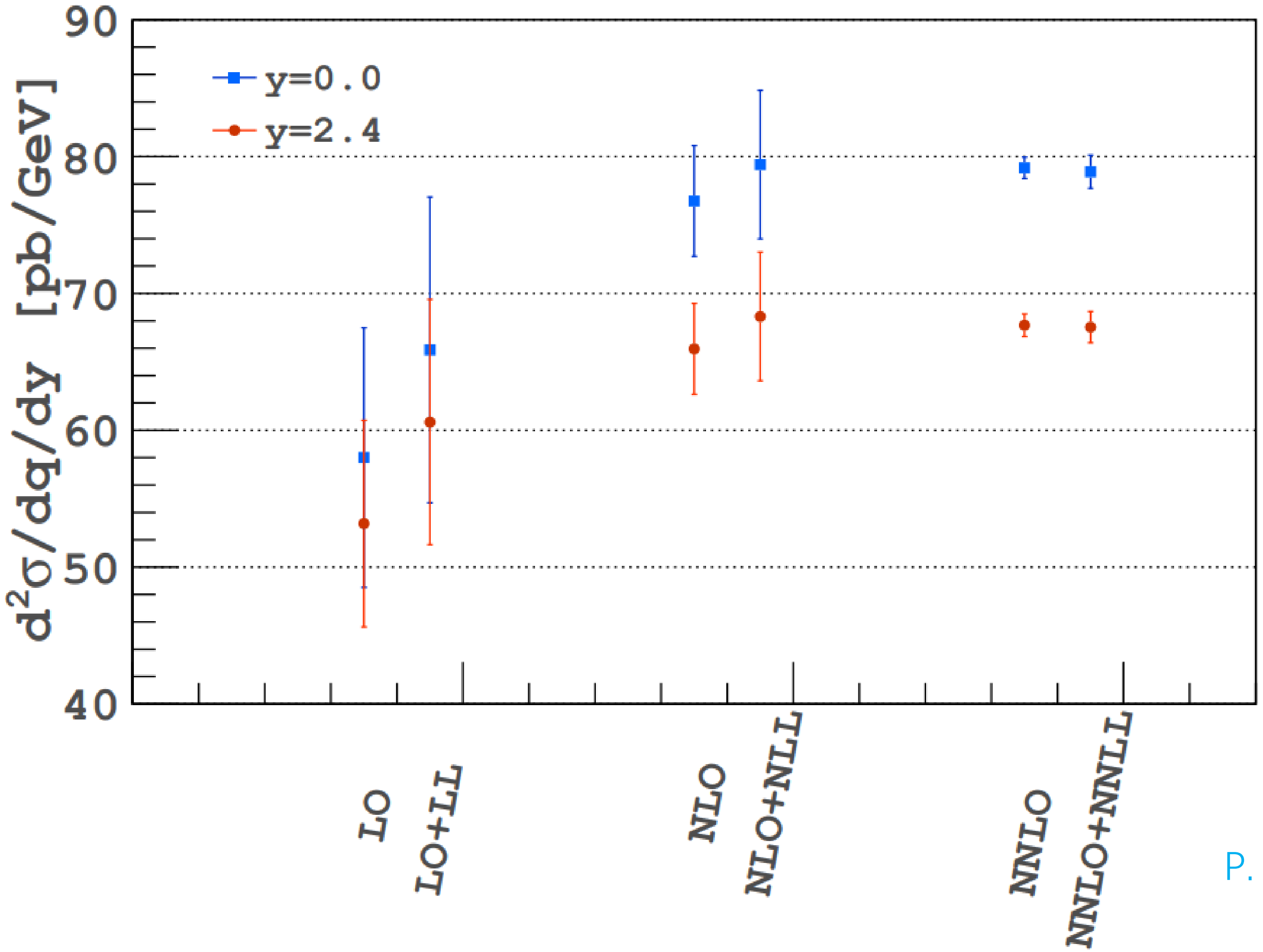
$$\text{LP} \qquad + \qquad \text{NLP} \qquad + \qquad \dots$$

$$d\hat{\sigma} \sim \begin{array}{cc} \begin{array}{cccc} \text{LL} & \text{NLL} & \text{NNLL} & \dots \end{array} & \\ \begin{pmatrix} 1 & & & \\ \alpha_s L & \alpha_s & & \\ \alpha_s^2 L^3 & \alpha_s^2 L^2 & \alpha_s^2 L & \alpha_s^2 \\ \alpha_s^3 L^5 & \cdots & & \\ \vdots & & & \end{pmatrix} & \begin{array}{l} \text{LO} \\ \text{NLO} \\ \text{NNLO} \\ \dots \end{array} \end{array}$$

$$L \to \left(\frac{\log \xi}{\xi}\right)_+ \quad \text{or} \quad \log \xi$$

$$L \sim \frac{1}{\alpha_s}$$

Threshold resummation

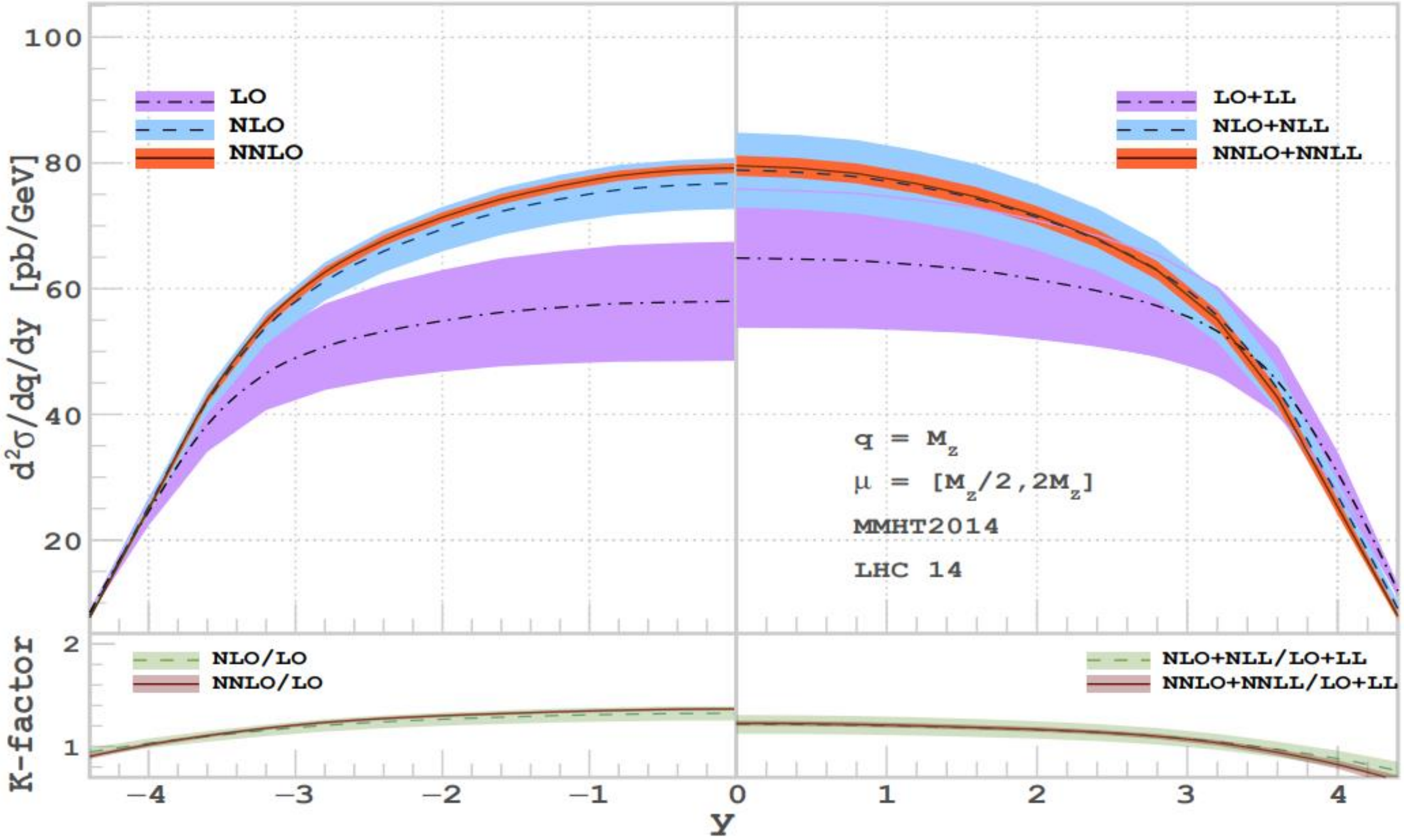


Fixed order and resummed predictions for rapidity y=0 and y=2.4 at each order

P. Banerjee *et.al* . (2018), arXiv:1805.01186

Threshold resummation

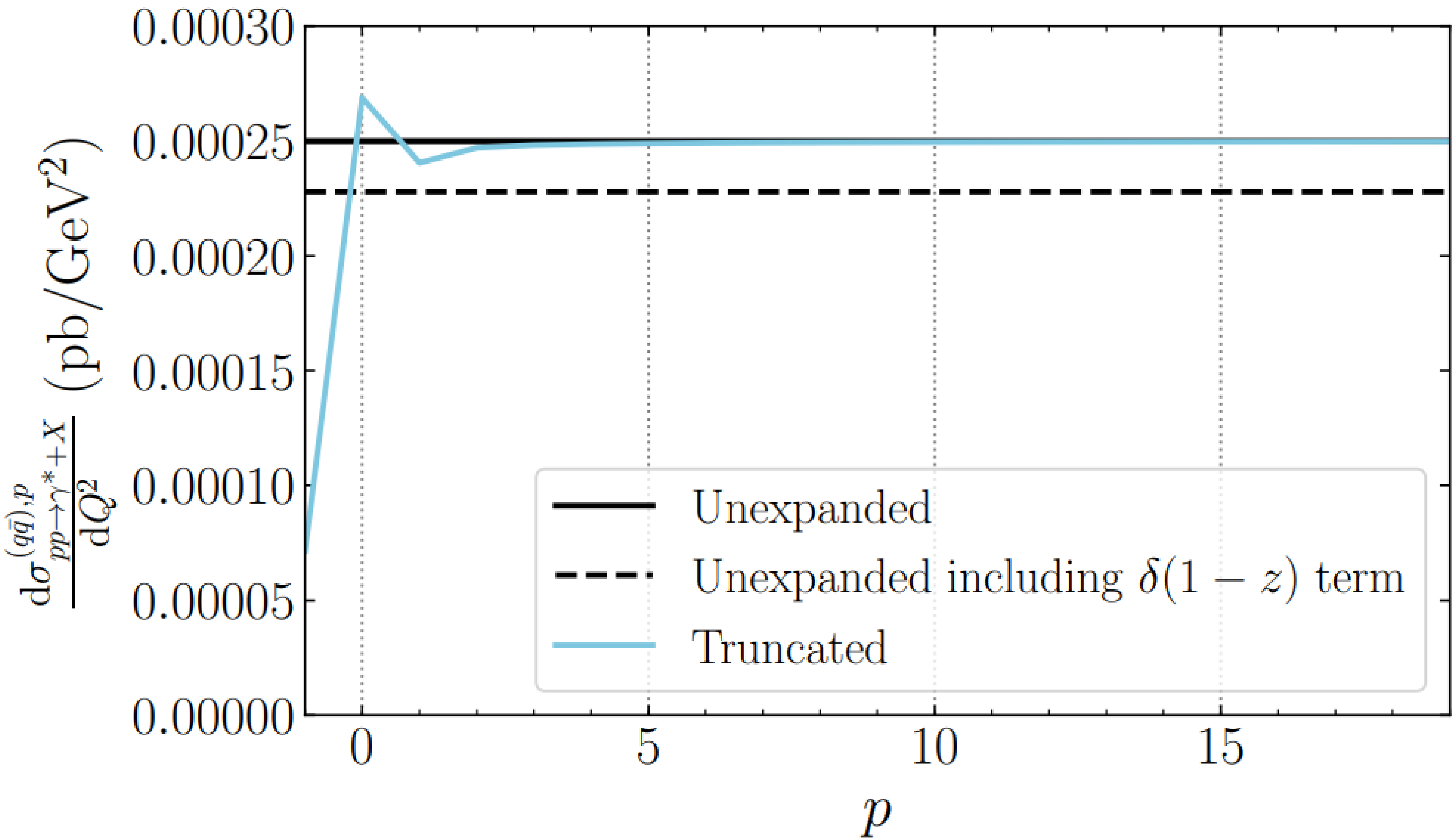
P. Banerjee, *et.al*. (2018), arXiv:1805.01186



Drell-Yan rapidity distribution for 14 TeV LHC at q = Mz

Threshold resummation

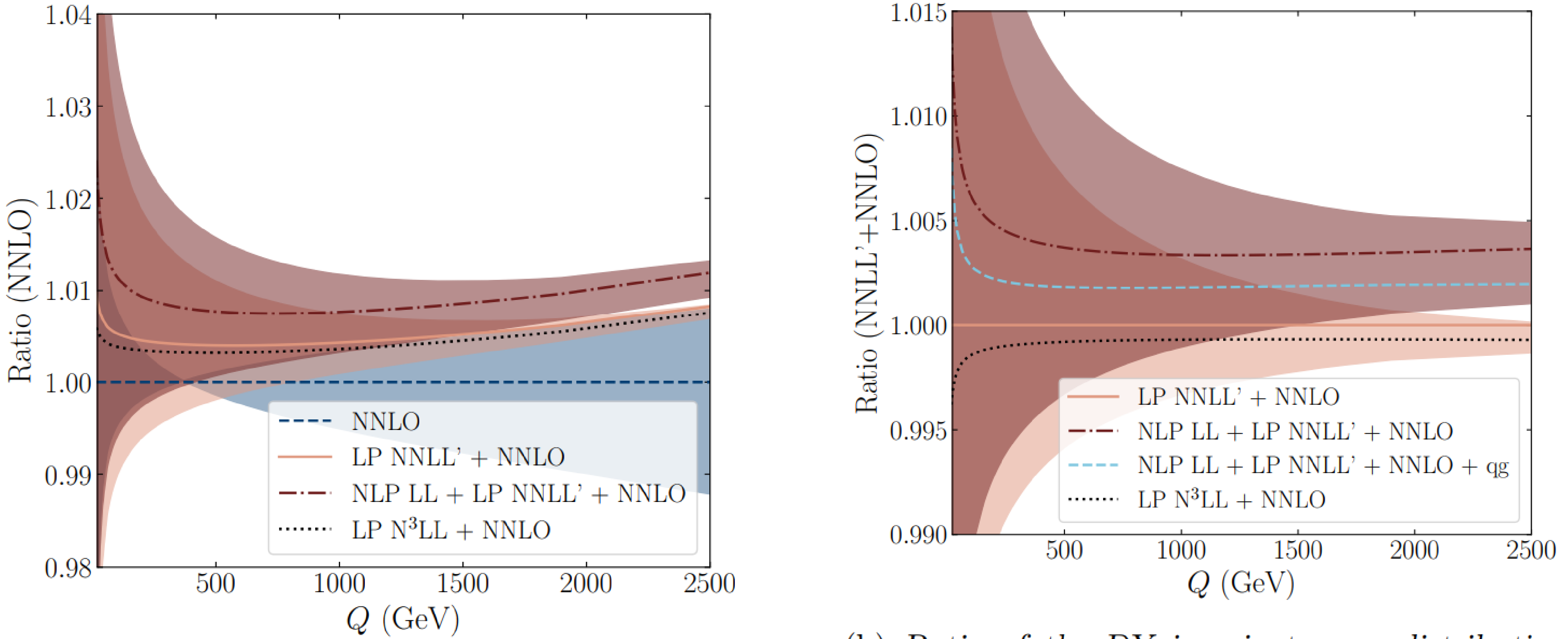
M. Beekveld, *et.al.*, arXiv:2101.07270



Power expansions of the NLO+NNLO hadronic cross sections for Drell-Yan.

Threshold resummation

M. Beekveld, *et.al.*, arXiv:2101.07270



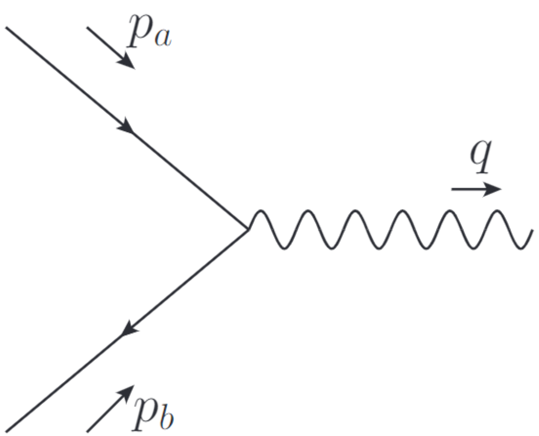
(b) *Ratio with respect to NNLO*

(b) *Ratio of the DY invariant mass distribution with respect to NNLL′ + NNLO.*

Ratio plot for the DY invariant mass distribution

Rapidity distribution for fixed-order Drell-Yan

$A(p_A) + B(p_B) \to \gamma^*(q) + X(p_X)$



- Hadronic differential cross section

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$

$$z \equiv \frac{Q^2}{\hat{s}} = \frac{Q^2}{x_a x_b s} = \frac{\tau}{x_a x_b}, \qquad y = \frac{e^{-2\hat{Y}} - z}{(1-z)\left(1+e^{-2\hat{Y}}\right)},$$

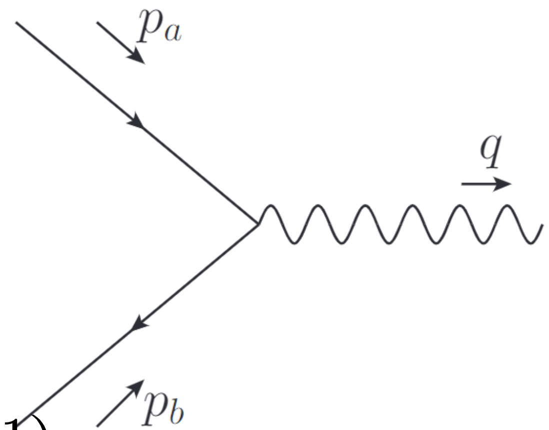
$$\mathcal{L}(z,y) = f_{q/A}\left(e^{Y}\sqrt{\frac{\tau}{z}\frac{1-(1-y)(1-z)}{1-y(1-z)}}\right) f_{\bar{q}/B}\left(e^{-Y}\sqrt{\frac{\tau}{z}\frac{1-y(1-z)}{1-(1-y)(1-z)}}\right) + (q \leftrightarrow \bar{q}).$$

$$\Delta(z,y) = \frac{1}{(2\pi)^d}\frac{(-g^{\mu\nu})W_{\mu\nu}}{1-\epsilon}\,\delta(q^2-Q^2)\,\delta\left[y - \frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)\,(p_a\cdot q + p_b\cdot q)}\right],$$

$$\Delta(z,y) = \Delta_{\mathrm{LP}}(z,y) + \Delta_{\mathrm{NLP}}(z,y) + \mathcal{O}(1-z), \qquad \Delta_{\mathrm{LP}}(z,y) = \frac{\delta(y)+\delta(1-y)}{2}\Delta_{\mathrm{LP}}(z)$$

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$



- LO

$$\Delta^{(0)}(z,y) = \delta(1-z)\,\delta\left(y-\frac{1}{2}\right) \quad \Rightarrow \quad \frac{d\sigma}{dQ^2\,dY} = \sigma_0 \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(z,\frac{1}{2}\right) \Delta^{(0)}(z)$$

$$\Delta_{\mathrm{LP}}(z,y) = \frac{\delta(y)+\delta(1-y)}{2} \Delta_{\mathrm{LP}}(z) \quad \Rightarrow \quad \frac{d\sigma}{dQ^2\,dY} = \sigma_0 \int_\tau^1 \frac{dz}{z}\, \frac{\mathcal{L}(z,0)+\mathcal{L}(z,1)}{2} \Delta^{(0)}(z)$$

$$\frac{\mathcal{L}(z,0)+\mathcal{L}(z,1)}{2} = \mathcal{L}\left(z,\frac{1}{2}\right) + \mathcal{O}[(1-z)^2]$$

M. Bonvini, arXiv:1212.0480

- NLO

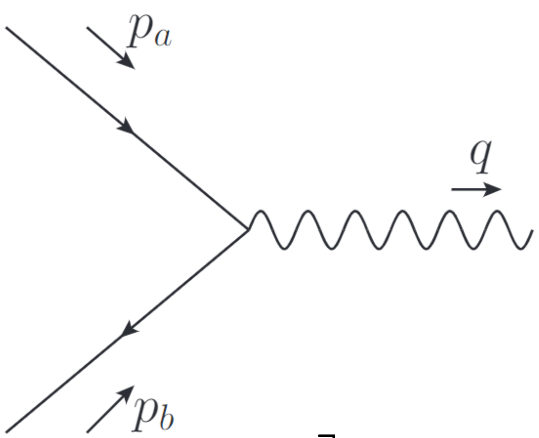
$$\Delta^{(1)}(z,y)|_{\mathrm{virtual}} = \frac{1}{4N_c}\frac{1}{2\pi} \int d\Phi_{\gamma^*} \sum_{\mathrm{s,c,p}} 2\,\mathrm{Re}\left[\mathcal{M}^{(0)*}_{q\bar{q}\to\gamma^*} \mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*}\right],$$

$$\Delta^{(1)}(z,y)|_{\mathrm{real}} = \frac{1}{4N_c}\frac{1}{2\pi} \int d\Phi_{\gamma^* g} \sum_{\mathrm{s,c,p}} \left|\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^* g}\right|^2,$$

$$\int d\Phi_{\gamma^*} = \int \frac{d^d q}{(2\pi)^{d-1}}\, (2\pi)^d \delta^{(d)}(p_a+p_b-q)\, \delta(q^2-Q^2)\, \delta\left[y - \frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)\,(p_a\cdot q + p_b\cdot q)}\right]$$

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_{\tau}^{1} \frac{dz}{z} \int_{0}^{1} dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$



- NLO real $\quad \Delta^{(1)}(z,y)\big|_{\mathrm{real}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^* g} \sum_{\mathrm{s,c,p}} \left|\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^* g}\right|^2,$

$$\sum_{\mathrm{s,c,p}} \left|\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^* g}\right|^2 = \frac{\alpha_s}{4\pi} 64\pi^2 N_c C_F (1-\epsilon) \frac{\hat{s}^2}{k\cdot p_a\, k\cdot p_b}\left[1 - \frac{2}{\hat{s}}(k\cdot p_a + k\cdot p_b) + \mathcal{O}[(1-z)^2]\right]$$

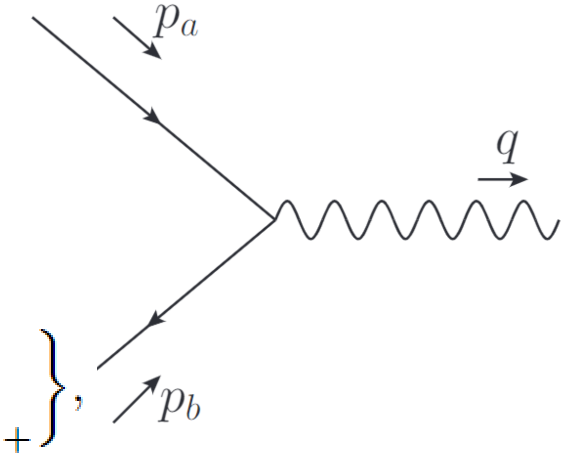
$$\Delta^{(1)}(z,y)\big|^{\mathrm{LP}}_{\mathrm{real}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} (1-z)^{-1-2\epsilon} y^{-1-\epsilon} (1-y)^{-1-\epsilon} [1-\epsilon(1-z)]\,[4 - 2\zeta_2 \mathrm{eps}^2 + \mathcal{O}(\epsilon^3)],$$

$$\Delta^{(1)}(z,y)\big|^{\mathrm{NLP}}_{\mathrm{real}} = -\frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} (1-z)^{-2\epsilon} y^{-1-\epsilon} (1-y)^{-1-\epsilon} [4 - 2\zeta_2 \epsilon^2 + \mathcal{O}(\epsilon^3)].$$

$$\xi^{-1+a\epsilon} = \frac{\delta(\xi)}{a\epsilon} + \left(\frac{1}{\xi}\right)_{+} + a\epsilon\left(\frac{\log\xi}{\xi}\right)_{+} + \frac{(a\epsilon)^2}{2!}\left(\frac{\log^2\xi}{\xi}\right)_{+} + \mathcal{O}(\epsilon^3),$$

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_{\tau}^{1} \frac{dz}{z} \int_{0}^{1} d\,y\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$



- NLO

$$\Delta^{(1)}(z,y)\big|^{\mathrm{LP}}_{\mathrm{ren}} = \frac{\alpha_s C_F}{4\pi} \Big\{ (6\zeta_2 - 8)\big[\delta(y) + \delta(1-y)\big]\delta(1-z) + \big[\delta(y)+\delta(1-y)\big]\Big(8\frac{\log(1-z)}{1-z}\Big|_+ + 4\Big) + \frac{4}{y(1-y)}\Big|_+ \frac{1}{1-z}\Big|_+ \Big\},$$

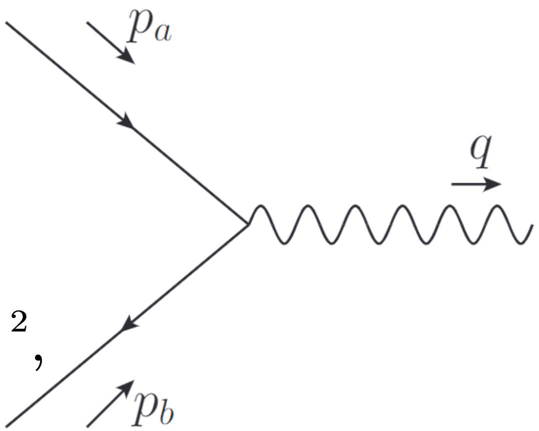
$$\Delta^{(1)}(z,y)\big|^{\mathrm{NLP}}_{\mathrm{ren}} = \frac{\alpha_s C_F}{4\pi} \Big\{ \big[\delta(y) + \delta(1-y)\big]\Big(-8\log(1-z)\Big) - \frac{4}{y(1-y)}\Big|_+ \Big\}.$$

$$\mathcal{L}(z,y) = f_{q/A}\left(e^{Y}\sqrt{\frac{\tau}{z}\,\frac{1-(1-y)\,(1-z)}{1-y\,(1-z)}}\right) f_{\bar{q}/B}\left(e^{-Y}\sqrt{\frac{\tau}{z}\,\frac{1-y\,(1-z)}{1-(1-y)\,(1-z)}}\right) + (q \leftrightarrow \bar{q}).$$

$$\Delta(z,y) = \frac{\delta(y)+\delta(1-y)}{2}\left[\Delta_{\mathrm{LP}}(z) + \Delta_{\mathrm{NLP,LLs}}(z)\right] + \Delta_{\mathrm{NLP,rest}}(z,y) + \mathcal{O}(1-z).$$

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\, dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$



- Shift kinematics

$$\sum_{\mathrm{s,c,p}} |\mathcal{M}|^2_{\mathrm{NLO,NLP}} = g_s^2\, C_F\, \frac{\hat{s}}{(p_a\cdot k)\,(p_b\cdot k)} \sum_{\mathrm{s,c,p}} \left|\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}(p_a+\delta p_a, p_b+\delta p_b)\right|^2,$$

$$\delta p_a^\mu = -\frac{1}{2}\left(\frac{k\cdot p_b}{p_a\cdot p_b}\,p_a^\mu - \frac{k\cdot p_a}{p_a\cdot p_b}\,p_b^\mu + k^\mu\right), \quad \delta p_b^\mu = -\frac{1}{2}\left(\frac{k\cdot p_a}{p_a\cdot p_b}\,p_b^\mu - \frac{k\cdot p_b}{p_a\cdot p_b}\,p_a^\mu + k^\mu\right),$$

V. Duca, *et.al.*, arXiv:1706.04018

$$\sum_{\mathrm{s,c,p}} |\mathcal{M}|^2_{\mathrm{NLO,NLP}} = \frac{\alpha_s}{4\pi} 64\pi^2 N_c C_F (1-\epsilon)\, \frac{\hat{s}^2}{k\cdot p_a\, k\cdot p_b} [1 - \boxed{(1-z)}].$$

$$\sum_{\mathrm{s,c,p}} \left|\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^* g}\right|^2 = \frac{\alpha_s}{4\pi} 64\pi^2 N_c C_F (1-\epsilon)\, \frac{\hat{s}^2}{k\cdot p_a k\cdot p_b} \left[1 - \boxed{\frac{2}{\hat{s}}(k\cdot p_a + k\cdot p_b)} + \mathcal{O}[(1-z)^2]\right]$$

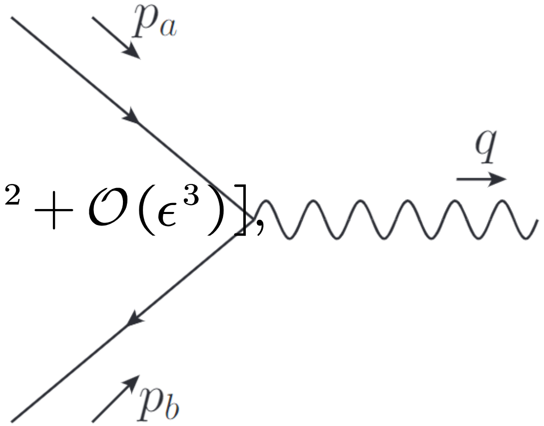
$$(k\cdot p_a + k\cdot p_b) = \frac{(\hat{s}-Q^2)}{2} = \frac{\hat{s}(1-z)}{2}$$

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 d\,y\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$

$$\Delta^{(1)}(z,y)|^{\rm LP}_{\rm real} = \frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} (1-z)^{-1-2\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon}[1-\epsilon(1-z)]\,[4-2\zeta_2 \mathrm{eps}^2+\mathcal{O}(\epsilon^3)],$$

$$\Delta^{(1)}(z,y)|^{\rm NLP}_{\rm real} = -\frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} (1-z)^{-2\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon}[4-2\zeta_2\epsilon^2+\mathcal{O}(\epsilon^3)].$$



- LP+NLP, LL @ NLO

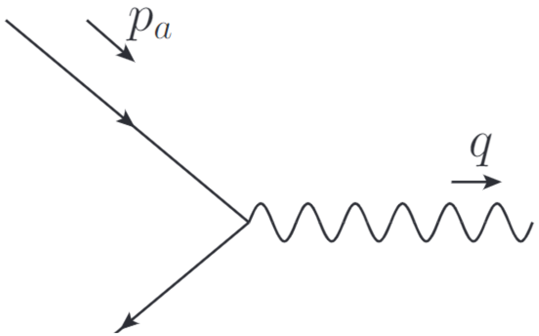
$$\begin{aligned}\Delta^{(1)}(z,y)|^{\rm LP}_{\rm real} &= \frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} (1-z)^{-1-2\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon}[1-\ (1-z)]\,[4+\mathcal{O}(\epsilon)]\\ &= \left[\frac{\delta(y)+\delta(1-y)}{2}+\mathcal{O}(\epsilon)\right] K_{\rm NLP}(z,\epsilon)\end{aligned}$$

$$K_{\rm NLP}(z,\epsilon) = \frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon} z(1-z)^{-1-2\epsilon}\,\frac{e^{\epsilon\gamma_E}\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)\Gamma(1-\epsilon)},$$

N. Bahjat-Abbas,, *et.al.*, arXiv:1905.13710

Rapidity distribution for fixed-order Drell-Yan

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y),$$



- NNLO

$$\Delta^{(2)}(z,y)|_{2\mathrm{v}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*} \sum_{\mathrm{s,c,p}} \Big\{ 2\,\mathrm{Re}\Big[\mathcal{M}^{(0)*}_{q\bar{q}\to\gamma^*}\mathcal{M}^{(4)}_{q\bar{q}\to\gamma^*}\Big] + \Big|\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*}\Big|^2 \Big\},$$

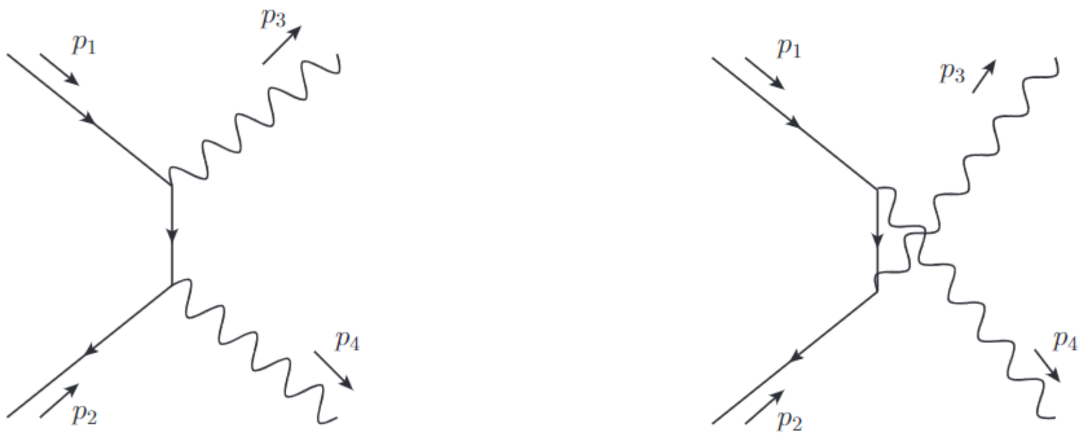
$$\Delta^{(2)}(z,y)|_{1\mathrm{v}1\mathrm{r}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^* g} \sum_{\mathrm{s,c,p}} 2\,\mathrm{Re}\Big[\mathcal{M}^{(1)*}_{q\bar{q}\to\gamma^* g}\mathcal{M}^{(3)}_{q\bar{q}\to\gamma^* g}\Big],$$

$$\Delta^{(2)}(z,y)|_{2\mathrm{r}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*(gg+q\bar{q})} \sum_{\mathrm{s,c,p}} \Big|\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*(gg+q\bar{q})}\Big|^2.$$

$$\Delta(z,y) = \frac{\delta(y)+\delta(1-y)}{2}\left[\Delta_{\mathrm{LP}}(z) + \Delta_{\mathrm{NLP,LLs}}(z)\right] + \Delta_{\mathrm{NLP,rest}}(z,y) + \mathcal{O}(1-z).$$

Rapidity distribution for fixed-order diphoton

$$q(p_1) + \overline{q}(p_2) \to \gamma(p_3) + \gamma(p_4)$$



$$\frac{d\sigma}{dQ^2 dY} = \frac{1}{s}\int_{\tau}^{1}\frac{dz}{z}\int_{\log\left(\sqrt{\frac{\tau}{z}}e^Y\right)}^{\log\left(\sqrt{\frac{z}{\tau}}e^Y\right)} d\eta\, \mathcal{L}(z,\eta)\,\frac{d\hat{\sigma}}{dzd\eta}(\hat{s},z,\eta)$$

$$\eta = \frac{1}{2}\log\left(\frac{p_2\cdot p_3}{p_1\cdot p_3}\right), \quad Y = \eta + \frac{1}{2}\log\left(\frac{x_a}{x_b}\right)$$

- LO

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(\hat{s},z,\eta,\epsilon,\bar{\mu}^2) = \frac{d\overline{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(\hat{s},\eta,\epsilon,\bar{\mu}^2)\,\delta(1-z)\,,$$

$$\Downarrow\ d\to 4$$

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(\hat{s},z,\eta) = \frac{\pi\alpha^2}{N_c\,\hat{s}}\,(1+\tanh^2\eta)\,\delta(1-z)\,.$$

$$\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(\hat{s},\eta,\epsilon,\bar{\mu}^2) = \frac{\pi\alpha^2}{N_c\,\hat{s}}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon}(4\cosh^2\eta)^{\epsilon} \times\left[(1+\tanh^2\eta)(1-\epsilon)^2 - \frac{\epsilon(1-\epsilon)}{\cosh^2\eta}\right].$$

Rapidity distribution for fixed-order diphoton

$$q(p_1) + \overline{q}(p_2) \to \gamma(p_3) + \gamma(p_4) + g(k)$$

$$s_{ij} = (\sigma_i p_i + \sigma_j p_j)^2, \qquad i,j = 1, \cdots, 4,$$
$$s_i = (\sigma_i p_i - k)^2, \qquad i = 1, \cdots, 4,$$
$$p_1 + p_2 - p_3 - p_4 - k = 0,$$
$$p_4^2 = (p_1 + p_2 - p_3 - k)^2 = 0$$

- NLO, direct calculation

$$\overline{|\mathcal{M}|^2_{\text{real}}}(s_{12}, s_{13}, s_{23}, s_1, s_2) = \sum_l c_l(\epsilon) s_{12}^{\alpha_l} s_{13}^{\beta_l} s_{23}^{\gamma_l} s_1^{\kappa_l} s_2^{-1-\alpha_l-\beta_l-\gamma_l-\kappa_l}.$$

$$\int dR_3 (k^+)^\alpha (k^-)^\beta (p_3^+)^\gamma (p_3^-)^\kappa = \bar{\mu}^{4\epsilon} \int dz d\eta \frac{2^{-6-\frac{1}{2}(\alpha+\beta+\gamma+\kappa)} z^{1+\gamma+\kappa-2\epsilon}}{\pi^3 \hat{s}^{-1+2\epsilon-\frac{1}{2}(\alpha+\beta+\gamma+\kappa)}}$$
$$\times \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (1-z)^{1+\alpha+\beta-2\epsilon+2j+n} (e^{2\eta})^{1+\gamma-\epsilon+j} (e^{2\eta}-1)^n (e^{2\eta}+z)^{-2-\gamma-\kappa+2\epsilon-2j-n}$$
$$\times \frac{e^{2\epsilon\gamma_E} \Gamma(1+\alpha-\epsilon+j) \Gamma(1+\beta-\epsilon+j+n) \Gamma(2+\gamma+\kappa-2\epsilon+2j+n)}{j! n! \Gamma(1-\epsilon) \Gamma(2+\gamma+\kappa-2\epsilon) \Gamma(1-\epsilon+j) \Gamma(2+\alpha+\beta-2\epsilon+2j+n)}.$$

Rapidity distribution for fixed-order diphoton

$$q\,(p_1)+\overline{q}\,(p_2)\to\gamma\,(p_3)+\gamma\,(p_4)+g\,(k)$$

- NLO, shift kinematics

$$\sum_{\mathrm{s,c,p}}\,|\mathcal{M}|^2_{\mathrm{NLO,NLP}}=g_s^2\,C_F\,\frac{\hat{s}}{(p_1\cdot k)\,(p_2\cdot k)}\sum_{\mathrm{s,c,p}}\Big|\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}(p_1+\delta p_1,p_2+\delta p_2)\Big|^2,$$

$$\delta p_1^\mu=-\frac{1}{2}\left(\frac{k\cdot p_2}{p_1\cdot p_2}p_1^\mu-\frac{k\cdot p_1}{p_1\cdot p_2}p_2^\mu+k^\mu\right),\quad\delta p_2^\mu=-\frac{1}{2}\left(\frac{k\cdot p_1}{p_1\cdot p_2}p_2^\mu-\frac{k\cdot p_2}{p_1\cdot p_2}p_1^\mu+k^\mu\right),$$

V. Duca, *et.al.*, arXiv:1706.04018

$$\begin{aligned}\eta&=\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)=\frac{1}{2}\log\left(\frac{(p_1+p_2)^+-p_4^+}{(p_1+p_2)^--p_4^-}\right)\\&\to\frac{1}{2}\log\left(\frac{(p_1+\delta p_1+p_2+\delta p_2)^+-p_4^+}{(p_1+\delta p_1+p_2+\delta p_2)^--p_4^-}\right)\\&=\frac{1}{2}\log\left(\frac{(p_1+p_2)^+-p_4^+-k^+}{(p_1+p_2)^--p_4^--k^-}\right)\end{aligned}$$

Using mometum conservation:

$$\eta=\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)=\frac{1}{2}\log\left(\frac{(p_1+p_2)^+-p_4^+-k^+}{(p_1+p_2)^--p_4^--k^-}\right)$$

Rapidity distribution for fixed-order diphoton

$q(p_1) + \overline{q}\,(p_2) \rightarrow \gamma(p_3) + \gamma(p_4) + g(k)$

- NLO, shift of the rapidity

$$\eta = \frac{1}{2}\log\left(\frac{(p_1+p_2-p_4)^+}{(p_1+p_2-p_4)^-}\right) + \frac{1}{2}\left(\frac{k^-}{p_3^-} - \frac{k^+}{p_3^+}\right) + \mathcal{O}(k^2)$$
$$\equiv \overline{\eta} + \delta\eta + \mathcal{O}((1-z)^2),$$

$$\delta\left(\eta - \frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right) = \delta(\eta - \overline{\eta}) + \delta\eta\frac{\partial}{\partial\eta}\delta(\eta-\overline{\eta}) + \mathcal{O}(\delta\eta^2).$$

$$\frac{d\hat{\sigma}^{(1)}_{q\overline{q}}(Q^2, z, \eta, \epsilon, \overline{\mu}^2)}{dzd\eta}\Bigg|^{\text{NLP}}_{\text{real}} = \frac{\alpha_s}{4\pi}C_F\frac{4\pi\alpha^2}{N_c z\hat{s}}\left(\frac{\overline{\mu}^2}{\text{s}}\right)^{\epsilon}\left(\frac{\overline{\mu}^2}{z\hat{s}}\right)^{\epsilon}\frac{2e^{2\epsilon\gamma_E}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}$$
$$\times\,(1+\tanh^2\eta)\,z(1-z)^{-1-2\epsilon}$$
$$= K_{\text{NLP}}(z,\epsilon)\frac{d\overline{\sigma}^{(0)}_{q\overline{q}}}{dzd\eta}(z\hat{s},\eta),$$

Rapidity distribution for fixed-order diphoton

$q(p_1) + \overline{q}\,(p_2) \rightarrow \gamma(p_3) + \gamma(p_4) + g(k)$

- NLO, beyond NLP

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta)}{dz d\eta}\Bigg|^{\mathrm{N^3LP}}_{\mathrm{ren}} = \frac{\alpha_s C_F}{\pi}\frac{\pi\alpha^2}{N_c Q^2}\Bigg\{4\left(1+\tanh^2\eta\right)\frac{\log(1-z)}{1-z}\Bigg|_+ + 2\left(1+\tanh^2\eta\right)\left[1-2\log(1-z)\right] + \frac{1-z}{8\cosh^4\eta}\left[4\left(5+\cosh 4\eta\right)\log(1-z) - 14 + 23\cosh 2\eta + 2\cosh 4\eta + \cosh 6\eta\right] + \frac{(1-z)^2}{6\cosh^4\eta}\left[7-\cosh 2\eta + \cosh 4\eta\right] + \mathcal{O}\left[(1-z)^3\right]\Bigg\}.$$

Resummation for rapidity distribution

- Diphoton:

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon) = K_{\mathrm{NLP}}(z,\epsilon)\,\frac{d\overline{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon)$$
$$= z\mathcal{S}_{\mathrm{LP}}(z,\epsilon)\,\frac{d\overline{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon).$$

$$\int d\,z\,z^{N-1}\,\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon) = \mathcal{S}_{\mathrm{LP}}(N+1)\,\frac{d\overline{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon).$$

$$\mathcal{S}_{\mathrm{LP}}(N) = \left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{2\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon}\left(\log N - \frac{1}{2N}\right) + \log^2 N - \frac{\log N}{N}\right].$$

$$\int d\,z\,z^{N-1}\,\frac{d\hat{\sigma}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon) = \frac{\pi\alpha^2 C_F}{N_c Q^2}(1+\tanh^2\eta)\exp\left[\frac{2\alpha_s C_F}{\pi}\frac{1}{\epsilon}\log N\right]$$
$$\times\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{1}{\epsilon}\frac{1}{2N}+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right).$$

Resummation for rapidity distribution

- Diphoton:

$$\int_0^1 dz\, z^{N-1} \frac{d\hat{\sigma}_{q\bar{q}}}{dz d\eta} = \frac{\pi\alpha^2}{N_c Q^2}(1+\tanh^2\eta)\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right)$$

- Drell-Yan

$$\int_0^1 dz\, z^{N-1} \frac{d\hat{\sigma}_{q\bar{q}}}{dz d\eta}(Q^2,\eta) = \frac{\hat{\sigma}_0(Q^2)}{2\cosh^2\eta}\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right).$$

$$\int dz\, z^{N-1} \frac{d\hat{\sigma}_{q\bar{q}}}{dz dy}(z,y) = \hat{\sigma}_0\left(\frac{\delta(y)+\delta(1-y)}{2}\right)\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right).$$

Summary and outlook

- NNLO NLP fixed order calculation for Drell-Yan process differential in both invariant mass and rapidity variables.
- NLO including all powers in (1-z) calculation for diphoton production process differential in both invariant mass and rapidity variables.
- Factorization of rapidity variable and threshold variable at NLP LL.
- NLP LL resummation for both processes.

- Resummation beyond LL at NLP.
- Including off-diagonal channels.

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Thanks for your attention!