Witten diagrams in AdS/QCD with applications to scattering amplitudes and GPD moment parametrization

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AdS/CFT correspondence can be used to compute correlation functions of local operators

\[ Z_{gauge}(J_0, N_c, \lambda) \equiv Z_{gravity}(\phi_0, g_5, \alpha'/R^2) \]

with \( J \equiv \phi_0 \)

- the computation of the correlation functions is done using Witten diagrams in AdS
- for non-conformal theories with mass gap dual to some gravity theory in deformed AdS background, we can use Witten diagrams to compute scattering amplitudes
The Dirac and Pauli form factors of proton are defined by

\[ \langle p_2 | J_{em}^\mu(0) | p_1 \rangle = \bar{u}(p_2) \left( F_1(q^2) \gamma^\mu + F_2(q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2m_N} \right) u(p_1) \]

In holographic QCD they can be computed using the Witten diagram in AdS

**Figure:** Witten diagram for spin-1 (electromagnetic) form factors of proton \( \langle p_2 | J_{em}^\mu(0) | p_1 \rangle \) due to the exchange of spin-1 (vector) meson resonances.
Spin-1 (Electromagnetic) Form Factors of Proton

- the bulk-to-boundary propagator for virtual photon

\[ \mathcal{V}(Q, z) = g_5 \sum_n F_n \phi_n(z) = \Gamma \left( 1 + \frac{Q^2}{4\kappa_V^2} \right) \kappa_V^2 z^2 \mathcal{U} \left( 1 + \frac{Q^2}{4\kappa_V^2}; 2; \kappa_V^2 z^2 \right) \]

- the scattering amplitude is given by

\[
S_{Dirac}^{EM}[i, f] = \frac{1}{2g_5^2} \int \frac{dz d^4 y}{\sqrt{g}} \frac{e^{-\phi} z}{R} \times \left( \bar{\psi}_1 f \gamma^N \left( \frac{1}{3} V^0_N T^0 + V^3_N T^3 \right) \psi_1 i + \bar{\psi}_2 f \gamma^N \left( \frac{1}{3} V^0_N T^0 + V^3_N T^3 \right) \psi_2 i \right) \\
= (2\pi)^4 \delta^4(p' - p - q) F_N(p') F_N(p) \times \frac{1}{2g_5^2} \times 2g_5^2 \times e_N \times \bar{u}_s f (p') \epsilon(q) u_s i (p) \times \frac{1}{2} \int \frac{dz}{z^{2M}} e^{-\phi} \mathcal{V}(Q, z) \left( \psi_L^2(z) + \psi_R^2(z) \right)
\]
the gravitational form factors of proton defined as
\[
\langle p_2 | T_{\mu\nu}(0) | p_1 \rangle = \bar{u}(p_2) \left( A(k) \gamma^{(\mu} p^{\nu)} + B(k) \frac{ip^{(\mu} \sigma^{\nu)} k_\alpha}{2m_N} + C(k) \frac{k^\mu k^\nu - \eta^{\mu\nu} k^2}{m_N} \right) u(p_1)
\]
can also be computed by the corresponding Witten diagrams:
Figure: Witten diagram for the spin-2 gravitational form factor due to the exchange of spin-2 glueball resonances.
Spin-2 (Gravitational) Form Factor of Proton

Figure: Witten diagram for the spin-0 gravitational form factor $\langle p_2 | T_{\mu}^{\mu}(0) | p_1 \rangle$ due to the exchange of spin-0 glueball resonances.
the gravitational form factors of proton follow from the coupling of the irreducible representations of the metric fluctuations $h_{\mu\nu}$ to a bulk Dirac fermion

and the bulk metric fluctuations can be decomposed in terms of the $2 \oplus 0$ invariant tensors [Kanitscheider:2008]

$$h_{\mu\nu}(k, z) \sim \left[ \epsilon^{TT}_{\mu\nu} h(k, z) + \frac{1}{3} \eta_{\mu\nu} f(k, z) \right]$$

which is the spin-2 made of the transverse-traceless part $h$ plus the spin-0 tracefull part $f$
the scattering amplitudes are given by

\[ \epsilon^{TT}_{\alpha\beta} \gamma_{h\bar{\psi}\psi}(p_1, p_2, k_z) = - \frac{1}{2} \int dz \sqrt{g} e^{-\phi(z)} \bar{\psi}(p_2, z) \epsilon^{TT}_{\alpha\beta} \gamma^\alpha p^\beta \psi(p_1, z) \mathcal{H}(K, z) \]

and

\[ \frac{1}{3} \eta_{\alpha\beta} \gamma_{f\bar{\psi}\psi}(p_1, p_2, k_z) = - \frac{1}{2} \int dz \sqrt{g} e^{-\phi(z)} \bar{\psi}(p_2, z) \eta_{\alpha\beta} \gamma^\alpha p^\beta \psi(p_1, z) \mathcal{F}(K, z) \]
the spin-j form factors of proton can be defined as

\[
\langle p_2 | T^{\mu_1 \mu_2 \ldots \mu_j} (0) | p_1 \rangle = \bar{u}(p_2) \left( A(k, j) \gamma^{\mu_1} p^{\mu_2} \ldots p^{\mu_j} + B(k, j) \frac{ip^{\mu_1} p^{\mu_3} \ldots p^{\mu_j} \sigma^{\mu_2 \alpha} k_\alpha}{2m_N} \right. \\
+ \left. C(k, j) \frac{k^{\mu_1} k^{\mu_2} \ldots k^{\mu_j} - \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \ldots \eta^{\mu_{j-1} \mu_j} k^2}{m_N} \right) u(p_1)
\]
Spin-j Form Factor of Proton

- the spin-j form factor, can be computed using the Witten diagram

Figure: Witten diagram for the spin-j form factor due to the exchange of spin-j glueball resonances.
Four-point correlation functions $\langle \bar{P} J_\mu J_\nu P \rangle$

electroproduction of a vector meson probing the \textbf{gluon GPD}: (a) leading QCD contribution in the Regge limit; (b) leading Witten diagram in the large-$N_c$ limit.
the scattering amplitude is given by

\[ A^h_{\gamma_{L/T}p \to \nu p(j, s, t)} \sim - \frac{1}{g^5} \times 2\kappa^2 \times \mathcal{V}_{h\gamma_{L/T}^*}^{\mu\nu} v(j, Q, M_V) \times \frac{1}{m_N} \times \bar{u}(p_2)u(p_1) \]

\[ \times \sum_{n=0}^{\infty} [q^{\mu_1} q^{\mu_2} ... q^{\mu_j} P_{\mu_1 \mu_2 ... \mu_j; \nu_1 \nu_2 ... \nu_j} (m_n, k) p_1^{\nu_1} p_1^{\nu_2} ... p_1^{\nu_j}] \]

\[ \times \mathcal{V}_{h\bar{\psi}\psi} (j, K, m_n) \]
we use the identity

\[ q^{\mu_1} q^{\mu_2} \cdots q^{\mu_j} P_{\mu_1 \mu_2 \cdots \mu_j; \nu_1 \nu_2 \cdots \nu_j}(m_n, k) p_1^{\nu_1} p_1^{\nu_2} \cdots p_1^{\nu_j} = (p_1 \cdot q)^j \times \hat{d}_j(\eta, m_n^2) \]

with

\[ \hat{d}_j(\eta, m_n^2) = 2 F_1 \left( -\frac{j}{2}, \frac{1 - j}{2}, \frac{1}{2} - j; \frac{4m_n^2}{-m_n^2} \times \eta^2 \right) \]

where the skewness parameter is given by \( \eta \sim \frac{k \cdot q}{2 p_1 \cdot q} \). Also note that, for \( j = 2 \), we have the massive spin-2 projection operator \( P_{\mu_1 \mu_2; \nu_1 \nu_2}(m_n, k) \equiv P_{\mu \nu; \alpha \beta}(m_n, k) \) defined as

\[ P_{\mu \nu; \alpha \beta}(m_n, k) = \frac{1}{2} \left( P_{\mu; \alpha} P_{\nu; \beta} + P_{\mu; \beta} P_{\nu; \alpha} - \frac{2}{3} P_{\mu; \nu} P_{\alpha; \beta} \right) \]

which is written in terms of the massive spin-1 projection operator

\[ P_{\mu; \alpha}(m_n, k) = \eta_{\mu \alpha} - k_{\mu} k_{\alpha} / m_n^2 \]
Four-point correlation functions $< \bar{P} J_{\mu} J_{\nu} P >$

- separating the skewness $\eta$ dependent and independent part, we can rewrite the scattering amplitude as

$$A_{h \gamma_{L/T} \to V_{p}}^h(j, s, t) \sim -\frac{1}{g_5} \times 2\kappa^2 \times \mathcal{V}_{\mu\nu}^{\gamma_1} \nu(j, Q, M_V) \times (p_1 \cdot q)^j \times \frac{1}{m_N} \times \bar{u}(p_2)u(p_1) \times \left[ A(j, t) + D_\eta(j, \eta, t) \right]$$

where

$$D_\eta(j, \eta, t) = \left( \hat{d}_j(\eta, t) - 1 \right) \times \left[ A(j, t) - A_S(j, t) \right]$$

with

$$A_S(j, t) \equiv A(j, \kappa_T \to \kappa_S)$$
Holographic parametrization of conformal moments of gluon GPDs

- our holographic parametrization of conformal moments of gluon GPD at finite skewness, and at input scale $\mu = \mu_0$, is given by

$$F_g^{(+)}(j, \eta, t; \mu_0) = A_g(j, t; \mu_0) + D_{g\eta}(j, \eta, t; \mu_0)$$

for even $j = 2, 4, ...$

- we use

$$A_g(j, t; \mu_0) = \int_0^1 dx \, x^{j-2} \, xg(x; \mu_0) \, x^{a_T}$$

with the input gluon PDF $xg(x; \mu_0)$ at $\mu = \mu_0$, and $a_T \equiv -\alpha'_T t$
the skewness or $\eta$-dependent terms are fixed by holography as

$$D_{g\eta}(j, \eta, t; \mu_0) = \left(\hat{d}_j(\eta, t) - 1\right) \times \left[ A_g(j, t; \mu_0) - A_{gs}(j, t; \mu_0) \right]$$

where

$$A_{gs}(j, t; \mu_0) \equiv A_g(j, t; \mu_0, \alpha'_T \rightarrow \alpha'_S)$$

the A-form factor of the gluon gravitational form factor of the proton is given by

$$A_g(t; \mu_0) = A_g(j = 2, t; \mu_0)$$

and the D-form factor (or the D-term) of the gluon gravitational form factor of the proton is

$$\eta^2 D_g(t; \mu_0) = D_{g\eta}(j = 2, \eta, t; \mu_0)$$
the holographic parametrization of the conformal moments of the non-singlet (valence) quark GPDs at $\mu = \mu_0$ is given by

$$F_q^{(-)}(j, \eta, t; \mu_0) = F_q^{(-)}(j, t; \mu_0) + F_q^{(-)}(j, \eta, t; \mu_0)$$

for odd $j = 1, 3, \cdots$, where we have defined

$$F_q^{(-)}(j, t; \mu_0) = \int_0^1 dx x^{j-1} q_v(x; \mu_0) x^{a_q}$$

with the input valence (non-singlet) quark PDF

$$q_v(x; \mu_0) = q(x; \mu_0) - \bar{q}(x; \mu_0) = H_q^{(-)}(x, \eta = 0, t = 0; \mu_0) \text{ at } \mu = \mu_0, \text{ and } a_q \equiv -\alpha_q' t,$$

where $\alpha_q'$ is a Regge slope parameter.
Holographic parametrization of conformal moments of non-singlet quark GPDs

- The skewness or $\eta$-dependent $\mathcal{F}_{q\eta}^{(-)}$-terms are given by

$$
\mathcal{F}_{q\eta}^{(-)}(j, t; \mu_0) = \left( \hat{d}_{j-1}(\eta, t) - 1 \right) \times \left[ \mathcal{F}_q^{(-)}(j, t; \mu_0) - \mathcal{F}_{qS}^{(-)}(j, t; \mu_0) \right]
$$

where

$$
\mathcal{F}_{qS}^{(-)}(j, t; \mu_0) \equiv \mathcal{F}_q^{(-)}(j, t; \mu_0, \alpha'_q \rightarrow \alpha'_{qs})
$$

- The Dirac electromagnetic form factor of the proton is given by

$$
F_1(t) = \sum_{q=1}^{2} e_q \mathcal{F}_q^{(-)}(j = 1, t; \mu_0)
$$
the holographic parametrization of the conformal moments of the singlet (sea) quark GPDs at $\mu = \mu_0$ are given by

$$\sum_{q=1}^{N_f} \mathcal{F}^{(+)}_q(j, \eta, t; \mu_0) = \sum_{q=1}^{N_f} \mathcal{F}^{(+)}_q(j, t; \mu_0) + \mathcal{F}^{(+)}_{q\eta}(j, \eta, t; \mu_0)$$

for even $j = 2, 4, \cdots$, where we have defined

$$\sum_{q=1}^{N_f} \mathcal{F}^{(+)}_q(j, t; \mu_0) = \int_0^1 dx x^{j-1} \sum_{q=1}^{N_f} q^{(+)}(x; \mu_0) x^{a_q}$$

with the input singlet quark PDF

$$\sum_{q=1}^{N_f} q^{(+)}(x; \mu_0) = \sum_{q=1}^{N_f} q(x; \mu_0) + \bar{q}(x; \mu_0) = \sum_{q=1}^{N_f} H^{(+)}_q(x, \eta = 0, t = 0; \mu_0)$$

at $\mu = \mu_0$, and $a_q \equiv -\alpha'_q t$. 

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the skewness or $\eta$-dependent terms are given by

$$\sum_{q=1}^{N_f} F_{q\eta}^{(+)}(j, \eta, t; \mu_0) = \left( \hat{d}_j(\eta, t) - 1 \right) \times \left[ \sum_{q=1}^{N_f} F_{q}^{(+)}(j, t; \mu_0) - F_{qS}^{(+)}(j, t; \mu_0) \right]$$

where

$$\sum_{q=1}^{N_f} F_{qS}^{(+)}(j, t; \mu_0) \equiv \sum_{q=1}^{N_f} F_{q}^{(+)}(j, \mu_0, \alpha'_q \rightarrow \alpha'_{qs})$$
the A-form factor of the quark gravitational form factor of proton is given by

$$\sum_{q=1}^{N_f} A_q (t; \mu_0) = \sum_{q=1}^{N_f} F_q^{(+)} (j = 2, t; \mu_0)$$

and the D-form factor (or the D-term) of the quark gravitational form factor of the proton is given by

$$\eta^2 \sum_{q=1}^{N_f} D_q (t; \mu_0) = \sum_{q=1}^{N_f} F_{q\eta}^{(+)} (j = 2, \eta, t; \mu_0)$$
Comparison of the conformal moments to lattice QCD

Figure: Our evolved moments of $u-d$ quark GPD $H^{u-d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
Comparison of the conformal moments to lattice QCD

Figure: Our evolved moments of $u - d$ quark GPD $H_{u-d}^{x;\eta,t;\mu}$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
Figure: Our evolved moments of $u-d$ quark GPD $H^{u-d}(x, \eta, t; \mu)$ at $\mu = 2 \text{ GeV}$, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
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**Figure:** Our evolved moments of $u + d$ quark GPD $H^{u+d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
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Comparison of the conformal moments to lattice QCD

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1a.png}
\caption{(a)}
\end{subfigure} \hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1b.png}
\caption{(b)}
\end{subfigure}
\end{figure}

Figure: Our evolved moments of $u + d$ quark GPD $H^{u+d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
Comparison of the conformal moments to lattice QCD

Figure: Our evolved moments of gluon GPD $H^g_{symmetric}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.
Reconstructing quark and gluon GPDs from their conformal moments

- the gluon GPD series expansion in terms of their conformal (Gegenbauer) moments is extended to a Mellin-Barnes-type integral to facilitate the incorporation of complex-valued conformal spins [Mueller:2005]:

\[
H_g(x, \eta, t; \mu) = \frac{1}{2i} \int_{\mathbb{C}} dj \frac{(-1)}{\sin(\pi j)} g_{p_j}(x, \eta) \mathbb{F}_g(j, \eta, t; \mu^2)
\]

with

\[
g_{p_j}(x, \eta) = \theta(\eta - |x|) \frac{1}{\eta^{j-1}} g_{P_j} \left( \frac{x}{\eta} \right) + \theta(x - \eta) \frac{1}{x^{j-1}} g_{Q_j} \left( \frac{x}{\eta} \right)
\]

where the functions \( g_{P_j} \) and \( g_{Q_j} \) are given by

\[
g_{P_j} \left( \frac{x}{\eta} \right) = \left( 1 + \frac{x}{\eta} \right)^2 _2 F_1 \left( -j, j + 1, 3 \left| \frac{1}{2} \left( 1 + \frac{x}{\eta} \right) \right. \right) \frac{2^{j-1} \Gamma(3/2 + j)}{\Gamma(1/2) \Gamma(j - 1)}
\]

\[
g_{Q_j} \left( \frac{x}{\eta} \right) = _2 F_1 \left( \frac{j - 1}{2}, \frac{j}{2} + j; \frac{3}{2} + j; \frac{\eta^2}{x^2} \right) \frac{\sin(\pi [j + 1])}{\pi}
\]
for quark GPDs we have

\[ H_q(x, \eta, t; \mu) = \frac{1}{2i} \int_{\mathbb{C}} dj \frac{1}{\sin(\pi j)} p_j(x, \eta) F_q(j, \eta, t; \mu) \]

with

\[ p_j(x, \eta) = \theta(\eta - |x|) \frac{1}{\eta} \mathcal{P}_j \left( \frac{x}{\eta} \right) + \theta(x - \eta) \frac{1}{x} \mathcal{Q}_j \left( \frac{x}{\eta} \right) \]

where

\[ \mathcal{P}_j \left( \frac{x}{\eta} \right) = \left( 1 + \frac{x}{\eta} \right) {}_2F_1 \left( -j, j + 1, 2 \left| \frac{1}{2} \left( 1 + \frac{x}{\eta} \right) \right. \right) \times \frac{2^j \Gamma(3/2 + j)}{\Gamma(1/2) \Gamma(j)} \]

\[ \mathcal{Q}_j \left( \frac{x}{\eta} \right) = {}_2F_1 \left( \frac{j}{2}, \frac{j + 1}{2}; \frac{3}{2} + j \left| \eta^2 \right| x^2 \right) \times \frac{\sin(\pi j)}{\pi} \]
Comparison of our reconstructed $u - d$ quark GPDs to lattice

**Figure:** Comparison of our reconstructed $u - d$ quark GPD $H_{u-d}^{(-)}(x, \eta, t; \mu)$ with lattice.
Thank You!