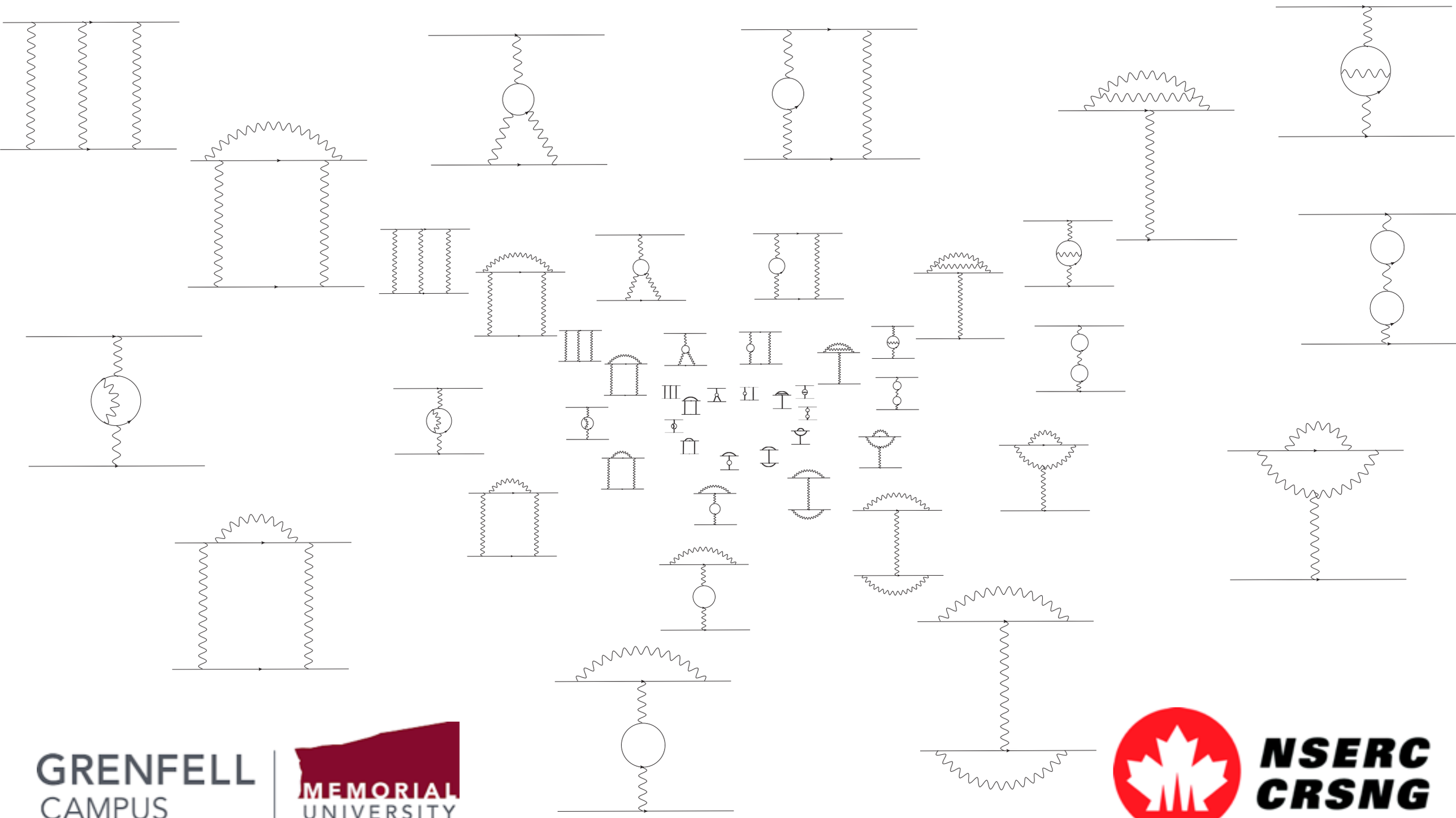
Dispersive Treatment of Two-Loop Processes

A. Aleksejevs and S. Barkanova, Grenfell Campus of Memorial University



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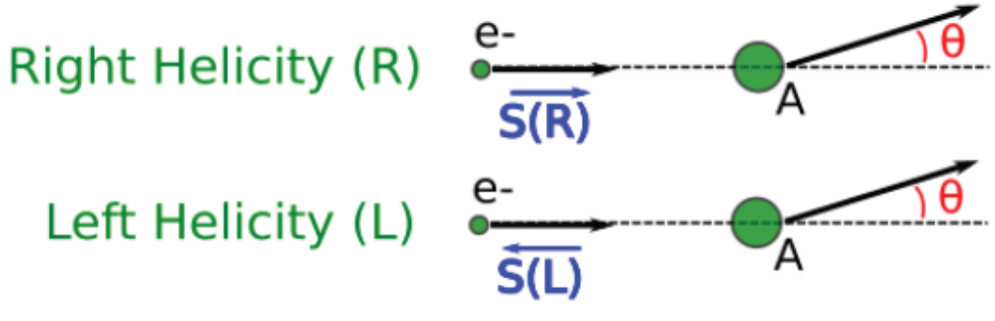
NSERC CRSNG

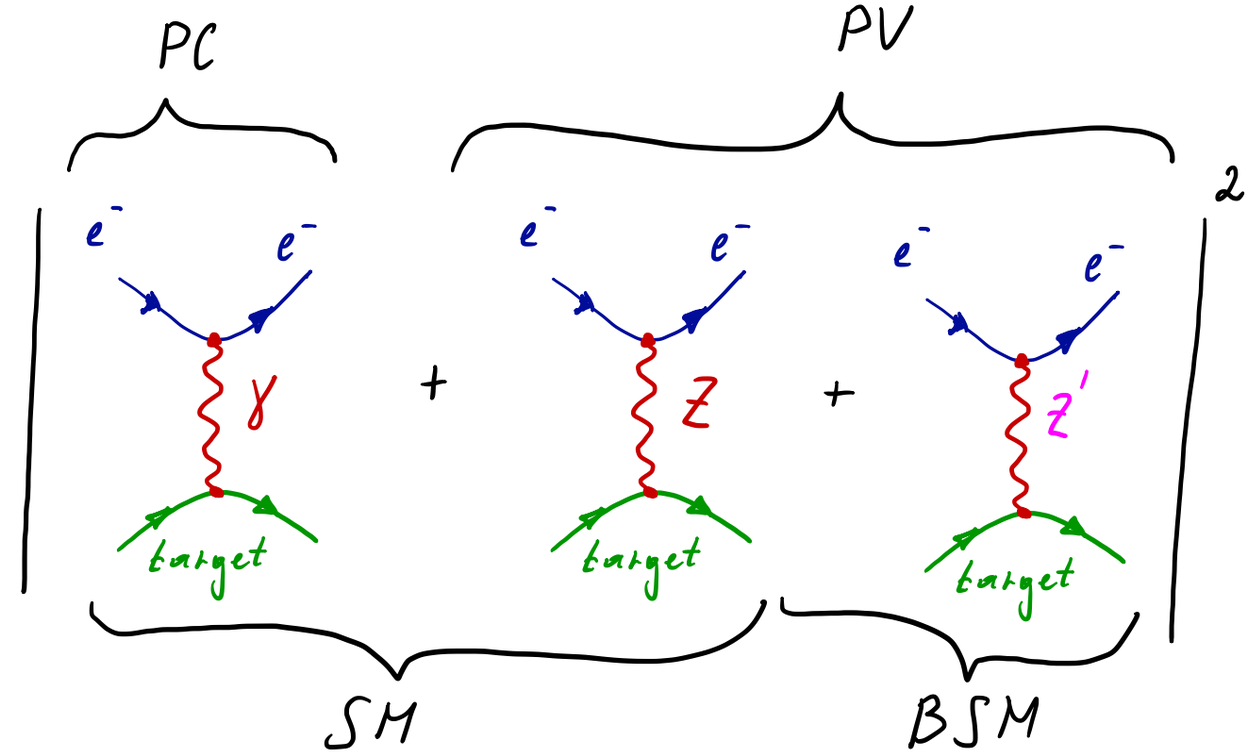
Precision PV Scattering

• To access the scale of the new physics at TeV level, we need to push one or more experimental parameters to the extreme precision.



$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$$

•New Physics/Weak-Electromagnetic Interference



$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \simeq \frac{2Re(M_\gamma M_Z^+ + M_\gamma M_{NP}^+ + M_Z M_{NP}^+)_{LR}}{\sigma_L + \sigma_R}$$

Precision Scattering: MOLLER

Asymmetry is an observable which is directly related to the interference term:

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \simeq \frac{2Re(M_\gamma M_Z^+ + M_\gamma M_{NP}^+ + M_Z M_{NP}^+)_{LR}}{\sigma_L + \sigma_R} \sim (10^{-5}\ to\ 10^{-4}) \cdot Q^2$$

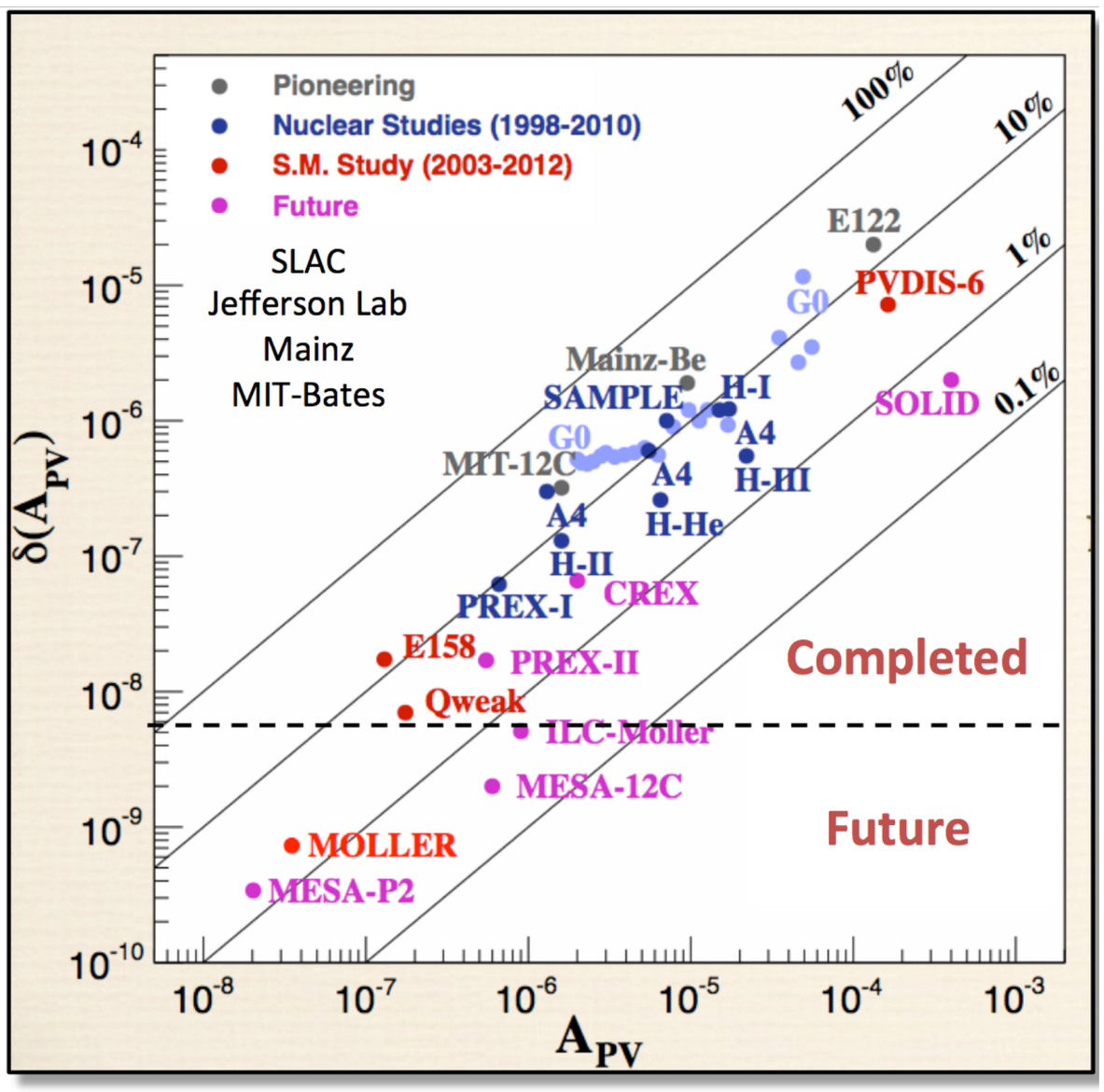


Figure courtesy of Kent Paschke

To access multi-TeV electron scale it is required to measure:

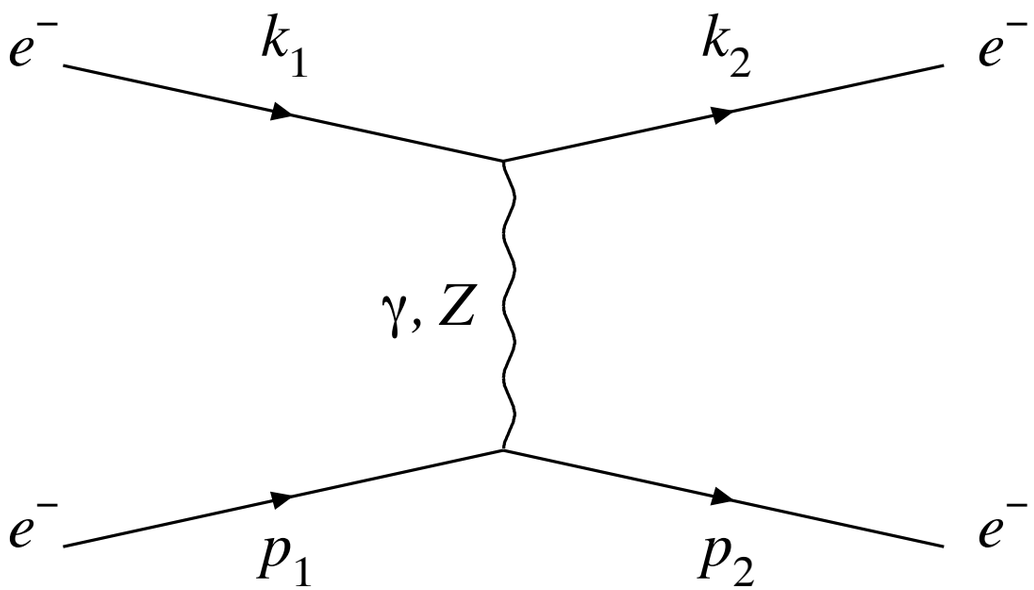
$$\delta(\sin^2\theta_W) < 0.002$$

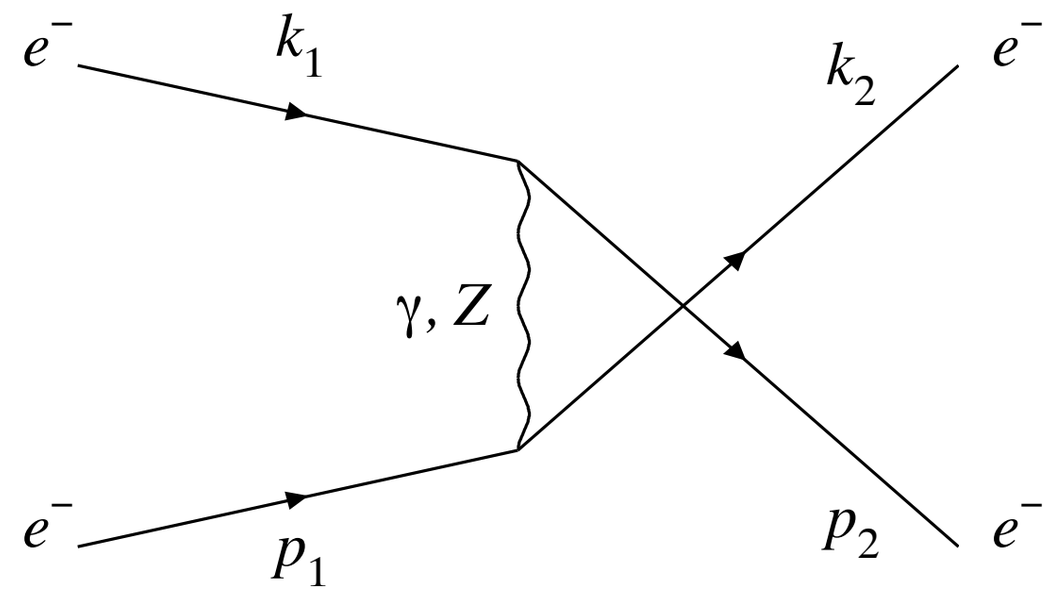
MOLLER experiment offers an unique opportunity to reach multi-TeV scale and will become complimentary to the LHC direct searches of the New Physics

Møller scattering at the tree level

The process of electron–electron scattering (Møller process)

C. Møller, Annalen der Physik 406, 531 (1932)





$$A_{LR} = \frac{\sigma_{LL} + \sigma_{LR} - \sigma_{RL} - \sigma_{RR}}{\sigma_{LL} + \sigma_{LR} + \sigma_{RL} + \sigma_{RR}} = \frac{\sigma_{LL} - \sigma_{RR}}{\sigma_{LL} + 2\sigma_{LR} + \sigma_{RR}}$$

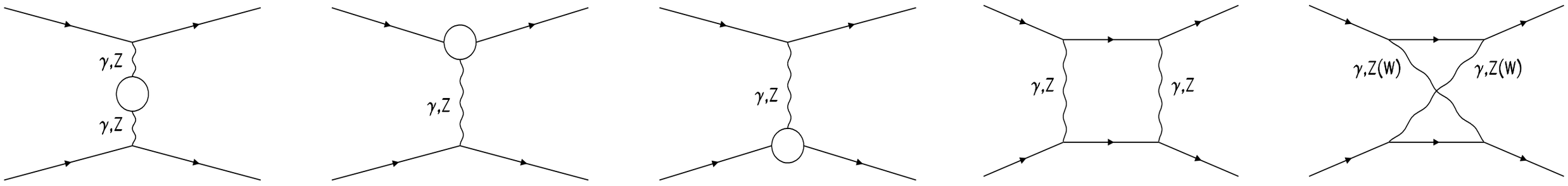
$$A^0_{LR} = \frac{s}{2m_W^2} \frac{y(1-y)}{1 + y^4 + (1-y)^4} \frac{1 - 4s_W^2}{s_W^2}, \quad y = -t/s$$

Møller scattering at one-loop

Although PV asymmetry ($A_{LR} \sim 10^{-7}$) is very small, the accuracy of modern experiments exceeds the accuracy of the theoretical result in Born approximation.
One–loop contribution was found to be rather big in the previous works:

A. Czarnecki, W. J. Marciano, Phys. Rev. D53, 1066 (1996)
A. Denner, S. Pozzorini, Eur. Phys. J. C7, 185 (1999)
A. A, S. Barkanova, A. Ilyichev, V. Zykunov, Phys. Rev. D82, 093013 (2010)

First Stage:One-Loop Corrections for MOLLER



(1) (2) (3) (4) (5)

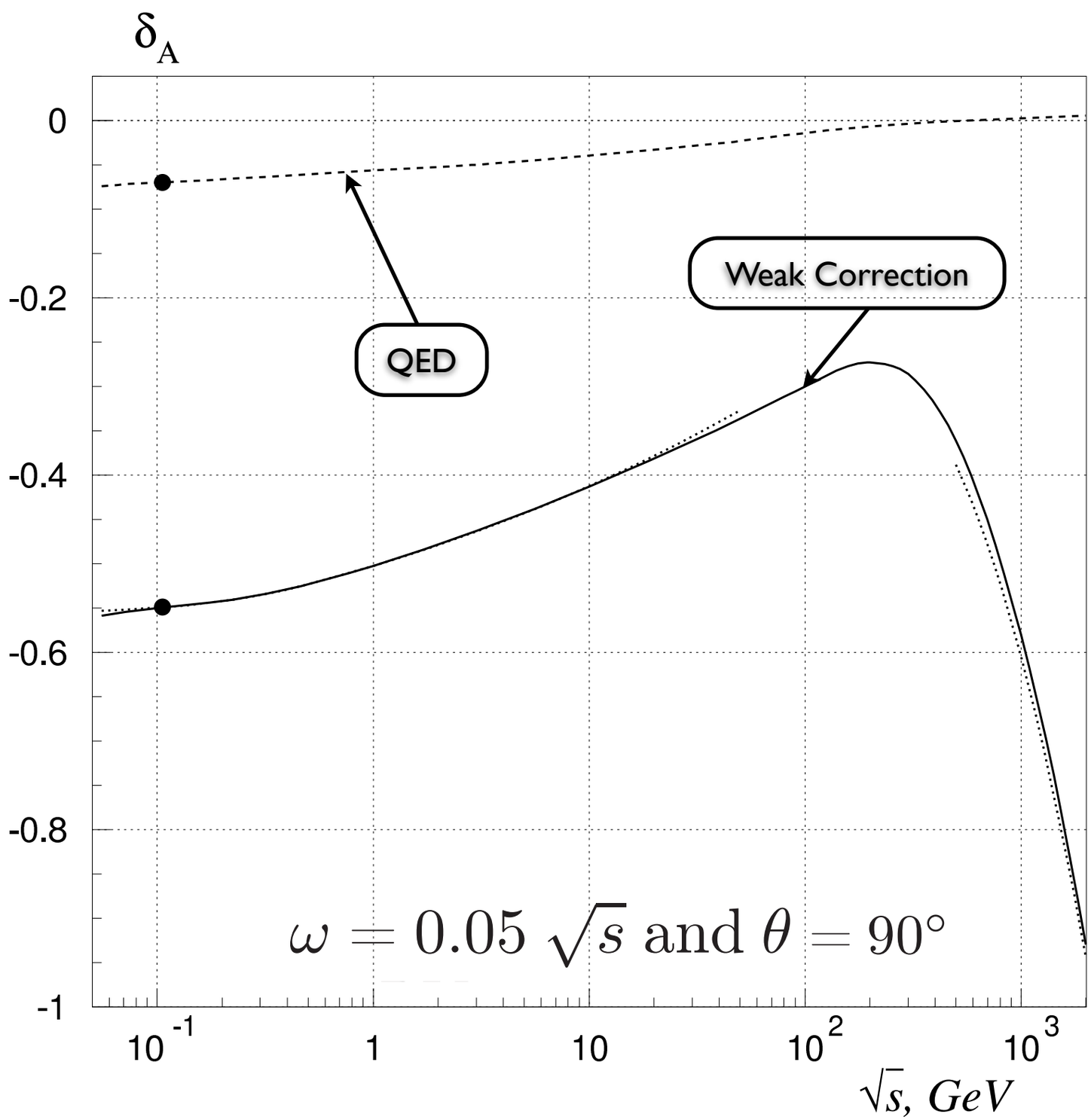
$$\sigma = \frac{\pi^3}{2s}|M_0 + M_1|^2 = \frac{\pi^3}{2s}(\underbrace{M_0 M_0^+}_{\propto \alpha^2} + \underbrace{2\mathrm{Re} M_1 M_0^+}_{\propto \alpha^3} + \underbrace{M_1 M_1^+}_{\propto \alpha^4}) = \sigma_0 + \sigma_1 + \sigma_Q$$

$$\sigma_1 = \sigma_1^{BSE} + \sigma_1^{Ver} + \sigma_1^{Box}$$

- Calculated in the on-shell renormalization, using both:
  - Computer-based approach, with Feynarts, FormCalc, LoopTools and Form
    T. Hahn, Comput. Phys. Commun. 140 418 (2001);
    T. Hahn, M. Perez-Victoria, Comput. Phys. Commun. 118, 153 (1999);
    J. Vermaseren, (2000) [arXiv:math-ph/0010025]
  - "On paper", with approximations in small energy region $\frac{\{t,u\}}{m_{Z,W}^2} \ll 1$, for $\sqrt{s} \ll 30\ GeV$ and high energy approximation for $\sqrt{s} \gg 500\ GeV$

A. Aleksejevs, S. Barkanova, A. Ilyichev, V. Zykunov, Phys. Rev. D82 (2010) 093013

One-Loop Corrections for MOLLER



$$\delta_A = \frac{A^C_{LR} - A^0_{LR}}{A^0_{LR}}$$

The relative weak (solid line in DRC (semi-automated) and dotted line in HRC ("on paper")) and QED (dashed line) corrections to the Born asymmetry A^0_{LR} versus $\sqrt{s}$ at $\theta = 90°$.

The filled circle corresponds to our predictions for the MOLLER experiment.

A. Aleksejevs, S. Barkanova, A. Ilyichev, Y. Kolomensky, V. Zykunov, Phys. Part. Nucl. 44(2013) 161-174

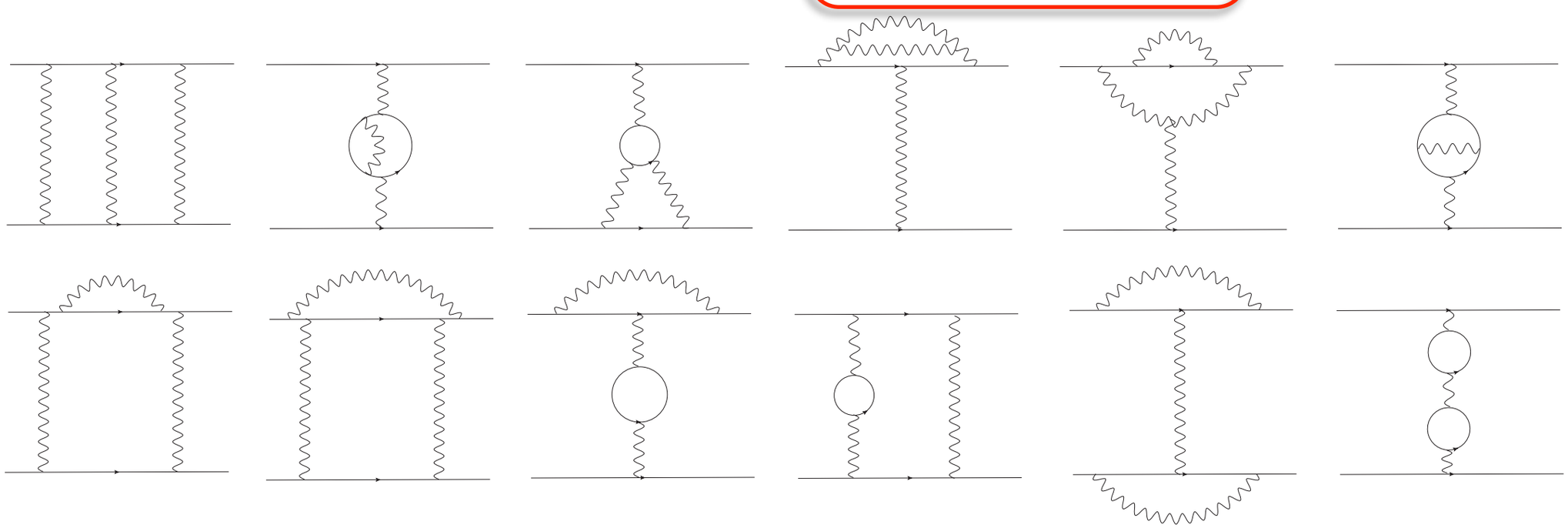
Second Stage: NNLO Corrections for MOLLER

The Next-to-Next-to-Leading Order (NNLO) EWC to the Born (~ $M_0M_0^+$) cross section can be divided into two classes:

- Q-part induced by quadratic one-loop amplitudes ~ $M_1M_1^+$, and
- T-part – the interference of Born and two-loop diagrams ~ $2\mathrm{Re}M_0M_{2\text{-loop}}^+$.

$$\sigma = \frac{\pi^3}{2s}|M_0 + M_1|^2 = \frac{\pi^3}{2s}(\underbrace{M_0M_0^+}_{\propto\,\alpha^2} + \underbrace{2\mathrm{Re}M_1M_0^+}_{\propto\,\alpha^3} + \underbrace{M_1M_1^+}_{\propto\,\alpha^4}) = \sigma_0 + \sigma_1 + \sigma_Q$$

$$\sigma_T = \frac{\pi^3}{s}\,\mathrm{Re}M_2M_0^+ \propto \alpha^4$$



+ …

Quadratic correction: IR part

Differential quadratic cross section σ_Q written as sums of λ-dependent (IRD-terms) and λ-independent (infrared-finite) parts:

$$\sigma_Q = \frac{\pi^3}{2s} M_1 M_1^+ = \underline{\sigma_Q^\lambda} + \underline{\sigma_Q^f}$$

$$\frac{\pi^3}{2s} M_1^{\lambda +} \left(M_1^\lambda + 2 M_1^f \right) = \frac{1}{4} \left(\frac{\alpha}{\pi} \right)^2 \mathrm{Re} \left[\delta_1^{\lambda *} (\delta_1^\lambda + 2 \delta_1^f) \right] \sigma_0 .$$

$$\left(\frac{\alpha}{\pi} \right)^2 \delta_Q^f \ \sigma_0$$

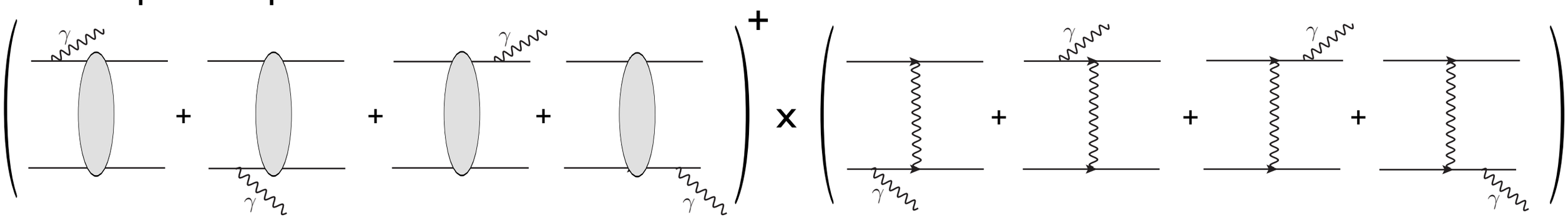
$$\delta_1^\lambda = 4B \log \frac{\lambda}{\sqrt{s}}$$

$$B = \log \frac{tu}{m^2 s} - 1 + i\pi$$

Quadratic correction: photon emission

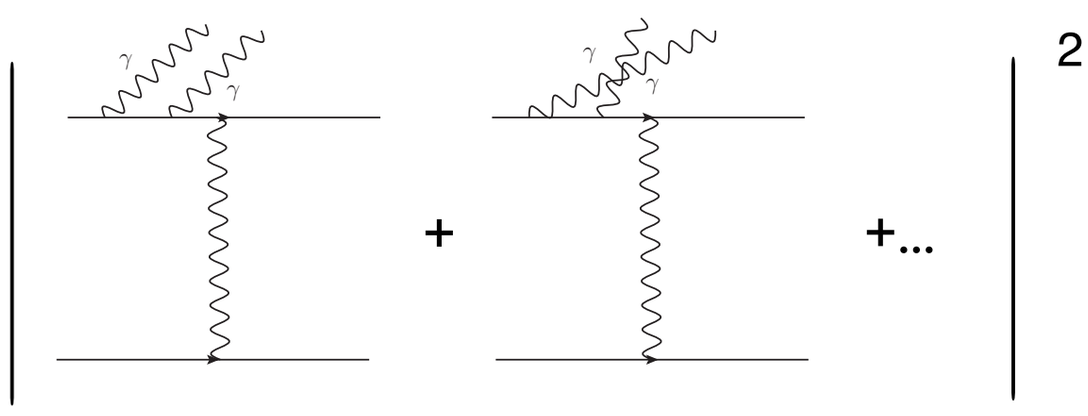
In order to remove the IR-divergent terms in quadratic cross section, wee need to consider:

1. Photon emission from one-loop diagrams
2. Two photon photon emission



$$\sigma_Q^\gamma = \frac{1}{2}\sigma^\gamma = \frac{\pi^2}{s}\ \mathrm{Re}\ [(-\delta_1^\lambda + R_1)^* M_1^+ M_0]$$

$$R_1 = -4B\log\frac{\sqrt{s}}{2\omega} - \log^2\frac{s}{em^2} + 1 - \frac{\pi^2}{3} + \log^2\frac{u}{t}$$



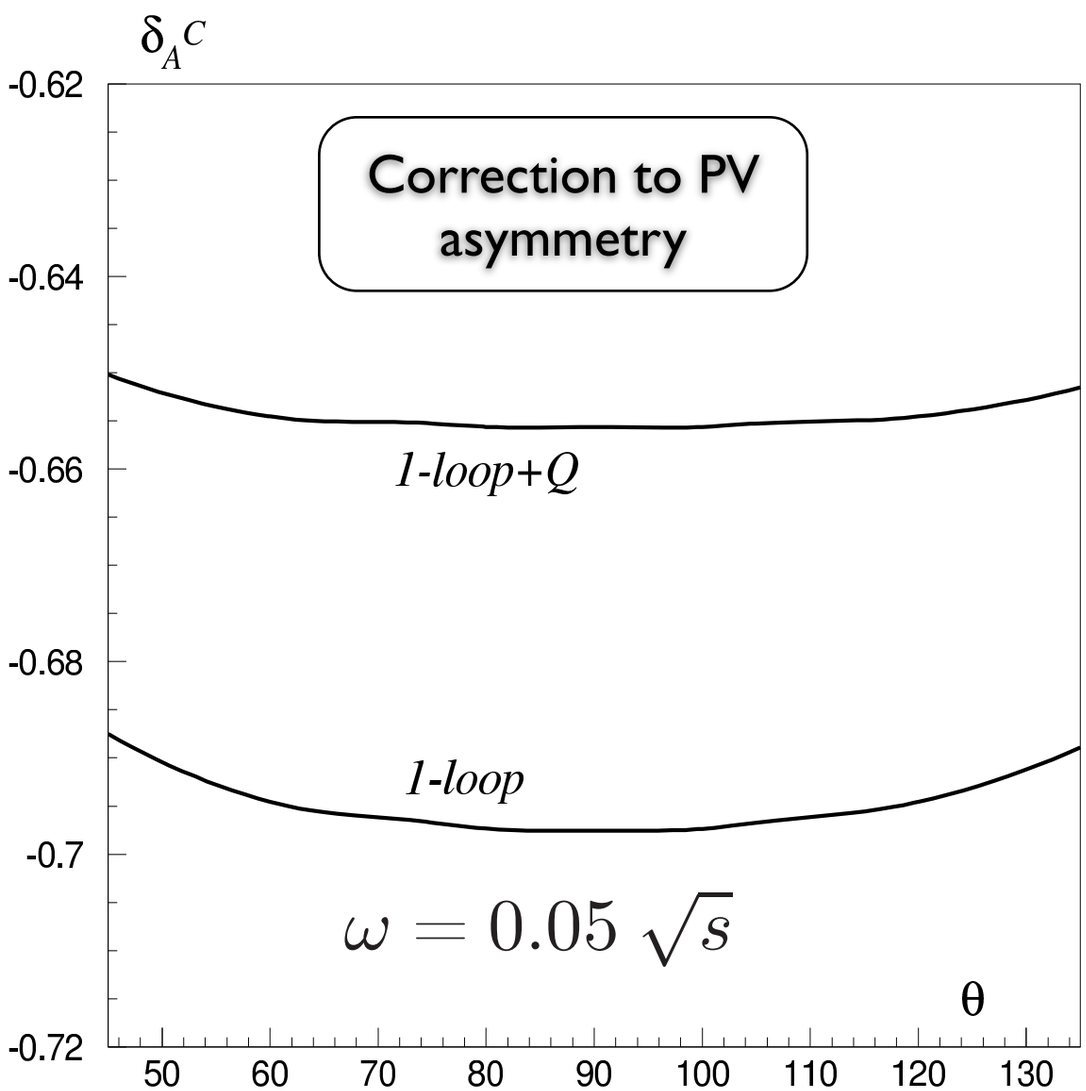
$$\sigma_Q^{\gamma\gamma} = \frac{1}{2}\sigma^{\gamma\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\left(\left|-\delta_1^\lambda + R_1\right|^2 - R_2\right)\sigma_0$$

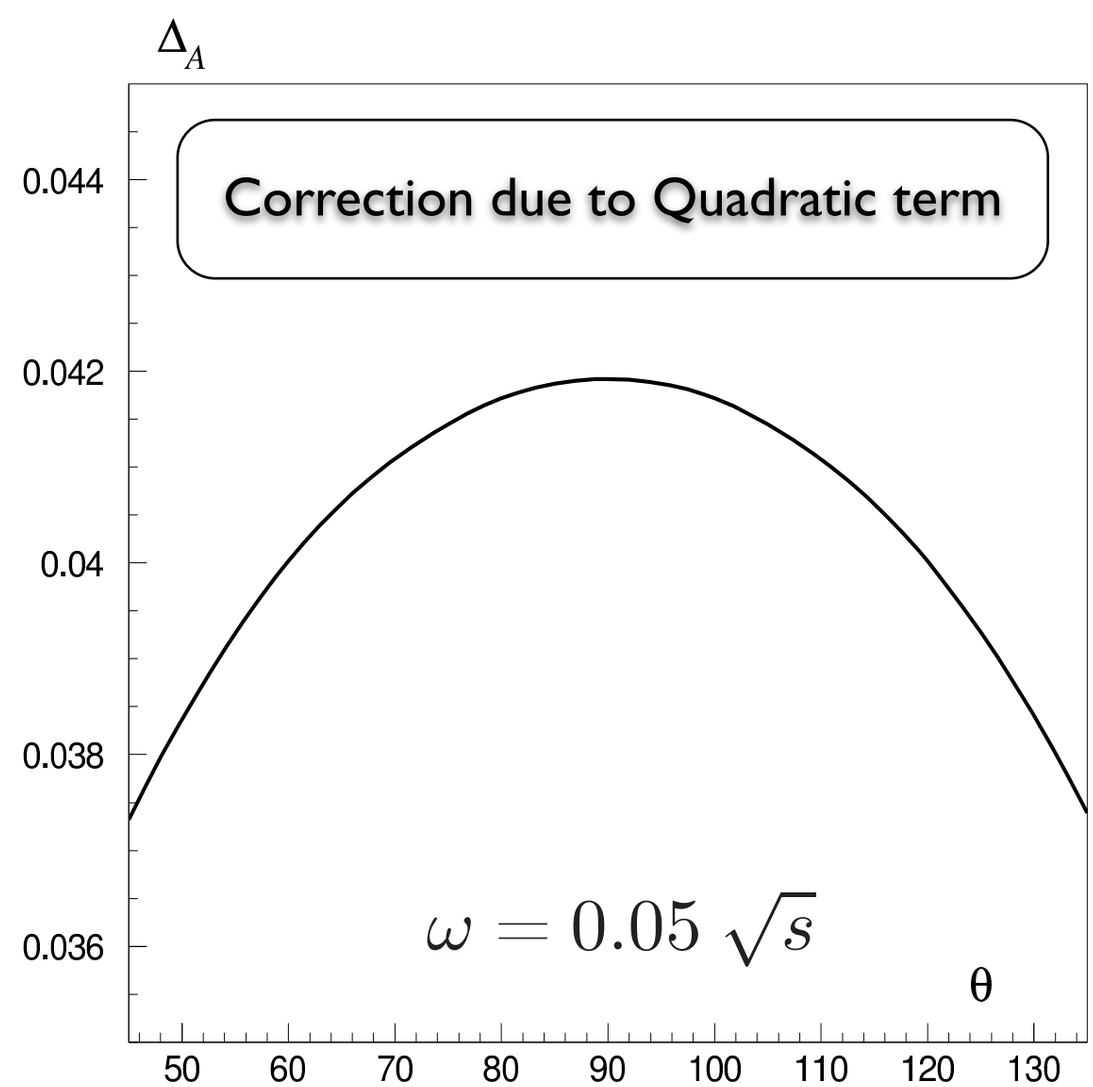
$$R_2 = \frac{8}{3}\pi^2(\log\frac{tu}{m^2 s} - 1)^2$$

Quadratic correction: results

$$\delta_A^{\mathrm{C}} = (A_{LR}^{\mathrm{C}} - A_{LR}^{0})/A_{LR}^{0}$$

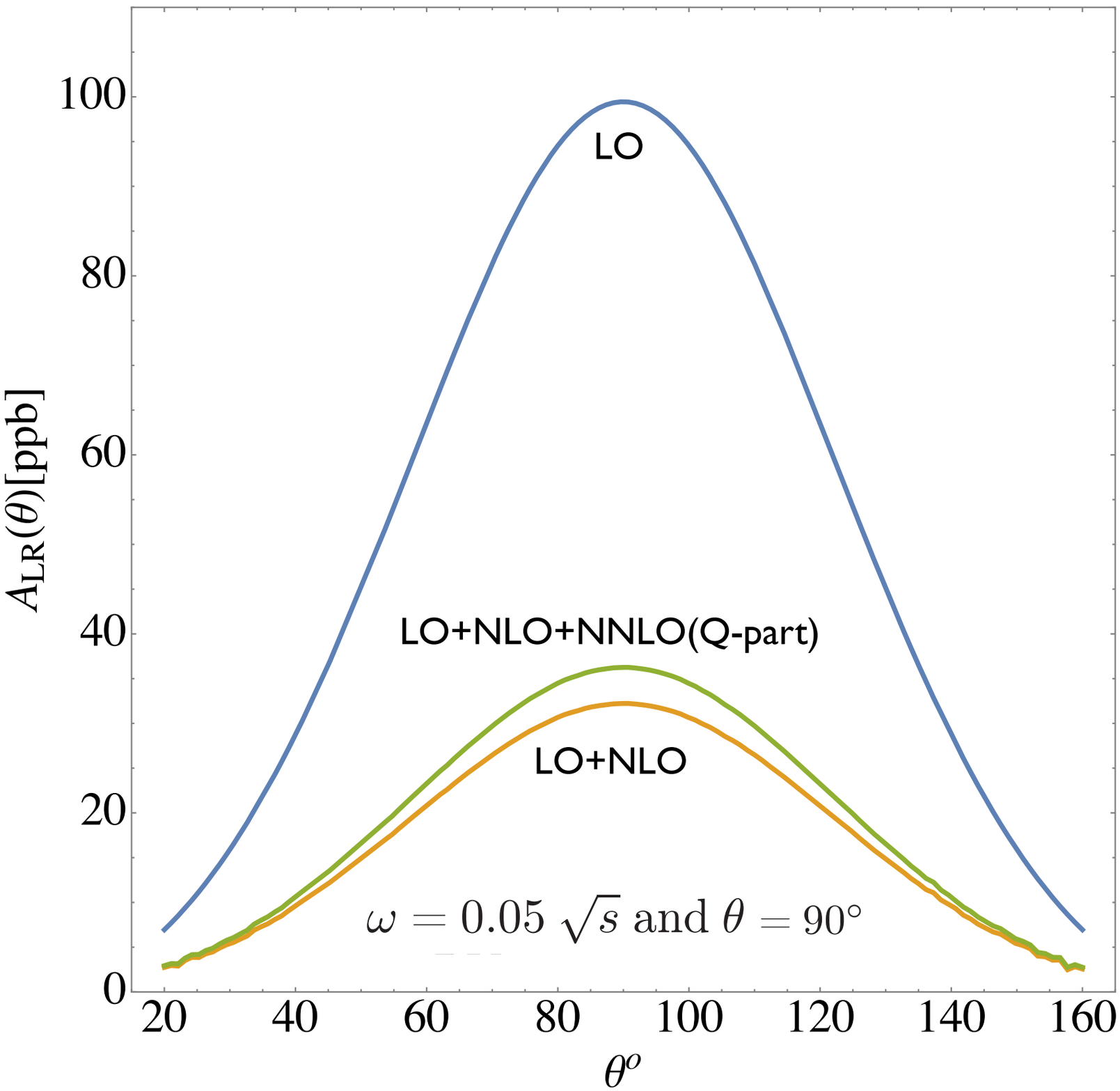
$$\Delta_A = (A_{LR}^{1-\mathrm{loop+Q}} - A_{LR}^{1-\mathrm{loop}})/A_{LR}^{0}$$





E_{lab} = 11 GeV

PV Asymmetry



Predicted PV asymmetry (E_{lab} = 11GeV):

$A_{PV}^{(LO+NLO)}$ (90°) = 32.2 (ppb)

$A_{PV}^{(LO+NLO+Q\text{-part})}$ (90°) = 36.2 (ppb)

$$m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu \sin^2\theta_W (1-\Delta r)}$$

Two-Loops Contribution

We split the two-loops contribution into subsets of the gauge invariant classes:

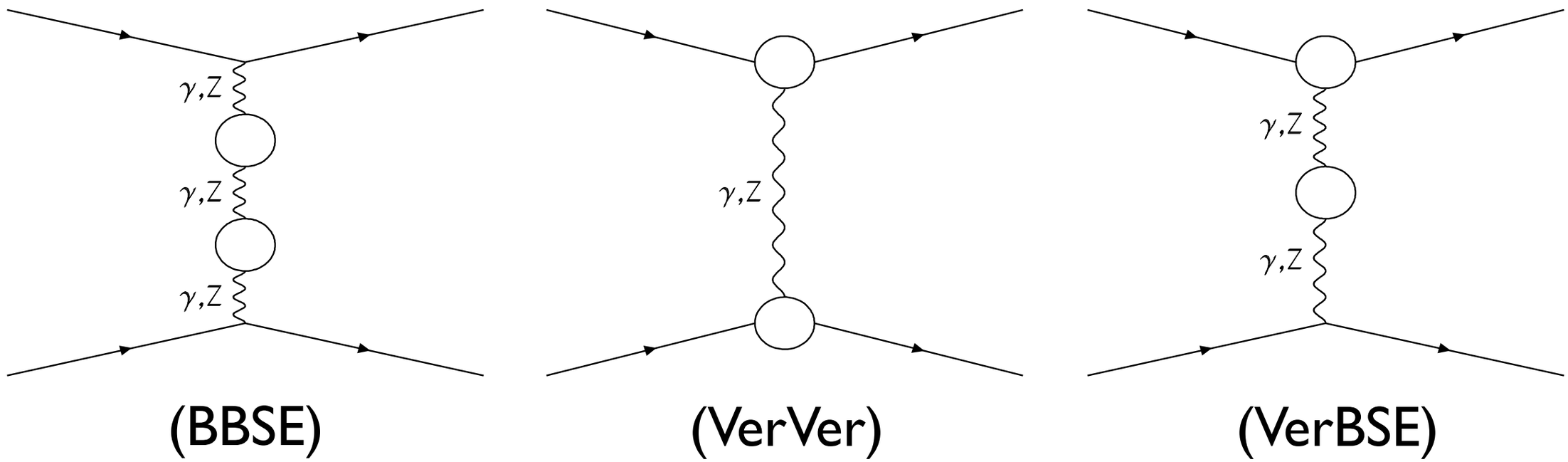
- Reducible contribution (BSE + Ver)^2.

- Irreducible ladder, decorated boxes and boxes with electron self-energies.

- Irreducible two-loops vertex correction (double vertices) and self energy diagrams.

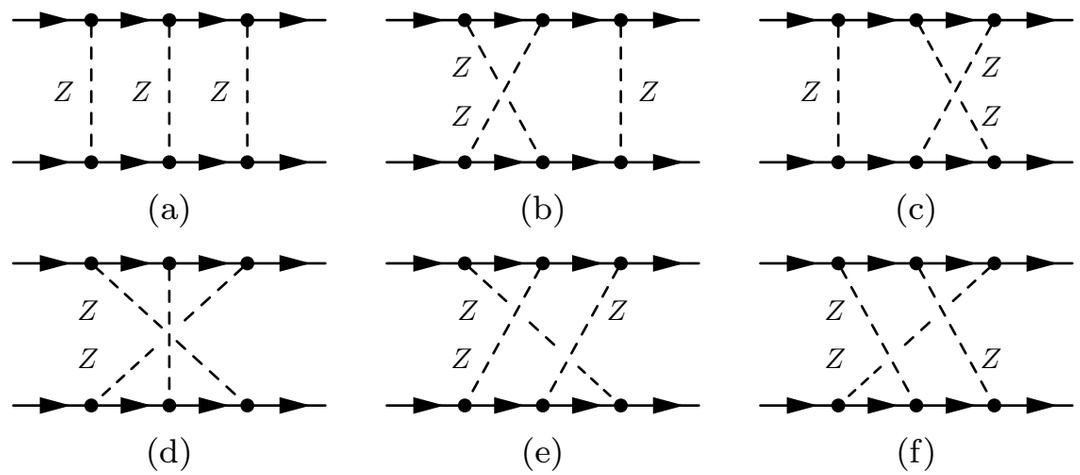
$$\sigma_T = \frac{\pi^3}{s}\ \mathrm{Re} M_2 M_0^+ \propto \alpha^4$$

(BSE+Ver)^2 Two-Loops Contribution

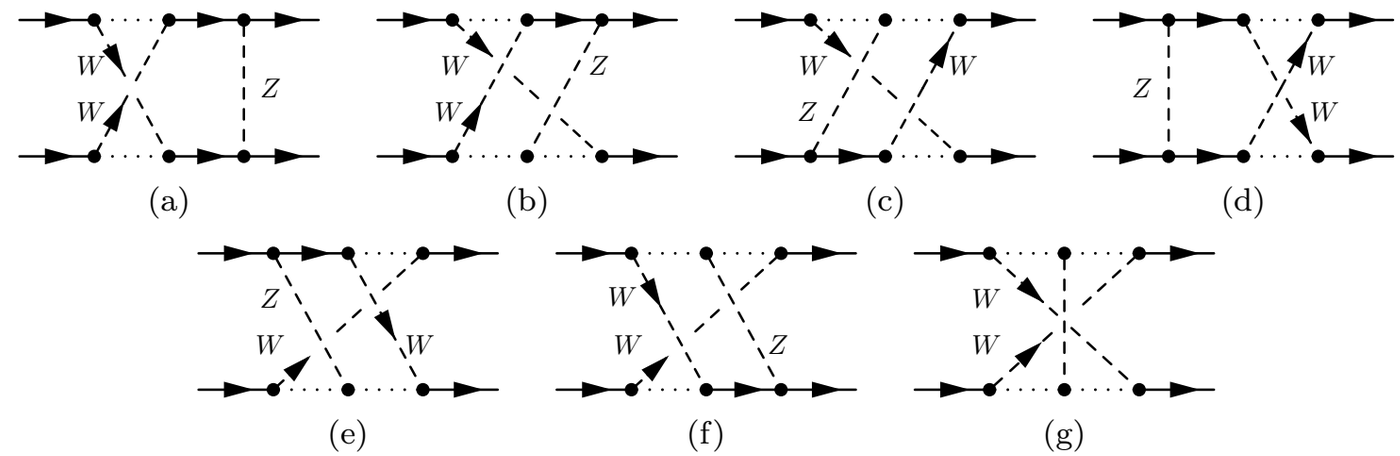


Two-loops t-channel diagrams from the gauge-invariant set of vertices and boson self-energies. Here, the circles represent the contributions of self-energies and vertex functions.

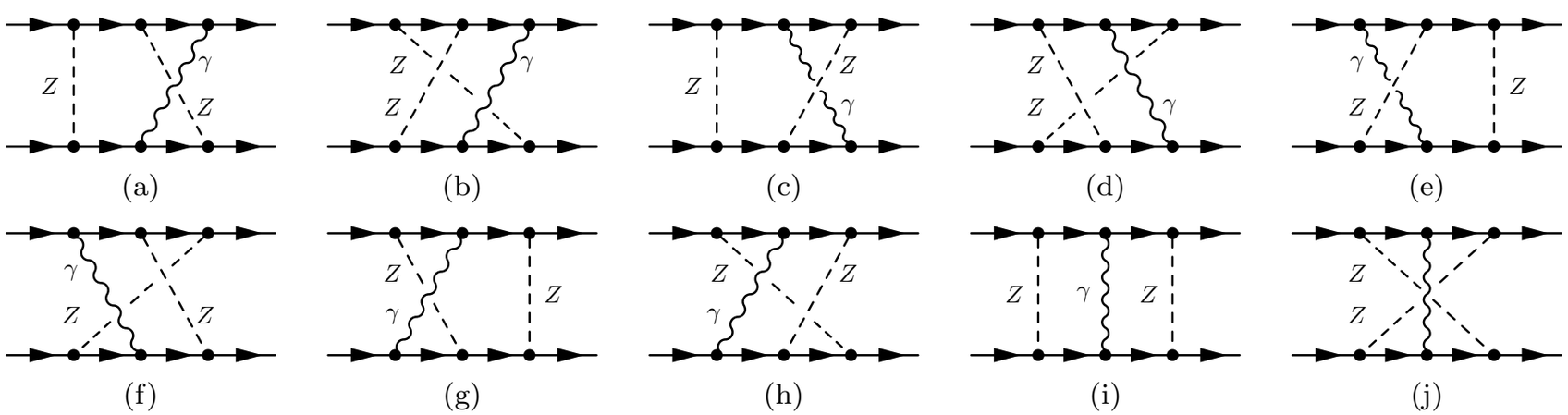
Ladder-Box Diagrams



Diagrams with ZZZ exchange.



Diagrams with WWZ exchange.



Diagrams with ZZγ exchange.

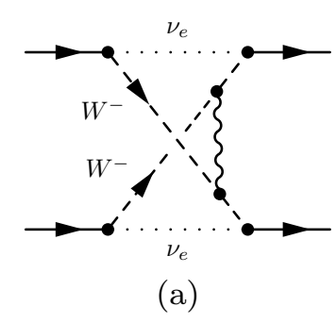
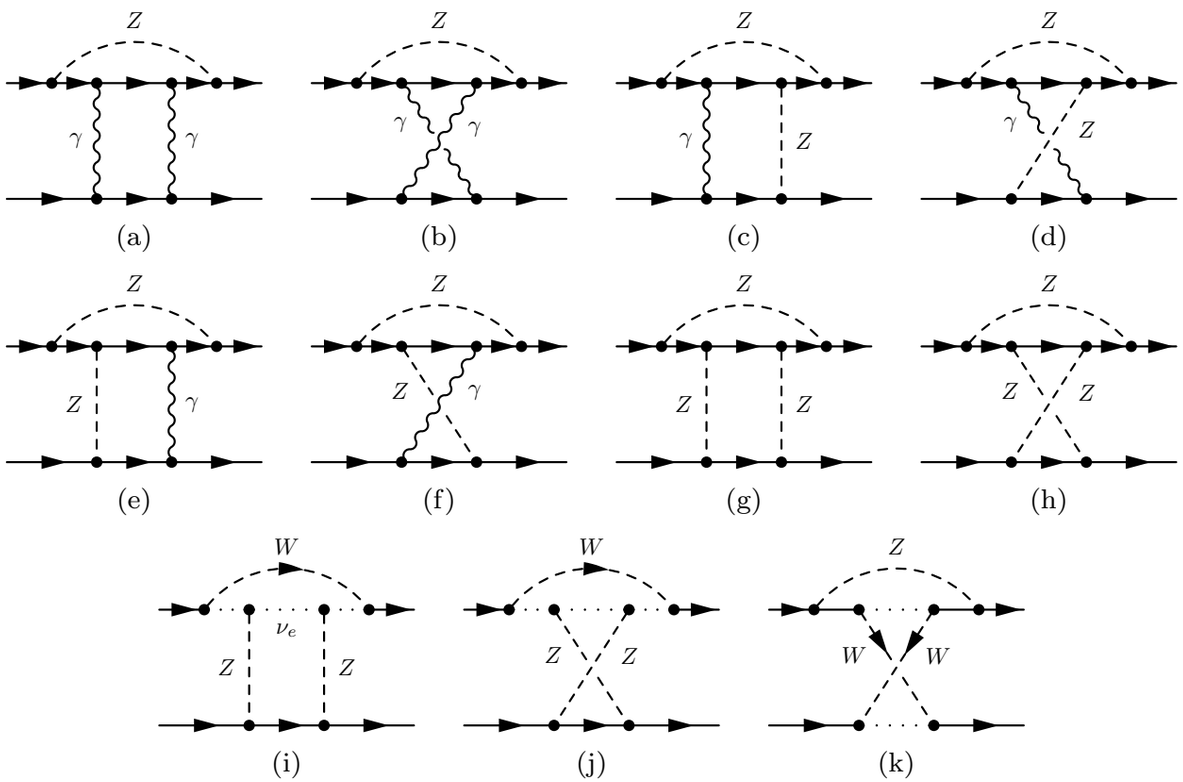


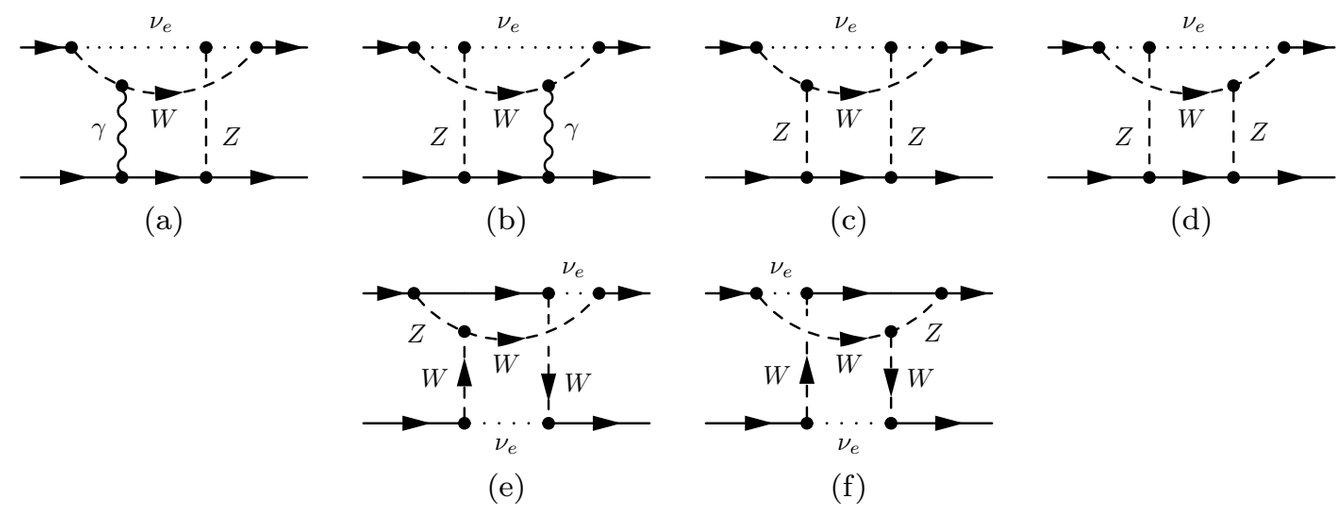
Diagram with W W γ exchange.

(Double Boxes)

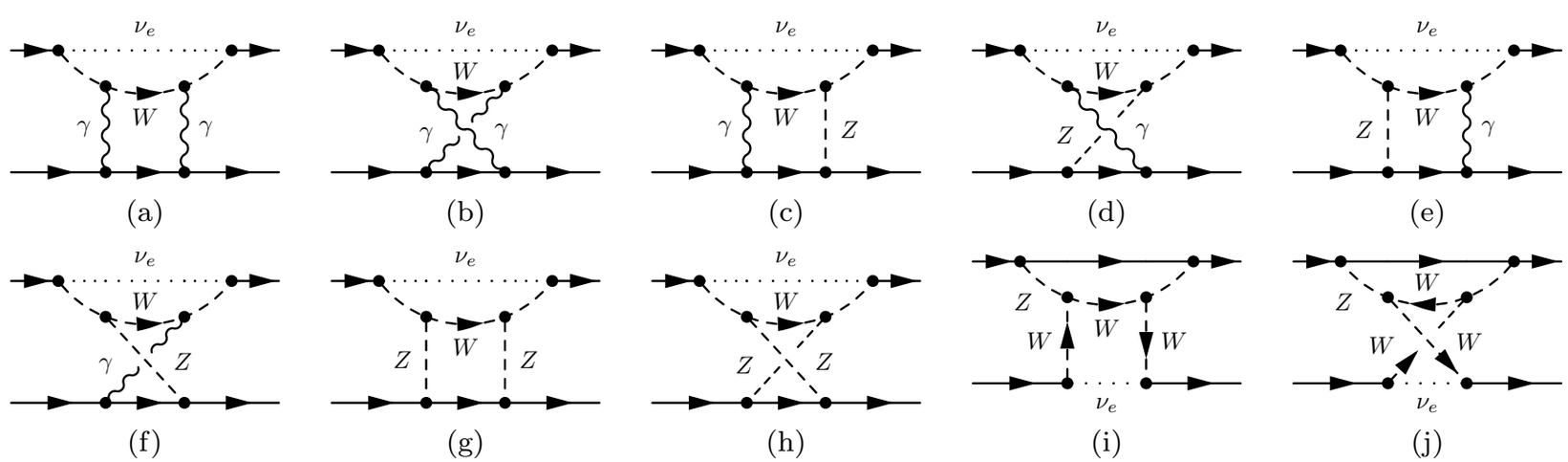
Decorated-Box Diagrams



Decorated boxes of type I.



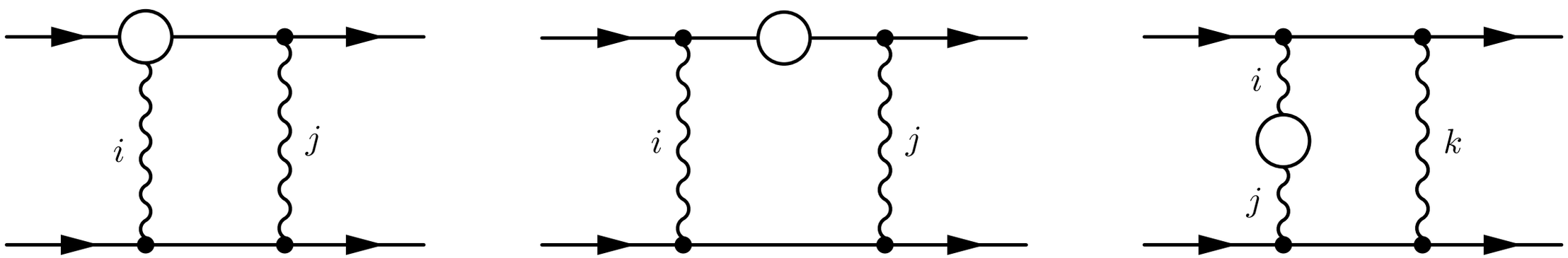
Decorated boxes of type III.



Decorated boxes of type II.

(Double Boxes)

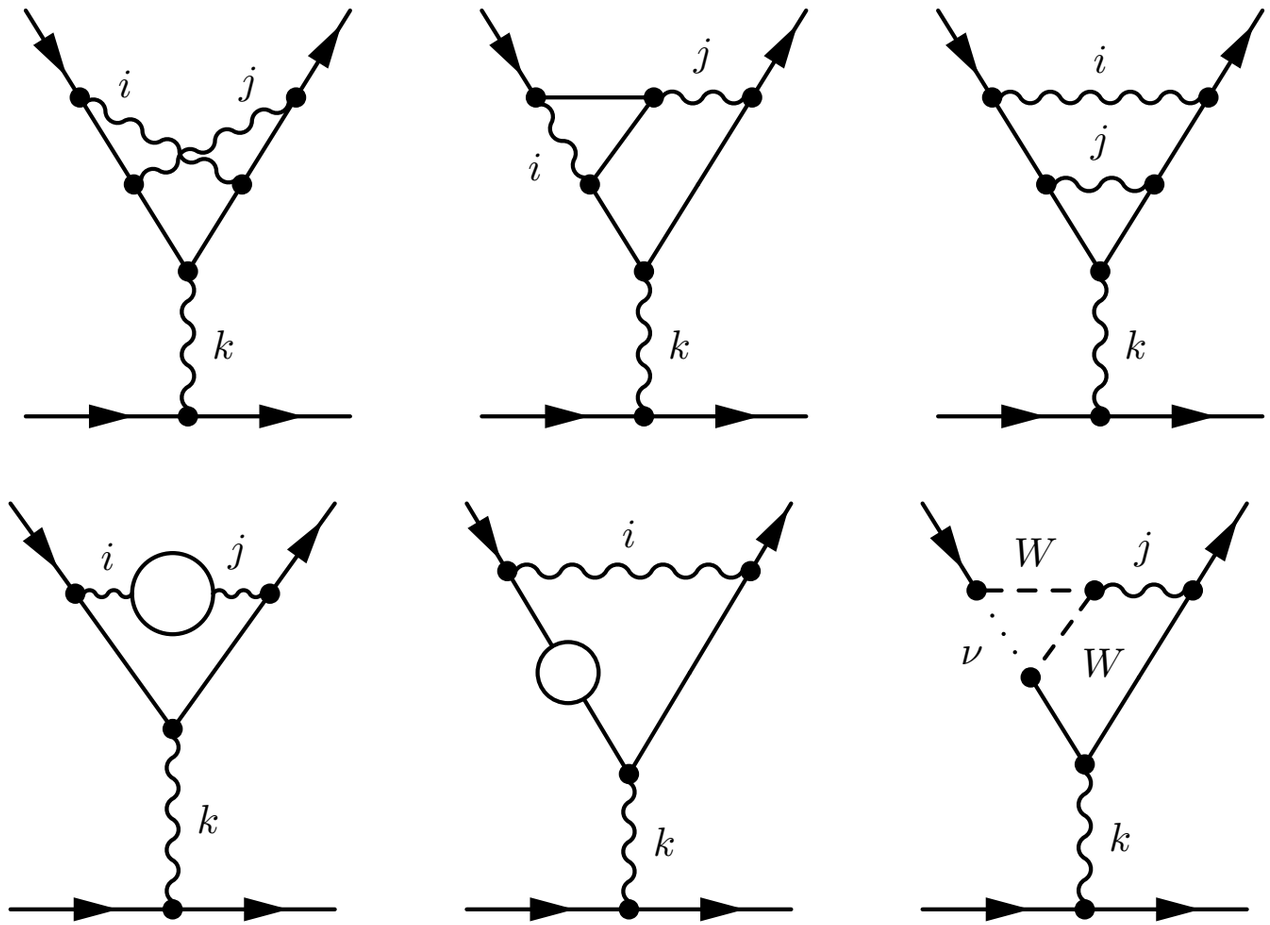
Boxes with Lepton Self-Energy and Vertex Insertions



Boxes with vertices (VB), fermion self-energy boxes FSEB and boson self-energy boxes BSEB.

(SE and Ver in boxes)

Double Vertices



Two loops electron vertices
(NNLO EW Vert)

Combination of Corrections

For the orthogonal kinematics: $\theta = 90^\circ$

Type of contribution	$\delta_A{}^C$
NLO	-0.6953
...+Q+ BBSE+VVer+	-0.6420
...+ double boxes	-0.6534
...+NNLO QED	-0.6500
...+SE and Ver in boxes	-0.6539
...+NNLO EW Ver	-0.6574

Correction to PV asymmetry:

$$\delta_A^C = \frac{A_{LR}^C - A_{LR}^0}{A_{LR}^0}$$

Soft-photon bremsstrahlung cut:

$$\omega = 0.05\sqrt{s}$$

"..." means all contributions from the lines above

A. Aleksejevs, S. Barkanova, Y. Kolomensky, E. Kuraev, V. Zykunov, Phys. Rev. D 85 (2012) 013007
A. Aleksejevs, S. Barkanova, Y. Kolomensky, E. Kuraev, V. Zykunov, Nuovo Cim. C035N04 (2012) 192-197
A. Aleksejevs, S. Barkanova, V. Zykunov, Phys. Atom. Nucl., 75(2012) 209-226
A. Aleksejevs, S. Barkanova, Y. Bystritskiy, A. Ilyichev, E. Kuraev, V. Zykunov, Phys. Rev. D 85 (2012) 013007
A. Aleksejevs, S. Barkanova, Y. Bystritskiy, E. Kuraev, V. Zykunov, Phys. Part. Nucl. Lett. 12(2015) 5 645-656
A. Aleksejevs, S. Barkanova, Y. Bystritskiy, E. Kuraev, V. Zykunov, Phys. of Part. and Nucl. Letters, (2016), 13-3, 310–317

Third Stage: Computer Algebra

- The most of the leading two-loop EWC corrections to Moller process has been completed.

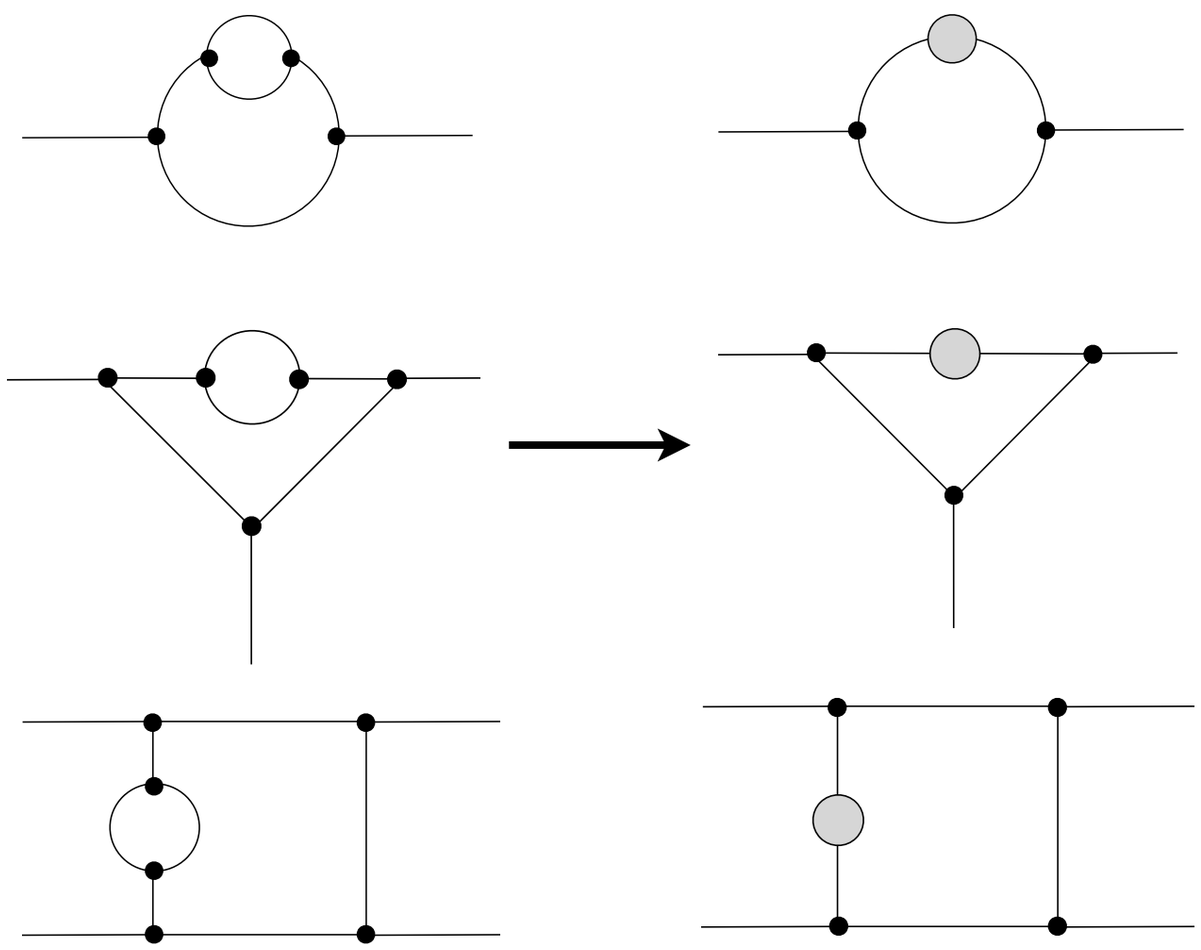
- It is essential to apply alternative approaches in two-loop EWC calculations for the cross-check purposes.

- We develop the third stage method which is based on the dispersive representation of many-point Passarino-Veltman functions.

- Advantages include not only cross checking previous results, but also our ability to retain kinematical dependence of two-loop EWC and inclusion of broader sets of two-loops graphs.

Sub-Loop Insertions: Self-Energy

W. Hollik, U. Meier, S. Uccirati, Nucl.Phys. B731 213-224 (2005)

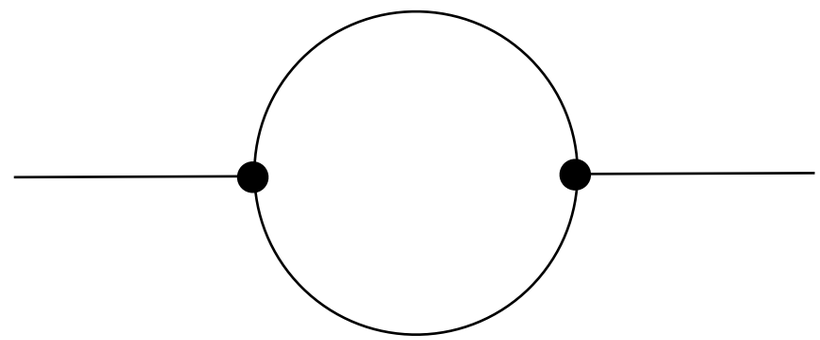


$$L(q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\Im L(s)}{s - q^2 - i\epsilon}$$

- Replace self-energy insertion by effective propagator

- Dispersive representation of self-energy sub-loop has propagator like structure with mass s

Self-Energy Sub-Loop



Vector boson: $\Sigma^{V-V}_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Sigma^{V-V}_T\left(q^2\right) + \frac{q_\mu q_\nu}{q^2}\Sigma^{V-V}_L\left(q^2\right)$

Fermion: $\Sigma^f(q) = \not{q}\omega_-\Sigma^f_L\left(q^2\right) + \not{q}\omega_+\Sigma^f_R\left(q^2\right) + m_f\Sigma^f_S\left(q^2\right)$

Each of the Σ terms are functions of:

$$B_{i,ij,ijk}\left(q^2, m_\alpha^2, m_\beta^2\right) = \frac{1}{\pi}\int_{\left(m_\alpha+m_\beta\right)^2}^{\infty} ds\frac{\Im B_{i,ij,ijk}\left(s, m_\alpha^2, m_\beta^2\right)}{s - q^2 - i\epsilon}$$

Sub-Loop: Vector Boson SE

First loop insertion:

$$\Sigma^{V-V}_{\mu\nu}(q) = \frac{1}{\pi} \sum_{\alpha,\beta} \int_{(m_\alpha+m_\beta)^2}^{\infty} ds \frac{1}{s-q^2-i\epsilon} \left[\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \Im\Sigma^{V-V}_T \left(s, m_\alpha^2, m_\beta^2\right) + \frac{q_\mu q_\nu}{q^2} \Im\Sigma^{V-V}_L \left(s, m_\alpha^2, m_\beta^2\right) \right]$$

Second loop integration:

$$I^{1,M,1}_{\mu_1\mu_2,\nu_1\ldots\nu_R} = \frac{1}{\pi} \frac{\mu^{(4-D)}}{\left(i\pi^{D/2}\right)} \sum_{\alpha,\beta} \int_{(m_\alpha+m_\beta)^2}^{\infty} ds \int d^D q_2 \cdot \frac{q_{2,\nu_1} \ldots q_{2,\nu_R}}{(s-q_2^2-i\epsilon) \prod_{j=0}^{M} \left[(q_2+k_{j,M})^2 - m_{j,M}^2\right]} F_{\mu_1\mu_2}\left(q_2, s, m_\alpha, m_\beta\right)$$

$$F_{\mu_1\mu_2}\left(q_2, s, m_\alpha, m_\beta\right) = \left(g_{\mu_1\mu_2} - \frac{q_{2\mu_1} q_{2\mu_2}}{q_2^2} \right) \Im\Sigma^{V-V}_T \left(s, m_\alpha^2, m_\beta^2\right) + \frac{q_{2\mu_1} q_{2\mu_2}}{q_2^2} \Im\Sigma^{V-V}_L \left(s, m_\alpha^2, m_\beta^2\right)$$

Sub-Loop: Fermion SE

First loop insertion:

$$\Sigma^f(q) = \frac{1}{\pi} \sum_{\alpha,\beta} \int_{(m_\alpha+m_\beta)^2}^{\infty} ds \frac{1}{s - q^2 - i\epsilon} \left[\not{q}\omega_- \Im\Sigma_L^f\left(s, m_\alpha^2, m_\beta^2\right) + \not{q}\omega_+ \Im\Sigma_R^f\left(s, m_\alpha^2, m_\beta^2\right) + m_f \Im\Sigma_S^f\left(s, m_\alpha^2, m_\beta^2\right) \right]$$

Second loop integration:

$$I_{\nu_1 \ldots \nu_R}^{1,M,1} = \frac{1}{\pi} \frac{\mu^{(4-D)}}{\left(i\pi^{D/2}\right)} \sum_{\alpha,\beta} \int_{(m_\alpha+m_\beta)^2}^{\infty} ds \int d^D q_2 \cdot \frac{q_{2,\nu_1} \ldots q_{2,\nu_R}}{(s - q_2^2 - i\epsilon) \prod_{j=0}^{M} \left[(q_2 + k_{j,M})^2 - m_{j,M}^2\right]} G\left(q_2, s, m_\alpha, m_\beta\right)$$

$$G\left(q_2, s, m_\alpha, m_\beta\right) = \left[\not{q}_2\omega_- \Im\Sigma_L^f\left(s, m_\alpha^2, m_\beta^2\right) + \not{q}_2\omega_+ \Im\Sigma_R^f\left(s, m_\alpha^2, m_\beta^2\right) + m_f \Im\Sigma_S^f\left(s, m_\alpha^2, m_\beta^2\right) \right]$$

Self-Energy Sub-Loop: General Structure

<u>Vector boson SE sub-loop insertion two-loop result:</u>

$$I^{1,M,1}_{\mu_1\mu_2,\nu_1\ldots\nu_R} = \frac{1}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\cdot\Big[L^{1,M,1}_{a,\mu_1\mu_2,\nu_1\ldots\nu_R}(B,C,D)\,\Im\Sigma^{V-V}_T\left(s,m_\alpha^2,m_\beta^2\right)+$$

$$L^{1,M,1}_{b,\mu_1\mu_2,\nu_1\ldots\nu_R}(B,C,D)\,\Im\Sigma^{V-V}_L\left(s,m_\alpha^2,m_\beta^2\right)\Big].$$

<u>Fermion SE sub-loop insertion two-loop result:</u>

$$I^{1,M,1}_{\nu_1\ldots\nu_R} = \frac{1}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\cdot\Big[N^{1,M,1}_{a,\nu_1\ldots\nu_R}(B,C,D)\,\Im\Sigma^{f}_L\left(s,m_\alpha^2,m_\beta^2\right)\omega_-+$$

$$N^{1,M,1}_{b,\nu_1\ldots\nu_R}(B,C,D)\,\Im\Sigma^{f}_R\left(s,m_\alpha^2,m_\beta^2\right)\omega_+ + N^{1,M,1}_{c,\nu_1\ldots\nu_R}(B,C,D)\,\Im\Sigma^{f}_S\left(s,m_\alpha^2,m_\beta^2\right)\Big]$$

Vector Boson SE Subtractions

{Z-Z} or {W-W} mixings:

$$\hat{\Sigma}^{V-V}(q^2) = \Sigma^{V-V}(q^2) - \Sigma^{V-V}(m_V^2) - \frac{\partial}{\partial q^2}\Sigma^{V-V}(q^2)\Big|_{q^2=m_V^2}\left(q^2 - m_V^2\right) =$$

$$\frac{\left(q^2 - m_V^2\right)^2}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\,\frac{\Im\Sigma^{V-V}(s, m_\alpha, m_\beta)}{\left(s - m_V^2\right)^2\left(s - q^2 - i\epsilon\right)}$$

ϒ-Z mixing:

$$\hat{\Sigma}^{\gamma-Z}(q^2) = \Sigma^{\gamma-Z}(q^2) - \frac{1}{m_Z^2}\left[\Sigma^{\gamma-Z}(0)q^2 - \Sigma^{\gamma-Z}(m_Z^2)\left(q^2 - m_Z^2\right)\right] =$$

$$\frac{q^2\left(q^2 - m_Z^2\right)}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\,\frac{\Im\Sigma^{\gamma-Z}(s, m_\alpha, m_\beta)}{s\left(s - m_Z^2\right)\left(s - q^2 - i\epsilon\right)}$$

ϒ-ϒ mixing:

$$\hat{\Sigma}^{\gamma-\gamma}(q^2) = \frac{q^4}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\,\frac{\Im\Sigma^{\gamma-\gamma}(s, m_\alpha, m_\beta)}{s^2\left(s - q^2 - i\epsilon\right)}$$

Fermion SE Subtractions

$$\hat{\Sigma}^f(q) = \Sigma^f(\not{q}) - \Sigma^f(m_f) - \frac{\partial}{\partial \not{q}}\Sigma^f(\not{q})\Big|_{\not{q}=m_f}(\not{q} - m_f) =$$

$$\not{q}\omega_-\left(I_L + a_L\right) + \not{q}\omega_+\left(I_R + a_R\right) + m_f\left(I_S + a_S\right)$$

$$I_{L,R,S} = \frac{q^2 - m_f^2}{\pi}\sum_{\alpha,\beta}\int_{(m_\alpha+m_\beta)^2}^{\infty} ds\,\frac{\Im\Sigma^f_{L,R,S}(s, m_\alpha, m_\beta)}{\left(s - m_f^2\right)\left(s - q^2 - i\epsilon\right)}$$

$$a_{L,R} = -2m_f^2\left(\Sigma'_{L,R}\left(m_f^2\right) + \Sigma'_S\left(m_f^2\right)\right)$$

$$a_S = m_f^2\left(\Sigma'_L\left(m_f^2\right) + \Sigma'_R\left(m_f^2\right) + 2\Sigma'_S\left(m_f^2\right)\right)$$

Effective SE Propagators

Vector boson effective propagator:

$$\Pi^{V-V}_{\mu\nu}(q) = \Pi^{V-V}_{T,\mu\nu} + \Pi^{V-V}_{L,\mu\nu}$$

$$\Pi^{V-V}_{T,\mu\nu} = \frac{-ig_{\rho\mu}}{q^2 - m_V^2}\left[\frac{g^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}}{s - q^2 - i\epsilon}\Im\Sigma^{V-V}_T\left(s, m_\alpha^2, m_\beta^2\right)\right]\frac{-ig_{\sigma\nu}}{q^2 - m_V^2}$$

$$\Pi^{V-V}_{L,\mu\nu} = \frac{-ig_{\rho\mu}}{q^2 - m_V^2}\left[\frac{\frac{q^\rho q^\sigma}{q^2}}{s - q^2 - i\epsilon}\Im\Sigma^{V-V}_L\left(s, m_\alpha^2, m_\beta^2\right)\right]\frac{-ig_{\sigma\nu}}{q^2 - m_V^2}$$

Fermion effective propagator:

$$\Pi^f(q) = \frac{1}{\not{q} - m_f}\left[\frac{G\left(q, s, m_\alpha, m_\beta\right)}{s - q^2 - i\epsilon}\right]\frac{1}{\not{q} - m_f}$$

Effective Propagators: Subtracted VB

$$\hat{\Pi}^{V-V}_{\mu\nu}(q) = \hat{\Pi}^{V-V}_{T,\mu\nu} + \hat{\Pi}^{V-V}_{L,\mu\nu}$$

$$\hat{\Pi}^{V-V}_{T,\mu\nu} = -T^{V-V}\left(s, m_V^2\right)\left[\frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}}{s - q^2 - i\epsilon}\right]\Im\Sigma_T^{V-V}\left(s, m_\alpha^2, m_\beta^2\right)$$

$$\hat{\Pi}^{V-V}_{L,\mu\nu} = -T^{V-V}\left(s, m_V^2\right)\left[\frac{\frac{q_\mu q_\nu}{q^2}}{s - q^2 - i\epsilon}\right]\Im\Sigma_L^{V-V}\left(s, m_\alpha^2, m_\beta^2\right)$$

	$\gamma - \gamma$	$\{Z, W\} - \{Z, W\}$	$\gamma - Z$
T^{V-V}	$\frac{1}{s^2}$	$\frac{1}{\left(s - m^2_{\{Z,W\}}\right)^2}$	$\frac{1}{s\left(s - m_z^2\right)}$

Effective Propagators: Subtracted FF

$$\hat{\Pi}^f(q) = \hat{\Pi}_1^f(q) + \hat{\Pi}_2^f(q)$$

$$\hat{\Pi}_1^f(q) = (\not{q} + m_f)\left[\frac{y_L \not{q}\omega_- + y_R \not{q}\omega_+ + m_f y_S}{\left(q^2 - m_f^2\right)\left(s - q^2 - i\epsilon\right)}\right](\not{q} + m_f)$$

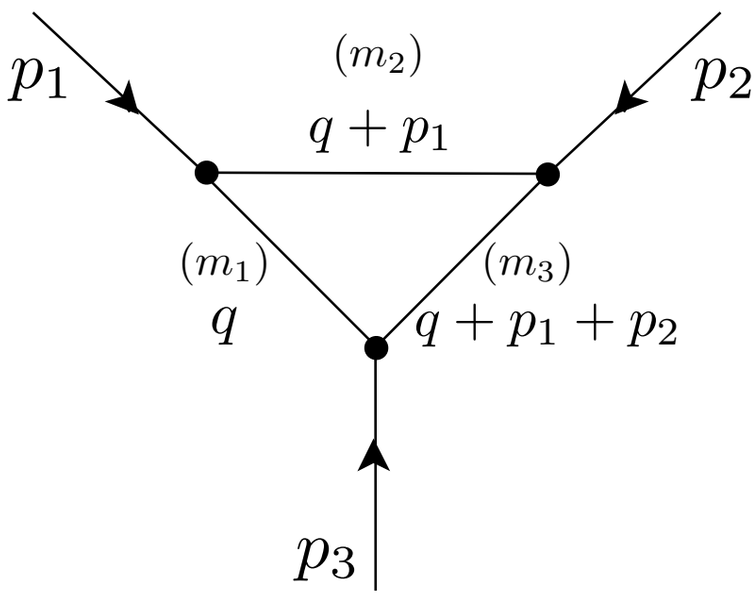
$$\hat{\Pi}_2^f(q) = \frac{1}{\not{q} - m_f}\left[d_L \not{q}\omega_- + d_R \not{q}\omega_+ + m_f d_S\right]\frac{1}{\not{q} - m_f},$$

$$y_{L,R,S} \equiv y_{L,R,S}\left(s, m_\alpha^2, m_\beta^2\right) = \frac{\Im\Sigma_{L,R,S}^f}{s - m_f^2}$$

$$d_{L,R} \equiv d_{L,R}\left(s, m_\alpha^2, m_\beta^2\right) = -2m_f^2 \frac{\Im\Sigma_{L,R}^f + \Im\Sigma_S^f}{\left(s - m_f^2\right)^2}$$

$$d_S \equiv d_S\left(s, m_\alpha^2, m_\beta^2\right) = m_f^2 \frac{\Im\Sigma_L^f + \Im\Sigma_R^f + 2\Im\Sigma_S^f}{\left(s - m_f^2\right)^2}$$

Second Loop Integration: Many-Point PV



$$C_0 \equiv C_0\left(p_1^2, p_2^2, (p_1+p_2)^2, m_1^2, m_2^2, m_3^2\right) =$$

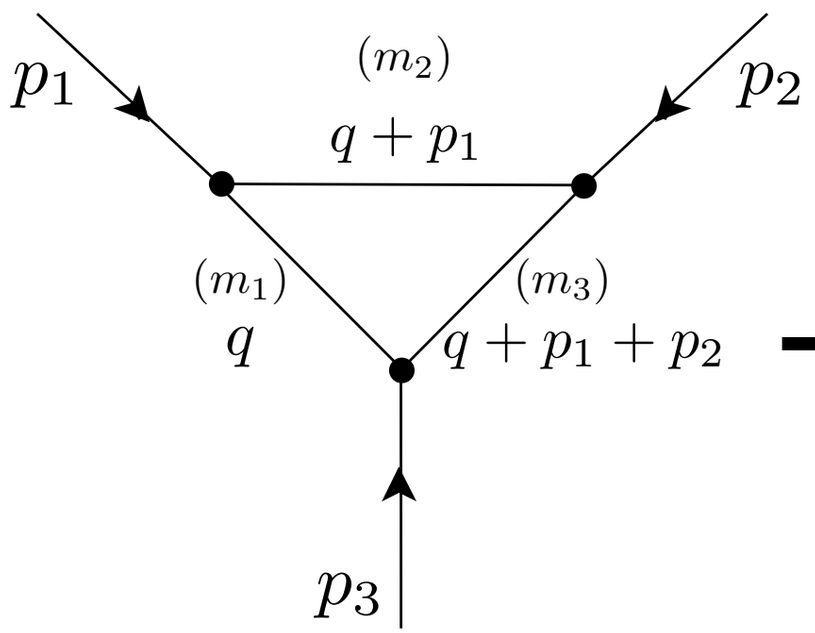
$$\frac{\mu^{4-D}}{i\pi^{D/2}} \int d^D q \frac{1}{\left[q^2 - m_1^2\right]\left[(q+p_1)^2 - m_2^2\right]\left[(q+p_1+p_2)^2 - m_3^2\right]}$$

$$C_0 = \frac{\mu^{4-D}}{i\pi^{D/2}} \int_0^1 dx \int d^D\tau \frac{1}{\left[(\tau - (p_1\bar{x} + p_2))^2 - m_{12}^2\right]^2 \left[\tau^2 - m_3^2\right]}, \text{ here } \quad m_{12}^2 = m_1^2\bar{x} + m_2^2 x - p_1^2 x\bar{x}$$

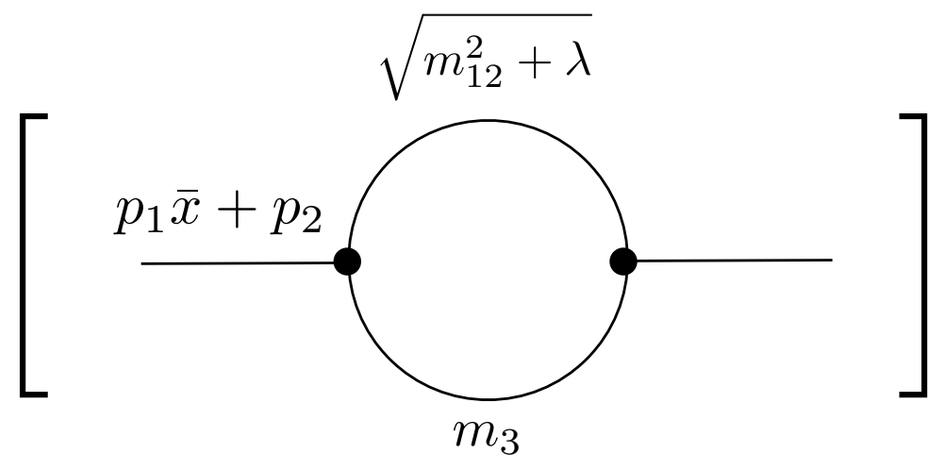
Using: $$\frac{1}{\left[(\tau - (p_1\bar{x} + p_2))^2 - m_{12}^2\right]^2} = \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \frac{1}{\left[(\tau - (p_1\bar{x} + p_2))^2 - (m_{12}^2 + \lambda)\right]}$$

$$C_0 = \frac{\mu^{4-D}}{i\pi^{D/2}} \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \int_0^1 dx \int d^D\tau \frac{1}{\left[(\tau - (p_1\bar{x} + p_2))^2 - (m_{12}^2 + \lambda)\right]\left[\tau^2 - m_3^2\right]} = \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \int_0^1 dx\ B_0\left((p_1\bar{x} + p_2)^2, m_3^2, m_{12}^2 + \lambda\right)$$

Second Loop Integration: Many-Point PV



$$\lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx$$



Second Loop Integration: Many-Point PV

$$C_{\mu_1 \ldots \mu_N} = \frac{\mu^{4-D}}{i\pi^{D/2}} \int d^D q \frac{q_{\mu_1} \ldots q_{\mu_N}}{\left[q^2 - m_1^2\right]\left[(q+p_1)^2 - m_2^2\right]\left[(q+p_1+p_2)^2 - m_3^2\right]}$$

<u>Vector case:</u>

Applying tensor decomposition to both sides of above equation:

$$p_{1\mu} C_1 + (p_{1\mu} + p_{2\mu}) C_2 = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \left[B_\mu - (p_{1\mu} + p_{2\mu}) B_0\right]$$

$$\downarrow$$

$$B_\mu = -(p_{1\mu}\bar{x} + p_{2\mu}) B_1$$

Matching coefficients in front of $p_{1,2}$, we get:

$$C_1 = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx B_1 x \qquad C_2 = -\lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \left[B_0 + B_1\right]$$

Second Loop Integration: Many-Point PV

$$\hat{\mathbf{I}}_C = \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \int_0^1 dx ...$$

$C_{\mu\nu}$:

$$C_{00} = \hat{\mathbf{I}}_C \left[B_{00}\right] \qquad C_{12} = -\hat{\mathbf{I}}_C \left[\left(B_1 + B_{11}\right) x\right]$$

$$C_{11} = \hat{\mathbf{I}}_C \left[B_{11} x^2\right] \qquad C_{22} = \hat{\mathbf{I}}_C \left[B_0 + 2B_1 + B_{11}\right]$$

$C_{\mu\nu\alpha}$:

$$C_{001} = \hat{\mathbf{I}}_C \left[B_{001} x\right] \qquad C_{112} = -\hat{\mathbf{I}}_C \left[\left(B_{11} + B_{111}\right) x^2\right]$$

$$C_{002} = -\hat{\mathbf{I}}_C \left[B_{00} + B_{001}\right] \qquad C_{122} = \hat{\mathbf{I}}_C \left[\left(B_1 + 2B_{11} + B_{111}\right) x\right]$$

$$C_{111} = \hat{\mathbf{I}}_C \left[B_{111} x^3\right] \qquad C_{222} = -\hat{\mathbf{I}}_C \left[B_0 + 3\left(B_1 + B_{11}\right) + B_{111}\right]$$

$$B_{i,ij,ijk} \equiv B_{i,ij,ijk} \left[\left(p_1 \bar{x} + p_2\right)^2, m_3^2, m_{12}^2 + \lambda\right]$$

Second Loop Integration: Many-Point PV

$$D_0 \equiv D_0\left(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2\right) =$$

$$\frac{\mu^{4-D}}{i\pi^{D/2}} \int d^D q \frac{1}{\left[q^2 - m_1^2\right]\left[(q+p_1)^2 - m_2^2\right]\left[(q+p_1+p_2)^2 - m_3^2\right]\left[(q+p_1+p_2+p_3)^2 - m_4^2\right]}$$

$$D_0 = 2\frac{\mu^{4-D}}{i\pi^{D/2}} \int_0^1 dx \int_0^{1-x} dy \int d^D\tau \frac{1}{\left[\left(\tau - \left(p_1\left(\bar{x}-y\right) + p_2\bar{y} + p_3\right)\right)^2 - m_{123}^2\right]^3 \left[\tau^2 - m_4^2\right]}$$

$$m_{123}^2 = m_1^2\left(\bar{x}-y\right) + m_2^2 x + m_3^2 y - p_1^2 x\bar{x} - p_{12}^2 y\bar{y} + 2xy\left(p_1 p_{12}\right)$$

$$\downarrow$$

$$p_{12} = p_1 + p_2$$

$$D_0 = \lim_{\lambda\to 0} \frac{\partial^2}{\partial\lambda^2} \int_0^1 dx \int_0^{\bar{x}} dy B_0\left[\left(p_1\left(\bar{x}-y\right) + p_2\bar{y} + p_3\right)^2, m_4^2, m_{123}^2 + \lambda\right]$$

Second Loop Integration: Many-Point PV

$$\hat{\mathbf{I}}_D = \lim_{\lambda\to 0} \frac{\partial^2}{\partial\lambda^2} \int_0^1 dx \int_0^{\bar{x}} dy ...$$

$$B_{i,ij,ijk,ijkl} \equiv B_{i,ij,ijk,ijkl}\left[\left(p_1\left(\bar{x}-y\right)+p_2\bar{y}+p_3\right)^2, m_4^2, m_{123}^2+\lambda\right]$$

D_μ :

$$D_1 = \hat{\mathbf{I}}_D\left[B_1 x\right]$$
$$D_2 = \hat{\mathbf{I}}_D\left[B_1 y\right]$$
$$D_3 = -\hat{\mathbf{I}}_D\left[B_0 + B_1\right]$$

$D_{\mu\nu}$:

$$D_{00} = \hat{\mathbf{I}}_D\left[B_{00}\right]$$
$$D_{11} = \hat{\mathbf{I}}_D\left[B_{11}x^2\right]$$
$$D_{22} = \hat{\mathbf{I}}_D\left[B_{11}y^2\right]$$
$$D_{33} = \hat{\mathbf{I}}_D\left[B_0 + 2B_1 + B_{11}\right]$$
$$D_{12} = \hat{\mathbf{I}}_D\left[B_{11}xy\right]$$
$$D_{13} = -\hat{\mathbf{I}}_D\left[\left(B_1 + B_{11}\right)x\right]$$
$$D_{23} = -\hat{\mathbf{I}}_D\left[\left(B_1 + B_{11}\right)y\right].$$

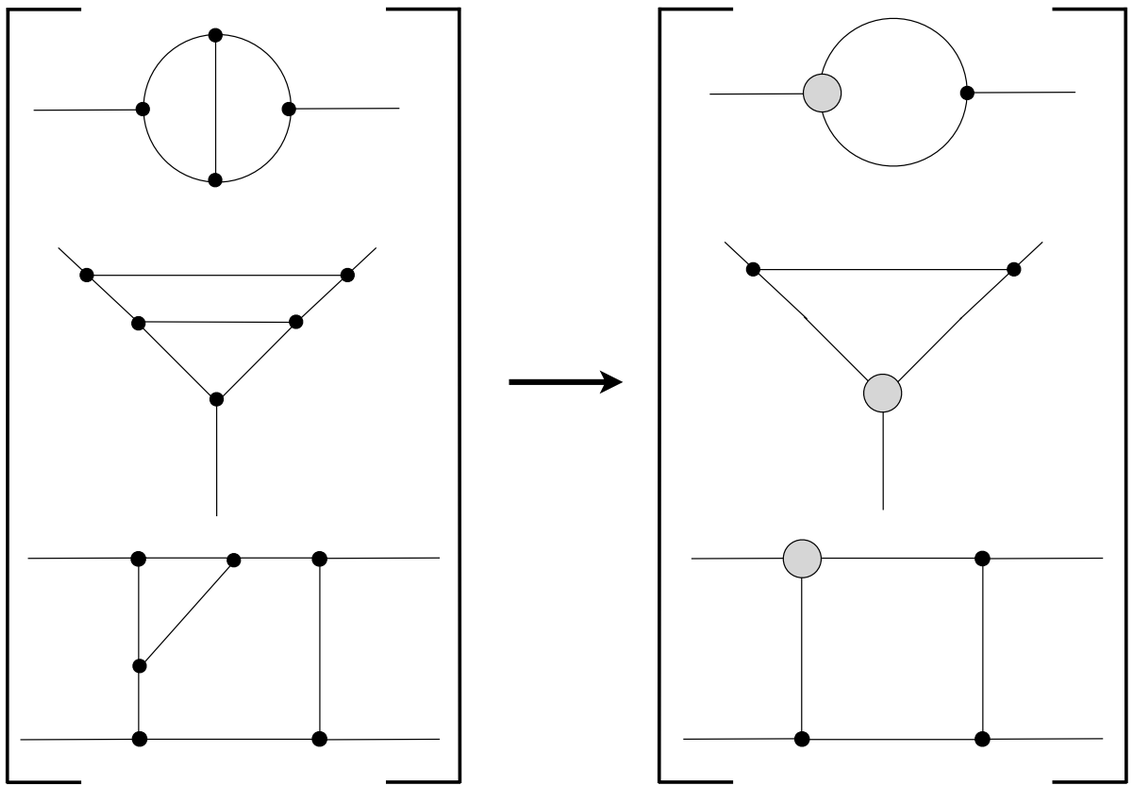
$D_{\mu\nu\rho}$:

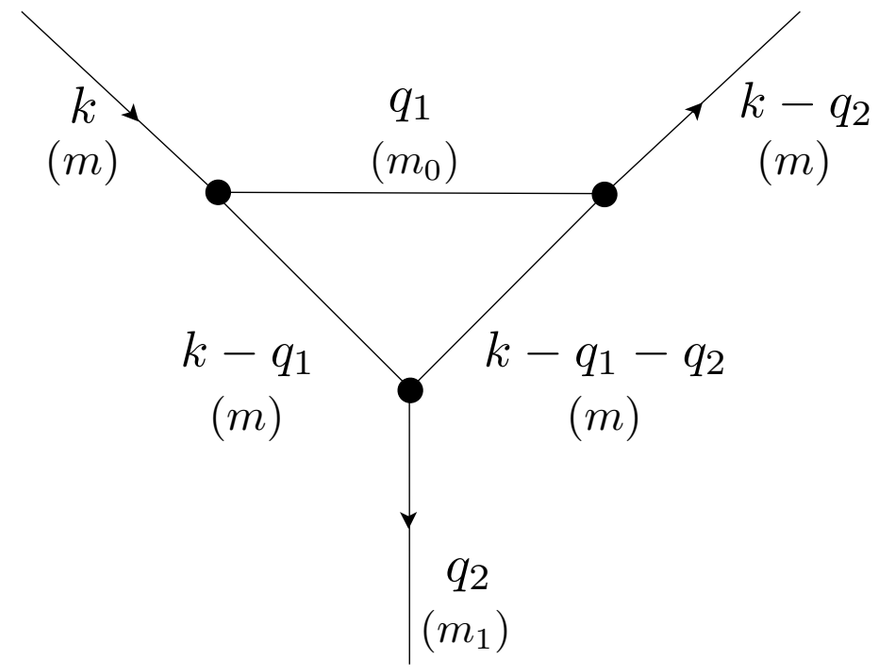
$$D_{001} = \hat{\mathbf{I}}_D\left[B_{001}x\right]$$
$$D_{002} = \hat{\mathbf{I}}_D\left[B_{001}y\right]$$
$$D_{003} = -\hat{\mathbf{I}}_D\left[B_{00} + B_{001}\right]$$
$$D_{111} = \hat{\mathbf{I}}_D\left[B_{111}x^3\right]$$
$$D_{112} = \hat{\mathbf{I}}_D\left[B_{111}x^2y\right]$$
$$D_{113} = -\hat{\mathbf{I}}_D\left[\left(B_{11} + B_{111}\right)x^2\right]$$
$$D_{122} = \hat{\mathbf{I}}_D\left[B_{111}xy^2\right]$$
$$D_{123} = -\hat{\mathbf{I}}_D\left[\left(B_{11} + B_{111}\right)xy\right]$$
$$D_{222} = \hat{\mathbf{I}}_D\left[B_{111}y^3\right]$$
$$D_{223} = -\hat{\mathbf{I}}_D\left[\left(B_{11} + B_{111}\right)y^2\right]$$
$$D_{233} = \hat{\mathbf{I}}_D\left[\left(B_1 + 2B_{11} + B_{111}\right)y\right]$$
$$D_{333} = -\hat{\mathbf{I}}_D\left[B_0 + 3\left(B_1 + B_{11}\right) + B_{111}\right]$$

$D_{\mu\nu\rho\sigma}$:

$$D_{0000} = \hat{\mathbf{I}}_D\left[B_{0000}\right]$$
$$D_{0011} = \hat{\mathbf{I}}_D\left[B_{0011}x^2\right]$$
$$D_{0012} = \hat{\mathbf{I}}_D\left[B_{0011}xy\right]$$
$$D_{0013} = -\hat{\mathbf{I}}_D\left[\left(B_{001} + B_{1111}\right)x\right]$$
$$D_{0022} = \hat{\mathbf{I}}_D\left[B_{0011}x^2\right]$$
$$D_{0023} = -\hat{\mathbf{I}}_D\left[\left(B_{001} + B_{1111}\right)y\right]$$
$$D_{0033} = \hat{\mathbf{I}}_D\left[B_{00} + 2B_{001} + B_{0011}\right]$$
$$D_{1111} = \hat{\mathbf{I}}_D\left[B_{1111}x^4\right]$$
$$D_{1112} = \hat{\mathbf{I}}_D\left[B_{1111}x^3y\right]$$
$$D_{1113} = -\hat{\mathbf{I}}_D\left[\left(B_{111} + B_{1111}\right)x^3\right]$$
$$D_{1122} = \hat{\mathbf{I}}_D\left[B_{1111}x^2y^2\right]$$
$$D_{1123} = -\hat{\mathbf{I}}_D\left[\left(B_{111} + B_{1111}\right)x^2y\right]$$
$$D_{1133} = \hat{\mathbf{I}}_D\left[\left(B_{11} + 2B_{111} + B_{1111}\right)x^2\right]$$
$$D_{1222} = \hat{\mathbf{I}}_D\left[B_{1111}xy^3\right]$$
$$D_{1223} = -\hat{\mathbf{I}}_D\left[\left(B_{111} + B_{1111}\right)xy^2\right]$$
$$D_{1233} = \hat{\mathbf{I}}_D\left[\left(B_{11} + 2B_{111} + B_{1111}\right)xy\right]$$
$$D_{1333} = -\hat{\mathbf{I}}_D\left[\left(B_1 + 3\left(B_{11} + B_{111}\right) + B_{1111}\right)x\right]$$
$$D_{2222} = \hat{\mathbf{I}}_D\left[B_{1111}y^4\right]$$
$$D_{2223} = -\hat{\mathbf{I}}_D\left[\left(B_{111} + B_{1111}\right)y^3\right]$$
$$D_{2233} = \hat{\mathbf{I}}_D\left[\left(B_{11} + 2B_{111} + B_{1111}\right)y^2\right]$$
$$D_{2333} = -\hat{\mathbf{I}}_D\left[\left(B_1 + 3\left(B_{11} + B_{111}\right) + B_{1111}\right)y\right]$$
$$D_{3333} = \hat{\mathbf{I}}_D\left[\left(B_0 + 4\left(B_1 + B_{111}\right) + 6B_{11} + B_{1111}\right)\right]$$

Triangle Insertion: Dispersive Approach



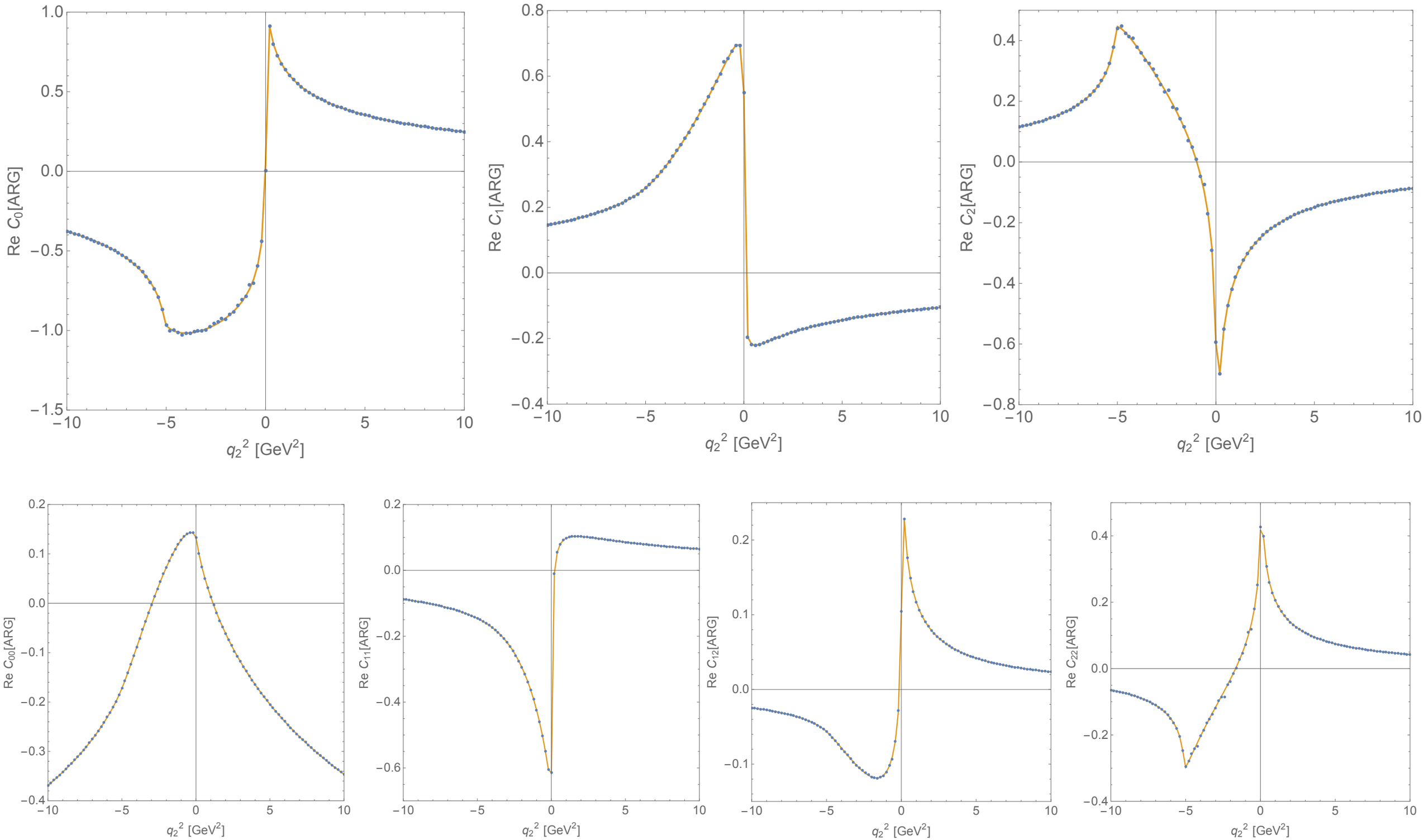


$$C_0\left(m^2, q_2^2, (q_2-k)^2, m_0^2, m^2, m^2\right) = \frac{\mu^{4-D}}{i\pi^{D/2}} \int \frac{d^D q_1}{\underbrace{\left[q_1^2 - m_0^2\right]\left[(q_1-k)^2 - m^2\right]}_{\text{join two propagators without } q_2}\left[(q_1-k+q_2)^2 - m^2\right]}$$

$$C_0\left(m^2, q_2^2, (q_2-k)^2, m_0^2, m^2, m^2\right) = \frac{1}{\pi} \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \int_0^1 dx \int_{\left(m_3+\left(m_{12}^2+\lambda\right)^{1/2}\right)^2}^{\Lambda^2} ds\, \frac{\Im B_0\left(s, m_3^2, m_{12}^2+\lambda\right)}{s-(q_2-k\bar{x})^2 - i\epsilon}.$$

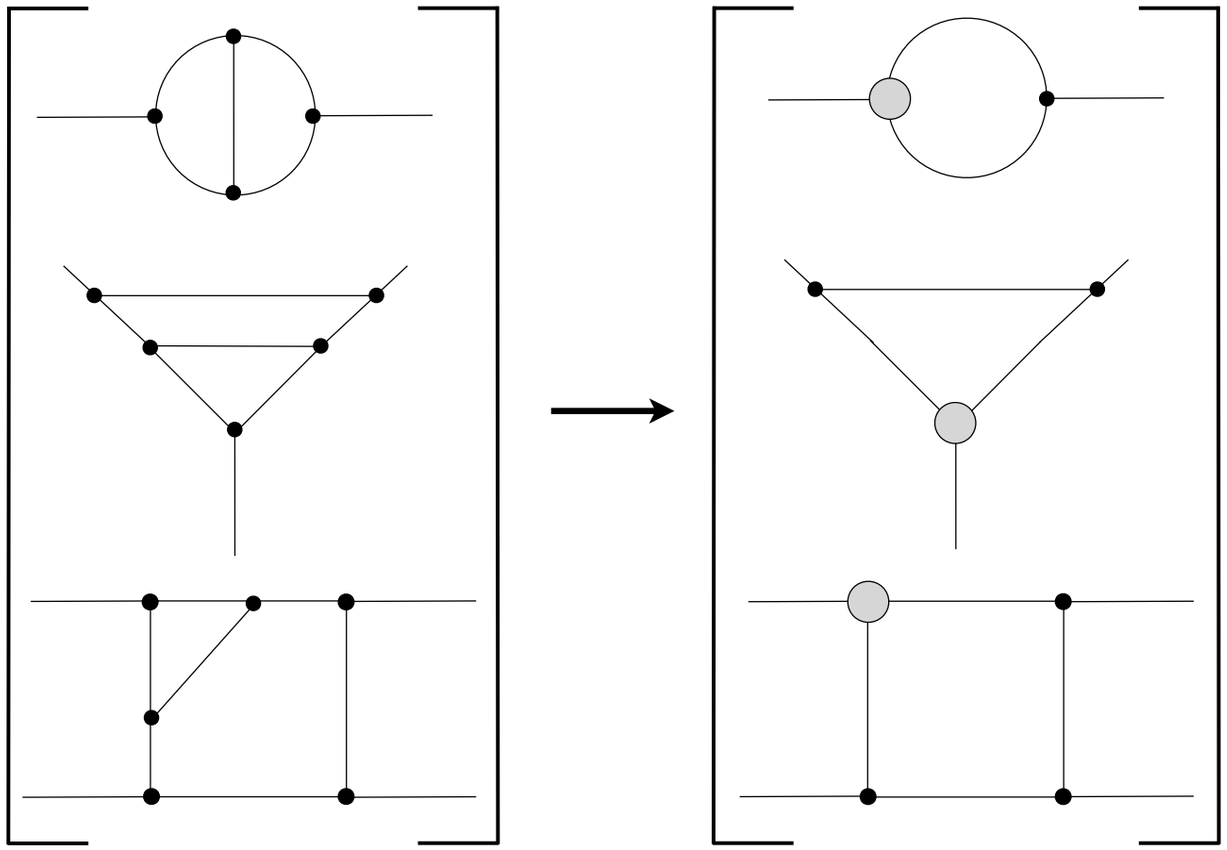
$$m_{12}^2 = m_0^2\bar{x} + m^2 x^2$$

Triangle Insertion: Dispersive Approach



$$m_0 = 1.2 \text{ GeV}, \; m = 0.1 \text{ GeV and } (k \cdot q_2) = -3.4 \text{ GeV}^2$$

Triangle Insertion: Dispersive Approach



$$\hat{\mathbf{D}} = \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} \int_0^1 dx \int_{\left(m_3+\left(m_{12}^2+\lambda\right)^{1/2}\right)^2}^{\Lambda^2} ds \ldots \qquad \Gamma = \hat{\mathbf{D}} \left[\frac{\Im F\left(s, m_3^2, m_{12}^2+\lambda\right)}{s-\left(p_2+p_1\bar{x}\right)^2 - i\epsilon} \right]$$

$$m_{12}^2 = m_1^2\bar{x} + m_2^2 x - p_1^2 x\bar{x}$$

Subtracted vertex at zero momentum: $\hat{\Gamma} = \hat{\mathbf{D}} \left[\frac{\Im F\left(s, m_3^2, m_{12}^2+\lambda\right)\left[\left(p_2+p_1\bar{x}\right)^2 - p_1^2\bar{x}^2\right]}{\left[s-\left(p_2+p_1\bar{x}\right)^2 - i\epsilon\right]\left[s-p_1^2\bar{x}^2\right]} \right]$

Triangle Insertion: Dispersive Approach

$$C_0\left(m^2, q_2^2, (q_2-k)^2, m_0^2, m^2, m^2\right) = \frac{1}{\pi}\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\int_0^1 dx \int_{\left(m_3+\left(m_{12}^2+\lambda\right)^{1/2}\right)^2}^{\Lambda^2} ds\, \frac{\Im B_0\left(s, m_3^2, m_{12}^2+\lambda\right)}{s-\left(q_2-k\bar{x}\right)^2 - i\epsilon}.$$

$$m_{12}^2 = m_1^2\bar{x} + m_2^2 x - p_1^2 x\bar{x}$$

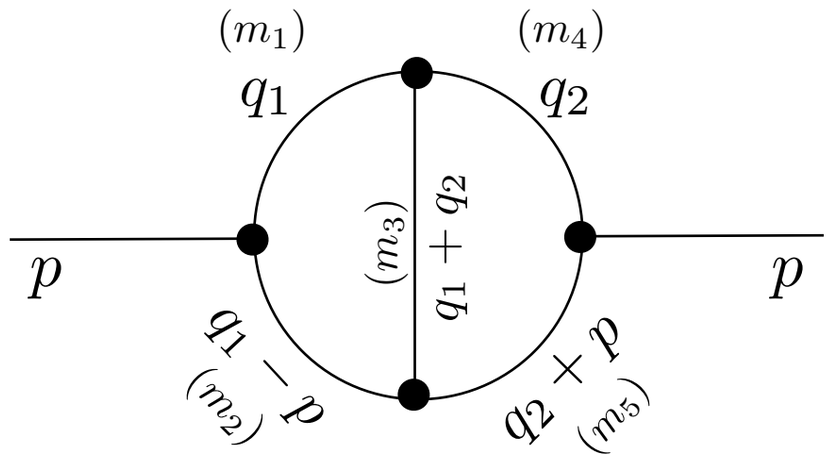
Mass, m_{12}^2 could become negative.

$$\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}B_0\left(p^2, m_1^2, -|m_2^2|+\lambda\right) = \frac{1}{2\pi i}\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\int_{-\Lambda^2}^{\Lambda^2} ds \frac{B_0\left(s, m_1^2, -|m_2^2|+\lambda\right)}{s-p^2-i\epsilon}$$

$$\Lambda^2 = 10^{10}\ GeV^2,\ m_1^2 = 1\ GeV^2 \text{ and } |m_2^2| = 9\ GeV^2$$

p^2 (GeV)^2	Eq.(36)	LoopTools
-5.0	0.147987 - 0.079999 i	0.147988 - 0.079999 i
-1.0	0.127835 - 0.037405 i	0.127836 - 0.037405 i
-0.5	0.124874 - 0.034242 i	0.124875 - 0.034242 i
0.5	0.119143 - 0.028889 i	0.119144 - 0.028889 i
1.0	0.116402 - 0.026626 i	0.116403 - 0.026626 i
5.0	0.097877 - 0.014968 i	0.097878 - 0.014968 i
10.0	0.081819 - 0.008497 i	0.081820 - 0.008497 i
50.0	0.039036 - 0.000911 i	0.039037 - 0.000911 i

Numerical Example



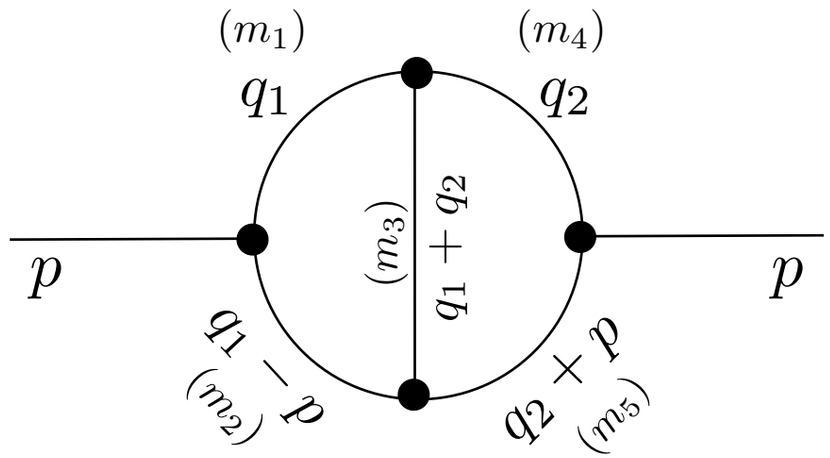
$$I = -\frac{1}{\pi^4}\int \frac{d^4q_1 d^4q_2}{\left[q_1^2 - m_1^2\right]\left[(q_1-p)^2 - m_2^2\right]\left[(q_1+q_2)^2 - m_3^2\right]\left[q_2^2 - m_4^2\right]\left[(q_2+p)^2 - m_5^2\right]}$$

$$I = \frac{1}{2\pi^3}\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\int_0^1 dx \int_{r(x,\lambda)}^{\Lambda^2} ds\ \left(2i\Im B_0\left[s, m_3^2, m_{12}^2+\lambda\right]\theta\left(m_{12}^2\right) + B_0\left[s, m_3^2, m_{12}^2+\lambda\right]\theta\left(-m_{12}^2\right)\right)$$

$$\int \frac{d^4q_2}{\left[q_2^2 - m_4^2\right]\left[(q_2+xp)^2 - s\right]\left[(q_2+p)^2 - m_5^2\right]},$$

$$r(x,\lambda) = \left(m_3 + (m_{12}^2+\lambda)^{1/2}\right)^2\theta(m_{12}^2) - \Lambda^2\theta(-m_{12}^2)$$

Numerical Example



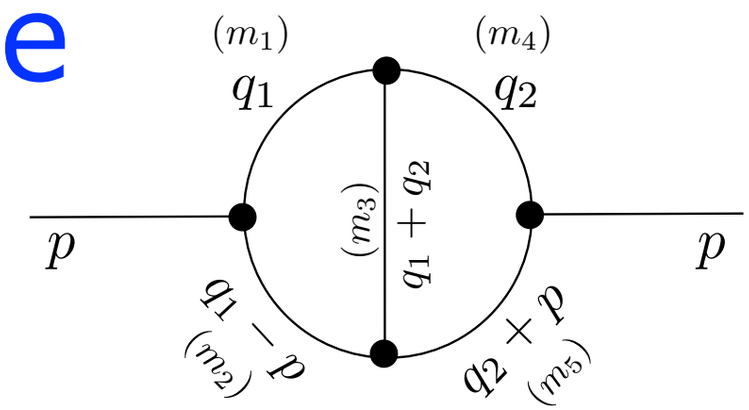
$$I = -\frac{1}{2\pi i} \lim_{\{\lambda,\xi\}\to 0} \frac{\partial^2}{\partial\lambda\partial\xi} \int_0^1 dxdy \int_{r(x,\lambda)}^{\Lambda^2} ds \Big(2i\Im B_0 \left[s, m_3^2, m_{12}^2 + \lambda\right] \theta\left(m_{12}^2\right) +$$

$$B_0 \left[s, m_3^2, m_{12}^2 + \lambda\right] \theta\left(-m_{12}^2\right) \Big) B_0 \left[p^2 (x-y)^2, s, m_{45}^2 + \xi\right],$$

$$m_{12}^2 = m_1^2 \bar{x} + m_2^2 x - p^2 \bar{x} x$$

$$m_{45}^2 = m_4^2 \bar{y} + m_5^2 y - p^2 \bar{y} y$$

Numerical Example



[This work] A. A, Phys. Rev. D 98, 036021 (2018)

[13] S. Bauberger, M. Bohm, Nucl. Phys. B 445, 25-46 (1995)

p^2 (GeV)^2	This work	$\Delta t_{\text{This Work}}$	[13]	$\Delta t_{[13]}$
-50.0	-0.08296	75	-	-
-10.0	-0.18399	22	-	-
-5.0	-0.22178	17	-	-
-1.0	-0.26919	8	-	-
-0.5	-0.27712	9	-	-
-0.1	-0.28360	9	-	-
0.1	-0.28714	9	-0.28701	84
0.5	-0.29443	9	-0.29479	85
1.0	-0.30449	10	-0.30493	85
5.0	-0.45230	14	-0.45241	86
10.0	-0.48810 - 0.35318 i	30	-0.48825 - 0.35333 i	86
50.0	0.17335 - 0.11781 i	1120	0.17391 - 0.11807 i	85

$$m_1^2 = 1,\ m_2^2 = 2,\ m_3^2 = 3,\ m_4^2 = 4 \text{ and } m_5^2 = 5\ (\text{GeV})^2$$

Crossed Two-Loop Topologies

A. A, arXiv:1809.05592

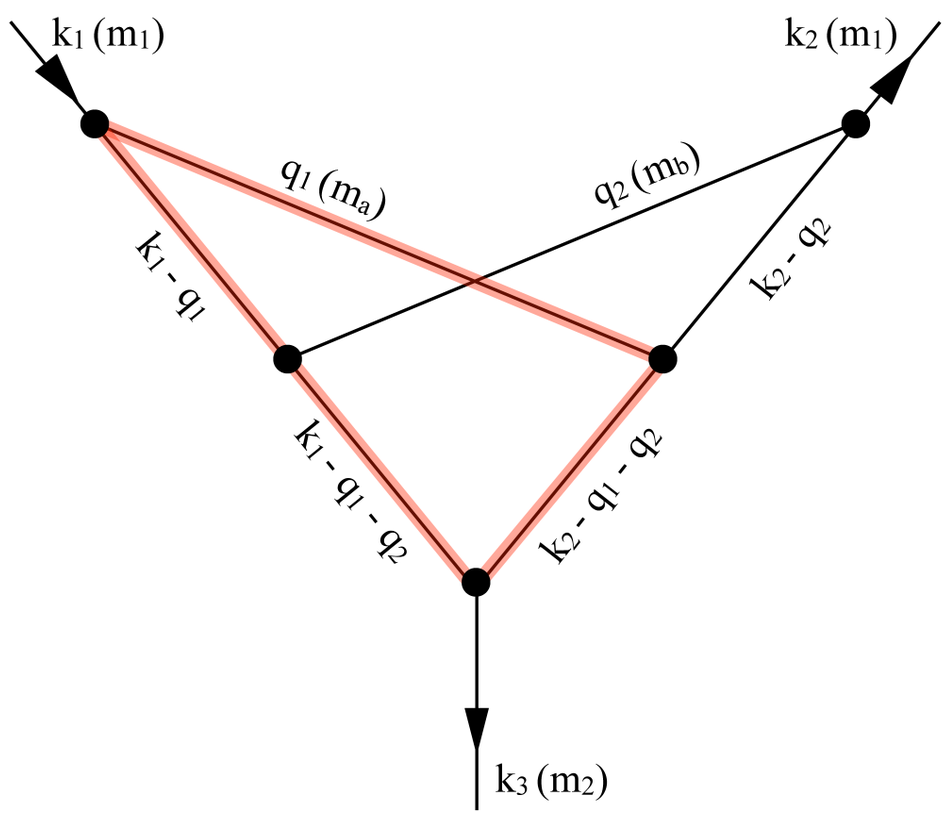


Figure 3: Crossed two-loop vertex topology.

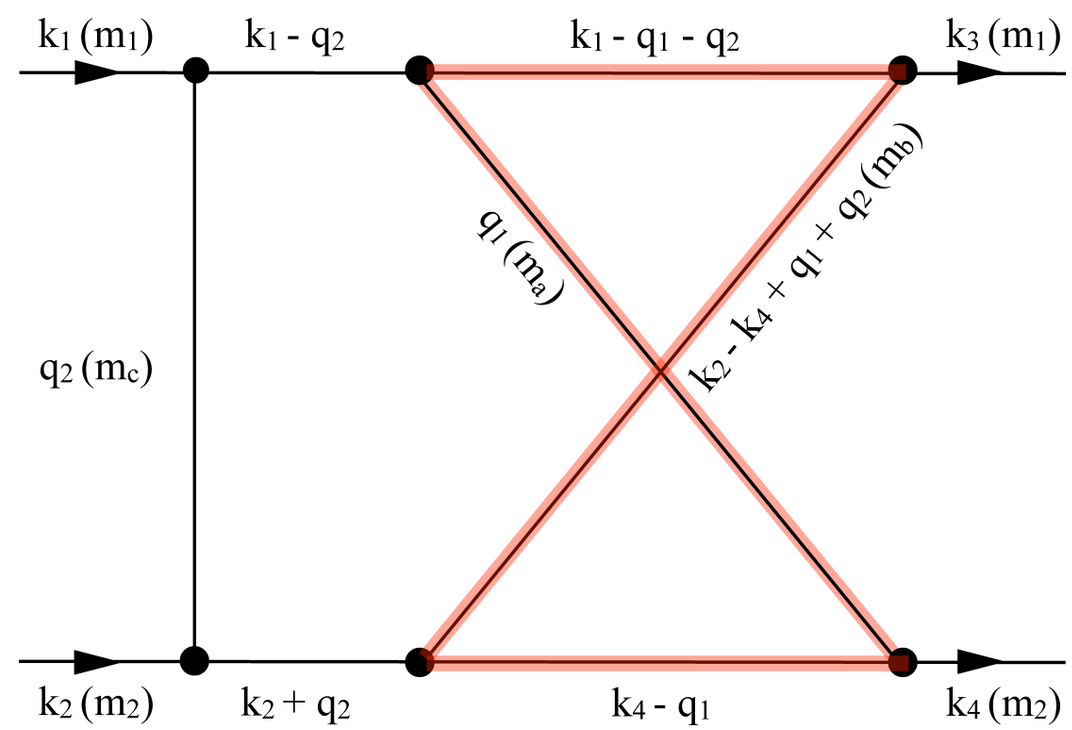


Figure 4: Crossed two-loop box topology.

Crossed Two-Loop Triangle

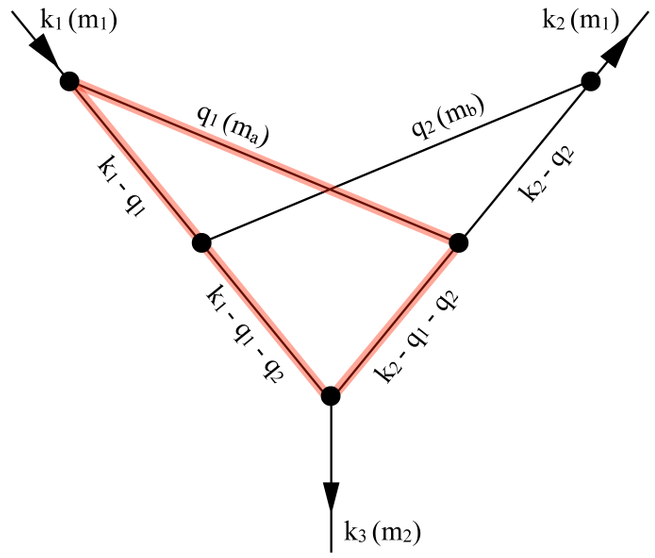


Figure 3: Crossed two-loop vertex topology.

$$I_{\Delta_2} = -\frac{1}{\pi^4}\int \frac{d^4q_1 d^4q_2}{\underline{\left[q_1^2 - m_a^2\right]\left[(k_1-q_1)^2 - m_1^2\right]}\ \underline{\left[(k_1-q_1-q_2)^2 - m_1^2\right]\left[(k_2-q_1-q_2)^2 - m_1^2\right]}}$$

$$\times \frac{1}{\left[q_2^2 - m_b^2\right]\left[(k_2-q_2)^2 - m_1^2\right]}.$$

$$I_{\Delta_2} = -\frac{1}{\pi^4}\lim_{\{\xi,\,\lambda\}\to 0}\frac{\partial^2}{\partial\xi\partial\lambda}\int_0^1 dxdy\int\frac{d^4q_2}{\left[q_2^2 - m_b^2\right]\left[(k_2-q_2)^2 - m_1^2\right]}\int\frac{d^4q_1}{\left[(q_1-k_1x)^2 - m_\xi^2\right]\left[(q_1+q_2-\bar{y}k_2-yk_1)^2 - m_\lambda^2\right]}$$

$$m_\xi^2 = m_a^2\bar{x} + m_1^2x^2 + \xi \qquad\qquad m_\lambda^2 = m_1^2 - \bar{y}yk_3^2 + \lambda$$

Crossed Two-Loop Triangle

$$I_{\Delta_2} = -\frac{1}{\pi^4} \lim_{\{\xi,\,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dxdy \int \frac{d^4q_2}{\left[q_2^2 - m_b^2\right]\left[\left(k_2 - q_2\right)^2 - m_1^2\right]} \int \frac{d^4q_1}{\left[\left(q_1 - k_1 x\right)^2 - m_\xi^2\right]\left[\left(q_1 + q_2 - \bar{y}k_2 - yk_1\right)^2 - m_\lambda^2\right]}$$

$$I_{\Delta_2} = -\frac{i}{\pi^2} \lim_{\{\xi,\,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dxdy \int d^4q_2 \frac{B_0\left[\left(q_2 - \bar{y}k_2 + k_1\left(x - y\right)\right)^2, m_\xi^2, m_\lambda^2\right]}{\left[q_2^2 - m_b^2\right]\left[\left(k_2 - q_2\right)^2 - m_1^2\right]}$$

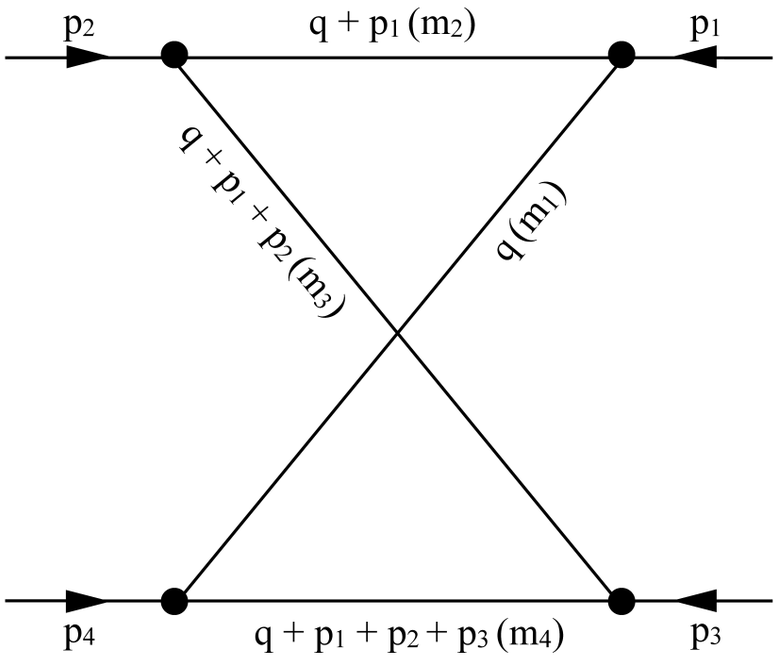
$$I_{\Delta_2} = \frac{i}{\pi^3} \lim_{\{\xi,\,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dxdy \int_{\left(m_\xi + m_\lambda\right)^2}^{\Lambda^2} ds\, \Im B_0\left[s, m_\xi^2, m_\lambda^2\right]$$

$$\times \int \frac{d^4q_2}{\left[q_2^2 - m_b^2\right]\left[\left(k_2 - q_2\right)^2 - m_1^2\right]\left[\left(q_2 - \bar{y}k_2 + k_1\left(x - y\right)\right)^2 - s - i\epsilon\right]}$$

$$I_{\Delta_2} = -\frac{1}{\pi} \lim_{\{\xi,\,\lambda,\,\phi\}\to 0} \frac{\partial^3}{\partial\xi\partial\lambda\partial\phi} \int_0^1 dxdydz \int_{\left(m_\xi + m_\lambda\right)^2}^{\Lambda^2} ds\, \Im B_0\left[s, m_\xi^2, m_\lambda^2\right] B_0\left[\left(\left(z - \bar{y}\right)k_2 + k_1\left(x - y\right)\right)^2, m_\phi^2, s\right]$$

$$m_\phi^2 \;=\; m_b^2\bar{z} + m_1^2 z^2 + \phi$$

Crossed Two-Loop: Generalization



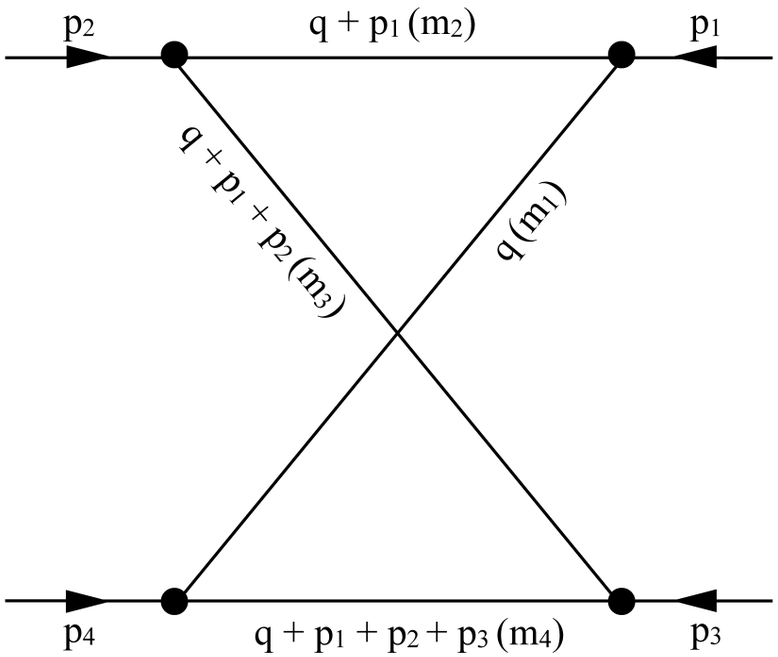
$$I_{\boxtimes} = \frac{1}{i\pi^2} \int \frac{d^4 q}{\underline{\left[q^2 - m_1^2\right]\left[(q+p_1)^2 - m_2^2\right]}\ \underline{\left[(q+p_1+p_2)^2 - m_3^2\right]\left[(q+p_1+p_2+p_3)^2 - m_4^2\right]}}$$

external momenta p_2 and p_4 would depend on the momentum of the second loop

$$I_{\boxtimes} = \lim_{\{\xi,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dx dy\, B_0\left[(\bar{x}p_1 + p_2 + y p_3)^2, m_\xi^2, m_\lambda^2\right]$$

$$m_\xi^2 = \bar{x}m_1^2 + x m_2^2 - x\bar{x}p_1^2 + \xi \qquad m_\lambda^2 = \bar{y}m_3^2 + y m_4^2 + p_1^2 + y p_3^2 + \lambda$$

Crossed Two-Loop: Generalization



$$I_{\boxtimes} = -\frac{1}{\pi} \lim_{\{\xi,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dxdy \int_{\left(m_\xi+m_\lambda\right)^2}^{\Lambda^2} ds \, \frac{\Im B_0\left[s, m_\xi^2, m_\lambda^2\right]}{\left(\bar{x}p_1 + p_2 + yp_3\right)^2 - s - i\epsilon}$$

$$\Gamma_{\boxtimes} = \hat{\boldsymbol{D}} \left[\frac{\Im B_0\left[s, m_\xi^2, m_\lambda^2\right]}{\left(\bar{x}p_1 + p_2 + yp_3\right)^2 - s - i\epsilon} \right]$$

$$\hat{\boldsymbol{D}} \; = \; \lim_{\{\xi,\lambda\}\to 0} \frac{\partial^2}{\partial\xi\partial\lambda} \int_0^1 dxdy \int_{\left(m_\xi+m_\lambda\right)^2}^{\Lambda^2} ds$$

Conclusion

- We are now in the last stage of the NNLO EWC calculations for the MOLLER experiment.

- Automatization of the NNLO EWC calculations for MOLLER is currently under way.

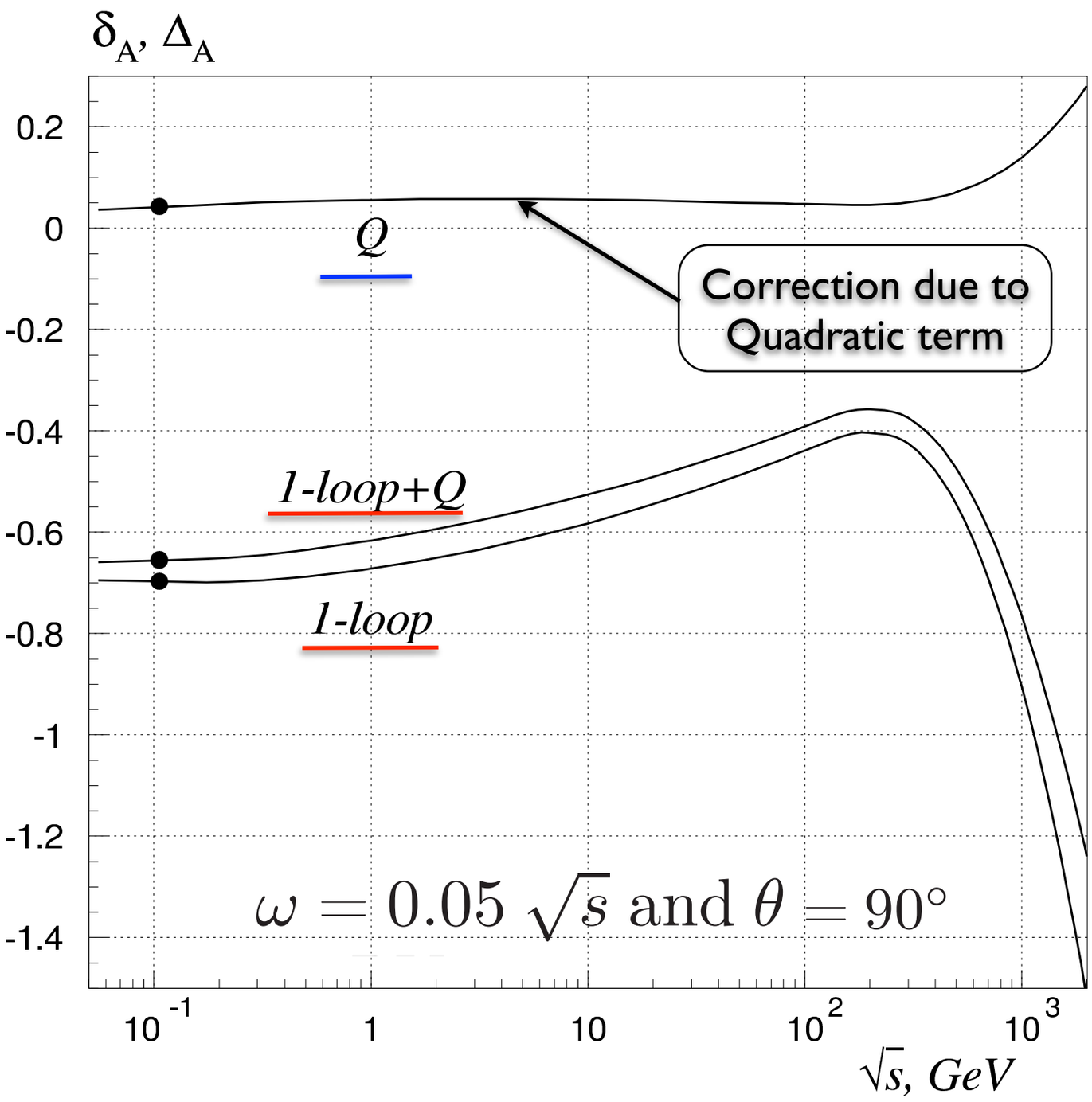
- Our next goal is a full gauge-invariant set of two-loop EW graphs with SE and triangles insertions.

- Results to be obtained will be cross checked with our previous calculations and other literature.

- We are looking for additional collaborative projects in two-loops calculations for various processes.

Additional Slides

Quadratic correction: results



$$\delta_A^{\mathrm{C}} = (A_{LR}^{\mathrm{C}} - A_{LR}^{0})/A_{LR}^{0}$$

$$\Delta_A = (A_{LR}^{1-\mathrm{loop+Q}} - A_{LR}^{1-\mathrm{loop}})/A_{LR}^{0}$$

The scale of the Q-part contribution in the low-energy region is approximately constant, but starting from $\sqrt{s} \geq m_Z$, where the weak contribution becomes comparable with electromagnetic, the effect of Q-part grows sharply.

This effect of increasing importance of two-loop contribution at higher energies may have a significant effect on the asymmetry measured at the future $e^- e^-$ -colliders.

Final Expressions

Combining all the terms together, we get the infrared-finite result at one-loop level

$$\sigma_{NLO} = \frac{\alpha}{\pi}\Re[R_1 + \delta_1^f]\sigma_0$$

and NNLO level

$$\sigma_{NNLO} = \sigma_Q + \sigma_T + \sigma^\gamma + \sigma^{\gamma\gamma} = \left(\frac{\alpha}{\pi}\right)^2 \Re\left[R_1^* \delta_1^f + \frac{1}{2}|R_1|^2 - \frac{1}{2}R_2 + \delta_Q^f + \delta_T^f\right]\sigma_0$$

$$= \quad \underline{\sigma_O^f} + \underline{\sigma_B^f} + \underline{\sigma_Q^f} + \underline{\sigma_T^f},$$

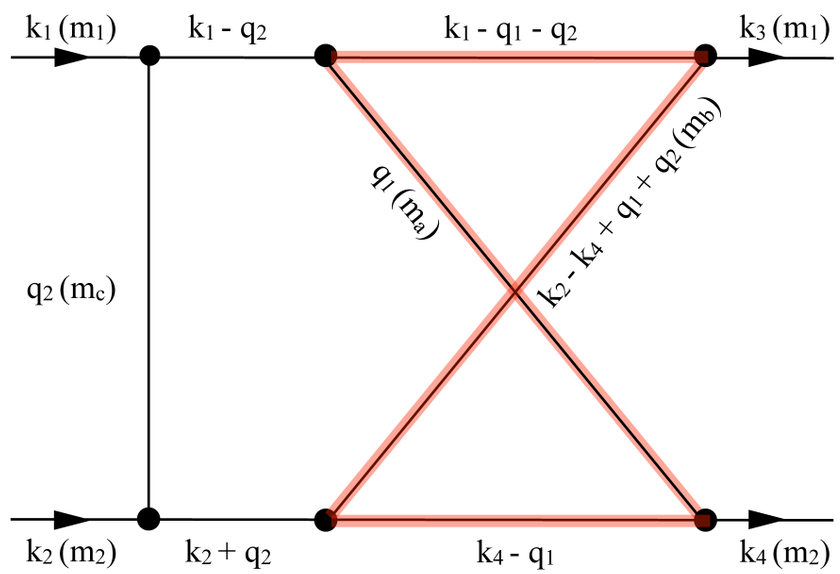
where:

$$\sigma_O^f = \frac{\alpha}{\pi}\Re[R_1^* \sigma_{NLO}] \qquad \sigma_B^f = -\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 \Re(|R_1|^2 + R_2)\sigma_0$$

$$\sigma_Q^f = \left(\frac{\alpha}{\pi}\right)^2 \delta_Q^f \sigma_0 \qquad \sigma_T^f = \left(\frac{\alpha}{\pi}\right)^2 \delta_T^f \sigma_0$$

$$R_1 = -4B\log\frac{\sqrt{s}}{2\omega} - \left(\log\frac{s}{m^2} - 1\right)^2 + 1 - \frac{\pi^2}{3} + \log^2\frac{u}{t},\ R_2 = \frac{8}{3}\pi^2 B^2,\ B = \log\frac{tu}{m^2 s} - 1 + i\pi$$

Crossed Two-Loop Box



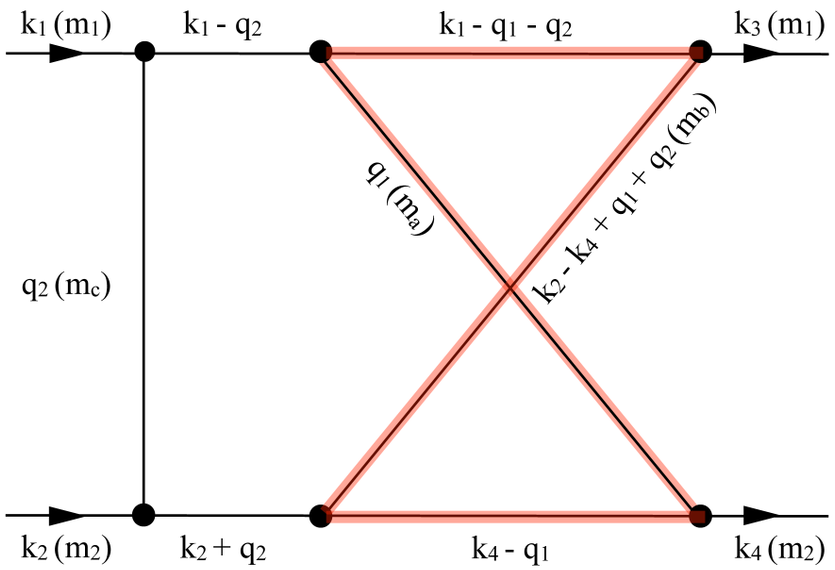
$$I_{\square_2} = -\frac{1}{\pi^4}\int \frac{d^4q_1 d^4q_2}{\left[q_1^2 - m_a^2\right]\left[(k_4-q_1)^2 - m_2^2\right]\left[(k_2-k_4+q_1+q_2)^2 - m_b^2\right]\left[(k_1-q_1-q_2)^2 - m_1^2\right]}$$

$$\times \frac{1}{\left[q_2^2 - m_c^2\right]\left[(k_1-q_2)^2 - m_1^2\right]\left[(k_2+q_2)^2 - m_2^2\right]}.$$

$$I_{\square_2} = -\frac{i}{\pi^2}\lim_{\{\xi,\lambda\}\to 0}\frac{\partial^2}{\partial\xi\partial\lambda}\int_0^1 dxdy\int d^4q_2 \frac{B_0\left[(q_2 - xk_4 + yk_3 - k_1)^2, m_\xi^2, m_\lambda^2\right]}{\left[q_2^2 - m_c^2\right]\left[(k_1-q_2)^2 - m_1^2\right]\left[(k_2+q_2)^2 - m_2^2\right]}$$

$$m_\xi^2 = \bar{x}m_a^2 + x^2 m_2^2 + \xi \qquad m_\lambda^2 = \bar{y}^2 m_1^2 + y m_b^2 + \lambda$$

Crossed Two-Loop Box



$$I_{\square_2} = -\frac{1}{\pi} \lim_{\{\xi,\lambda\}\to 0} \frac{\partial^4}{\partial\xi\partial\lambda\partial\delta^2} \int_0^1 dxdydz \int_0^{1-z} dw \int_{(m_\xi+m_\lambda)^2}^{\Lambda^2} ds\, \Im B_0\left[s, m_\xi^2, m_\lambda^2\right]$$

$$\times\, B_0\left[\left(\bar{z}k_1 + \omega k_2 + xk_4 - yk_3\right)^2, m_\delta^2, s\right],$$

$$m_\delta^2 \;=\; (\bar{z}-\omega)\, m_c^2 + \left(\omega k_2 - zk_1\right)^2 + \delta$$

Sensitivity to effective mixing angle

Sensitivity of Asymmetry to effective mixing angle

Representation of effective Born amplitude:

$$\mathfrak{M}_\gamma = \frac{\alpha(t)Q_e^2}{t}\left(\bar{u}_e\gamma_\mu u_e\right)\left(\bar{u}_e\gamma^\mu u_e\right),$$

D. Binosi, J. Papavassiliou, arXiv:0909.2536
D. Kennedy, B. Lynn, Nucl. Phys. B322 (1989)1
W. Hollik, DESY Report, DESY 88-188 (1988)

$$\mathfrak{M}_Z = \frac{G_\mu}{\sqrt{2}}\kappa\frac{m_Z^2}{t-m_Z^2+i\frac{t}{m_Z}\Gamma_Z}\left(\bar{u}_e\gamma_\mu\left[I_3^e-2\bar{s}_W^2(t)Q_e-I_3^e\gamma_5\right]u_e\right)\left(\bar{u}_e\gamma^\mu\left[I_3^e-2\bar{s}_W^2(t)Q_e-I_3^e\gamma_5\right]u_e\right).$$

$$\kappa = \frac{1-\Delta r}{1+\Re\left[\frac{\partial}{\partial t}\hat{\Sigma}_{ZZ}(t)\right]} \qquad \alpha(t) = \frac{\alpha}{1+\Re\left[\hat{\Sigma}_{\gamma\gamma}(t)\right]/t}$$

$$\Delta r = \frac{\Re[\hat{\Sigma}_{WW}(0)]}{m_W^2}+\frac{\alpha}{4\pi s_W^2}\left(6+\frac{7-4s_W^2}{2s_W^2}\ln c_W^2\right)+\frac{c_W^2}{m_Z^2 s_W^2}\Re\left[\frac{\hat{\Sigma}_{\gamma Z}^2(m_Z^2)}{m_Z^2+\hat{\Sigma}_{\gamma\gamma}(m_Z^2)}\right]$$

Sensitivity of Asymmetry to effective mixing angle

Effective Weinberg mixing angle up to NLO:

$$\bar{s}_W^2(t) = s_W^2 - s_W c_W \frac{\Re\left[\hat{\Sigma}_{\gamma Z}(t)\right]}{t + \Re\left[\hat{\Sigma}_{\gamma\gamma}(t)\right]}.$$

$$\sin^2\theta_W \equiv s_W^2 = 1 - \frac{m_W^2}{m_Z^2} \qquad m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu \sin^2\theta_W (1-\Delta r)}$$

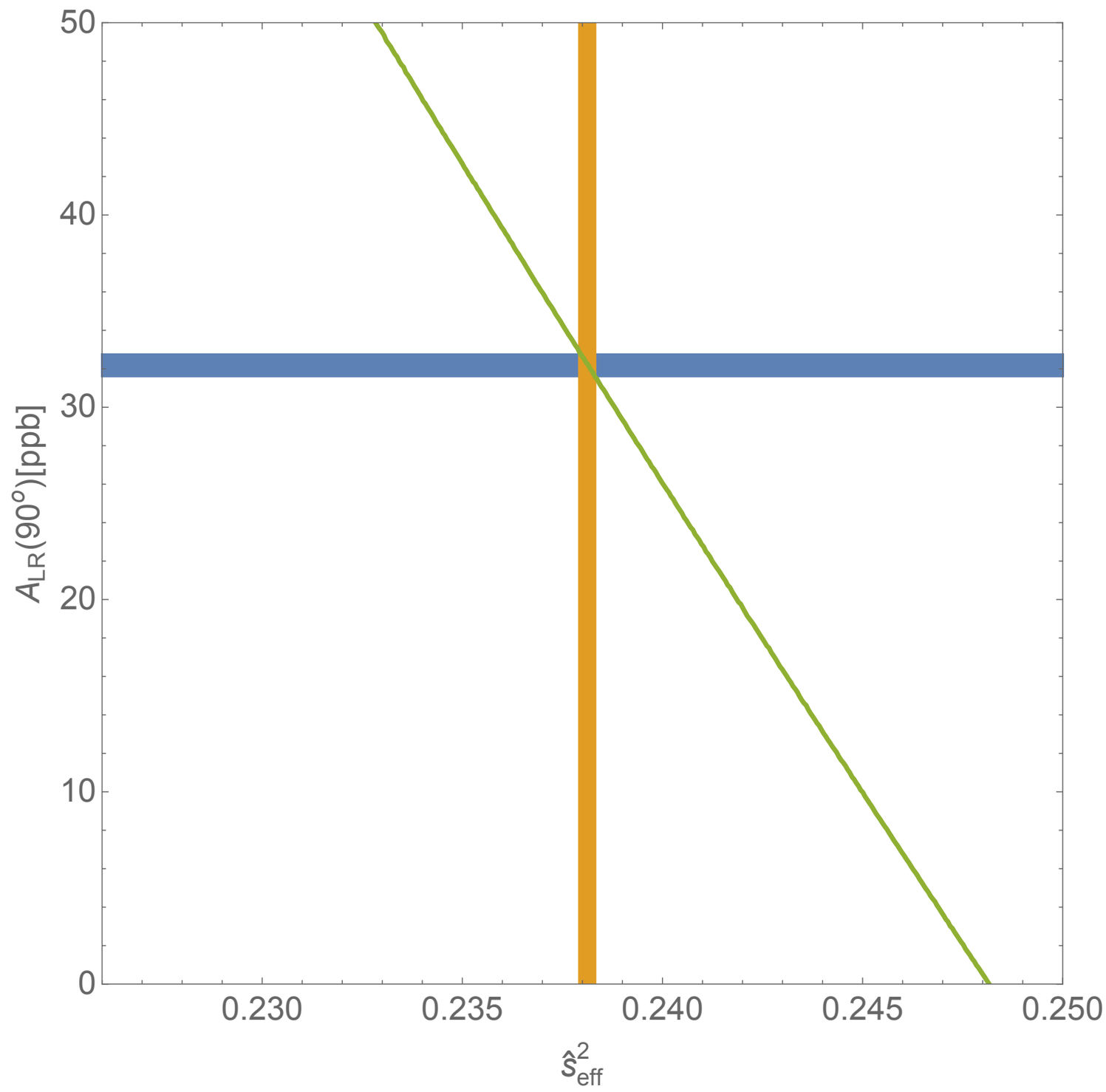
$Q^2(GeV^2)$	$\bar{s}^2_{W(ON-SHELL)}$	$\bar{s}^2_{W(\bar{M}S-PT)}$[1]	$\bar{s}^2_{W(\bar{M}S)}$PDG(2015)
0	0.2383	0.2387	0.2386
m_z	0.2313	0.2320	0.2313

[1] A. Ferroglia, G. Ossola, A. Sirlin, EPJC., 10.1140, July, (2003).

Sensitivity of Polarization Asymmetry to effective mixing angle

$$\bar{s}_W^2(t) = s_W^2 - s_W c_W \frac{\Re\left[\hat{\Sigma}_{\gamma Z}(t)\right]}{t + \Re\left[\hat{\Sigma}_{\gamma\gamma}(t)\right]}.$$

Value of the effective s_W^2 for MOLLER kinematics:

$$\bar{s}_W^2(Q^2 = 0.0056\ GeV^2) = 0.2382$$

The 2% uncertainty on polarization asymmetry translates to 0.1% uncertainty for weak mixing angle