# Witten diagrams in AdS/QCD with applications to scattering amplitudes and GPD moment parametrization

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# AdS/CFT

- AdS/CFT correspondence can be used to compute correlation functions of local operators

$$Z_{gauge}(J\mathcal{O}, N_c, \lambda) \equiv Z_{gravity}(\phi_0, g_5, \alpha'/R^2)$$

  with $J \equiv \phi_0$
- the computation of the correlation functions is done using Witten diagrams in AdS
- for non-conformal theories with mass gap dual to some gravity theory in deformed AdS background, we can use Witten diagrams to compute scattering amplitudes

# Spin-1 (Electromagnetic) Form Factors of Proton

- the Dirac and Pauli form factors of proton are defined by

$$\langle p_2 | J^{\mu}_{em}(0) | p_1 \rangle = \overline{u}(p_2) \left( F_1(q^2)\gamma^{\mu} + F_2(q^2) \frac{i\sigma^{\mu\nu} q_{\nu}}{2m_N} \right) u(p_1)$$

- in holographic QCD they can be computed using the Witten diagram in AdS

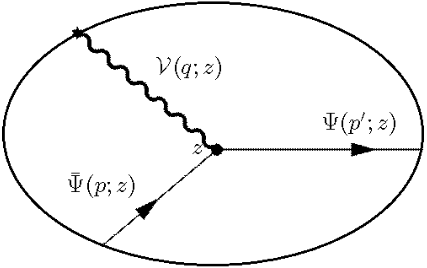



Figure: Witten diagram for spin-1 (electromagnetic) form factors of proton $\langle p_2 | J^{\mu}_{em}(0) | p_1 \rangle$ due to the exchange of spin-1 (vector) meson resonances.

# Spin-1 (Electromagnetic) Form Factors of Proton

- the bulk-to-boundary propagator for virtual photon

$$\mathcal{V}(Q,z) = g_5 \sum_n \frac{F_n \phi_n(z)}{Q^2 + m_n^2} = \Gamma\left(1 + \frac{Q^2}{4\kappa_V^2}\right) \kappa_V^2 z^2 \, \mathcal{U}\left(1 + \frac{Q^2}{4\kappa_V^2}; 2; \kappa_V^2 z^2\right)$$

- the scattering amplitude is given by

$$\begin{aligned} S_{Dirac}^{EM}[i,f] &= \frac{1}{2g_5^2} \int dz d^4y \sqrt{g} e^{-\phi} \frac{z}{R} \\ &\times \left(\bar{\Psi}_{1f} \gamma^N \left(\frac{1}{3} V_N^0 T^0 + V_N^3 T^3\right) \Psi_{1i} + \bar{\Psi}_{2f} \gamma^N \left(\frac{1}{3} V_N^0 T^0 + V_N^3 T^3\right) \Psi_{2i}\right) \\ &= (2\pi)^4 \delta^4(p' - p - q) \, F_N(p') \, F_N(p) \times \frac{1}{2g_5^2} \times 2g_5^2 \times e_N \times \bar{u}_{s_f}(p') \epsilon(q) u_{s_i}(p) \\ &\times \frac{1}{2} \int \frac{dz}{z^{2M}} e^{-\phi} \mathcal{V}(Q,z) \left({\psi_L}^2(z) + {\psi_R}^2(z)\right) \end{aligned}$$

# Spin-2 (Gravitational) Form Factor of Proton

- the gravitational form factors of proton defined as

$$\langle p_2 | T^{\mu\nu}(0) | p_1 \rangle = \overline{u}(p_2) \left( A(k)\gamma^{(\mu} p^{\nu)} + B(k) \frac{ip^{(\mu}\sigma^{\nu)\alpha} k_\alpha}{2m_N} + C(k) \frac{k^\mu k^\nu - \eta^{\mu\nu} k^2}{m_N} \right) u(p_1)$$

can also be computed by the corresponding Witten diagrams:

# Spin-2 (Gravitational) Form Factor of Proton

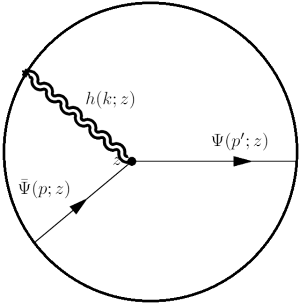



Figure: Witten diagram for the spin-2 gravitational form factor due to the exchange of spin-2 glueball resonances.

# Spin-2 (Gravitational) Form Factor of Proton

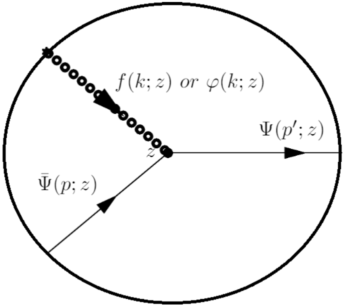



Figure: Witten diagram for the spin-0 gravitational form factor $\langle p_2 | T^{\mu}_{\mu}(0) | p_1 \rangle$ due to the exchange of spin-0 glueball resonances.

# Spin-2 (Gravitational) Form Factor of Proton

- the gravitational form factors of proton follow from the coupling of the irreducible representations of the metric fluctuations $h_{\mu\nu}$ to a bulk Dirac fermion
- and the bulk metric fluctuations can be decomposed in terms of the $2 \oplus 0$ invariant tensors [Kanitscheider:2008]

$$h_{\mu\nu}(k,z) \sim \left[\epsilon^{TT}_{\mu\nu} h(k,z) + \left[\frac{1}{3}\eta_{\mu\nu} f(k,z)\right]\right]$$

which is the spin-2 made of the transverse-traceless part $h$ plus the spin-0 tracefull part $f$

# Spin-2 and Spin-0 (Gravitational) Form Factor of Proton

- the scattering amplitudes are given by

$$\epsilon^{TT}_{\alpha\beta}\mathcal{V}^{\alpha\beta}_{h\bar{\Psi}\Psi}(p_1,p_2,k_z) = -\frac{1}{2}\int dz\,\sqrt{g}\,e^{-\phi}z\,\bar{\Psi}(p_2,z)\epsilon^{TT}_{\alpha\beta}\gamma^{\alpha}p^{\beta}\,\Psi(p_1,z)\mathcal{H}(K,z)$$

and

$$\frac{1}{3}\eta_{\alpha\beta}\mathcal{V}^{\alpha\beta}_{f\bar{\Psi}\Psi}(p_1,p_2,k_z) = -\frac{1}{2}\int dz\,\sqrt{g}\,e^{-\phi}z\,\bar{\Psi}(p_2,z)\eta_{\alpha\beta}\gamma^{\alpha}p^{\beta}\,\Psi(p_1,z)\mathcal{F}(K,z)$$

# Spin-j Form Factor of Proton

- the spin-j form factors of proton can be defined as

$$\langle p_2 | T^{\mu_1\mu_2\ldots\mu_j}(0) | p_1 \rangle = \overline{u}(p_2)\Bigg(\mathcal{A}(k,j)\gamma^{\mu_1}p^{\mu_2}\ldots p^{\mu_j} + \mathcal{B}(k,j)\frac{ip^{\mu_1}p^{\mu_3}\ldots p^{\mu_j}\sigma^{\mu_2\alpha}k_\alpha}{2m_N}$$
$$+\,\mathcal{C}(k,j)\frac{k^{\mu_1}k^{\mu_2}\ldots k^{\mu_j} - \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\ldots\eta^{\mu_{j-1}\mu_j}k^2}{m_N}\Bigg)u(p_1)$$

# Spin-j Form Factor of Proton

- the spin-j form factor, can be computed using the Witten diagram

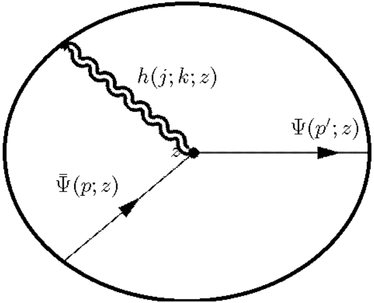
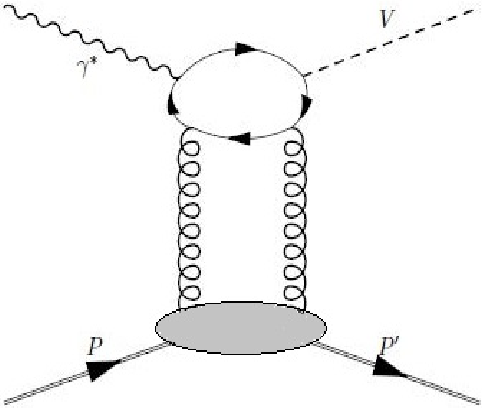



Figure: Witten diagram for the spin-j form factor due to the exchange of spin-j glueball resonances.

# Four-point correlation functions $< \bar{P} J_\mu J_\nu P >$

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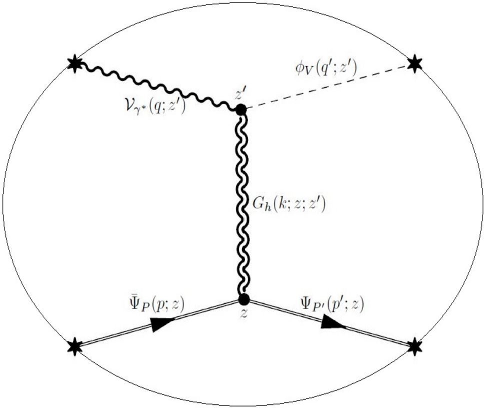


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electroproduction of a vector meson probing the **gluon GPD**: (a) leading QCD contribution in the Regge limit; (b) leading Witten diagram in the large-$N_c$ limit.

# Four-point correlation functions $< \bar{P} J_\mu J_\nu P >$

- the scattering amplitude is given by

$$\begin{aligned}
\mathcal{A}^h_{\gamma^*_{L/T} p \to V p}(j, s, t) \sim & -\frac{1}{g_5} \times 2\kappa^2 \times \mathcal{V}^{\mu\nu}_{h\gamma^*_{L/T} V}(j, Q, M_V) \times \frac{1}{m_N} \times \bar{u}(p_2) u(p_1) \\
& \times \sum_{n=0}^{\infty} \left[ q^{\mu_1} q^{\mu_2} ... q^{\mu_j} \, P_{\mu_1\mu_2...\mu_j;\nu_1\nu_2...\nu_j}(m_n, k) \, p_1^{\nu_1} p_1^{\nu_2} ... p_1^{\nu_j} \right] \\
& \times \mathcal{V}_{h\bar{\Psi}\Psi}(j, K, m_n)
\end{aligned}$$

- we use the identity

$$q^{\mu_1} q^{\mu_2} ... q^{\mu_j} \, P_{\mu_1 \mu_2 ... \mu_j ; \nu_1 \nu_2 ... \nu_j}(m_n, k) \, p_1^{\nu_1} p_1^{\nu_2} ... p_1^{\nu_j} = (p_1 \cdot q)^j \times \hat{d}_j(\eta, m_n^2)$$

with

$$\hat{d}_j(\eta, m_n^2) = {}_2F_1\left(-\frac{j}{2}, \frac{1-j}{2}; \frac{1}{2} - j; \frac{4 m_N^2}{-m_n^2} \times \eta^2\right)$$

where the skewness parameter is given by $\eta \sim \frac{k \cdot q}{2 p_1 \cdot q}$. Also note that, for $j = 2$, we have the massive spin-2 projection operator $P_{\mu_1 \mu_2 ; \nu_1 \nu_2}(m_n, k) \equiv P_{\mu\nu;\alpha\beta}(m_n, k)$ defined as

$$P_{\mu\nu;\alpha\beta}(m_n, k) = \frac{1}{2}\left(P_{\mu;\alpha} P_{\nu;\beta} + P_{\mu;\beta} P_{\nu;\alpha} - \frac{2}{3} P_{\mu;\nu} P_{\alpha;\beta}\right)$$

which is written in terms of the massive spin-1 projection operator

$$P_{\mu;\alpha}(m_n, k) = \eta_{\mu\alpha} - k_\mu k_\alpha / m_n^2$$

# Four-point correlation functions $< \bar{P} J_\mu J_\nu P >$

- separating the skewness $\eta$ dependent and independent part, we can rewrite the scattering amplitude as

$$\begin{aligned} \mathcal{A}^h_{\gamma^*_{L/T} p \to V p}(j,s,t) &\sim -\frac{1}{g_5} \times 2\kappa^2 \times \mathcal{V}^{\mu\nu}_{h\gamma^*_{L/T} V}(j,Q,M_V) \times (p_1 \cdot q)^j \times \frac{1}{m_N} \times \bar{u}(p_2) u(p_1) \\ &\times \left[\mathcal{A}(j,t) + \mathcal{D}_\eta(j,\eta,t)\right] \end{aligned}$$

where

$$\mathcal{D}_\eta(j,\eta,t) = \left(\hat{d}_j(\eta,t) - 1\right) \times \left[\mathcal{A}(j,t) - \mathcal{A}_S(j,t)\right]$$

with

$$\mathcal{A}_S(j,t) \equiv \mathcal{A}(j, ; \kappa_T \to \kappa_S)$$

# Holographic parametrization of conformal moments of gluon GPDs

- our holographic parametrization of conformal moments of gluon GPD at finite skewness, and at input scale $\mu = \mu_0$, is given by

$$\mathbb{F}_g^{(+)}(j, \eta, t; \mu_0) = \mathcal{A}_g(j, t; \mu_0) + \mathcal{D}_{g\eta}(j, \eta, t; \mu_0)$$

  for even $j = 2, 4, \ldots$

- we use

$$\mathcal{A}_g(j, t; \mu_0) = \int_0^1 dx\, x^{j-2}\, xg(x; \mu_0)\, x^{a_T}$$

  with the input gluon PDF $xg(x; \mu_0)$ at $\mu = \mu_0$, and $a_T \equiv -\alpha'_T t$

# Holographic parametrization of conformal moments of gluon GPDs

- the skewness or $\eta$-dependent terms are fixed by holography as

$$\mathcal{D}_{g\eta}(j,\eta,t;\mu_0) = \left(\hat{d}_j(\eta,t) - 1\right) \times [\mathcal{A}_g(j,t;\mu_0) - \mathcal{A}_{gS}(j,t;\mu_0)]$$

where

$$\mathcal{A}_{gS}(j,t;\mu_0) \equiv \mathcal{A}_g(j,t;\mu_0,\alpha'_T \to \alpha'_S)$$

- the A-form factor of the gluon gravitational form factor of the proton is given by

$$A_g(t;\mu_0) = \mathcal{A}_g(j=2,t;\mu_0)$$

and the D-form factor (or the D-term) of the gluon gravitational form factor of the proton is

$$\eta^2 D_g(t;\mu_0) = \mathcal{D}_{g\eta}(j=2,\eta,t;\mu_0)$$

# Holographic parametrization of conformal moments of non-singlet quark GPDs

- the holographic parametrization of the conformal moments of the non-singlet (valence) quark GPDs at $\mu = \mu_0$ is given by

$$\mathbb{F}_q^{(-)}(j, \eta, t; \mu_0) = \mathcal{F}_q^{(-)}(j, t; \mu_0) + \mathcal{F}_{q\eta}^{(-)}(j, \eta, t; \mu_0)$$

for odd $j = 1, 3, \cdots$, where we have defined

$$\mathcal{F}_q^{(-)}(j, t; \mu_0) = \int_0^1 dx\, x^{j-1}\, q_v(x; \mu_0)\, x^{a_q}$$

with the input valence (non-singlet) quark PDF $q_v(x; \mu_0) = q(x; \mu_0) - \bar{q}(x; \mu_0) = H_q^{(-)}(x, \eta = 0, t = 0; \mu_0)$ at $\mu = \mu_0$, and $a_q \equiv -\alpha'_q t$, where $\alpha'_q$ is a Regge slope parameter

# Holographic parametrization of conformal moments of non-singlet quark GPDs

- the skewness or $\eta$-dependent $\mathcal{F}_{q\eta}^{(-)}$-terms are given by

$$\mathcal{F}_{q\eta}^{(-)}(j,t;\mu_0) = \left(\hat{d}_{j-1}(\eta,t) - 1\right) \times \left[\mathcal{F}_{q}^{(-)}(j,t;\mu_0) - \mathcal{F}_{qS}^{(-)}(j,t;\mu_0)\right]$$

where

$$\mathcal{F}_{qS}^{(-)}(j,t;\mu_0) \equiv \mathcal{F}_{q}^{(-)}(j,t;\mu_0,\alpha'_q \to \alpha'_{qs})$$

- the Dirac electromagnetic form factor of the proton is given by

$$F_1(t) = \sum_{q=1}^{2} e_q\, \mathcal{F}_{q}^{(-)}(j=1,t;\mu_0)$$

# Holographic parametrization of conformal moments of singlet quark GPDs

- the holographic parametrization of the conformal moments of the singlet (sea) quark GPDs at $\mu = \mu_0$ are given by

$$\sum_{q=1}^{N_f} \mathbb{F}_q^{(+)}(j,\eta,t;\mu_0) = \sum_{q=1}^{N_f} \mathcal{F}_q^{(+)}(j,t;\mu_0) + \mathcal{F}_{q\eta}^{(+)}(j,\eta,t;\mu_0)$$

for even $j = 2, 4, \cdots$, where we have defined

$$\sum_{q=1}^{N_f} \mathcal{F}_q^{(+)}(j,t;\mu_0) = \int_0^1 dx\, x^{j-1} \sum_{q=1}^{N_f} q^{(+)}(x;\mu_0)\, x^{a_q}$$

with the input singlet quark PDF
$\sum_{q=1}^{N_f} q^{(+)}(x;\mu_0) = \sum_{q=1}^{N_f} q(x;\mu_0) + \bar{q}(x;\mu_0) = \sum_{q=1}^{N_f} H_q^{(+)}(x,\eta=0,t=0;\mu_0)$ at $\mu = \mu_0$, and $a_q \equiv -\alpha'_q t$.

# Holographic parametrization of conformal moments of singlet quark GPDs

- the skewness or $\eta$-dependent terms are given by

$$\sum_{q=1}^{N_f} \mathcal{F}_{q\eta}^{(+)}(j,\eta,t;\mu_0) = \left(\hat{d}_j(\eta,t) - 1\right) \times \left[\sum_{q=1}^{N_f} \mathcal{F}_{q}^{(+)}(j,t;\mu_0) - \mathcal{F}_{qS}^{(+)}(j,t;\mu_0)\right]$$

where

$$\sum_{q=1}^{N_f} \mathcal{F}_{qS}^{(+)}(j,t;\mu_0) \equiv \sum_{q=1}^{N_f} \mathcal{F}_{q}^{(+)}(j,t;\mu_0,\alpha_q' \to \alpha_{qs}')$$

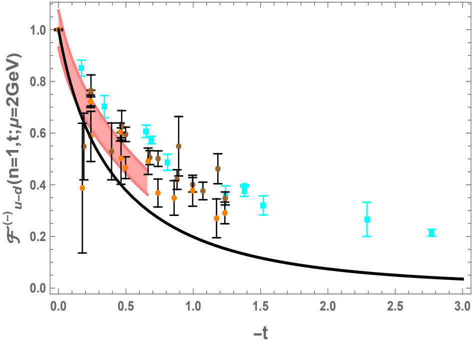
# Holographic parametrization of conformal moments of singlet quark GPDs

- the A-form factor of the quark gravitational form factor of proton is given by

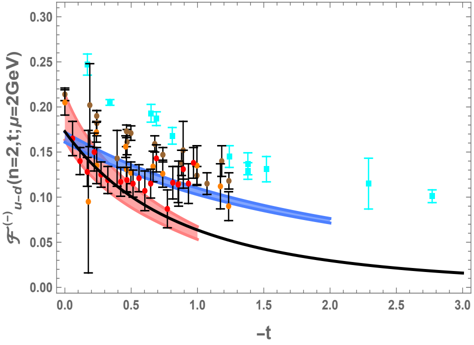
$$\sum_{q=1}^{N_f} A_q(t;\mu_0) = \sum_{q=1}^{N_f} \mathcal{F}_q^{(+)}(j=2,t;\mu_0)$$

and the D-form factor (or the D-term) of the quark gravitational form factor of the proton is given by

$$\eta^2 \sum_{q=1}^{N_f} D_q(t;\mu_0) = \sum_{q=1}^{N_f} \mathcal{F}_{q\eta}^{(+)}(j=2,\eta,t;\mu_0)$$



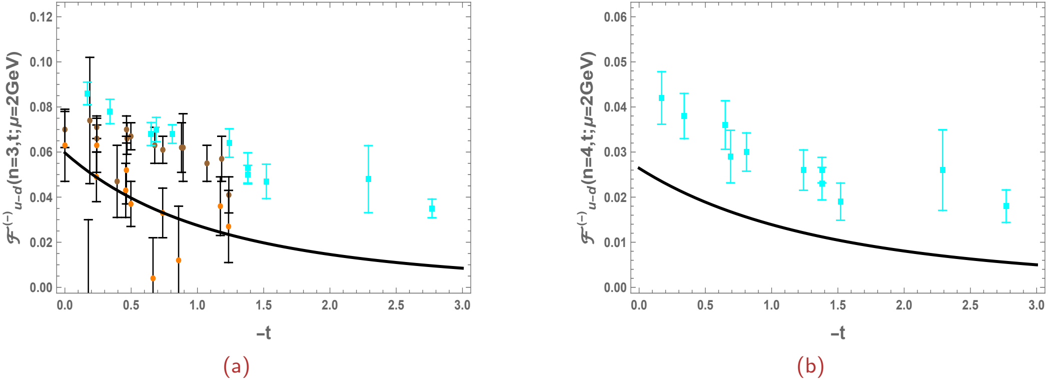

(a)




(b)

Figure: Our evolved moments of $u - d$ quark GPD $H^{u-d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.

# Comparison of the conformal moments to lattice QCD

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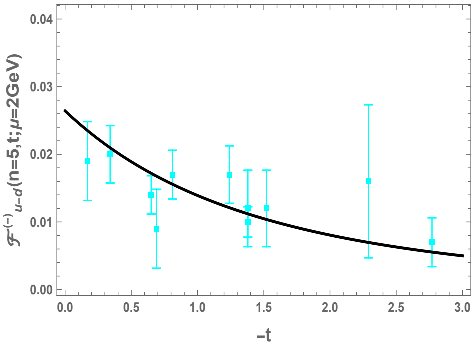


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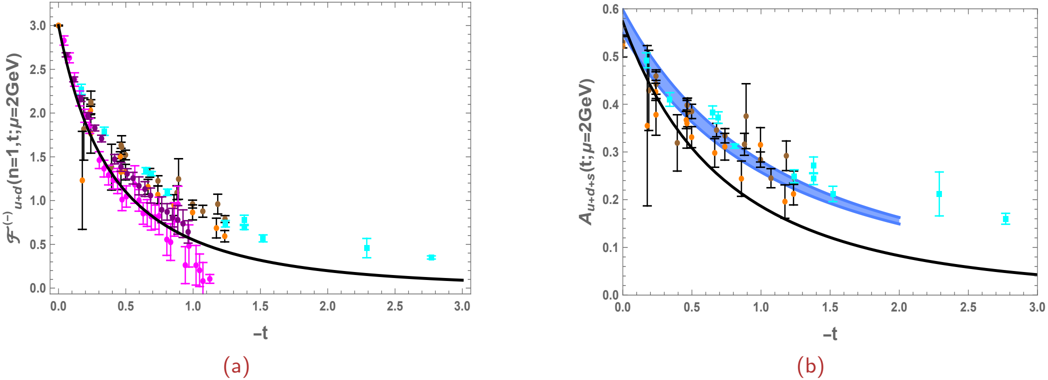


Figure: Our evolved moments of $u+d$ quark GPD $H^{u+d}(x,\eta,t;\mu)$ at $\mu = 2\ \mathrm{GeV}$, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.

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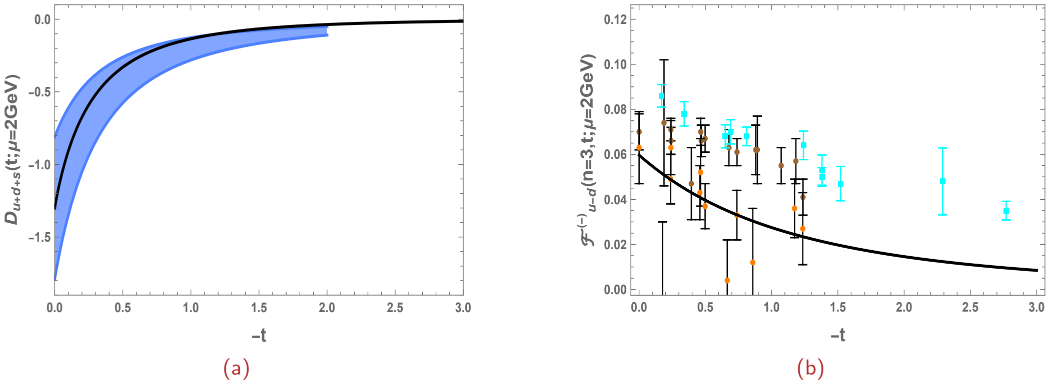


Figure: Our evolved moments of $u + d$ quark GPD $H^{u+d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.

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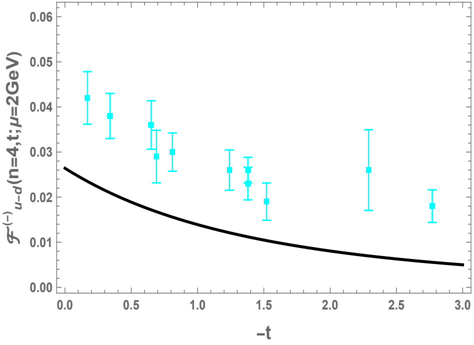
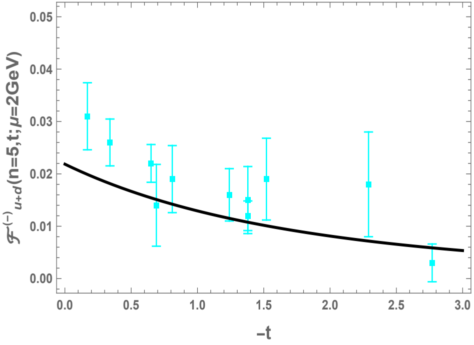



(a)




(b)

Figure: Our evolved moments of $u + d$ quark GPD $H^{u+d}(x, \eta, t; \mu)$ at $\mu = 2$ GeV, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.

# Comparison of the conformal moments to lattice QCD

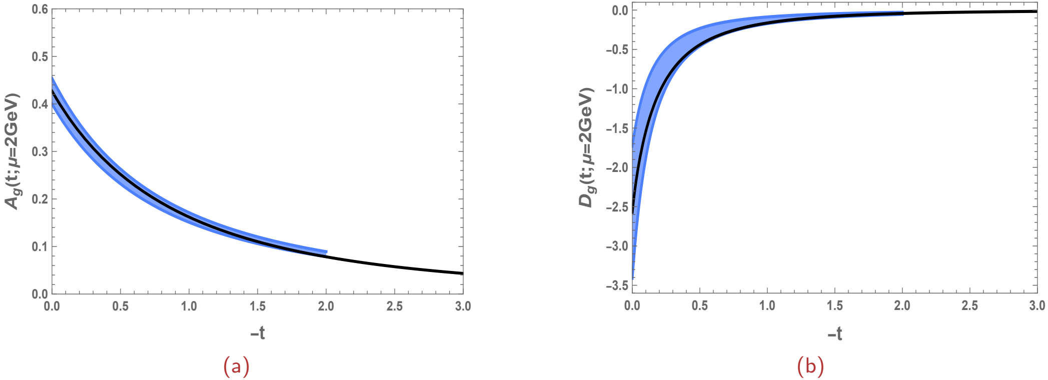



Figure: Our evolved moments of gluon GPD $H^g_{symmetric}(x,\eta,t;\mu)$ at $\mu = 2\ \mathrm{GeV}$, are represented by black line. The other colored curves and data points, corresponding to lattice data, from various sources are shown for comparison.

# Reconstructing quark and gluon GPDs from their conformal moments

- the gluon GPD series expansion interms of their conformal (Gegenbauer) moments is extended to a Mellin-Barnes-type integral to facilitate the incorporation of complex-valued conformal spins [Mueller:2005]:

$$H_g(x,\eta,t;\mu) = \frac{1}{2i}\int_{\mathbb{C}} dj \, \frac{(-1)}{\sin(\pi j)} {}^{g}p_j(x,\eta)\mathbb{F}_g(j,\eta,t;\mu^2)$$

with

$${}^{g}p_j(x,\eta) = \theta(\eta-|x|)\frac{1}{\eta^{j-1}}{}^{g}\mathcal{P}_j\left(\frac{x}{\eta}\right) + \theta(x-\eta)\frac{1}{x^{j-1}}{}^{g}\mathcal{Q}_j\left(\frac{x}{\eta}\right)$$

where the functions ${}^{g}\mathcal{P}_j$ and ${}^{g}\mathcal{Q}_j$ are given by

$$\begin{aligned}
{}^{g}\mathcal{P}_j\left(\frac{x}{\eta}\right) &= \left(1+\frac{x}{\eta}\right)^2 {}_2F_1\left(-j, j+1, 3 \middle| \frac{1}{2}\left(1+\frac{x}{\eta}\right)\right)\frac{2^{j-1}\Gamma(3/2+j)}{\Gamma(1/2)\Gamma(j-1)} \\
{}^{g}\mathcal{Q}_j\left(\frac{x}{\eta}\right) &= {}_2F_1\left(\frac{j-1}{2}, \frac{j}{2}; \frac{3}{2}+j; \frac{\eta^2}{x^2}\right)\frac{\sin(\pi[j+1])}{\pi}
\end{aligned}$$

# Reconstructing quark and gluon GPDs from their conformal moments

- for quark GPDs we have

$$H_q(x,\eta,t;\mu) = \frac{1}{2i}\int_{\mathbb{C}} dj\,\frac{1}{\sin(\pi j)}p_j(x,\eta)\mathbb{F}_q(j,\eta,t;\mu)$$

with

$$p_j(x,\eta) = \theta(\eta-|x|)\frac{1}{\eta^j}\mathcal{P}_j\left(\frac{x}{\eta}\right) + \theta(x-\eta)\frac{1}{x^j}\mathcal{Q}_j\left(\frac{x}{\eta}\right)$$

where

$$\mathcal{P}_j\left(\frac{x}{\eta}\right) = \left(1+\frac{x}{\eta}\right){}_2F_1\left(-j,j+1,2\middle|\frac{1}{2}\left(1+\frac{x}{\eta}\right)\right)\times\frac{2^j\Gamma(3/2+j)}{\Gamma(1/2)\Gamma(j)}$$

$$\mathcal{Q}_j\left(\frac{x}{\eta}\right) = {}_2F_1\left(\frac{j}{2},\frac{j+1}{2};\frac{3}{2}+j\middle|\frac{\eta^2}{x^2}\right)\times\frac{\sin(\pi j)}{\pi}$$

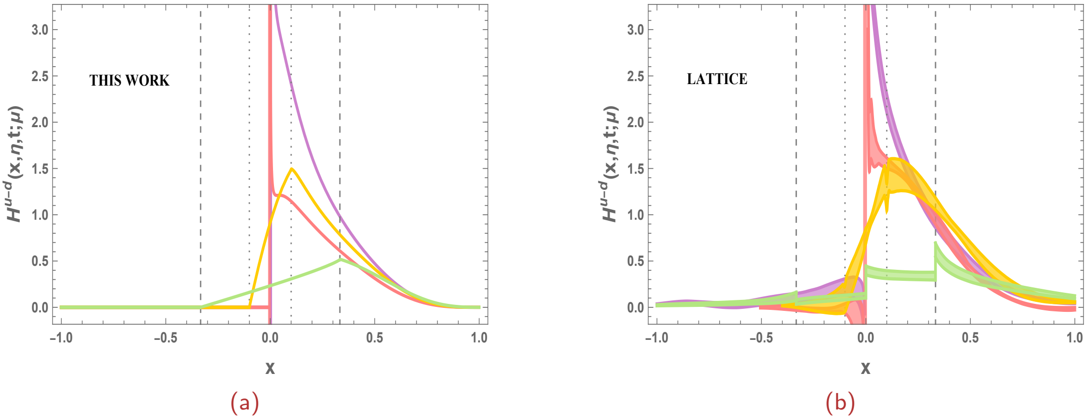



Figure: Comparsion of our reconstructed $u-d$ quark GPD $H_{u-d}^{(-)}(x,\eta,t;\mu)$ with lattice.

results

# Thank You!