Towards a unification of gluon distributions at small and moderate $x$

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August 31, 2020

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Introduction
Proton structure and QCD evolution

- at large $Q^2$ and moderate $x \sim Q^2/s$ the proton can be described as a collection of weakly interacting partons
- partonic interpretation breaks down beyond leading twist
- at small $x$ gluon occupation number increases and eventually saturates due to non-linear effects (gluon recombination)

$Q_s^2(x) \sim \frac{Q^2}{s} x$ at large $Q^2$ and moderate $x$

DGLAP

BFKL

BK-JIMWLK

saturation boundary

non perturbative

$\log \frac{1}{x}$

$\log Q^2$

DIS kinematics

- DIS kinematics: hadronic part
  \[ Q^2 = -q^2 > 0 \]
  \[ x_{Bj} \equiv \frac{Q^2}{2P \cdot q} = \frac{Q^2}{s} \]

- Mass condition
  \[ p = xP + k_\perp \]
  \[ k_\perp \ll Q \]
  \[ (p + q)^2 = 0 \]
  \[ \Rightarrow -Q^2 + xP^-q^+ \approx 0 \]
  \[ \Rightarrow x = x_{Bj} \]

- At leading order we recover the Parton model:
  \[ F_2(x) \sim xq(x) \]

- Light-cone variables:
  \[ q^+ = \frac{q^0 + q^3}{\sqrt{2}} \]
  \[ q^- = \frac{q^0 - q^3}{\sqrt{2}} \]
DGLAP: Bjorken limit $Q^2 \sim s$  

- DGLAP evolution resums large collinear logarithms  
  \[ \bar{\alpha} \log Q/\mu = \bar{\alpha} \int_{\mu}^{Q} \frac{d k_\perp}{k_\perp} \]

- successive emissions are strongly ordered in transverse momentum  
  \[ Q \gg k_{\perp 1} \gg k_{\perp 2} \gg \ldots \gg \mu \]

- The variable related to $x = k^-/P^- \sim 1$  
  \[ k^-_1 \sim k^-_2 \sim \ldots \sim P^- \sim \sqrt{s} \]

- $k^- \sim t_f^{-1}$ is the inverse formation time scale

- It follows that $k^+ = k^2_{\perp}/k^-$ 's are strongly ordered  
  \[ q^+ \gg k^+_1 \gg k^+_2 \gg \ldots \gg \mu^2 \frac{\mu^2}{P^-} \]

$P \equiv (0, P^-, 0)$  
$s = q^+ P^-$
BFKL: Regge limit $Q^2 \ll s$

- **BFKL evolution resums large soft logarithms**
  \[ \bar{\alpha} \log s^{-1} = \bar{\alpha} \int_{q_0^+}^{q^+} \frac{dk^+}{k^+} \]

- **successive emissions are strongly ordered in longitudinal momenta**
  \[ \sqrt{s} \sim q^+ \gg k_1^+ \gg k_2^+ \gg \ldots \gg q_0^+ \]

- And the variable $x = k^-/P^- \ll 1$
  \[ k_1^- \ll k_2^- \ll \ldots \ll P^- \sim \sqrt{s} \]

- $k^- \sim t_f^{-1}$ is the inverse formation time scale

- **Transverse momenta are of same order**
  \[ Q \sim k_{\perp 1} \sim k_{\perp 2} \sim \ldots \sim \mu \]

\[ q \equiv (q^+, Q^2/2q^+, 0) \]
\[ k_{\perp 1} \]
\[ k_{\perp 2} \]
\[ k_{\perp 3} \]
\[ P \equiv (0, P^-, 0_\perp) \]
\[ s = q^+P^- \]
The dipole model in DIS

\[ p \equiv [+\infty, -\infty] \equiv P \exp \left[ ig \int_{-\infty}^{+\infty} dx^+ A^-(x^+, x) \right] \]

- the relevant d.o.f. in the saturation regime are strong classical fields \( g A^- \sim 1 \)
- eikonal lines: \( p^+ \gg k^+ \) and \( k^- \ll P^- \)
- path ordered Wilson line

\[ U_x \equiv [+\infty, -\infty]_x = P \exp \left[ ig \int_{-\infty}^{+\infty} dx^+ A^-(x^+, x) \right] \]

- dominant contribution in DIS: dipole scattering

\[ \langle P | \text{Tr} \ U_{x_1} \ U_{x_2}^\dagger | P \rangle \]

shock wave limit: \( t_f \equiv Q^2/q^+ \gg 1/P^- \rightarrow x = Q^2/s \ll 1 \)
BK and the NLO crisis
Balitsky-Kovchegov (1996-1999) equation describes the non-linear evolution of the dipole scattering amplitude as function of the rapidity

\[ Y \equiv \log \frac{q^+}{\Lambda^+} \]

\[ S_Y(x - y) \equiv \frac{1}{N_c} \left\langle \text{Tr} \, U(x) \, U^+(y) \right\rangle_Y \]

\[ \frac{\partial}{\partial Y} S_Y(x - y) = \bar{\alpha} \int dz \frac{(x - y)^2}{(x - z)^2(z - y)^2} \left[ S_Y(x - z)S_Y(z - y) - S_Y(x - y) \right] \]
BK and the NLO crisis

[Balitsky and Chirilli (2008)]

- At NLO BK equation was found to be numerically unstable

[Lappi and Mäntysaari (2015)]
• This is due to the fact that the rapidity variable $a$

\[ Y \equiv \log \frac{q^+}{\Lambda^+} \]

evolves independently from $x$

• As a result small dipoles that radiate larger
dipoles generate large collinear logarithms when

\[ k_1^- \sim k_2^- \Rightarrow \text{NLO} > \text{LO} \]

formation ime ordering

\[ k_1^- = x_1^2 k_1^+ \quad \Rightarrow \quad \frac{x_1^2}{x_2^2} < \frac{k_2^+}{k_1^+} \ll 1 \]

BK and the NLO crisis


• Several solutions have been proposed: implementing kinematic bound, resummation of double logarithms. However,
  
  ▸ they spoil the renormalization picture established at LO
  ▸ exhibit scheme dependence
  ▸ no operator definition for systematic order by order calculations
  ▸ not discussed at the level of observables: inclusive DIS
Where is x hiding?
Where is x hiding?

- In the shock wave [Balitsky (1996)] approach there is no explicit dependence on the longitudinal momentum fraction of the gluons in the target.
- The standard trick is to identify rapidity variables $Y = \eta$

\[
\eta \equiv \log \frac{1}{x_{Bj}}
\]

\[
Y \equiv \log \frac{q^+}{q_0^+}
\]

transverse photon wave function

\[
\varphi(r) = \frac{r \cdot \epsilon^\lambda}{|r|} K_1 \left( \sqrt{z(1-z)} |r|^2 Q^2 \right)
\]

cross-section

\[
\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 dz \; P(z) \; |\varphi(r)|^2 \langle \text{Tr} U(r) U^\dagger(0) \rangle_Y
\]

- It is instructive to check the collinear limit.
Where is $x$ hiding?

- In the leading twist: expanding for small dipoles $r_\perp \sim 1/Q \ll b_\perp$

  $$U(b + \frac{r}{2}) = U(b) + r^i \partial^i U(b)$$

  $$\partial^i U(b) = \int_{-\infty}^{+\infty} dx^+ \left( [+\infty, x^+] \partial^i A^-(x^+) [+\infty, x^+] \right)_b \quad \text{where} \quad \partial^i A^- \equiv F^{i-}$$

- we recover the gluon PDF …

  $$\int d\mathbf{b} \left\langle \text{Tr} \left. U(b + \frac{r}{2}) U^\dagger (b - \frac{r}{2}) \right\rangle_Y \approx r^2 x g \left( \frac{1}{r^2}, x \right) + O(r^4)$$
Where is $x$ hiding?

- ...except for the phase that encodes to $x = k^-/P^-$ dependence

$$xg(x, \mu^2) = \frac{2}{P^-} \int_b \int_{x^+, y^+} \left\langle \text{Tr} [y^+, x^+] F^i-(x^+)[y^+, x^+] F^i-(y^+) \right\rangle_Y$$

- the complete gluon PDF writes (the integral over $x^+$ and $b$ cancel against $\langle P | P \rangle$)

$$xg(x, \mu^2) = 2 \int \frac{d\xi^+}{(2\pi)P^-} e^{ixP^-\xi^+} \langle P | \text{Tr} [0, \xi^+] F^i-(\xi^+)[\xi^+, 0] F^i-(0) | P \rangle$$

![Diagram of quark and gluon exchange](image-url)
Where is \( x \) hiding?

- Parametrically we have: \( \xi^+ = x^+ - y^+ \sim 1/P^- \)

- Then the phase scales as: \( i x P^- \xi^+ \sim x \)

- In shock wave one assumes: \( x_{Bj} \ll x \ll 1 \Rightarrow \) one can neglect the phase.
  
  This is the BFKL-BK phase space

- On the other hand a potentially large collinear logarithm lives in the region
  \( x_{Bj} \ll x \sim 1 \)

- This might not be so relevant at leading order but at NLO dilute spots in the proton wave function may generate large collinear logarithms
Digression: background field and transverse gauge links
consider a target boosted along the $-z$ direction close to the light cone. Due to time dilation, the target color sources are "frozen" in the $-z$ direction.

- Yang-Mills equations $[D_\mu, F^{\mu\nu}] = J^\nu$ can be solved exactly (together with the continuity equation $[D_\mu, J^\mu] = 0$) in covariant gauge $\partial \cdot A = 0$ (or light-cone gauge $A^+ = 0$)

\[ A_{\text{cov}}^- = - \frac{1}{\partial_{\perp}^2} J^- \quad \text{and} \quad A^+ = A_{\perp} = 0 \]
Background field and transverse gauge links

• under an arbitrary gauge rotation $\Omega(x^+, x_\perp)$ the target field transforms as

$$A^- \rightarrow \Omega_x(x^+) A_{\text{cov}}^-(x^+, x) \Omega_x^{-1}(x^+) - \frac{1}{ig} \Omega_x(x^+) \partial^- \Omega_x^{-1}(x^+)$$

$$A^i \rightarrow -\frac{1}{ig} \Omega_x(x^+) \partial^i \Omega_x^{-1}(x^+)$$

• exploiting the residual gauge freedom we can generate a transverse pure gauge

• N.B.: the partonic picture is manifest in the LC-gauge $A^- = 0$ (with $A_\perp \neq 0$)

• small x observables are (in the dilute/dense limit) more naturally expressed in the wrong LC-gauge $A^- \neq 0$ (with $A_\perp = 0$).

• in order to connect to the partonic interpretation one needs to deal with transverse fields
Background field and transverse gauge links

- geometric interpretation of the all twist resummation

\[ U_{x_1} = U_{x_2} - r^i \int_0^1 ds \left( \partial^i U_{x_2 + sr} \right) \]

- and noticing that \( \frac{1}{ig} (\partial^i U_x) U_x \equiv A^i(x) \), one can express the dipole operator (in the background field \( A^- \)) as a transverse gauge link:

\[ U_{x_1} U_{x_2}^\dagger = [x_1, x_2] = 1 - ig \int_{x_2}^{x_1} dz A^i(z) [z, x_2] \]

- dipole operator can be expressed in terms of tranverse link operators
Background field and transverse gauge links

- **non-Abelian Stokes theorem:** more generally, the dipole operator can be written as a path ordered tower of "twisted" field strength tensor (i.e. dressed with future pointing Wilson lines)

\[ U_{x_2} U_{x_1}^\dagger \equiv P \exp \left[ -ig \int_S dt dz \left[ +\infty, x^+ \right]_x F_i^-(x^+, x) \left[ x^+, + \infty \right]_x \right] \]

Revisiting the Shock Wave Approximation
Revisiting the Shock Wave Approximation

- Under the assumption that all transverse momenta are of same order along the ladder, the leading power $1/s$ is obtained letting $p^+ \to \infty$ for any particle propagating inside the shock wave.

\[ \mathcal{G}_{p^+}(x^+, y^+) = \left[ i \frac{\partial}{\partial x^+} - \frac{\hat{p}_\perp^2}{2p^+} - gA^- \right]^{-1} \]

sock wave limit

\[ \lim_{p^+ \to +\infty} \mathcal{G}_{p^+}(x^+, x; y^+, y) = \delta(x - y) U_x(x^+, y^+) \]
Revisiting the Shock Wave Approximation

• This limit neglects quantum diffusion

\[ \mathcal{C}_p^0(x^+, x; y^+, y) = \frac{p^+}{2i\pi \Delta x^+} e^{i\frac{(x-y)^2 p^+}{\Delta x^+}} \]

• It is important when \((\Delta x)^2 \sim \Delta x^+/p^+ \sim s^{-1}\)

• In effect, the phase relates the transverse dynamics to longitudinal dynamics, this is the phase that appears in the definition of PDF’s

• It encodes the information about \(k^-\)’s in the target. It is expected to be non-negligible away from the strongly ordered region in \(k^-\).
Revisiting the Shock Wave Approximation

• Instead of taking the limit $p^+ \to +\infty$ we can simply expand around the constant transverse coordinate (classical trajectory)

$$X \equiv \frac{x + y}{2}$$

• At leading power we recover a Wilson line multiplied by a phase

$$\mathcal{G}_{p^+}(x^+, x; y^+, y) = \mathcal{G}_0(x - y, x^+ - y^+) U_X(x^+, y^+) + \ldots$$

• Taking again the limit $p^+ \to +\infty$ we recover the shock wave

• The ellipses are subleading terms suppressed as $(x - y)^i (\partial^i U)_X$
Road Test: Inclusive DIS
Inclusive DIS beyond shock wave

\[ q \equiv (q^+, q^-, 0) \]

1. ordering in \( k^+ \)

2. gluonic target

3. leading power in \( k_{\perp}^2/s \)

\[ P \equiv (0^+, P^-, 0) \]
Inclusive DIS beyond shock wave

• Working assumptions:

1. We shall assume a large separation between the target modes and the quantum fluctuations (here the quark dipole)

\[ k^+ > \Lambda^+ \quad \text{quantum} \quad k^+ < \Lambda^+ \quad \text{classical} \]

\[ \psi_q(x) \quad A_c^-(x^+, x) \sim \delta(k^+) \]

NB: strong ordering in \( k^+ \) is common to small \( x \) and leading twist DGLAP

\[ k_1^+ = \frac{k_1^-}{k_{1\perp}^2} \ll k_2^+ = \frac{k_2^-}{k_{2\perp}^2} \]

[Balitsky and Braun (1988)]
[Balitsky and Trasov (2015)]
Inclusive DIS beyond shock wave

• Working assumptions:

2. We shall neglect power suppressed terms of the form

\[ O\left(\frac{k_\perp^2}{s}\right) \]

• Bjorken limit: If \( s \sim Q^2 \) it is higher twist \( k_\perp/Q \)

• Regge limit: If \( s \gg Q^2 \) it is suppressed by a powers of the energy
In the shock wave approximation the times of the photon splitting into quark antiquark pair are integrated form $0 < x^+ < +\infty$ and $-\infty < y^+ < 0$ which yields the photon wave functions

- What are the integration limits of the vertices if one relaxes the shock wave approximation?
- What is the longitudinal extent of the shock wave?
Inclusive DIS beyond shock wave

- **Solution:** extracting the first and last interactions provides a physical boundary to the shock wave
- we have 4 contributions
- This seems involved at first glance
Inclusive DIS beyond shock wave

- Consider the left side of the diagram first
- The two gluon fields combine to generate field strength tensor

\[ A^{-}(x) - A^{-}(y) = \int_{0}^{1} ds \, r^{i} \partial^{i} A^{-}(y + sr) \]

\[ = \int_{0}^{1} dz^{i} F^{i-}(z) \]

- Where \( z(s) \equiv s \, x + (1 - s) \, y \) is a straight line trajectory in the transverse plane
• applying the same trick to the r.h.s. we obtain the following hadronic operator

\[ O_{\text{dipole}}(x_1, x_2, y_1, y_2, x_1^+, x_2^+) \equiv \int_{y_1}^{x_1} \int_{y_2}^{x_2} dz_1^i dz_2^j \]

\[ \langle P | \text{Tr} \mathcal{G}(y_2, x_2^+; y_1, x_1^+ | (1 - z)q^+) F_{j^-}^i(x_1^+, z_1) \mathcal{G}(x_1, x_1^+; x_2, x_2^+ | zq^+) F_{i^-}^j(x_2^+, z_2) | P \rangle \]

• performing a gauge rotation leads to transverse gauge links: explicit gauge invariance

• explicit dependence on $+$ momenta of the dipole $zq^+$ and $(1 - z)q^+$
Inclusive DIS beyond shock wave

• DIS cross-section takes now a form similar to that of the shock wave. The wave functions are identical

$$\sigma(x_B, Q^2) \sim e^2 \int_0^1 dz \, \mathcal{P}(z) \int_{r,b} dr \, |\varphi(z(1-z) | \mathbf{r} \mid^2 Q^2)|^2 \langle \text{Tr} U(\mathbf{r}) U^\dagger(0) \rangle_Y$$

\[
\sigma(x_B, Q^2) \sim e^2 \int_0^1 dz \, \mathcal{P}(z) \int_{r_1,r_2,b_1,b_2} \varphi(z(1-z) | \mathbf{r}_1 \mid^2 Q^2) \varphi^*(z(1-z) | \mathbf{r}_2 \mid^2 Q^2) \\
\times e^{i x_B P \cdot (\mathbf{x}_2^+ - \mathbf{x}_1^+)} \langle P \mid O_{\text{dipole}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{b}_1, \mathbf{b}_2, x_1^+, x_2^+ | z) \mid P \rangle
\]
We have constructed a gauge invariant hadronic operator that has explicit dependence on longitudinal variables.

The hard matrix element is composed of the same wave functions as in shock wave albeit non-local in transverse position space: dipole transverse size $a_{1^+}$ and $a_{2^+}$ are not equal $r_1 \neq r_2$.

Bialas, Navlet and Peschanski (2000) noticed that the dipole model (with locality in transverse space) is inconsistent with $x$ dependence.
A Novel Unintegrated Gluon Distribution
Unintegrated Gluon Distribution

• Now we apply the third working assumption, that is the (modified) eikonal propagation

\[ \mathcal{G}_{p^+}(x^+, x_2^+; y^+, x_1) = \mathcal{G}_0(x_2 - x_1, x_2^+ - y_1^+) \, U_X(x_2^+, x_1^+) + \ldots \]

• On may Fourier transform w.r.t. \( u = x_2 - x_1 \)

\[ \mathcal{G}_{p^+}(x_2, x_1^+, X; \ell) = e^{i \frac{\ell^2}{2q^+} \Delta x^+} \, U_X(x_2^+, x_1^+) + \ldots \]

• Note the factorization of the phase that encodes the dependence on the longitudinal momentum

• \( \ell \) is the average transverse momentum of the quark
Factorization formula for DIS at arbitrary $x$

- Combining all three phases we obtain

$$ik^- \Delta x^+ \equiv i \frac{\not{\ell}^2 + z\bar{z}Q^2}{2z\bar{z}q^+} \Delta x^+$$

- This is nothing but Feynman $x$ that we encountered when deriving the DGLAP limit

$$x_F \equiv \frac{\not{\ell}^2 + z\bar{z}Q^2}{2z\bar{z}s}$$
Factorization formula for DIS at arbitrary $x$

- After integration over the factorized free propagators that lead to the Feynman $x$ phase, we obtain the factorization formula (for the transverse photon cross-section), in momentum space,

$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 dz \, P(z) \int_0^1 dx \int_{\mathcal{E}_k} \partial^i \phi \left( \mathcal{E} - \frac{k}{2} \right) \partial^j \phi^* \left( \mathcal{E} + \frac{k}{2} \right) \delta \left( x - x_{Bj} - \frac{\mathcal{E}^2}{2z\bar{z}q^+} \right)$$

$$\times xG^{ij}(x, k) + O \left( k_{\perp}^2 / s \right)$$

- $\phi(\mathcal{E})$ is the Fourier transform of the photon wave function (shock wave)

- The delta function relates $x$ in the gluon distribution to $x_{Bj}$
Factorization formula for DIS at arbitrary $x$

- The unintegrated gluon distribution $xG^{ij}(x, k)$ writes

$$xG^{ij}(x, k_{\perp}) \equiv 2 \int \frac{d\xi^+ d\xi}{(2\pi)^3 P^-} e^{ixP^-\xi^+-i k\cdot \xi} \langle P | \text{Tr} [0, \xi^+]_{\xi} F^{i-}(\xi^+, s' \xi) [\xi^+, 0]_0 F^{i-}(0, s\xi) | P \rangle$$

- Integrating over $k_{\perp}$ yields $\xi_{\perp} = 0$ and we recover the gluon PDF

- At small $x$ we recover shock wave

$$\xi_i \xi_j G^{ij}(x = 0, \xi) \rightarrow \langle P | \text{Tr} U_\xi U^\dagger_0 | P \rangle$$

\[ (\xi, \xi^+) \]

\[ (0, 0^+) \]
Summar and Outlook
Summary and outlook

- We revisited the shock wave approximation in high energy scattering by performing a gradient expansion around the classical trajectory of partons.
- The leading power accounts for both small and moderate $x$ limit.
- We have calculated in this framework gluon induced DIS. We have obtained in particular a new factorization formula involving a novel unintegrated gluon distribution.
- **Outlook:** quantum evolution, application to other observables such as DVCS.
- Potential probe of gluon saturation on the lattice.