Rapidity-only TMD factorization

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TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(ff \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

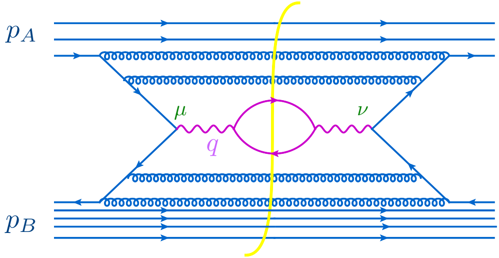
Examples: Drell-Yan process with Q being the mass of DY pair of Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A, p_B, q) \;=\; \frac{1}{(2\pi)^4}\int d^4x\, e^{-iqx}\langle p_A, p_B|J_\mu(x)J_\nu(0)|p_A, p_B\rangle$$



p_A, p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ "Y} - \text{terms''}$$

The quantities $\mathcal{D}_{f/A}(x_A, k_\perp)$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$, and $C(q, k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS approach. The TMDs $\mathcal{D}_{f/A}(x_A, k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS approach to small x ($\Leftrightarrow$ nobody tried)

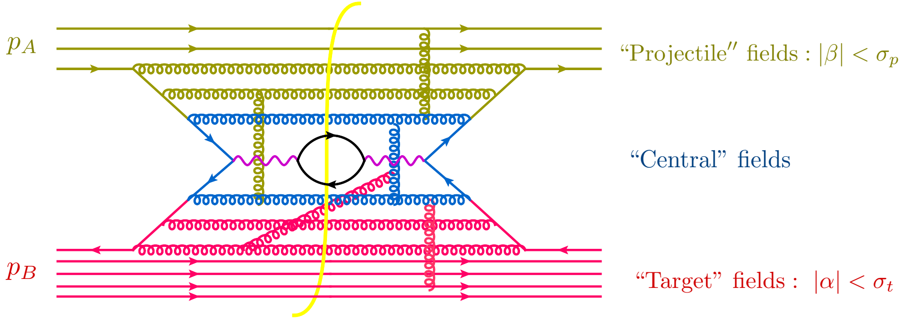
It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Visible success: power corrections $\sim \frac{1}{Q^2}$ for small-x DY hadronic tensor $\Rightarrow$ EM gauge invariance of DY tensor. It is not obtained (yet?) using CSS or SCET

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p \;=\; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A,\;\; p_2 \simeq p_B,\;\;\; p_1^2 = p_2^2 = 0$$



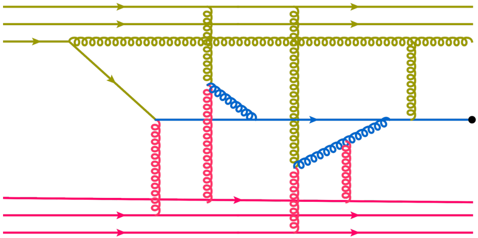
The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2}$ $\Rightarrow$ power corrections

In the tree approximation: classical YM field with sources

Tree approximation:

Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$

Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



ψ_C = sum of tree diagrams in external $A, \tilde{A}, \psi_A, \tilde{\psi}_A$ and $B, \tilde{B}, \psi_B, \tilde{\psi}_B$ fields with sources

$J_\psi = (\not{P} + m)(\psi_A + \psi_B)$, $\quad J_\nu = D^\mu F^{\mu\nu}(A + B)$

and

$\tilde{J}_\psi = (\not{P} + m)(\tilde{\psi}_A + \tilde{\psi}_B)$, $\quad \tilde{J}_\nu = D^\mu F^{\mu\nu}(\tilde{A} + \tilde{B})$

$\sum_X \Rightarrow$ Feynman diagrams with retarded propagators

The fields A, ψ and $\tilde{A}, \tilde{\psi}$ do not depend on $x^+ \Rightarrow$
if they coincide at $x^+ = \infty \Rightarrow$ they coincide everywhere.

Similarly,
B, ψ_b and $\tilde{B}, \tilde{\psi}_b$ do not depend on $x^- \Rightarrow$
if they coincide at $x^- = \infty$ they should be equal.

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

$\Rightarrow$

ψ_C and C_μ are given by the sum of tree diagrams with *retarded* Green functions
(F. Gelis, R. Venugopalan)

Classical solution

The sum of diagrams with retarded Green functions $\Leftrightarrow$ solution of classical YM equations

$$(\not{P}+m_f)\psi^f \;=\; 0, \quad D^\nu F^a_{\mu\nu} \;=\; \sum_f g\bar{\psi}^f t^a\gamma_\mu\psi^f$$

Boundary conditions :
$$A_\mu(x) \stackrel{x^+\to-\infty}{=} \bar{A}_\mu(x^-,x_\perp), \quad \psi(x) \stackrel{x^+\to-\infty}{=} \psi_a(x^-,x_\perp)$$
$$A_\mu(x) \stackrel{x^-\to-\infty}{=} \bar{B}_\mu(x^+,x_\perp), \quad \psi(x) \stackrel{x^-\to-\infty}{=} \psi_b(x^+,x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P}+m_f)\psi^f_a \;=\; 0, \quad D^\nu F^a_{\mu\nu} \;=\; g\bar{\psi}^f_a t^a\gamma_\mu\psi^f_a$$
$$(\not{P}+m_f)\psi^f_b \;=\; 0, \quad D^\nu F^a_{\mu\nu} \;=\; g\bar{\psi}^f_b t^a\gamma_\mu\psi^f_b$$

Projectile partons: $k=\alpha p_1+k_\perp$, target partons: $k=\beta p_1+k_\perp$ $\Rightarrow$ partons are *not* on the mass shell

Classical solution

The sum of diagrams with retarded Green functions ⇔ solution of classical YM equations

$$(\not{P} + m_f)\psi^f = 0, \quad D^\nu F^a_{\mu\nu} = \sum_f g\bar{\psi}^f t^a \gamma_\mu \psi^f$$

Boundary conditions :

$$A_\mu(x) \overset{x^+ \to -\infty}{=} \bar{A}_\mu(x^-, x_\perp), \quad \psi(x) \overset{x^+ \to -\infty}{=} \psi_a(x^-, x_\perp)$$
$$A_\mu(x) \overset{x^- \to -\infty}{=} \bar{B}_\mu(x^+, x_\perp), \quad \psi(x) \overset{x^- \to -\infty}{=} \psi_b(x^+, x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P} + m_f)\psi^f_a = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}^f_a t^a \gamma_\mu \psi^f_a$$
$$(\not{P} + m_f)\psi^f_b = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}^f_b t^a \gamma_\mu \psi^f_b$$

Projectile partons: $k = \alpha p_1 + k_\perp$, target partons: $k = \beta p_1 + k_\perp$ ⇒ partons are *not* on the mass shell

Method of solution:

- Start with $\psi_A + \psi_B$ and $\bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(U + V)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of YM equations in general case (scattering of two "color glass condensates") is yet unsolved problem.

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\psi, A \;=\; \text{series in } \frac{q_\perp}{Q}: \qquad \psi = \psi^{(0)} + \psi^{(1)} + ..., \qquad A = A^{(0)} + A^{(1)} + ...$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \;+ ...$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\left(\frac{q_\perp}{Q} \text{ or } \frac{q_\perp^2}{Q^2}\right) \times \left(1 + \frac{1}{N_c} + \frac{1}{N_c^2}\right)$.

(Pleasant) surprise: most of the terms not suppressed by $\frac{1}{N_c}$ are determined by the leading-twist TMDs due to QCD equations of motion

Leading twist:

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp \; e^{-i\alpha x^- + i(k,x)_\perp} \; \langle A|\hat{\bar{\psi}}_f(x^-, x_\perp) \; \not{p}_2 \hat{\psi}_f(0)|A\rangle \; = \; f_{1f}(\alpha, k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp \; e^{-i\alpha_q x^- + i(k,x)_\perp}$$
$$\times \; \langle A|\hat{\bar{\psi}}^f(x^-, x_\perp) \; \not{p}_2[\hat{U}_i(x^-, x_\perp) - i\gamma_5 \hat{\tilde{U}}_i(x^-, x_\perp)]\hat{\psi}^f(0)|A\rangle$$

$$= \; -k_i f_1(\alpha_q, k_\perp) + \alpha_q k_i\big[f_\perp(\alpha_q, k_\perp) + g^\perp(\alpha_q, k_\perp)\big],$$

(Mulders & Tangerman, 1996)

At small $\alpha_q \equiv x_A$ one can drop the second term

Result for $W_{\mu\nu}$ for unpolarized hadrons

Result:

$$W_{\mu\nu}(q) \;=\; W^1_{\mu\nu}(q) + W^2_{\mu\nu}(q)$$

The first, gauge-invariant, part is given by

$$W^1_{\mu\nu}(q) \;=\; W^{1F}_{\mu\nu}(q) + W^{1H}_{\mu\nu}(q),$$
$$W^{1F}_{\mu\nu}(q) \;=\; \sum_f e_f^2 W^{fF}_{\mu\nu}(q), \quad W^{fF}_{\mu\nu}(q) \;=\; \frac{1}{N_c}\int d^2k_\perp F^f(q,k_\perp)\mathcal{W}^F_{\mu\nu}(q,k_\perp),$$
$$W^{1H}_{\mu\nu}(q) \;=\; \sum_f e_f^2 W^{fH}_{\mu\nu}(q), \quad W^{fH}_{\mu\nu}(q) \;=\; \frac{1}{N_c}\int d^2k_\perp H^f(q,k_\perp)\mathcal{W}^H_{\mu\nu}(q,k_\perp)$$

where F^f and H^f are ($\alpha_q \equiv x_A$, $\beta_q \equiv x_B$)

$$F^f(q,k_\perp) \;=\; f_1^f(\alpha_q,k_\perp)\bar{f}_1^f(\beta_q,(q-k)_\perp) \;+\; f_1^f \leftrightarrow \bar{f}_1^f$$
$$H^f(q,k_\perp) \;=\; h_{1f}^\perp(\alpha_q,k_\perp)\bar{h}_{1f}^\perp(\beta_q,(q-k)_\perp) \;+\; h_{1f}^\perp \leftrightarrow \bar{h}_{1f}^\perp$$

Gauge-invariant structures $q^\mu W^F_{\mu\nu} = q^\mu W^H_{\mu\nu} = 0$

$$\mathcal{W}^F_{\mu\nu}(q,k_\perp)$$
$$= \ -g^\perp_{\mu\nu} + \frac{1}{Q_\parallel^2}(q^\parallel_\mu q^\perp_\nu + q^\parallel_\nu q^\perp_\mu) + \frac{q_\perp^2}{Q_\parallel^4}q^\parallel_\mu q^\parallel_\nu + \frac{\tilde q_\mu \tilde q_\nu}{Q_\parallel^2}[q_\perp^2 - 4(k,q-k)_\perp]$$
$$- \Big[\frac{\tilde q_\mu}{Q_\parallel^2}\Big(g^\perp_{\nu i} - \frac{q^\parallel_\nu q_i}{Q_\parallel^2}\Big)(q-2k)^i_\perp + \mu\leftrightarrow\nu\Big]$$

$$\tilde q \equiv \alpha_q p_1 - \beta_q p_2$$

$$m^2\mathcal{W}^H_{\mu\nu}(q,k_\perp)$$
$$= \ -k^\perp_\mu(q-k)^\perp_\nu - k^\perp_\nu(q-k)^\perp_\mu - g^\perp_{\mu\nu}(k,q-k)_\perp + 2\frac{\tilde q_\mu\tilde q_\nu - q^\parallel_\mu q^\parallel_\nu}{Q_\parallel^4}k_\perp^2(q-k)_\perp^2$$
$$- \Big(\frac{q^\parallel_\mu}{Q_\parallel^2}\big[k_\perp^2(q-k)^\perp_\nu + k^\perp_\nu(q-k)^2_\perp\big] + \ \frac{\tilde q_\mu}{Q_\parallel^2}\big[k_\perp^2(q-k)^\perp_\nu - k^\perp_\nu(q-k)^2_\perp\big] + \mu\leftrightarrow\nu\Big)$$
$$- \ \frac{\tilde q_\mu\tilde q_\nu + q^\parallel_\mu q^\parallel_\nu}{Q_\parallel^4}\big[q_\perp^2 - 2(k,q-k)_\perp\big](k,q-k)_\perp - \ \frac{q^\parallel_\mu\tilde q_\nu + \tilde q_\mu q^\parallel_\nu}{Q_\parallel^4}(2k-q,q)_\perp(k,q-k)_\perp$$

Angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and $Q_\perp$ varies from 0 to 120 GeV.

Our analysis is valid at $Q_\perp = 10 - 30$ GeV and $Y \simeq 0$ $(x_A \sim x_B \sim 0.1)$ so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi}\frac{d\sigma}{dQ^2 dy}\Big[(1+c_\theta^2) + \frac{A_0}{2}(1-3c_\theta^2) + A_1 s_{2\theta}c_\phi + \frac{A_2}{2}s_\theta^2 c_{2\phi} + A_3 s_\theta c_\phi + A_4 c_\theta + A_5 s_\theta^2 s_{2\phi} + A_6 s_{2\theta}s_\phi + A_7 s_\theta s_\phi\Big]$$

Easy-to-do approximations

- Large N_c
- Only TMD f_1 in the factorization approximation: $f_1(x, k_\perp^2) \simeq f(x)g(k_\perp^2)$
- Log accuracy: $f_1(x, k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$ and $Q^2 \gg k_\perp^2 \gg q_\perp^2$

With this approximations, only A_0 and A_2 can be calculated

Comparison of A_0 with LHC results

Logarithmic estimate of A_0

$$\frac{\mathcal{W}_L^{\rm Ff}}{\mathcal{W}^{\rm Ff}} \simeq 1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2} \quad \Rightarrow \quad A_0 = \frac{Q_\perp^2}{m_z^2}\frac{1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}}{1+\frac{Q_\perp^2}{m_z^2}\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (*)$$

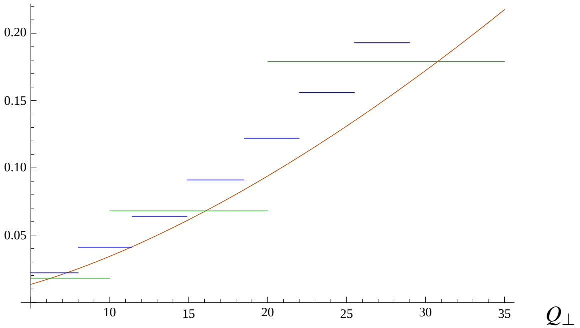


Figure: Comparison of prediction $(*)$ with lines depicting angular coefficient A_0 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$\frac{\mathcal{W}_L^{\rm Ff}}{\mathcal{W}^{\rm Ff}} \simeq 1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2} \quad\Rightarrow\quad A_2 = \frac{Q_\perp^2}{m_z^2}\frac{1}{1+\frac{Q_\perp^2}{m_z^2}\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (**)$$

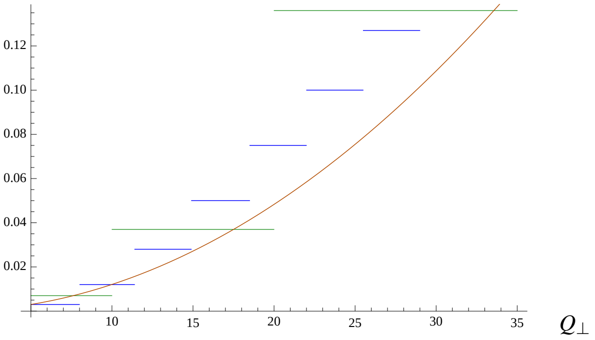
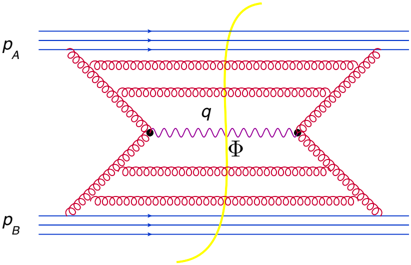


Figure: Comparison of prediction $(**)$ with lines depicting angular coefficient A_2 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Coefficient function for TMD factorization at one loop

Particle production by gluon-gluon fusion (point $gg\Phi$ vertex is a $\frac{m_H}{m_t} \ll 1$ approximation for Higgs production.)



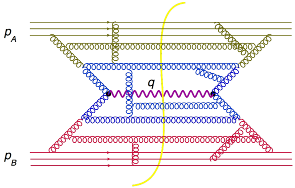
$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

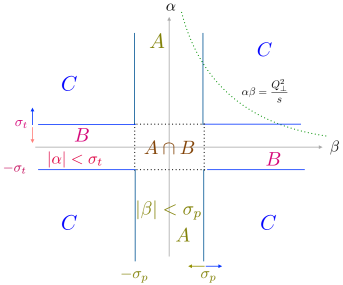
Goal: one-loop TMD factorizaton formula for hadronic tensor.

Result of calculations:

$$W(p_A, p_B; q) = \int db_\perp \ e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b)$$
$$\times \ \exp\left\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\big(\ln \frac{\alpha_q}{\sigma_t} + \gamma\big)\big(\ln \frac{\beta_q}{\sigma_p} + \gamma\big) + \frac{\pi^2}{2}\Big]\right\}$$
$$+ \text{ NLO terms} \sim O\big(\alpha_s^2\big) \ + \ \text{power corrections}$$

Reminder: rapidity factorization of functional integral





Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons

$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons

sG gluons cancel out

Formal rescaling: $s = \zeta s_0,\ \zeta \to \infty$, $Q_\perp^2$-fixed

$$\text{Rapidity cutoffs}: \ \alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s} \sim \zeta^{-1}, \quad \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s} \sim \zeta^{-1}, \quad \frac{\sigma_p \sigma_t s}{Q_\perp^2} \sim \zeta^{-1/2}$$

Coefficient function in the functional-integral language

After integration over central fields

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle\\
&=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p_A'}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p_B'}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\\
&\quad+\int dz_1^-dz_{1_\perp}dz_2^-dz_{2_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t)\\
&\quad\times\mathcal{O}^{\sigma_p}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,z_{2_\perp};z_1^+,z_{1_\perp})\ +...\Big]
\end{aligned}
$$

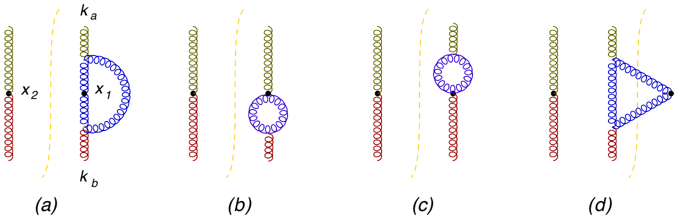
where $\mathscr{A}=A+B+sG$

Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A}=\bar{A}+\bar{B}+\bar{C}$

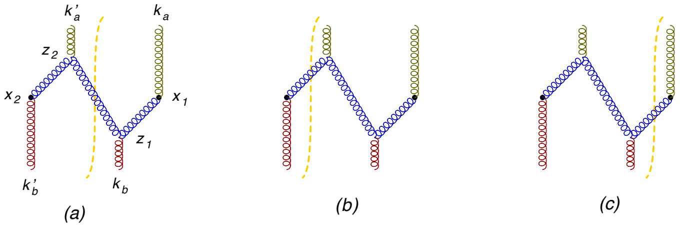
$$
\begin{aligned}
&\int dz_2^-dz_{2_\perp}dz_1^-dz_{1_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t)\\
&\qquad\times U^{-i,a}(z_2^+,z_{2_\perp})U^{-j,a}(z_1^+,z_{1_\perp})V^{+i,a}(z_2^-,z_{2_\perp})V^{+j,a}(z_1^-,z_{1_\perp})\\
&=\frac{N_c^2-1}{16}g^4\langle\tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}\\
&\qquad-\langle\hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\rangle_{\mathbb{A}}
\end{aligned}
$$

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



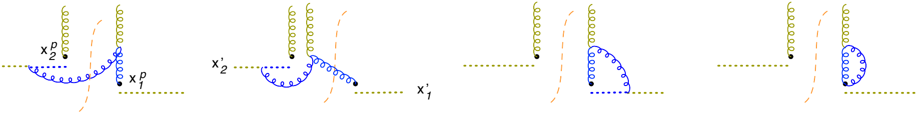
"Real" diagrams



Diagrams for subtracted TMD matrix elements

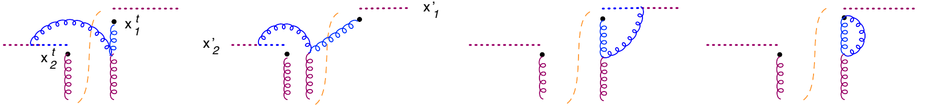
"Projectile" TMD matrix elements.

The rapidity-only $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting:
F^{+k} shown by dots stand at $x_1^p = x_{1_\perp} + x_1^-$ and $x_2^p = x_{2_\perp} + x_2^-$
Wilson lines start from $x_1' = x_2 + \delta^+$ and $x_2' = x_1 + \delta^+$ where $\delta^+ = \frac{1}{\varrho\sigma_p}$



(a) (b) (c) (d)

"Target" TMD matrix elements. The rapidity-only $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



(a) (b) (c) (d)

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\,{\mathchar'26\mkern-12mu d}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int {\mathchar'26\mkern-12mu d}\alpha {\mathchar'26\mkern-12mu d}\beta {\mathchar'26\mkern-12mu d}p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int \frac{{\mathchar'26\mkern-12mu d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{{\mathchar'26\mkern-12mu d}\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta-i\epsilon}$$

δ-regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta-i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int {\mathchar'26\mkern-12mu d}\alpha {\mathchar'26\mkern-12mu d}\beta {\mathchar'26\mkern-12mu d}p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int \frac{{\mathchar'26\mkern-12mu d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\infty}{\mathchar'26\mkern-12mu d}\alpha\frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

$$\mathcal{W}(x_1,x_2) \ - \ \mathcal{W}^{\text{tmd}}(x_1,x_2)$$
$$= \int d\!\!{}^-\alpha'_a d\!\!{}^-k'_{a_\perp} d\!\!{}^-\beta'_b d\!\!{}^-k_{b_\perp} d\!\!{}^-\alpha_a d\!\!{}^-k'_{a_\perp} d\!\!{}^-\beta_b d\!\!{}^-k'_{b_\perp} e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+}$$
$$\times\ e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} U_i^{+,b}(\alpha'_a,k'_{a_\perp}) V^{-i,a}(\beta'_b,k'_{b_\perp}) U_j^{+,b}(\alpha_a,k_{a_\perp}) V^{-j,a}(\beta_b,k_{b_\perp})$$
$$\times\ g^2[I - I_{\text{tmd}}^{\sigma_p,\sigma_t}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)$$

with

$$[I - I_{\text{tmd}}^{\sigma_p,\sigma_t}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1)$$
$$= -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}} \ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4}$$
$$- \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t} \ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t} \ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

(Intermediate) Result

$$\begin{aligned}
&\mathcal{W}(x_1,x_2) \ - \ \mathcal{W}^{\rm tmd}(x_1,x_2)\\
&= \int d\!\!{}^-\alpha'_a d\!\!{}^- k'_{a_\perp} d\!\!{}^-\beta'_b d\!\!{}^- k_{b_\perp} d\!\!{}^-\alpha_a d\!\!{}^- k'_{a_\perp} d\!\!{}^-\beta_b d\!\!{}^- k'_{b_\perp}\, e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-}\, e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp}\, U_i^{+,b}(\alpha'_a,k'_{a_\perp}) V^{-i,a}(\beta'_b,k'_{b_\perp}) U_j^{+,b}(\alpha_a,k_{a_\perp}) V^{-j,a}(\beta_b,k_{b_\perp})\\
&\quad\times g^2[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)
\end{aligned}$$

with

$$\begin{aligned}
&[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1)\\
&\qquad = -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}\\
&\qquad\quad - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

This formula is not yet the final result for the coefficient function. The coefficient function was defined as a result of integration over C-fields with $\alpha > \sigma_t$ and $\beta > \sigma_p$. Since we did not impose these restrictions while calculating the loop integrals, we need to subtract sG contributions (with $\alpha < \sigma_t, \beta < \sigma_p$) to these integrals.

Result for the coefficient function

Result of sG subtraction:

term $-\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}$ disappears $\Rightarrow$ no dynamics in the transverse plane

$$\begin{aligned}
&\mathcal{W}(x_1,x_2)\ -\ \mathcal{W}^{\rm tmd}(x_1,x_2)\ -\ \mathcal{W}^{\rm sG}(x_1,x_2)\\
&=\int đ\alpha'_a đ\beta'_b đ\alpha_a đ\beta_b e^{-i\alpha'_a\varrho x_2^- -i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ -i\beta_b\varrho x_1^+}\\
&\quad\times U^{+,b}_i(\alpha'_a,x_{2_\perp})V^{-i,a}(\beta'_b,x_{2_\perp})U^{+,b}_j(\alpha_a,x_{1_\perp})V^{-j,a}(\beta_b,x_{1_\perp})\\
&\quad\times g^2\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_1,x_2)
\end{aligned}$$

where

$$\begin{aligned}
&\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_2,x_1)\ =\ I-I^{\sigma_p,\sigma_t}_{\rm tmd}-I^{\sigma_p,\sigma_t}_{\rm sG}\\
&=\ \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}-\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p}-\ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2
\end{aligned}$$

The coefficient function in the coordinate space is made of $(+)$ - prescriptions since

$$\int d\alpha\, e^{i\alpha z}\Big[\ln\Big(-i\frac{\alpha}{\sigma}+\epsilon\Big)\ =\ \frac{\theta(-z)}{z}+\delta(z)\int_0^{1/\sigma}\frac{dz'}{z'}$$

Result for the coefficient function

Our formula

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2\perp};z_1^-,x_{1\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2\perp};x_1^+,x_{1\perp})$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})\ +\ O(\alpha_s^2)\Big]$$

is not yes TMD formula since $\mathscr{A}=A+B+sG$ and soft/Glauber gluons connect “projectile” and “target” gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections ⇒ TMD formula

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$=\ \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2\perp};x_1^-,x_{1\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2\perp};x_1^+,x_{1\perp})|p_B\rangle$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})|p_B\rangle$$

Matching of coefficient function and TMDs

TMD evolution equations

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi} \Big[2\ln\frac{s x_{12_\perp}^2}{4} + \ln(-i\alpha'_a \sigma_p + \epsilon) + \ln(-i\alpha_a \sigma_p + \epsilon) + 2\gamma \Big] \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi} \Big[2\ln\frac{s x_{12_\perp}^2}{4} + \ln(-i\beta'_b \sigma_t + \epsilon) + \ln(-i\beta_b \sigma_t + \epsilon) + 2\gamma \Big] \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

Matching of σ_p and σ_t evolutions $\Rightarrow$

$$\sigma_t \frac{d}{d\sigma_t} \mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi} \Big[2\ln\frac{s x_{12_\perp}^2}{4}$$
$$+ \ln(-i\beta'_b \sigma_t + \epsilon) + \ln(-i\beta_b \sigma_t + \epsilon) + 2\gamma \Big] \mathfrak{C}(x_1, x_2; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$
$$\sigma_p \frac{d}{d\sigma_p} \mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi} \Big[2\ln\frac{s x_{12_\perp}^2}{4}$$
$$+ \ln(-i\alpha'_a \sigma_p + \epsilon) + \ln(-i\alpha_a \sigma_p + \epsilon) + 2\gamma \Big] \mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \ = \ e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1_\perp}, x_{2_\perp}) \ = \ \int đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b \ e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \ \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle \ + \ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t)$$
$$= \ \ln^2 \frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} - \ \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

$$W(p_A,p_B;q) \;=\; \int db_\perp\; e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\,\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$
$$\times\; \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big)+\frac{\pi^2}{2}\Big]\Big\}$$
$$+\ \text{NLO terms} \sim O\big(\alpha_s^2\big)\ +\ \text{power corrections}$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,$$

$$\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-,$$

Matching of coefficient function and TMDs

For the "forward" case $p'_A = p_A,\ p'_B = p_B$ The r.h.s. of the evolution formula (1) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_q s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) = \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q, b_\perp; p_B)$$
$$\times \exp\Big\{ -\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} + O\big(\alpha_s^2\big) \text{ terms } + \text{ power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $\beta_B \sigma_0 s \simeq b_\perp^{-2}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

1. Conclusion: rapidity-only TMD factorization works!
   - Power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor $\Rightarrow$ EM gauge invariance of DY tensor.
   - Back-of-the-envelope estimates of angular distributions for DY Z-boson production are in good agreement with LHC data.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

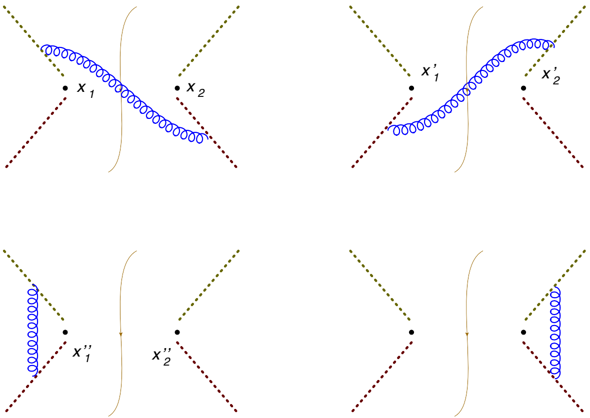
2. Outlook
   - Power corrections $\sim \frac{1}{Q^2}$ for SIDIS hadronic tensor.
   - Matching to DGLAP and BFKL/BK evolutions
   - Conformal invariance of rapidity-only factorization

Thank you for attention!

BACKUP SLIDES

Backup slide: soft factor

Leading-order diagrams



Result of calculation: $\frac{1}{4\pi^2}\mathrm{Li}_2\left(-\frac{x^2_{12\perp}}{2\delta^+\delta^-}\right) \sim O\left(\frac{\Delta^2_\perp}{2\delta^+\delta^-}\right) \sim O\left(\frac{\sigma_p\sigma_t s}{Q^2_\perp}\right) \sim O(\zeta^{-1/2})$

Soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution