Dispersive Treatment of Two-Loop Processes

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Precision PV Scattering

• To access the scale of the new physics at TeV level, we need to push one or more experimental parameters to the extreme precision.

\[ A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \]

• New Physics/Weak-Electromagnetic Interference

\[ A_{LR} = \frac{2Re(M_{\gamma M_Z}^+ + M_{\gamma M_{NP}^+}^+ + M_{Z M_{NP}^+})_{LR}}{\sigma_L + \sigma_R} \]
Asymmetry is an observable which is directly related to the interference term:

\[
A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} \approx \frac{2 \text{Re}(M_\gamma M_Z^+ + M_\gamma M_{NP}^+ + M_Z M_{NP}^+)_LR}{\sigma_L + \sigma_R} \sim (10^{-5} \text{ to } 10^{-4}) \cdot Q^2
\]

To access multi-TeV electron scale it is required to measure:

\[
\delta(\sin^2 \theta_W) < 0.002
\]

MOLLER experiment offers an unique opportunity to reach multi-TeV scale and will become complimentary to the LHC direct searches of the New Physics.
Møller scattering at the tree level

The process of electron–electron scattering (Møller process)
C. Møller, Annalen der Physik 406, 531 (1932)

\[
A_{LR} = \frac{\sigma_{LL} + \sigma_{LR} - \sigma_{RL} - \sigma_{RR}}{\sigma_{LL} + \sigma_{LR} + \sigma_{RL} + \sigma_{RR}} = \frac{\sigma_{LL} - \sigma_{RR}}{\sigma_{LL} + 2\sigma_{LR} + \sigma_{RR}}
\]

\[
A_{LR}^0 = \frac{s}{2m_W^2} \frac{y(1-y)}{1 + y^4 + (1-y)^4} \frac{1 - 4s_W^2}{s_W^2}, \quad y = -t/s
\]
Møller scattering at one-loop

Although PV asymmetry ($A_{LR} \sim 10^{-7}$) is very small, the accuracy of modern experiments exceeds the accuracy of the theoretical result in Born approximation. One–loop contribution was found to be rather big in the previous works:

The longitudinal parts of the boson self-energy make contributions that are proportional to
where

\[ |M_0 + M_1|^2 \]

renormalization conditions (within the same scheme) was addressed in our earlier paper [19].

A question of dependence of EWC on renormalization schemes and

We use the on-shell renormalization scheme from [21, 22], so there is no contributions from

vertex functions. The u-channel diagrams are obtained via the interchange

In order to get the electron vertex amplitude (2nd and 3rd diagrams in Fig. 2), we use the

Calculated in the on-shell renormalization, using both:

- Computer-based approach, with Feynarts, FormCalc, LoopTools and Form
  T. Hahn, Comput. Phys. Commun. 140 418 (2001);
  T. Hahn, M. Perez-Victoria, Comput. Phys. Commun. 118, 153 (1999);

- “On paper”, with approximations in small energy region \( \frac{t, u}{m_{Z,W}^2} \ll 1 \), for \( \sqrt{s} \ll 30 \, GeV \) and high energy approximation for \( \sqrt{s} \gg 500 \, GeV \)

One-Loop Corrections for MOLLER

The relative weak (solid line in DRC (semi-automated) and dotted line in HRC ("on paper")) and QED (dashed line) corrections to the Born asymmetry $A_{LR}^0$ versus $\sqrt{s}$ at $\theta = 90^\circ$.

The filled circle corresponds to our predictions for the MOLLER experiment.

\[ \delta_A = \frac{A_{LR}^C - A_{LR}^0}{A_{LR}^0} \]

The physical, NLO-corrected asymmetries, computed in both on-shell and CDR schemes, are compared in Fig. 9. Here, for consistency with the differential Renormalization Scheme (DRC) with the help of equations from [28], which is not mandated by the previously achievable experimental accuracy is an indication of the order of magnitude the higher-order, NNLO and weak correction becomes significant in the high-energy region the weak correction becomes important to PV asymmetry may become important to PV asymmetry.

To our predictions for the MOLLER experiment.

The Next-to-Next-to-Leading Order (NNLO) EWC to the Born ($\sim M_0 M_0^+$) cross section can be divided into two classes:

- **Q-part** induced by quadratic one-loop amplitudes $\sim M_1 M_1^+$, and
- **T-part** – the interference of Born and two-loop diagrams $\sim 2\text{Re}M_0 M_{2\text{-loop}}^+$.

\[
\sigma = \frac{\pi^3}{2s} |M_0 + M_1|^2 = \frac{\pi^3}{2s} \left( \frac{M_0 M_0^+}{\alpha^2} + 2\text{Re}M_1 M_0^+ \frac{1}{\alpha^3} + \frac{M_1 M_1^+}{\alpha^4} \right) = \sigma_0 + \sigma_1 + \sigma_Q
\]

\[
\sigma_T = \frac{\pi^3}{s} \text{Re}M_2 M_0^+ \propto \alpha^4
\]
**Quadratic correction: IR part**

Differential quadratic cross section $\sigma_Q$ written as sums of $\lambda$-dependent (IRD-terms) and $\lambda$-independent (infrared-finite) parts:

$$\sigma_Q = \frac{\pi^3}{2s} M_1 M_1^+ = \sigma^\lambda_Q + \sigma^f_Q$$

$$\frac{\pi^3}{2s} M_1^\lambda (M_1^\lambda + 2M_1^f) = \frac{1}{4} \left( \frac{\alpha}{\pi} \right)^2 \text{Re} \left[ \delta_1^{\lambda*} (\delta_1^\lambda + 2\delta_1^f) \right] \sigma_0.$$  

$$\delta_1^\lambda = 4B \log \frac{\lambda}{\sqrt{s}}$$

$$B = \log \frac{tu}{m^2 s} - 1 + i\pi$$
**Quadratic correction: photon emission**

In order to remove the IR-divergent terms in quadratic cross section, we need to consider:

1. Photon emission from one-loop diagrams
2. Two photon photon emission

\[
\sigma^\gamma_Q = \frac{1}{2} \sigma^\gamma = \frac{\pi^2}{s} \text{ Re } [(-\delta_1 + R_1) M_1^+ M_0]
\]

\[
R_1 = -4B \log \frac{\sqrt{s}}{2\omega} - \log^2 \frac{s}{em^2} + 1 - \frac{\pi^2}{3} + \log^2 \frac{u}{t}
\]

\[
\sigma^{\gamma\gamma}_Q = \frac{1}{2} \sigma^{\gamma\gamma} = \frac{1}{4} \left(\frac{\alpha}{\pi}\right)^2 \left(\left|-\delta_1 + R_1\right|^2 - R_2\right) \sigma_0
\]

\[
R_2 = \frac{8}{3} \pi^2 (\log \frac{tu}{m^2 s} - 1)^2
\]
The CDR (Constrained Diagram Reduction) results grows substantially as the dimension. The procedure has been implemented in FormCalc and LoopTools, which allows us to evaluate (scheme), which provides renormalized expressions.

The CDR (Constrained Diagram Reduction) mass is the electric charge of fermion on-shell and CDR results grows substantially as the dimension. Since the di

\[ \delta_A^c = (A_L^C - A_L^0)/A_L^0, \]

\[ \Delta_A = (A_L^{1-looop+Q} - A_L^{1-looop})/A_L^0 \]

\[ \omega = 0.05 \sqrt{s} \]

\[ \text{E}_{\text{lab}} = 11 \text{ GeV} \]
PV Asymmetry

Predicted PV asymmetry ($E_{\text{lab}} = 11 \text{GeV}$):

\[ A_{\text{PV}}^{(\text{LO+NLO})} (90^\circ) = 32.2 \text{ (ppb)} \]
\[ A_{\text{PV}}^{(\text{LO+NLO+Q-part})} (90^\circ) = 36.2 \text{ (ppb)} \]

\[ m_W^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu \sin^2 \theta_W (1 - \Delta r)} \]

\( \omega = 0.05 \sqrt{s} \) and \( \theta = 90^\circ \)
Two-Loops Contribution

We split the two-loops contribution into subsets of the gauge invariant classes:

- Reducible contribution \((\text{BSE} + \text{Ver})^2\).
- Irreducible ladder, decorated boxes and boxes with electron self-energies.
- Irreducible two-loops vertex correction (double vertices) and self energy diagrams.

\[
\sigma_T = \frac{\pi^3}{s} \Re M_2 M_0^+ \propto \alpha^4
\]
Two-loops t-channel diagrams from the gauge-invariant set of vertices and boson self-energies. Here, the circles represent the contributions of self-energies and vertex functions.
Ladder-Box Diagrams

Diagrams with ZZZ exchange.

Diagrams with WWZ exchange.

Diagrams with ZZγ exchange.

Diagram with WWγ exchange.

(Double Boxes)
Decorated Box Diagrams

(a)  (b)  (c)  (d)  (e)  (f)  (g)  (h)  (i)  (j)  (k)

Decorated boxes of type I.

(l)  (m)  (n)  (o)  (p)  (q)  (r)  (s)  (t)  (u)  (v)

Decorated boxes of type II.

(Double Boxes)
Boxes with Lepton Self-Energy and Vertex Insertions

Boxes with vertices (VB), fermion self-energy boxes FSEB and boson self-energy boxes BSEB.

(SE and Ver in boxes)
Double Vertices

Two loops electron vertices
( NNLO EW Vert )
**Combination of Corrections**

For the orthogonal kinematics: $\theta = 90^\circ$

<table>
<thead>
<tr>
<th>Type of contribution</th>
<th>$\delta_A^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLO</td>
<td>-0.6953</td>
</tr>
<tr>
<td>...+Q+ BBSE+VVer+</td>
<td>-0.6420</td>
</tr>
<tr>
<td>...+ double boxes</td>
<td>-0.6534</td>
</tr>
<tr>
<td>...+NNLO QED</td>
<td>-0.6500</td>
</tr>
<tr>
<td>...+SE and Ver in boxes</td>
<td>-0.6539</td>
</tr>
<tr>
<td>...+NNLO EW Ver</td>
<td>-0.6574</td>
</tr>
</tbody>
</table>

**Correction to PV asymmetry:**

$$\delta_A^C = \frac{A_{LR}^C - A_{LR}^0}{A_{LR}^0}$$

**Soft-photon bremsstrahlung cut:**

$$\omega = 0.05 \sqrt{s}$$

“...” means all contributions from the lines above

Third Stage: Computer Algebra

- The most of the leading two-loop EWC corrections to Moller process has been completed.

- It is essential to apply alternative approaches in two-loop EWC calculations for the cross-check purposes.

- We develop the third stage method which is based on the dispersive representation of many-point Passarino-Veltman functions.

- Advantages include not only cross checking previous results, but also our ability to retain kinematical dependence of two-loop EWC and inclusion of broader sets of two-loops graphs.
Sub-Loop Insertions: Self-Energy


\[
L(q^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\Im L(s)}{s - q^2 - i\epsilon}
\]

- Replace self-energy insertion by effective propagator
- Dispersive representation of self-energy sub-loop has propagator like structure with mass \( s \)
Each of the blocks fermion truncated self-energy graph. The $V$ mixing self-energies. In Eq.(4),

Here, in Eq.(3), terms:

or vector bosons, the self-energy sub-loop can be defined in form of the Lorentz covariant

the Fig.(1) could be reduced to graphs shown on the Fig.(2). More specifically, for fermion
generally, self-energy could be applied to any internal line.

Figure 1: Examples of self-energy sub-loops in the self-energy, triangle and box topologies. In

The self-energy sub-loop could be inserted into another self-energy, triangle or box topology (see Fig.(1)). After replacing self-energy sub-loop by the dispersion integral, graphs on

equivalent functions. Then, each of the two-point tensor coefficients in Eqs.(3) and (4) can be written in terms of Passarino-Veltman two-

represents transverse and longitudinal parts of truncated $V$-

represents left, right and scalar parts of the

are usual left/right chirality projectors.

Each of the $\Sigma$ terms are functions of:

Vector boson: $\Sigma_{\mu\nu}^{V-V}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \Sigma_{T}^{V-V}(q^2) + \frac{q_\mu q_\nu}{q^2} \Sigma_{L}^{V-V}(q^2)$

Fermion: $\Sigma^{f}(q) = q\omega_{-} \Sigma_{L}^{f}(q^2) + q\omega_{+} \Sigma_{R}^{f}(q^2) + m_f \Sigma_{S}^{f}(q^2)$

Each of the $\Sigma$ terms are functions of:

$$B_{i,ij,ijk} \left(q^2, m_\alpha^2, m_\beta^2\right) = \frac{1}{\pi} \int_{0}^{\infty} ds \frac{s \Sigma B_{i,ij,ijk} \left(s, m_\alpha^2, m_\beta^2\right)}{s - q^2 - i\epsilon} \left(m_\alpha + m_\beta\right)^2$$
Sub-Loop: Vector Boson SE

First loop insertion:

\[
\Sigma_{\mu\nu}^{V-V}(q) = \frac{1}{\pi} \sum_{\alpha,\beta} \int \frac{1}{(m_{\alpha}+m_{\beta})^2} ds \frac{1}{s-q^2-i\epsilon} \left[ \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2} \right) \Sigma_T^{V-V}(s, m_{\alpha}^2, m_{\beta}^2) + \frac{q_{\mu}q_{\nu}}{q^2} \Sigma_L^{V-V}(s, m_{\alpha}^2, m_{\beta}^2) \right]
\]

Second loop integration:

\[
I^{1,M,1}_{\mu_1\mu_2,\nu_1...\nu_R} = \frac{1}{\pi} \frac{\mu^{(4-D)}}{(i\pi D/2)} \sum_{\alpha,\beta} \int \frac{1}{(m_{\alpha}+m_{\beta})^2} ds \int d^D q_2 \cdot \frac{q_{2,\nu_1...q_{2,\nu_R}}}{(s-q_2^2-i\epsilon) \prod_{j=0}^{M} [(q_2 + k_{j,M})^2 - m_{j,M}^2]} \] 

\[
F_{\mu_1\mu_2}(q_2, s, m_\alpha, m_\beta) = \left( g_{\mu_1\mu_2} - \frac{q_{2,\mu_1}q_{2,\mu_2}}{q_2^2} \right) \Sigma_T^{V-V}(s, m_\alpha^2, m_\beta^2) + \frac{q_{2,\mu_1}q_{2,\mu_2}}{q_2^2} \Sigma_L^{V-V}(s, m_\alpha^2, m_\beta^2)
\]
First loop insertion:

\[
\Sigma^f(q) = \frac{1}{\pi} \sum_{\alpha,\beta} \int_0^\infty \frac{ds}{s-q^2 - i\epsilon} \left[ q_\omega \mathcal{S}_L^f(s,m_\alpha^2,m_\beta^2) + q_\omega \mathcal{S}_R^f(s,m_\alpha^2,m_\beta^2) + m_f \mathcal{S}_S^f(s,m_\alpha^2,m_\beta^2) \right]
\]

Second loop integration:

\[
I_{\nu_1...\nu_R}^{1,M,1} = \frac{1}{\pi} \frac{\mu^{(4-D)}}{(i\pi^{D/2})} \sum_{\alpha,\beta} \int_0^\infty ds \int d^D q_2 \frac{q_{2,\nu_1}...q_{2,\nu_R}}{(s-q_2^2 - i\epsilon) \prod_{j=0}^M [(q_2 + k_{j,M})^2 - m_{j,M}^2]} G(q_2, s, m_\alpha, m_\beta)
\]

\[
G(q_2, s, m_\alpha, m_\beta) = \left[ q_\omega \mathcal{S}_L^f(s,m_\alpha^2,m_\beta^2) + q_\omega \mathcal{S}_R^f(s,m_\alpha^2,m_\beta^2) + m_f \mathcal{S}_S^f(s,m_\alpha^2,m_\beta^2) \right]
\]
Vector boson SE sub-loop insertion two-loop result:

\[
I^{1,M,1}_{\mu_1\mu_2,\nu_1\ldots\nu_R} = \frac{1}{\pi} \sum_{\alpha,\beta} \int_{(m_{\alpha}+m_{\beta})^2}^{\infty} ds \cdot \left[ L^{1,M,1}_{a,\mu_1\mu_2,\nu_1\ldots\nu_R}(B,C,D) \otimes \Sigma^V_{\alpha} (s, m_{\alpha}^2, m_{\beta}^2) + \right.
\]

\[
\left. L^{1,M,1}_{b,\mu_1\mu_2,\nu_1\ldots\nu_R}(B,C,D) \otimes \Sigma^V_{\beta} (s, m_{\alpha}^2, m_{\beta}^2) \right].
\]

Fermion SE sub-loop insertion two-loop result:

\[
I^{1,M,1}_{\nu_1\ldots\nu_R} = \frac{1}{\pi} \sum_{\alpha,\beta} \int_{(m_{\alpha}+m_{\beta})^2}^{\infty} ds \cdot \left[ N^{1,M,1}_{a,\nu_1\ldots\nu_R}(B,C,D) \otimes \Sigma^f_{\alpha} (s, m_{\alpha}^2, m_{\beta}^2) \omega_- + \right.
\]

\[
\left. N^{1,M,1}_{b,\nu_1\ldots\nu_R}(B,C,D) \otimes \Sigma^f_{\beta} (s, m_{\alpha}^2, m_{\beta}^2) \omega_+ + N^{1,M,1}_{c,\nu_1\ldots\nu_R}(B,C,D) \otimes \Sigma^S (s, m_{\alpha}^2, m_{\beta}^2) \right].
\]
Vector Boson SE Subtractions

\{Z-Z\} or \{W-W\} mixings:

\[
\hat{\Sigma}^{\nu-\nu}(q^2) = \Sigma^{\nu-\nu}(q^2) - \Sigma^{\nu-\nu}(m_{\nu}^2) - \frac{\partial}{\partial q^2} \Sigma^{\nu-\nu}(q^2) \bigg|_{q^2 = m_{\nu}^2} (q^2 - m_{\nu}^2) =
\]

\[
\frac{(q^2 - m_{\nu}^2)^2}{\pi} \sum_{\alpha,\beta} \int_{q^2}^{\infty} ds \frac{\Im \Sigma^{\nu-\nu}(s, m_{\alpha}, m_{\beta})}{(s - m_{\nu}^2)^2 (s - q^2 - i\epsilon)}
\]

\[\gamma-Z\] mixing:

\[
\hat{\Sigma}^{\gamma-Z}(q^2) = \Sigma^{\gamma-Z}(q^2) - \frac{1}{m_{\gamma}^2} \left[ \Sigma^{\gamma-Z}(0)q^2 - \Sigma^{\gamma-Z}(m_{\gamma}^2) (q^2 - m_{\gamma}^2) \right] =
\]

\[
\frac{q^2 (q^2 - m_{\gamma}^2)}{\pi} \sum_{\alpha,\beta} \int_{q^2}^{\infty} ds \frac{\Im \Sigma^{\gamma-Z}(s, m_{\alpha}, m_{\beta})}{s (s - m_{\gamma}^2) (s - q^2 - i\epsilon)(m_{\alpha} + m_{\beta})^2}
\]

\[\gamma-\gamma\] mixing:

\[
\hat{\Sigma}^{\gamma-\gamma}(q^2) = \frac{q^4}{\pi} \sum_{\alpha,\beta} \int_{q^2}^{\infty} ds \frac{\Im \Sigma^{\gamma-\gamma}(s, m_{\alpha}, m_{\beta})}{s^2 (s - q^2 - i\epsilon)(m_{\alpha} + m_{\beta})^2}
\]
Fermion SE Subtractions

\[ \hat{\Sigma}^f(q) = \Sigma^f(q) - \Sigma^f(m_f) - \frac{\partial}{\partial q} \sum^f(q) \bigg|_{q=m_f} (q - m_f) = \]

\[ q\omega_- (I_L + a_L) + q\omega_+ (I_R + a_R) + m_f (I_S + a_S) \]

\[ I_{L,R,S} = \frac{q^2 - m_f^2}{\pi} \sum_{\alpha,\beta} \int_{(m_\alpha + m_\beta)^2}^{\infty} ds \frac{3 \Sigma^f_{L,R,S}(s, m_\alpha, m_\beta)}{(s - m_f^2) (s - q^2 - i\epsilon)} \]

\[ a_{L,R} = -2m_f^2 \left( \Sigma'_{L,R} (m_f^2) + \Sigma'_{S} (m_f^2) \right) \]

\[ a_S = m_f^2 \left( \Sigma'_{L} (m_f^2) + \Sigma'_{R} (m_f^2) + 2\Sigma'_{S} (m_f^2) \right) \]
Effective SE Propagators

Vector boson effective propagator:

\[ \Pi^{V-V}_{\mu\nu}(q) = \Pi_{T,\mu\nu}^{V-V} + \Pi_{L,\mu\nu}^{V-V} \]

\[ \Pi_{T,\mu\nu}^{V-V} = \frac{-i g_{\rho\mu}}{q^2 - m_V^2} \left[ \frac{g^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}}{s - q^2 - i\epsilon} \Im \Sigma_{T}^{V-V} \left( s, m_\alpha^2, m_\beta^2 \right) \right] \frac{-i g_{\sigma\nu}}{q^2 - m_V^2} \]

\[ \Pi_{L,\mu\nu}^{V-V} = \frac{-i g_{\rho\mu}}{q^2 - m_V^2} \left[ \frac{q^\rho q^\sigma}{q^2} \Im \Sigma_{L}^{V-V} \left( s, m_\alpha^2, m_\beta^2 \right) \right] \frac{-i g_{\sigma\nu}}{q^2 - m_V^2} \]

Fermion effective propagator:

\[ \Pi^f(q) = \frac{1}{q - m_f} \left[ \frac{G(q, s, m_\alpha, m_\beta)}{s - q^2 - i\epsilon} \right] \frac{1}{q - m_f} \]
The help of Eqs. (16) and (17), the following set of equations for the subtracted effective fermion propagators derived from specific

\[ \hat{\Pi}_{\mu\nu}^{V-V}(q) = \hat{\Pi}_{T,\mu\nu}^{V-V} + \hat{\Pi}_{L,\mu\nu}^{V-V} \]

\[ \hat{\Pi}_{T,\mu\nu}^{V-V} = -T^{V-V}(s, m_V^2) \left[ \frac{g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2}}{s - q^2 - i\epsilon} \right] \Re \Sigma_T^{V-V}(s, m_\alpha^2, m_\beta^2) \]

\[ \hat{\Pi}_{L,\mu\nu}^{V-V} = -T^{V-V}(s, m_V^2) \left[ \frac{q_{\mu} q_{\nu}}{q^2} \right] \Re \Sigma_L^{V-V}(s, m_\alpha^2, m_\beta^2) \]

| \( T^{V-V} \) | \( \frac{1}{s^2} \) | \( \frac{1}{(s - m_{\{Z,W\}}^2)^2} \) | \( \frac{1}{s(s - m_Z^2)} \) |
Effective Propagators: Subtracted FF

\[ \hat{\Pi}^f(q) = \hat{\Pi}_1^f(q) + \hat{\Pi}_2^f(q) \]

\[ \hat{\Pi}_1^f(q) = (q + m_f) \left[ \frac{y_L q \omega_- + y_R q \omega_+ + m_f y_S}{(q^2 - m_f^2) (s - q^2 - i\epsilon)} \right] (q + m_f) \]

\[ \hat{\Pi}_2^f(q) = \frac{1}{q - m_f} \left[ d_L q \omega_- + d_R q \omega_+ + m_f d_S \right] \frac{1}{q - m_f}, \]

\[ y_{L,R,S} \equiv y_{L,R,S} (s, m_{\alpha}^2, m_{\beta}^2) = \frac{\Im \Sigma_{L,R,S}^f}{s - m_f^2} \]

\[ d_{L,R} \equiv d_{L,R} (s, m_{\alpha}^2, m_{\beta}^2) = -2m_f^2 \frac{\Im \Sigma_{L,R}^f + \Re \Sigma_{S}^f}{(s - m_f^2)^2} \]

\[ d_S \equiv d_S (s, m_{\alpha}^2, m_{\beta}^2) = m_f^2 \frac{\Im \Sigma_{L}^f + \Re \Sigma_{R}^f + 2\Re \Sigma_{S}^f}{(s - m_f^2)^2} \]
Figure 3: Reduction of the triangle graph by the derivative representation of self-energy.

Second Loop Integration: Many-Point PV

\[ C_0 \equiv C_0 \left( p_1^2, p_2^2, (p_1 + p_2)^2, m_1^2, m_2^2, m_3^2 \right) = \]

\[ \frac{\mu^{A-D}}{i\pi^{D/2}} \int d^D q \frac{1}{[q^2 - m_1^2][(q + p_1)^2 - m_2^2][(q + p_1 + p_2)^2 - m_3^2]} \]

\[ C_0 = \frac{\mu^{A-D}}{i\pi^{D/2}} \int_0^1 dx \int d^D \tau \frac{1}{[(\tau - (p_1 \bar{x} + p_2))^2 - m_{12}^2][\tau^2 - m_3^2]}, \text{ here } m_{12}^2 = m_1^2 \bar{x} + m_2^2 x - p_1^2 x \bar{x} \]

Using:

\[ \frac{1}{[(\tau - (p_1 \bar{x} + p_2))^2 - m_{12}^2]^2} = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \frac{1}{[(\tau - (p_1 \bar{x} + p_2))^2 - (m_{12}^2 + \lambda)]} \]

\[ C_0 = \frac{\mu^{A-D}}{i\pi^{D/2}} \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \int d^D \tau \frac{1}{[(\tau - (p_1 \bar{x} + p_2))^2 - (m_{12}^2 + \lambda)][\tau^2 - m_3^2]} = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \ B_0 \left( (p_1 \bar{x} + p_2)^2, m_3^2, m_{12}^2 + \lambda \right) \]
Second Loop Integration: Many-Point PV

\[ \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_{0}^{1} dx \]

\[ \sqrt{m_{12}^2 + \lambda} \]

\[ p_1 \bar{x} + p_2 \]

\[ m_3 \]
In this case, we join the first three propagators, and after shifting momentum of $\cdots B$

For the four-point function we will use an analogous approach. We start with the general $0$\,$D$

1

$D_1 = 2$

For the vector $0$

$\cdots p$

shifting momenta as before, we get:

$c_w$ write:

for the three-point function in terms of the two-point basis. For the function

in the second-loop integration by the derivatives of two-point function. Taking into the

over the Feynman parameter, and then di

$\cdots$ different functions of the higher-rank tensors, we can derive the partial tensor reduction

$\cdots x$

trick on the right hand side of Eq.(25). After using derivative approach in Eq.(24), and

numerically. Using approach in Eq.(24), we can replace the many-point tensor functions

in the second-loop integration by the derivatives of two-point function. Taking into the

numerically. The final step would be a numerical evaluation of the derivative

$\cdots$.

structure of scalar $\cdots$ advantage of the partial tensor reduction is that we can substantially reduce a size of the

The same idea can be extrapolated to the higher orders of the three-point functions. The

of $\cdots$ and $\cdots m$ a t c h i n g t e r m s i n f r o n to $\cdots N$

$\cdots 1$

$\cdots 1$

$\cdots 2$

$\cdots 2$

$\cdots 0$

Vector case:

Applying tensor decomposition to both sides of above equation:

$p_1 \mu C_1 + (p_1 \mu + p_2 \mu) C_2 = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \left[ B_\mu - (p_1 \mu + p_2 \mu) B_0 \right]$

\[ B_\mu = - (p_1 \mu \bar{x} + p_2 \mu) B_1 \]

Matching coefficients in front of $p_1,2$, we get:

\[ C_1 = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx B_1 x \]

\[ C_2 = - \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx [B_0 + B_1] \]
Second Loop Integration: Many-Point PV

\[ \hat{I}_C = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \ldots \]

\[ C_{\mu \nu}: \]

\[ C_{00} = \hat{I}_C [B_{00}] \quad C_{12} = -\hat{I}_C [(B_1 + B_{11}) x] \]
\[ C_{11} = \hat{I}_C [B_{11} x^2] \quad C_{22} = \hat{I}_C [B_0 + 2B_1 + B_{11}] \]

\[ C_{\mu \nu \alpha}: \]

\[ C_{001} = \hat{I}_C [B_{001} x] \quad C_{112} = -\hat{I}_C [(B_{11} + B_{111}) x^2] \]
\[ C_{002} = -\hat{I}_C [B_{00} + B_{001}] \quad C_{122} = \hat{I}_C [(B_1 + 2B_{11} + B_{111}) x] \]
\[ C_{111} = \hat{I}_C [B_{111} x^3] \quad C_{222} = -\hat{I}_C [B_0 + 3 (B_1 + B_{11}) + B_{111}] \]

\[ B_{i,ij,ijk} \equiv B_{i,ij,ijk} \left( (p_1 \bar{x} + p_2)^2, m_3^2, m_{12}^2 + \lambda \right) \]
Second Loop Integration: Many-Point PV

\[ D_0 \equiv D_0 \left( p_1^2, p_2^2, p_3^2, p_4^2, (p_1 + p_2)^2, (p_2 + p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2 \right) = \]

\[ \frac{\mu^{A-D}}{i \pi^{D/2}} \int d^D q \frac{1}{[q^2 - m_1^2] [(q + p_1)^2 - m_2^2] [(q + p_1 + p_2)^2 - m_3^2] [(q + p_1 + p_2 + p_3)^2 - m_4^2]} \]

\[ D_0 = 2 \frac{\mu^{A-D}}{i \pi^{D/2}} \int_0^1 dx \int_0^{1-x} dy \int d^D \tau \frac{1}{[(\tau - (p_1(x - y) + p_2 y + p_3))^2 - m_{123}^2]^3 [\tau^2 - m_4^2]} \]

\[ m_{123}^2 = m_1^2 (\bar{x} - y) + m_2^2 x + m_3^2 y - p_1^2 x\bar{x} - p_{12}^2 y\bar{y} + 2xy(p_{12}) \]

\[ p_{12} = p_1 + p_2 \]

\[ D_0 = \lim_{\lambda \to 0} \frac{\partial^2}{\partial \lambda^2} \int_0^1 dx \int_0^{\bar{x}} dy B_0 \left[ (p_1 (\bar{x} - y) + p_2 \bar{y} + p_3)^2, m_4^2, m_{123}^2 + \lambda \right] \]
Second Loop Integration: Many-Point PV

\[ \hat{I}_D = \lim_{\lambda \to 0} \frac{\partial^2}{\partial \lambda^2} \int_0^1 dx \int_0^\bar{x} dy \ldots \]

\[ B_{i,j,ijk,ijkl} \equiv B_{i,j,ijk,ijkl} \left[ (p_1 (\bar{x} - y) + p_2 \bar{y} + p_3)^2, m_4^2, m_{123}^2 + \lambda \right] \]

\[ D_\mu : \]

\[ D_1 = \hat{I}_D [B_1 x] \]
\[ D_2 = \hat{I}_D [B_1 y] \]
\[ D_3 = -\hat{I}_D [B_0 + B_1] \]

\[ D_{\mu \nu} : \]

\[ D_{00} = \hat{I}_D [B_{00}] \]
\[ D_{11} = \hat{I}_D [B_{11} x^2] \]
\[ D_{22} = \hat{I}_D [B_{11} y^2] \]
\[ D_{33} = \hat{I}_D [B_0 + 2B_1 + B_{11}] \]

\[ D_{\mu \nu \rho} : \]

\[ D_{001} = \hat{I}_D [B_{001} x] \]
\[ D_{002} = \hat{I}_D [B_{001} y] \]
\[ D_{003} = -\hat{I}_D [B_0 + B_{001}] \]
\[ D_{011} = \hat{I}_D [B_{11} x^3] \]
\[ D_{112} = \hat{I}_D [B_{11} x^2 y] \]
\[ D_{113} = -\hat{I}_D [(B_1 + B_{111}) x^2] \]
\[ D_{012} = \hat{I}_D [B_{011} x^2] \]
\[ D_{013} = -\hat{I}_D [(B_1 + B_{111}) x] \]
\[ D_{111} = \hat{I}_D [B_{111} x^4] \]
\[ D_{1112} = \hat{I}_D [B_{111} x^3 y] \]
\[ D_{1113} = -\hat{I}_D [(B_1 + B_{111}) x^3] \]
\[ D_{11122} = \hat{I}_D [B_{111} x^2 y^2] \]
\[ D_{11123} = -\hat{I}_D [(B_1 + 3 (B_1 + B_{111}) + B_{111}) x] \]
\[ D_{11133} = -\hat{I}_D [(B_1 + 3 (B_1 + B_{111}) + B_{111}) y] \]
\[ D_{111222} = \hat{I}_D [B_{111} y^4] \]
\[ D_{111223} = -\hat{I}_D [(B_1 + B_{111}) y^3] \]
\[ D_{111233} = -\hat{I}_D [(B_1 + 2 B_{111} + B_{111}) y^2] \]
\[ D_{111333} = -\hat{I}_D [(B_1 + 3 (B_1 + B_{111}) + 6 B_{111} + B_{111}) y] \]

The author is grateful to S. Barkanova, A. Czarnecki and M. Vanderhaeghen for the
µ
\[ \in \mathbb{C} \]


\[ \rho \]


\[ \delta \]


\[ \sigma \]


\[ \alpha \]


\[ \beta \]


\[ \gamma \]


\[ \delta \]


\[ \epsilon \]


\[ \zeta \]


\[ \eta \]


\[ \theta \]


\[ \iota \]


\[ \kappa \]


\[ \lambda \]


\[ \mu \]


\[ \nu \]


\[ \xi \]


\[ \pi \]


\[ \rho \]


\[ \sigma \]


\[ \tau \]


\[ \upsilon \]


\[ \phi \]


\[ \chi \]


\[ \psi \]


\[ \omega \]


\[ \Delta \]


\[ \Theta \]


\[ \Lambda \]


\[ \Xi \]


\[ \Pi \]


\[ \Sigma \]


\[ \Upsilon \]


\[ \Phi \]


\[ \Psi \]


\[ \Omega \]
Triangle Insertion: Dispersive Approach

\[ C_0 \left( m^2, q_2^2, (q_2 - k)^2, m_0^2, m^2, m^2 \right) = \frac{\mu^{4-D}}{i\pi^{D/2}} \int \frac{d^D q_1}{\left[ q_1^2 - m_0^2 \right] \left[ (q_1 - k)^2 - m^2 \right] \left[ (q_1 - k + q_2)^2 - m^2 \right]} \]

join two propagators without \( q_2 \)

\[ C_0 \left( m^2, q_2^2, (q_2 - k)^2, m_0^2, m^2, m^2 \right) = \frac{1}{\pi} \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \int_{\Lambda^2} ds \frac{\Im B_0 \left( s, m_3^2, m_{12}^2 + \lambda \right)}{s - (q_2 - k \bar{x})^2 - i\epsilon} \]

\[ m_{12}^2 = m_0^2 \bar{x} + m^2 \bar{x}^2 \]
In general, we can write that triangle sub-loop can be replaced by the following function. With the help of Eq.(27) and Appendix A, we show in Figs.(7) results for Eq.(32), we can also derive dispersive representation for the higher order three-point functions calculated from dispersion integrals (dots) and LoopTools (solid line). Masses and external momenta have the same values as for Fig.(6)

\begin{align*}
\text{Figure 6: Comparison of the results obtained using LoopTools (solid line) and dispersion integral.} \\
\end{align*}

The derivative of the parameter will cancel. We checked numerically that final result in Eq.(32)

\begin{align*}
\text{Figure 8: Comparison of the } C_{1,2}^{\text{ARG}}(q) \text{ and } C_{1,2}^{\text{ARG}}(\vec{q}) \text{ functions calculated from dispersion integrals (dots) and LoopTools (solid line). Masses and external momenta have the same values as for Fig.(6).} \\
\end{align*}

\begin{align*}
m_0 = 1.2 \text{ GeV}, m = 0.1 \text{ GeV and } (k \cdot q_2) = -3.4 \text{ GeV}^2
\end{align*}
Triangle Insertion: Dispersive Approach

\[
\hat{D} = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \int_{\Lambda^2} \left( m_3 + (m_{12}^2 + \lambda)^{1/2} \right)^2 ds... \\
\hat{\Gamma} = \hat{D} \left[ \frac{\Im F(s, m_3^2, m_{12}^2 + \lambda)}{s - (p_2 + p_1 \bar{x})^2 - i\epsilon} \right]
\]

\[
m_{12}^2 = m_1^2 \bar{x} + m_2^2 x - p_1^2 x \bar{x}
\]

Subtracted vertex at zero momentum: \[
\hat{\Gamma} = \hat{D} \left[ \frac{\Im F(s, m_3^2, m_{12}^2 + \lambda) \left[ (p_2 + p_1 \bar{x})^2 - p_1^2 \bar{x}^2 \right]}{s - (p_2 + p_1 \bar{x})^2 - i\epsilon \left[ s - p_1^2 \bar{x}^2 \right]} \right]
\]
Triangle Insertion: Dispersive Approach

\[ C_0 \left( m^2, q_2^2, (q_2 - k)^2, m_0^2, m^2, m^2 \right) = \frac{1}{\pi} \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \int_{-\Lambda^2}^{\Lambda^2} ds \frac{\Im B_0 \left( s, m_3^2, m_{12}^2 + \lambda \right)}{s - (q_2 - kx)^2 - i\epsilon} \]

\[ m_{12}^2 = m_1^2 x + m_2^2 x - p_1^2 x \bar{x} \]

Mass, \( m_{12}^2 \) could become negative.

\[ \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} B_0 \left( p^2, m_1^2, -|m_2^2| + \lambda \right) = \frac{1}{2\pi i} \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_{-\Lambda^2}^{\Lambda^2} ds \frac{B_0 \left( s, m_1^2, -|m_2^2| + \lambda \right)}{s - p^2 - i\epsilon} \]

\[ \Lambda^2 = 10^{10} \text{ GeV}^2, \; m_1^2 = 1 \text{ GeV}^2 \text{ and } |m_2^2| = 9 \text{ GeV}^2 \]

<table>
<thead>
<tr>
<th>( p^2 ) (GeV)^2</th>
<th>Eq.(36)</th>
<th>LoopTools</th>
</tr>
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<td>0.147988 - 0.079999 i</td>
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<tr>
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<td>0.127835 - 0.037405 i</td>
<td>0.127836 - 0.037405 i</td>
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<td>0.124875 - 0.034242 i</td>
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<td>0.5</td>
<td>0.119143 - 0.028889 i</td>
<td>0.119144 - 0.028889 i</td>
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<td>0.116402 - 0.026626 i</td>
<td>0.116403 - 0.026626 i</td>
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<td>0.097878 - 0.014968 i</td>
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<td>10.0</td>
<td>0.081819 - 0.008497 i</td>
<td>0.081820 - 0.008497 i</td>
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<tr>
<td>50.0</td>
<td>0.039036 - 0.000911 i</td>
<td>0.039037 - 0.000911 i</td>
</tr>
</tbody>
</table>
Numerical Example

\[ I = -\frac{1}{\pi^4} \int \frac{d^4q_1 d^4q_2}{[q_1^2 - m_1^2] [(q_1 - p)^2 - m_2^2] [(q_1 + q_2)^2 - m_3^2] [q_2^2 - m_4^2] [(q_2 + p)^2 - m_5^2]} \]

\[ I = \frac{1}{2\pi^3} \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \int_0^1 dx \int_0^{\Lambda^2} ds \ (2i \Im B_0 [s, m_3^2, m_{12}^2 + \lambda] \theta (m_{12}^2) + B_0 [s, m_3^2, m_{12}^2 + \lambda] \theta (-m_{12}^2)) \]

\[ \int \frac{d^4q_2}{[q_2^2 - m_4^2] [(q_2 + xp)^2 - s] [(q_2 + p)^2 - m_5^2]}, \]

\[ r(x, \lambda) = (m_3 + (m_{12}^2 + \lambda)^{1/2})^2 \theta(m_{12}^2) - \Lambda^2 \theta(-m_{12}^2) \]
I = - \frac{1}{2\pi i} \lim_{\{\lambda, \xi\} \to 0} \frac{\partial^2}{\partial \lambda \partial \xi} \int_0^1 dx dy \int_{r(x,\lambda)}^{\Lambda^2} ds \left( 2i \Im B_0 [s, m_3^2, m_{12}^2 + \lambda] \theta (m_{12}^2) + \right.

B_0 \left[ s, m_3^2, m_{12}^2 + \lambda \right] \theta (-m_{12}^2) \left. \right) B_0 \left[ p^2 (x - y)^2, s, m_{45}^2 + \xi \right],

m_{12}^2 = m_1^2 x + m_2^2 x - p^2 \bar{x} x

m_{45}^2 = m_4^2 y + m_5^2 y - p^2 \bar{y} y
Numerical Example


<table>
<thead>
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<th>This work</th>
<th>$\Delta t_{This \ Work}$</th>
<th>[13]</th>
<th>$\Delta t_{[13]}$</th>
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<td>1120</td>
<td>0.17391 - 0.11807 i</td>
<td>85</td>
</tr>
</tbody>
</table>

$m_1^2 = 1, m_2^2 = 2, m_3^2 = 3, m_4^2 = 4$ and $m_5^2 = 5 \text{ (GeV)}^2$
Crossed Two-Loop Topologies

A. A, arXiv:1809.05592

Figure 3: Crossed two-loop vertex topology.

Figure 4: Crossed two-loop box topology.
Crossed Two-Loop Triangle

\[ I_{\Delta_2} = -\frac{1}{\pi^4} \int \frac{d^4 q_1 d^4 q_2}{[q_1^2 - m_a^2] \left[(k_1 - q_1)^2 - m_1^2\right] \left[(k_1 - q_1 - q_2)^2 - m_1^2\right] \left[(k_2 - q_1 - q_2)^2 - m_1^2\right]} \times \frac{1}{[q_2^2 - m_b^2] \left[(k_2 - q_2)^2 - m_2^2\right]} \cdot \]

\[ I_{\Delta_2} = -\frac{1}{\pi^4} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int \frac{d^4 q_2}{[q_2^2 - m_b^2] \left[(k_2 - q_2)^2 - m_2^2\right]} \int \frac{d^4 q_1}{[q_1^2 - k_1 x^2 - m_{\lambda_1}^2] \left[(q_1 + q_2 - \bar{y} k_2 - y k_1)^2 - m_\lambda^2\right]} \]

\[ m_{\xi}^2 = m_a^2 \bar{x} + m_1^2 x^2 + \xi \quad \quad m_\lambda^2 = m_1^2 - \bar{y} y k_3^2 + \lambda \]
Crossed Two-Loop Triangle

\[ I_{\Delta_2} = \frac{1}{\pi^4} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int \frac{d^4 q_2}{[q_2^2 - m_b^2][(k_2 - q_2)^2 - m_1^2]} \int \frac{d^4 q_1}{[(q_1 - k_1 x)^2 - m_\xi^2][(q_1 + q_2 - y k_2 - y k_1)^2 - m_\chi^2]} \]

\[ I_{\Delta_2} = -i \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int d^4 q_2 \frac{B_0}{[q_2^2 - m_b^2][(k_2 - q_2)^2 - m_1^2]} \frac{[(q_2 - y k_2 + k_1 (x - y))^2, m_\xi^2, m_\chi^2]}{[(q_1 - k_1 x)^2 - m_\xi^2][(q_1 + q_2 - y k_2 - y k_1)^2 - m_\chi^2]} \]

\[ I_{\Delta_2} = \frac{i}{\pi^3} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int \frac{\Lambda^2}{(m_\xi + m_\lambda)^2} ds \Im B_0 \left[ s, m_\xi^2, m_\chi^2 \right] \]

\[ \times \frac{d^4 q_2}{[q_2^2 - m_b^2][(k_2 - q_2)^2 - m_1^2][(q_2 - y k_2 + k_1 (x - y))^2 - s - i\epsilon]} \]

\[ I_{\Delta_2} = -\frac{1}{\pi} \lim_{\{\xi, \lambda, \phi\} \to 0} \frac{\partial^3}{\partial \xi \partial \lambda \partial \phi} \int_0^1 dx dy dz \int \frac{\Lambda^2}{(m_\xi + m_\lambda)^2} ds \Im B_0 \left[ s, m_\xi^2, m_\chi^2 \right] B_0 \left[ (((z - y) k_2 + k_1 (x - y))^2, m_\phi^2, s \right] \]

\[ m_\phi^2 = m_b^2 \bar{z} + m_1^2 z^2 + \phi \]
Crossed Two-Loop: Generalization

\[ I_{\otimes} = \frac{1}{i\pi^2} \int \frac{d^4q}{[q^2 - m_1^2] \[(q + p_1)^2 - m_2^2\]} \frac{d^4q}{[(q + p_1 + p_2)^2 - m_3^2] \[(q + p_1 + p_2 + p_3)^2 - m_4^2\]} \]

external momenta \( p_2 \) and \( p_4 \) would depend on the momentum of the second loop

\[ I_{\otimes} = \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy B_0 \left[(\bar{x} p_1 + p_2 + y p_3)^2, m_\xi^2, m_\lambda^2\right] \]

\[ m_\xi^2 = \bar{x} m_1^2 + x m_2^2 - x \bar{x} p_1^2 + \xi \quad m_\lambda^2 = \bar{y} m_3^2 + y m_4^2 + p_1^2 + y p_3^2 + \lambda \]
developed an approach where crossed subloop insertion can be replaced by two-point function in Eq.(20) by a dispersive representation, we arrive to the following result:

\[ I_\Box = -\frac{1}{\pi} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int_0^{\Lambda^2} ds \frac{B_0 [s, m_\xi^2, m_\lambda^2]}{\left(\bar{x}p_1 + p_2 + yp_3\right)^2 - s - i\epsilon} \]

\[ \Gamma_\Box = \hat{D} \left[ \frac{B_0 [s, m_\xi^2, m_\lambda^2]}{\left(\bar{x}p_1 + p_2 + yp_3\right)^2 - s - i\epsilon} \right] \]

\[ \hat{D} = \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int_0^{\Lambda^2} ds \left( m_\xi + m_\lambda \right)^2 \]
Conclusion

• We are now in the last stage of the NNLO EWC calculations for the MOLLER experiment.

• Automatization of the NNLO EWC calculations for MOLLER is currently under way.

• Our next goal is a full gauge-invariant set of two-loop EW graphs with SE and triangles insertions.

• Results to be obtained will be cross checked with our previous calculations and other literature.

• We are looking for additional collaborative projects in two-loops calculations for various processes.
Additional Slides
Although this mass is still to be determined experimentally, the dependence of EWC on the fine structure constant due to hadronic vacuum polarization mass is still to be determined experimentally, the dependence of EWC on the fine structure constant due to hadronic vacuum polarization

Let us define the relative corrections to the Born cross section due to specific type of contributions (labeled by C) as 

\[ \Delta A = \frac{A_{LR}^{1-loop+Q} - A_{LR}^{1-loop}}{A_{LR}^0} \]

\[ \delta_A^C = \frac{A_{LR}^C - A_{LR}^0}{A_{LR}^0} \]

The scale of the Q-part contribution in the low-energy region is approximately constant, but starting from \( \sqrt{s} \geq m_Z \), where the weak contribution becomes comparable with electromagnetic, the effect of Q-part grows sharply.

This effect of increasing importance of two-loop contribution at higher energies may have a significant effect on the asymmetry measured at the future e⁻ e⁻-colliders.
Final Expressions

Combining all the terms together, we get the infrared-finite result at one-loop level

\[
\sigma_{NLO} = \frac{\alpha}{\pi} \Re[R_1 + \delta_f] \sigma_0
\]

and NNLO level

\[
\sigma_{NNLO} = \sigma_Q + \sigma_T + \sigma^\gamma + \sigma^\gamma = \left(\frac{\alpha}{\pi}\right)^2 \Re\left[R_1^* \delta_f + \frac{1}{2} |R_1|^2 - \frac{1}{2} R_2 + \delta_f + \delta_T^f\right] \sigma_0
\]

\[
= \sigma^f_Y + \sigma^f_B + \sigma^f_Q + \sigma^f_T,
\]

where:

\[
\sigma^f_O = \frac{\alpha}{\pi} \Re[R_1^* \sigma_{NLO}]
\]

\[
\sigma^f_B = -\frac{1}{2} \left(\frac{\alpha}{\pi}\right)^2 \Re(|R_1|^2 + R_2) \sigma_0
\]

\[
\sigma^f_Q = \left(\frac{\alpha}{\pi}\right)^2 \delta_f \sigma_0
\]

\[
\sigma^f_T = \left(\frac{\alpha}{\pi}\right)^2 \delta_T^f \sigma_0
\]

\[
R_1 = -4B \log \frac{\sqrt{s}}{2\omega} - \left(\log \frac{s}{m^2} - 1\right)^2 + 1 - \frac{\pi^2}{3} + \log^2 \frac{u}{t}, \quad R_2 = \frac{8}{3} \pi^2 B^2, \quad B = \log \frac{tu}{m^2s} - 1 + i\pi
\]
\[ I_{\Box 2} = -\frac{1}{\pi^4} \int \frac{d^4 q_1 d^4 q_2}{[q_1^2 - m_a^2] [(k_4 - q_1)^2 - m_2^2] [(k_2 - k_4 + q_1 + q_2)^2 - m_b^2] [(k_1 - q_1 - q_2)^2 - m_1^2]} \times \frac{1}{[q_2^2 - m_c^2] [(k_1 - q_2)^2 - m_1^2] [(k_2 + q_2)^2 - m_2^2]} \]

\[ I_{\Box 2} = -\frac{i}{\pi^2} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^2}{\partial \xi \partial \lambda} \int_0^1 dx dy \int d^4 q_2 \frac{B_0 [(q_2 - xk_4 + yk_3 - k_1)^2, m_2^2, m_1^2]}{[q_2^2 - m_c^2] [(k_1 - q_2)^2 - m_1^2] [(k_2 + q_2)^2 - m_2^2]} \]

\[ m_2^2 = \bar{x}m_a^2 + x^2m_2^2 + \xi \quad m_1^2 = \bar{y}^2m_1^2 + ym_b^2 + \lambda \]
I_{\Box 2} = -\frac{1}{\pi} \lim_{\{\xi, \lambda\} \to 0} \frac{\partial^4}{\partial \xi \partial \lambda \partial \delta^2} \int_0^1 dx dy dz \int_0^{1-z} dw \int_0^{\Lambda^2} ds \, \mathcal{B}_0 [s, m^2_\xi, m^2_\lambda]

\times B_0 [(\bar{z}k_1 + \omega k_2 + x k_4 - y k_3)^2, m^2_\delta, s],

m^2_\delta = (\bar{z} - \omega) m^2_c + (\omega k_2 - z k_1)^2 + \delta
Sensitivity to effective mixing angle
Sensitivity of Asymmetry to effective mixing angle

Representation of effective Born amplitude:

\[ M_\gamma = \frac{\alpha(t)Q_e^2}{t} \left( \bar{u}_e \gamma_\mu u_e \right) \left( \bar{u}_e \gamma^\mu u_e \right), \]

\[ M_Z = \frac{G_\mu}{\sqrt{2}} \kappa \frac{m_Z^2}{t - m_Z^2 + i\frac{t}{m_Z} \Gamma_Z} \left( \bar{u}_e \gamma_\mu \left[ I_3^e - 2s_W^2(t)Q_e - I_3^e \gamma_5 \right] u_e \right) \left( \bar{u}_e \gamma^\mu \left[ I_3^e - 2s_W^2(t)Q_e - I_3^e \gamma_5 \right] u_e \right). \]

\[ \kappa = \frac{1 - \Delta r}{1 + \Re \left[ \frac{\partial}{\partial t} \hat{\Sigma}_{ZZ}(t) \right]}, \]

\[ \alpha(t) = \frac{\alpha}{1 + \Re \left[ \hat{\Sigma}_{\gamma\gamma}(t) \right] / t}, \]

\[ \Delta r = \Re[\hat{\Sigma}_{WW}(0)] + \frac{\alpha}{4\pi s_W^2} \left( 6 + \frac{7 - 4s_W^2}{2s_W^2} \ln c_W^2 \right) + \frac{c_W^2}{m_Z^2 s_W^2} \Re \left[ \frac{\hat{\Sigma}_{ZZ}(m_Z^2)}{m_Z^2 + \hat{\Sigma}_{\gamma\gamma}(m_Z^2)} \right]. \]
Effective Weinberg mixing angle up to NLO:

\[ \tilde{s}_W^2(t) = s_W^2 - s_W c_W \frac{\Re \left[ \hat{\Sigma}_{\gamma Z}(t) \right]}{t + \Re \left[ \hat{\Sigma}_{\gamma\gamma}(t) \right]} . \]

\[ \sin^2 \theta_W \equiv s_W^2 = 1 - \frac{m_W^2}{m_Z^2} \]

\[ m_W^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu \sin^2 \theta_W (1 - \Delta r)} \]

<table>
<thead>
<tr>
<th>( Q^2 (GeV^2) )</th>
<th>( \tilde{s}_W^2 (ON-SHELL) )</th>
<th>( \tilde{s}_W^2 (\tilde{M_S}-PT) [1] )</th>
<th>( \tilde{s}_W^2 (\tilde{M_S}) ) PDG(2015)</th>
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<td>( m_Z )</td>
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</tbody>
</table>

Sensitivity of Polarization Asymmetry to effective mixing angle

\[
\bar{s}_W^2(t) = s_W^2 - s_W c_W \frac{\Re \left[ \hat{\Sigma}_{\gamma Z}(t) \right]}{t + \Re \left[ \hat{\Sigma}_{\gamma \gamma}(t) \right]}
\]

Value of the effective \( s_W^2 \) for MOLLER kinematics:

\[
\bar{s}_W^2(Q^2 = 0.0056 \text{ GeV}^2) = 0.2382
\]

The 2\% uncertainty on polarization asymmetry translates to 0.1\% uncertainty for weak mixing angle.