Generalized parton distributions on the lattice

All views and misunderstandings are obviously personal.

Hervé Dutrieux - JLab Cake Seminar - Feb 4, 2025

Parton distributions on the lattice

- Parton distributions defined from non-local hadronic matrix elements with light-like separation

[Mu¨ller et al, 1994], [Radyushkin, 1996], [Ji, 1997]

$$\frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi} e^{ixP^+z^-} \langle N(p_f,\lambda_f)|\, \bar{\psi}^q\left(-\frac{z}{2}\right) \gamma^+ \hat{W}\left(-\frac{z}{2},\frac{z}{2};A\right) \psi^q\left(\frac{z}{2}\right) |N(p_i,\lambda_i)\rangle \,|_{z^+=0,\mathbf{z}_\perp=\mathbf{0}_\perp}$$

$$= \frac{1}{2P^+}\bar{u}(p_f,\lambda_f)\left[\gamma^+ H^q(x,\xi,t) + \frac{i\sigma^{+\nu}q_\nu}{2m} E^q(x,\xi,t)\right] u(p_i,\lambda_i)$$

Light-like distance z
Fourier conjugate of parton
momentum in light cone direction xP^+

OR

Lorentz invariant P.z Fourier conjugate
of parton momentum fraction x

Unpolarized spin-1/2 GPDs

Response to the momentum transfer
t = (p_f - p_i)^2 probes the spatial structure
of the hadron

- Interpreted as density of partons with a certain momentum fraction x and a certain radial distance to the center of momentum in appropriate frame. [Burkardt, 2000]

- Partonic picture particularly relevant in the Bjorken kinematic regime [high virtuality Q^2]. In this regime, light-like operators dominate the QCD factorization [leading-twist contribution]. Can we approximately extract the leading-twist contribution without light-like separation?

Parton distributions on the lattice

- Mellin moments of parton distributions are matrix elements of local twist-2 operators (symmetrized and with trace removal) [Ha¨gler et al, 2003, 2007]

$$\mathcal{O}^{\{\mu_1\ldots\mu_n\}} = \bar{q}(0)\gamma^{\{\mu_1} \quad i\overleftrightarrow{D}^{\mu_2}\cdots i\overleftrightarrow{D}^{\mu_n\}}q(0)$$

$$\langle P'|\mathcal{O}^{\mu_1}|P\rangle \;=\; \langle\!\langle\gamma^{\mu_1}\rangle\!\rangle A_{10}(t) + \frac{i}{2m}\langle\!\langle\sigma^{\mu_1\alpha}\rangle\!\rangle\Delta_\alpha B_{10}(t)$$

$$A_{10}(t) \;\rightarrow\; \int_{-1}^{1} dx \qquad H(x,\xi,t)$$

$$B_{10}(t) \;\rightarrow\; \int_{-1}^{1} dx \qquad E(x,\xi,t)$$

- Non-local operator can be connected to the previous picture through an operator product expansion (OPE) [Radyushkin, 2017]

$$\langle P|\bar{\psi}^q\left(-\frac{z}{2}\right)\gamma^\mu\hat{W}\psi^q\left(\frac{z}{2}\right)|P\rangle = P^\mu M(\nu\equiv P\cdot z, z^2) + z^\mu N(\nu, z^2)$$

$$M(\nu,z^2) = \int_{-1}^{1}\mathrm{d}x\, e^{-i\nu x}q(x,z^2) = \int_{-1}^{1}\mathrm{d}x\, e^{-i\nu x}q^2(x,z^2) + \mathcal{O}(z^2\Lambda_{QCD}^2) = A_{1,0}(z^2) - i\nu A_{2,0}(z^2) - \frac{\nu^2}{2}A_{3,0}(z^2) + \mathcal{O}(\nu^3) + \mathcal{O}(z^2\Lambda_{QCD}^2)$$

Parton distributions on the lattice

- q(x, z^2) is quite similar to a DIS structure function F(x_B, Q^2). It depends on a physical scale, can be in principle evaluated to arbitrarily large values of z^2, but is tainted by higher twist contaminations. q^2(x, z^2) and A_n,0(z^2) are twist-2 moments in a scheme which depends on this z^2.

$$M(\nu, z^2) = \int_{-1}^{1} \mathrm{d}x\, e^{-i\nu x} q(x, z^2) = \int_{-1}^{1} \mathrm{d}x\, e^{-i\nu x} q^2(x, z^2) + \mathcal{O}(z^2 \Lambda^2_{QCD}) = A_{1,0}(z^2) - i\nu A_{2,0}(z^2) - \frac{\nu^2}{2} A_{3,0}(z^2) + \mathcal{O}(\nu^3) + \mathcal{O}(z^2 \Lambda^2_{QCD})$$

- How far in z^2 can we go in the isolation of the twist-2 part? The only scale available to build a dimensionless quantity to characterize the size of power corrections is Lambda_QCD.

[Ji, 2022]

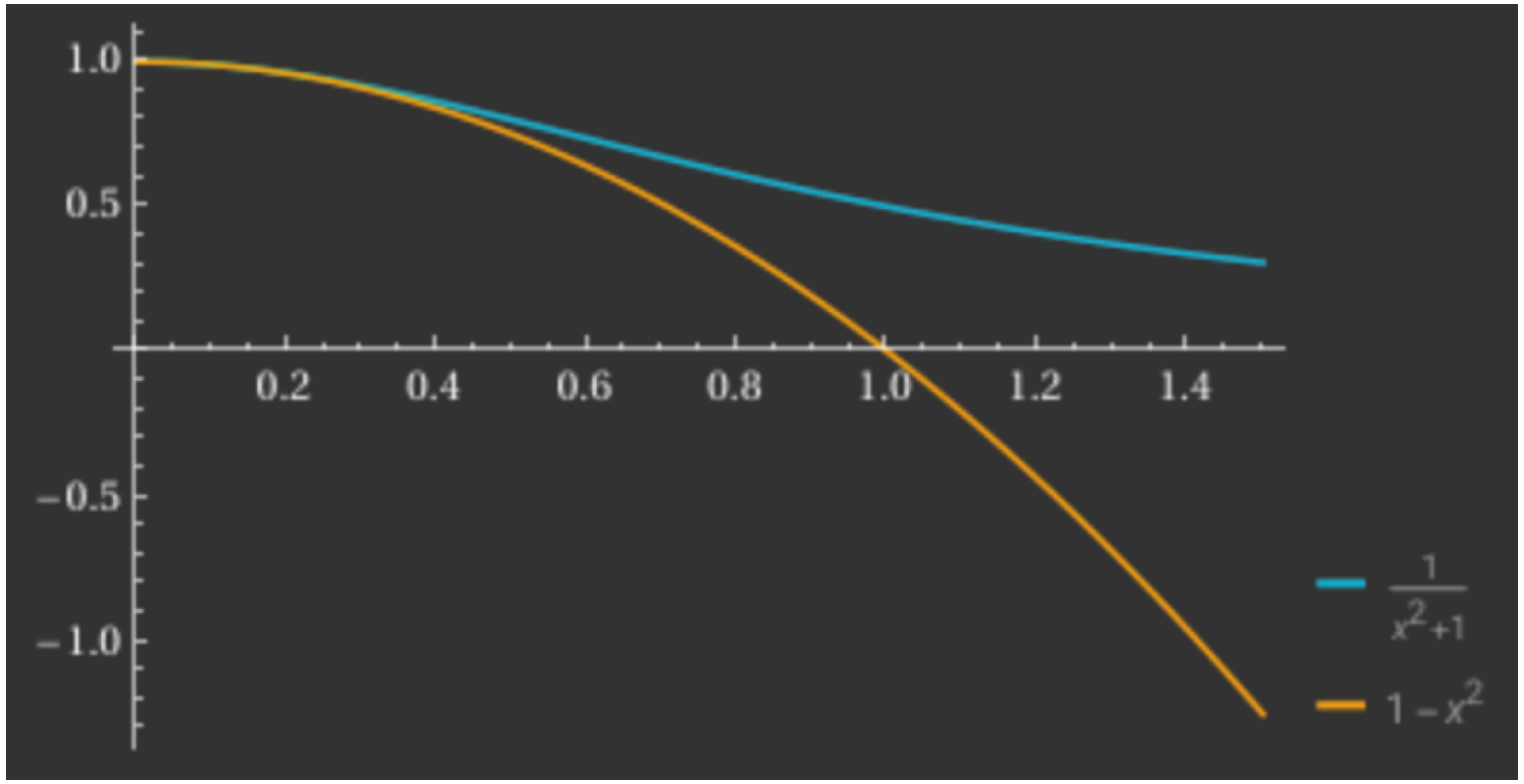
- Does it mean that nothing can be extracted from z^2 > (0.3 fm)^2? Number of theoretical and numerical works seem to envision that the size of power corrections is reduced by dividing the matrix element by the rest-frame one and allow to push to at least (0.6 fm)^2.

[Radyushkin, 2017] [Braun, Koller, Schoenleber, 2024]
[Hadstruc, ETMC, numerous publications]

Parton distributions on the lattice

- Detailed study remains to be performed. If true, the path towards precision will require non-perturbative Wilson coefficients in the large z^2 region.

- A simplistic devil's advocate argument: assume the OPE has a finite radius of convergence. May not transcribe into anything visible, and using values of the non-local operator outside of the radius of convergence to try to infer the Taylor coefficients will result in catastrophic meaningless-ness



Parton distributions on the lattice

- The LaMET formalism constructs a quasi-PDF [Ji, 2013]

$$q(y,P_z) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{iyP_z z} \langle P|\bar{\psi}^q\left(-\frac{z}{2}\right)\gamma^\mu \bar{\psi}^q\left(\frac{z}{2}\right)|P\rangle$$

$$q(x,\mu^2) = \int_{-\infty}^{\infty} \frac{dy}{y} C\left(\frac{x}{y}, \frac{\mu^2}{(xP_z)^2}, \frac{\mu^2}{((1-x)P_z)^2}\right) q(y,P_z)$$

[LaMET, quasi-PDF]

$$\langle P|\bar{\psi}^q\left(-\frac{z}{2}\right)\gamma^\mu \hat{W}\psi^q\left(\frac{z}{2}\right)|P\rangle = P^\mu M(\nu \equiv P\cdot z, z^2) + z^\mu N(\nu, z^2)$$

$$q(x,\mu^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \bar{C}(\nu x, \mu^2 z^2) M(\nu, z^2)$$

[Short-distance factorization, pseudo-PDF]

- Criticism of short-distance factorization on the basis that if z^2 ~ Lambda_QCD^2, the perturbative computation of C(mu^2 z^2) is unreliable (fair - although evolution effects are fairly small for pseudo-PDFs so may explain why most numerical studies get away with fixed order perturbation reasonably well), and that M(nu, z^2) is intrinsically spoiled by higher twists (debated).

- But how can LaMET use matrix elements with a separation z ~ Lambda_QCD? If those are truly unrelated to the physics of leading-twist parton distributions, how are they useful to construct the leading-twist parton distribution?

Parton distributions on the lattice

$$q(y, P_z) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{iyP_z z} \langle P|\bar{\psi}^q \left(-\frac{z}{2}\right) \gamma^\mu \bar{\psi}^q \left(\frac{z}{2}\right) |P\rangle$$

$$q(x, \mu^2) = \int_{-\infty}^{\infty} \frac{dy}{y} C\left(\frac{x}{y}, \frac{\mu^2}{(xP_z)^2}, \frac{\mu^2}{((1-x)P_z)^2}\right) q(y, P_z)$$

[LaMET, quasi-PDF]

$$\langle P|\bar{\psi}^q \left(-\frac{z}{2}\right) \gamma^\mu \hat{W} \psi^q \left(\frac{z}{2}\right) |P\rangle = P^\mu M(\nu \equiv P{\cdot}z, z^2) + z^\mu N(\nu, z^2)$$

$$q(x, \mu^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \bar{C}(\nu x, \mu^2 z^2) M(\nu, z^2)$$

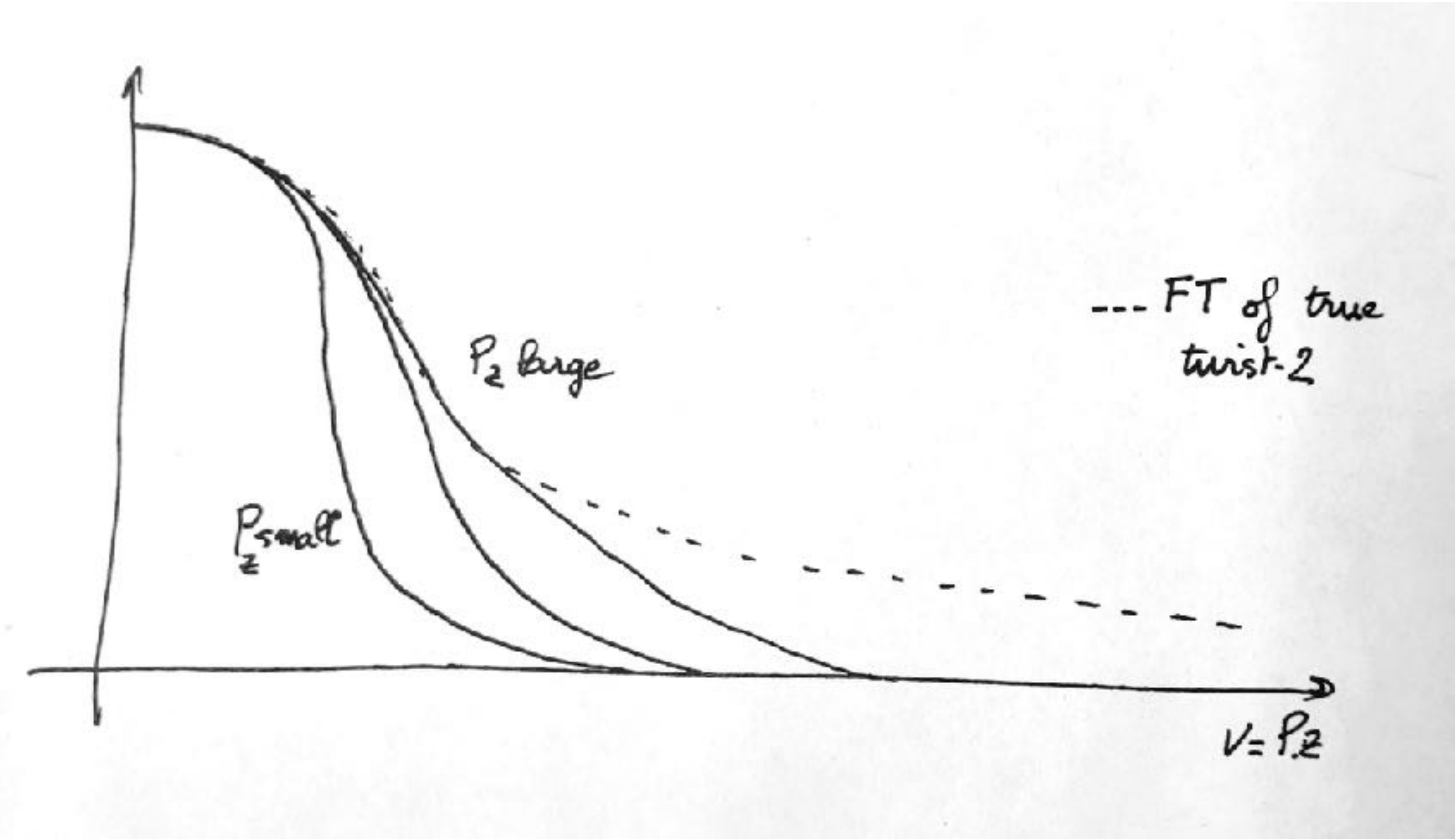
[Short-distance factorization, pseudo-PDF]

- Typical argument: z is not a relevant object to study. Once the quasi-PDF is constructed, it is a momentum-dependent object, and the limit of large momentum provides a proper approach to the leading-twist object. Vary P_z and check for yourself the convergence with P_z.

- Although I have no objection at all to this argument, I feel it does not answer the initial question of how does LaMET make use of large z matrix elements, which by the own admission of the authors, contain no relevant leading-twist physics.

Parton distributions on the lattice

Dumb schematic picture. There are many things not taken into account such as renormalization / ratio and matching effects.



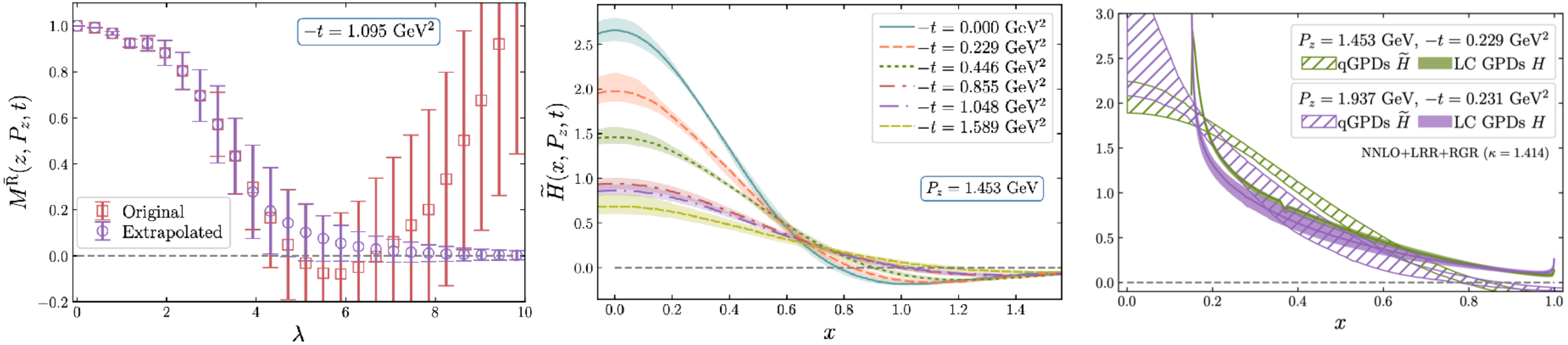
- The space-like correlation decreases exponentially with z when z ~ 1 fm so that most signal is lost by then. In the range of 0.3 fm to 1 fm, it may be tainted by large higher-twist -> but if ratio magic works, maybe it is better than it seems.

- As P_z increases, the region of reliable leading-twist signal increases in the Fourier space, producing a more reliable reconstruction. But that is true whether in LaMET or SDF. If the data is truly corrupted beyond a certain value of z [signal loss of the exponential decay or higher twist contamination], it seems that instead of using this corrupted data, any framework should stop at this upper z value, acknowledge an issue of missing / corrupted Fourier component and include it a theoretical error.

Parton distributions on the lattice

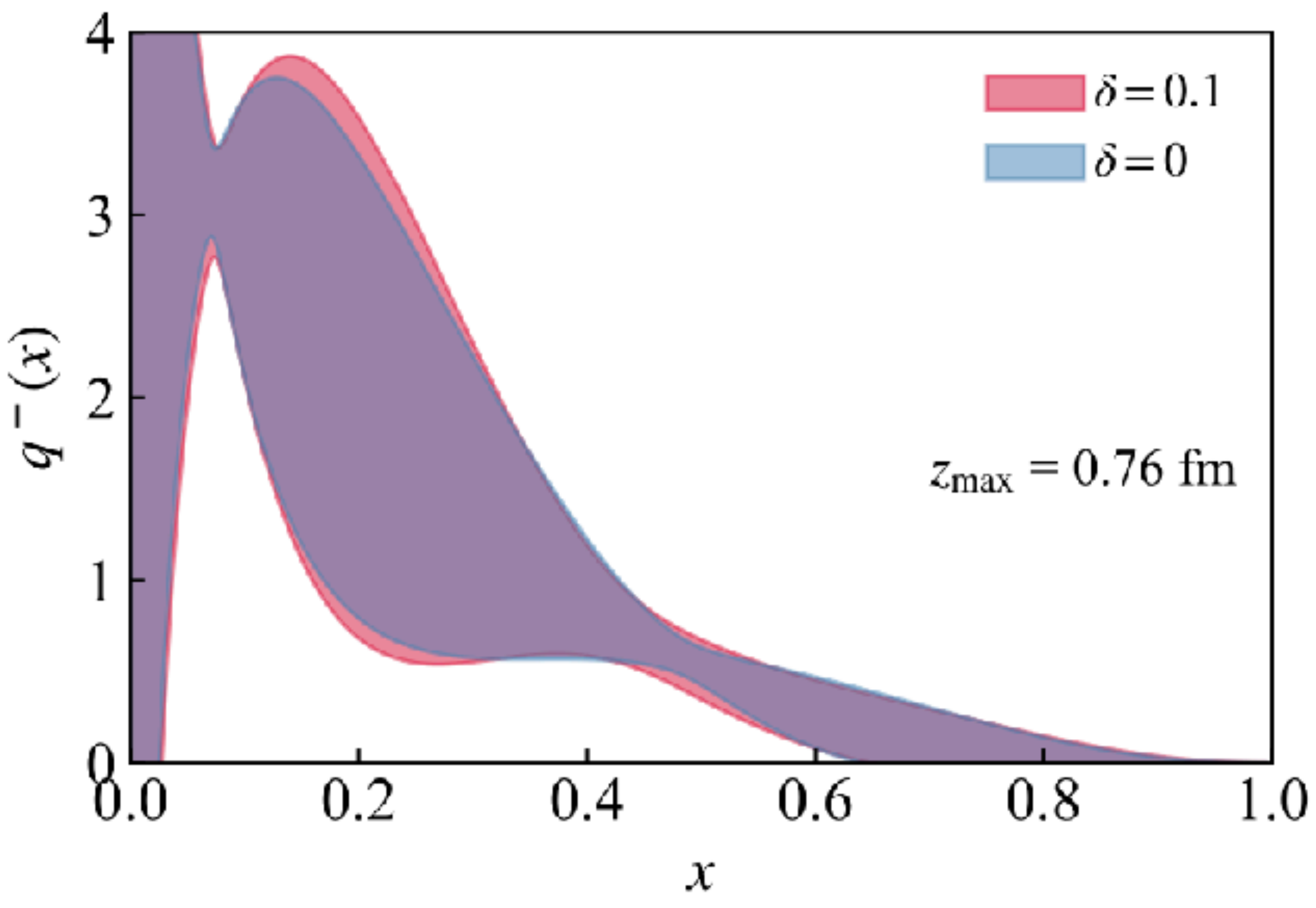
- Then both LaMET and SDF would max out the value of z and the value of P_z, and end up with exactly the same range of useful Fourier data. It would remain for SDF the issue of handling a Wilson coefficient at large z^2, and for LaMET the issue of its divergent power corrections on the boundaries (small or large x for PDF, with x ~ xi as an additional issue for GPDs).

- Last fundamental issue: the inverse problem of reconstructing missing Fourier data. Most common solution: fit by parametric form.



[Ding et al, 2024]

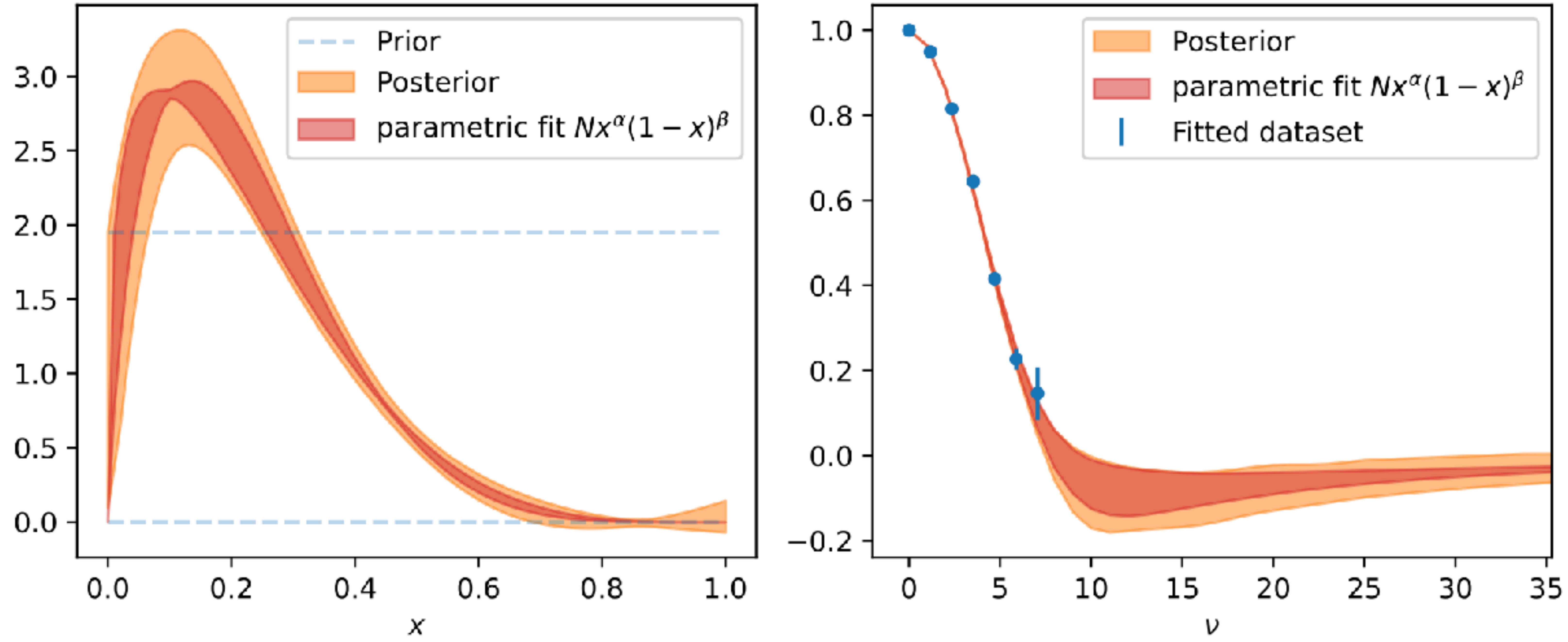
Parton distributions on the lattice



[Gao et al, 2023]

Parton distributions on the lattice

- Study of a method to reconstruct missing Fourier information, both more efficient numerically, with better control of the properties of the reconstruction and less artifacts



[HD, Karpie, Orginos, Zafeiropoulos, 2024]

- The degrees of freedom are the values of the reconstruction on a grid in x, and a prior covariance kernel is designed: controls correlation length and magnitude of uncertainty to the satisfaction of the physicist.

GPDs on the lattice

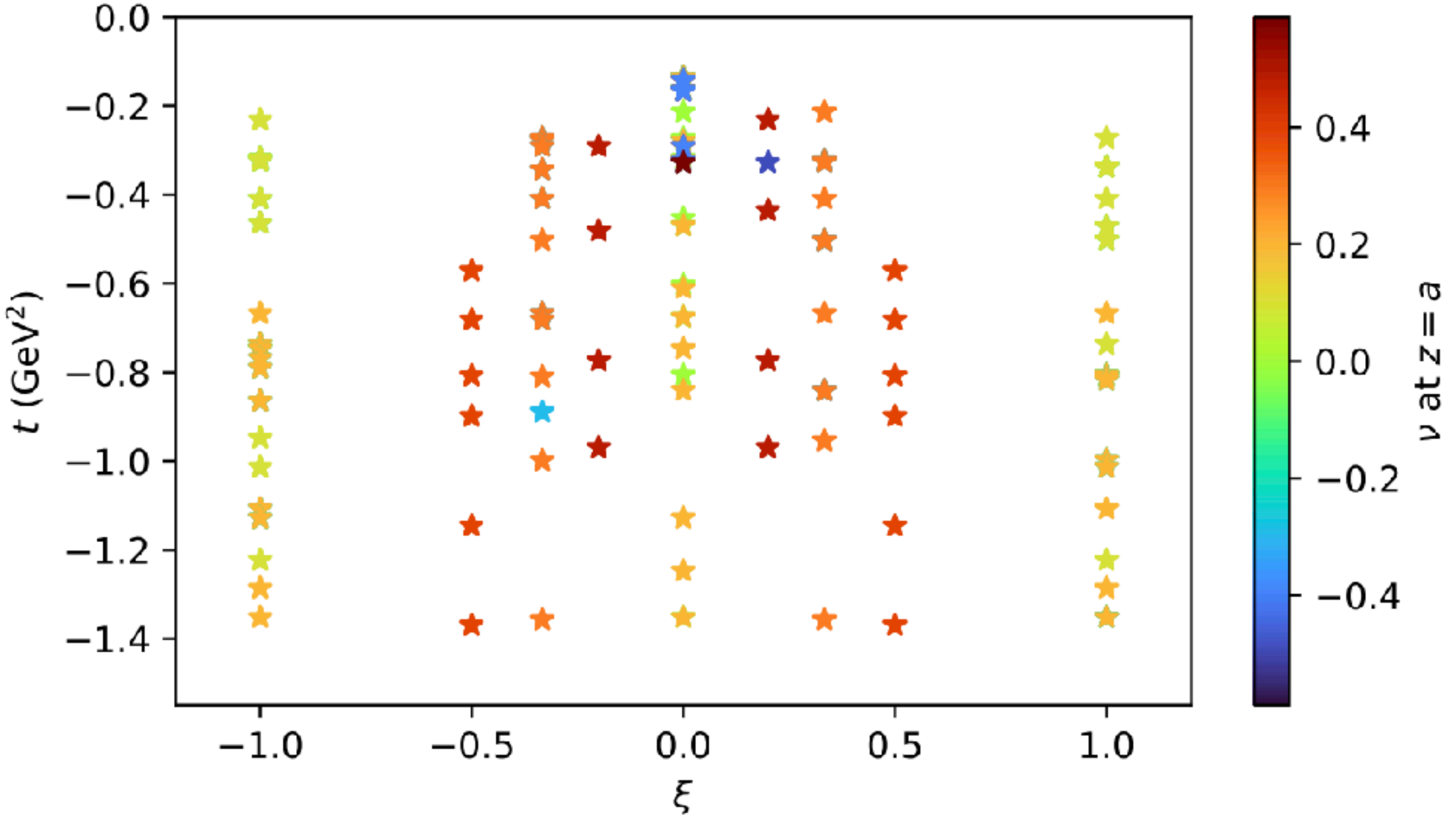
- GPDs are significantly more complicated due to the increase in Lorentz structures and kinematic domain. [Bhattacharya et al, 2022]

$$\langle N(p_f,\lambda_f)|\,\bar{\psi}^q\left(-\frac{z}{2}\right)\gamma^+\hat{W}\left(-\frac{z}{2},\frac{z}{2};A\right)\psi^q\left(\frac{z}{2}\right)|N(p_i,\lambda_i)\rangle$$

$$\begin{aligned}\mathcal{M}^\mu(p_f,p_i,z) = {}& \langle\langle\gamma^\mu\rangle\rangle\mathcal{A}_1\left(\nu,\xi,t,z^2\right) + z^\mu\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_2\left(\nu,\xi,t,z^2\right) + i\langle\langle\sigma^{\mu z}\rangle\rangle\mathcal{A}_3\left(\nu,\xi,t,z^2\right)\\ &+\frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle\mathcal{A}_4\left(\nu,\xi,t,z^2\right)+\frac{q^\mu}{2m}\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_5\left(\nu,\xi,t,z^2\right)\\ &+\frac{i}{2m}\langle\langle\sigma^{zq}\rangle\rangle\left[P^\mu\mathcal{A}_6\left(\nu,\xi,t,z^2\right)+q^\mu\mathcal{A}_7\left(\nu,\xi,t,z^2\right)+z^\mu\mathcal{A}_8\left(\nu,\xi,t,z^2\right)\right].\end{aligned} \quad (2.8)$$

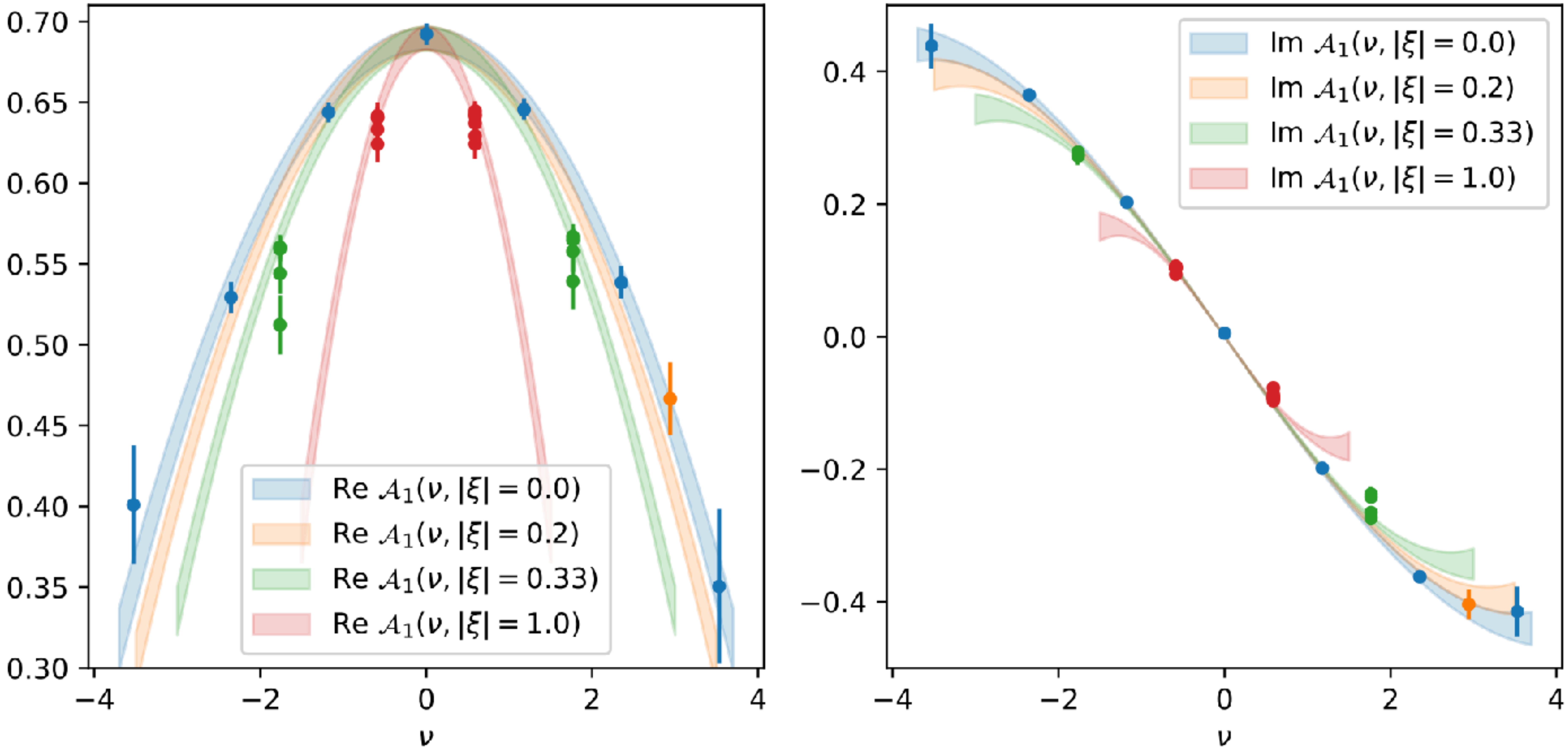
GPDs on the lattice

ID	a (fm)	m_π (MeV)	β	$m_\pi L$	$L^3 \times N_T$	$N_{\rm cfg}$	$N_{\rm srcs}$	rk ($\mathcal{D}$)
a094m358	0.094(1)	358(3)	6.3	5.4	$32^3 \times 64$	348	4	64



- With z = 6a (0.56 fm) and P_max = 1.4 GeV, we reach nu_max = 3.5. Opted for a simple moment extraction for the moment, waiting for extended momentum for a full reconstruction.

GPDs on the lattice



GPDs on the lattice

Pion mass = 0.36 GeV - Proton mass = 1.12 GeV
No continuum limit - signs of discretization errors / light-cone uncertainty
Matching at 2 GeV with leading logarithmic accuracy

Value at t = 0

GPD H^{u-d}		GPD E^{u-d}	
$A_{1,0}$: 0.974^{+12}_{-5}		$B_{1,0}$: 3.40^{+7}_{-1}	
$A_{2,0}$: 0.206^{+2}_{-6}		$B_{2,0}$: 0.370^{+9}_{-24}	
$A_{3,0}$: 0.064^{+2}_{-6}	$A_{3,2}$: 0.39^{+11}_{-3}	$B_{3,0}$: 0.063^{+24}_{-8}	$B_{3,2}$: 1.1^{+4}_{-8}
$A_{4,0}$: 0.065^{+5}_{-19}	$A_{4,2}$: 0.5^{+3}_{-3}	$B_{4,0}$: 0.06^{+16}_{-2}	$B_{4,2}$: > 1.1

Dipole mass (GeV)

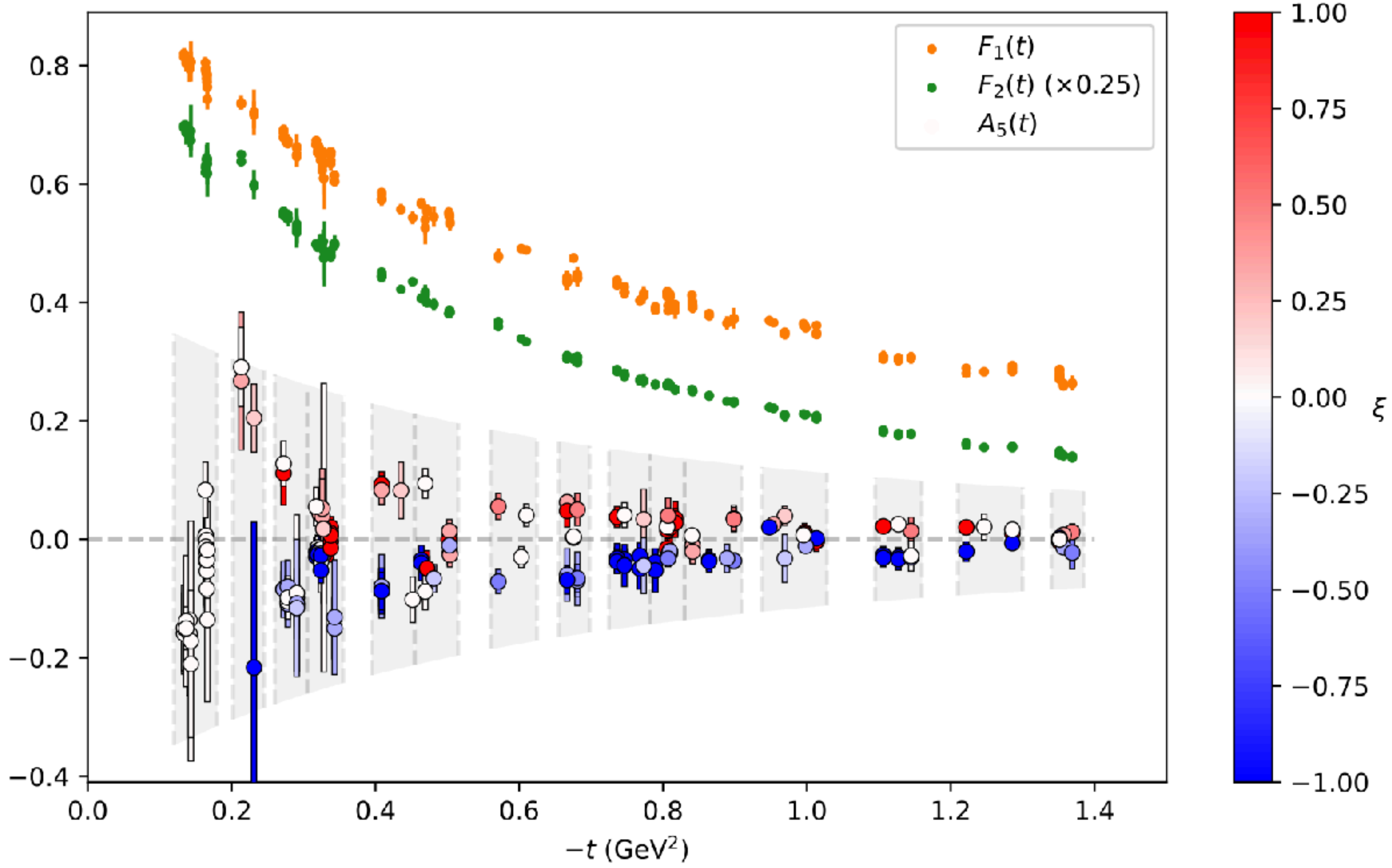
GPD H^{u-d}		GPD E^{u-d}	
$A_{1,0}$: 1.255^{+3}_{-29}		$B_{1,0}$: 0.987^{+2}_{-6}	
$A_{2,0}$: 1.83^{+9}_{-3}		$B_{2,0}$: 1.39^{+11}_{-5}	
$A_{3,0}$: 2.3^{+2}_{-5}	$A_{3,2}$: 1.10^{+7}_{-11}	$B_{3,0}$: 2.2^{+36}_{-5}	$B_{3,2}$: 0.78^{+77}_{-9}
$A_{4,0}$: > 3.5	$A_{4,2}$: > 0.9	$B_{4,0}$: > 0.6	$B_{4,2}$: 0.5^{+5}_{-2}

D-term^{u-d}	Value at t = 0	Dipole mass (GeV)
C_2	0.025^{+8}_{-8}	> 2.2

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} \begin{pmatrix} H^{u-d} \\ E^{u-d} \end{pmatrix} (x,\xi,t) = \sum_{k=0 \text{ even}}^{n-1} \begin{pmatrix} A_{n,k}(t) \\ B_{n,k}(t) \end{pmatrix} \xi^k \pm \mathrm{mod}(n+1,2)\xi^n C_n(t)\,,$$

$$A_{n,k}(t) = A_{n,k}(t=0)\left(1 - \frac{t}{\Lambda_{n,k}^2}\right)^{-2}$$

GPDs on the lattice

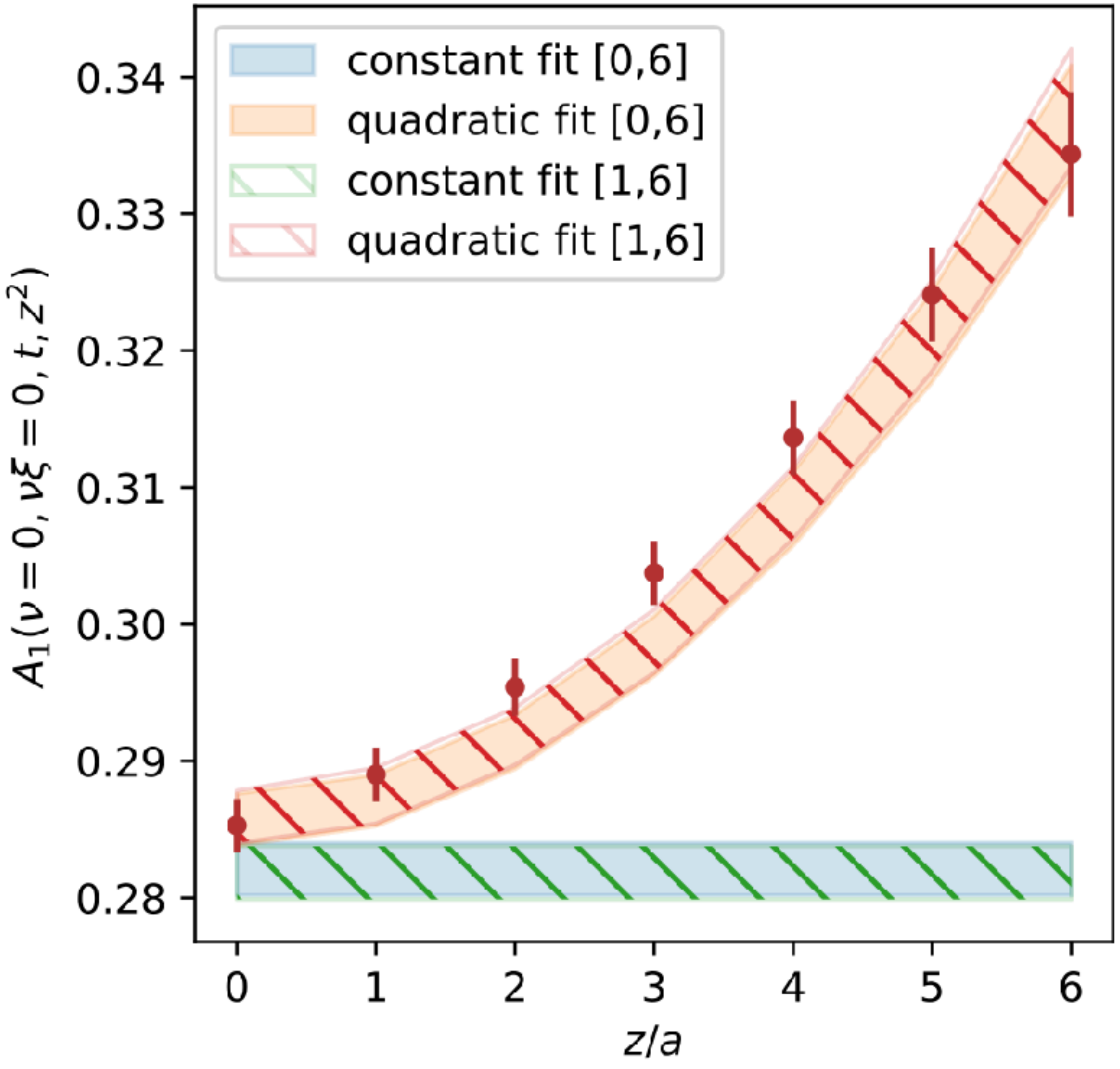


- Local matrix element z = 0

$$\langle\langle\gamma^\mu\rangle\rangle F_1(t) + \frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle F_2(t) + \frac{q^\mu}{2m}\langle\langle\mathbb{1}\rangle\rangle \mathcal{A}_5(0,\xi,t,0)$$

- Good consistency for the EFF (especially F_2)
- A_5 should be 0 by the Ward identity. Likely sign of discretization errors + a tad bit of excited state uncertainty.

GPDs on the lattice

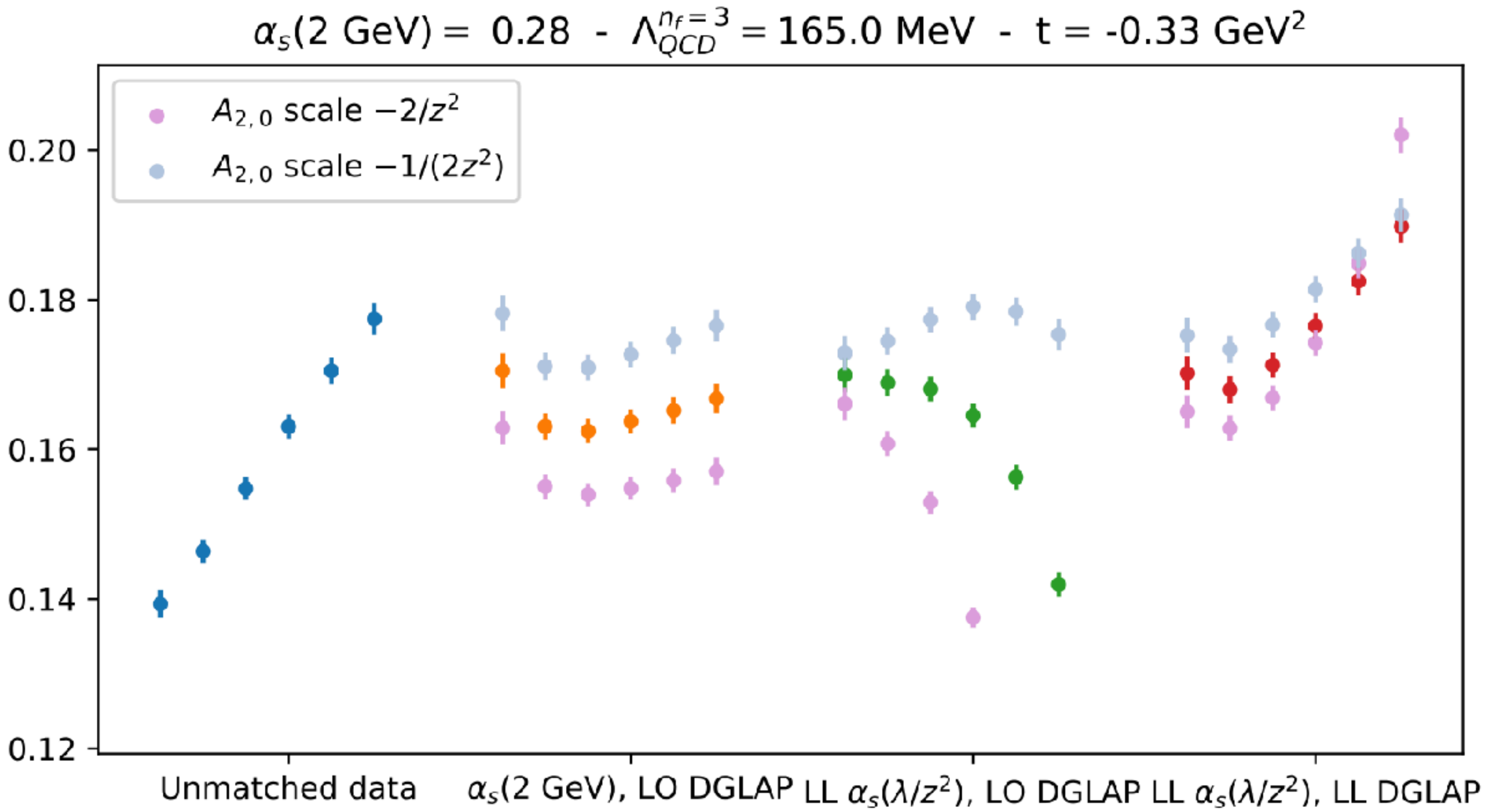


- The issue of drifting EFF with z: discretization error, higher-twist contamination?

$$\mathrm{Re}\,\mathcal{A}_1(\nu,\xi,t,z^2) = F_1(t) - \frac{\nu^2}{2}\Big(A_{3,0}(t,z^2) + \xi^2 A_{3,2}(t,z^2)\Big) + \mathcal{O}(\nu^4) + \mathcal{O}(\Lambda_{\mathrm{QCD}}^2 z^2, tz^2)\,,$$

$$\mathrm{Im}\,\mathcal{A}_1(\nu,\xi,t,z^2) = -\nu A_{2,0}(t,z^2) + \frac{\nu^3}{6}\Big(A_{4,0}(t,z^2) + \xi^2 A_{4,2}(t,z^2)\Big) + \mathcal{O}(\nu^5) + \mathcal{O}(\Lambda_{\mathrm{QCD}}^2 z^2, tz^2)\,,$$

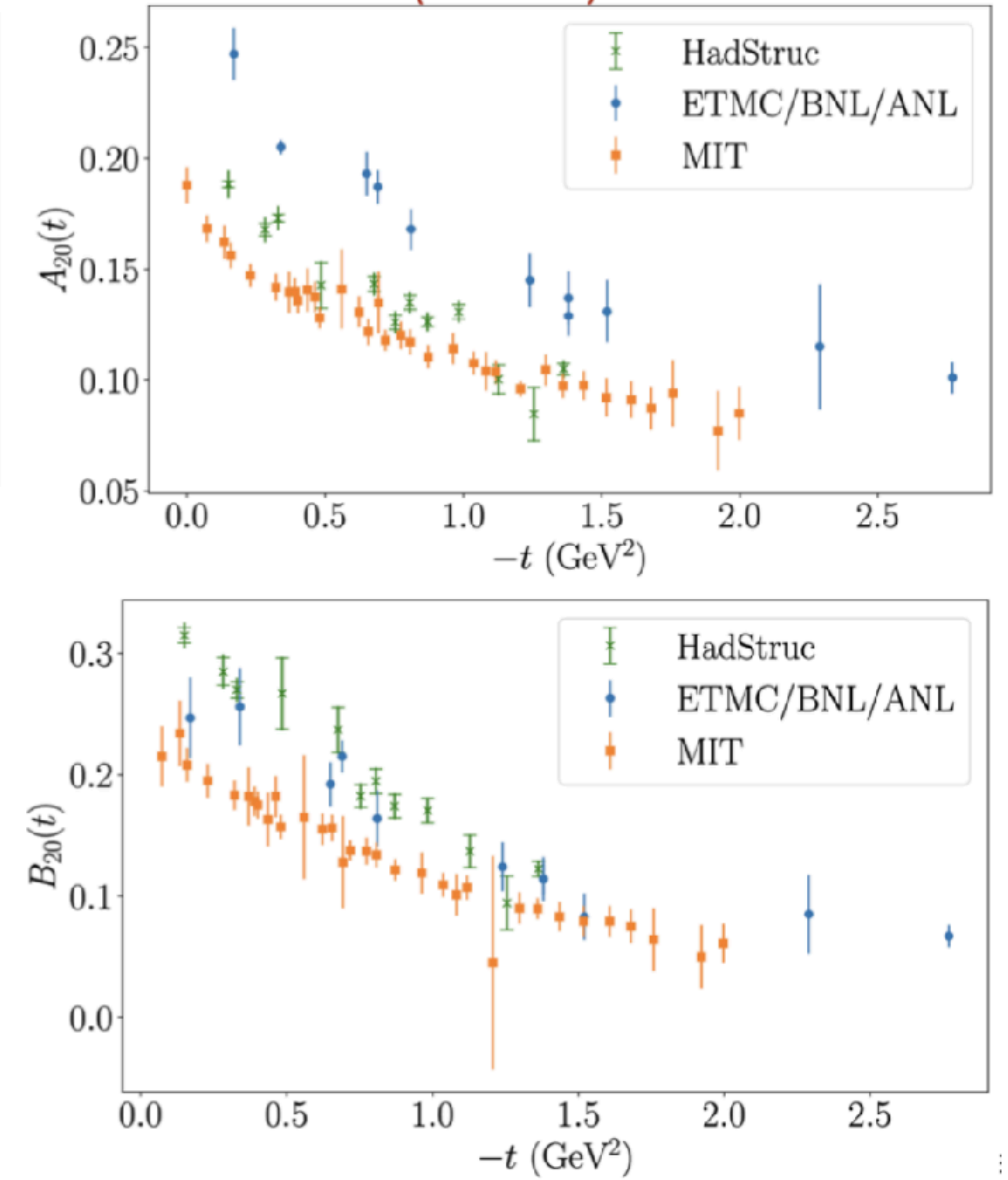
GPDs on the lattice



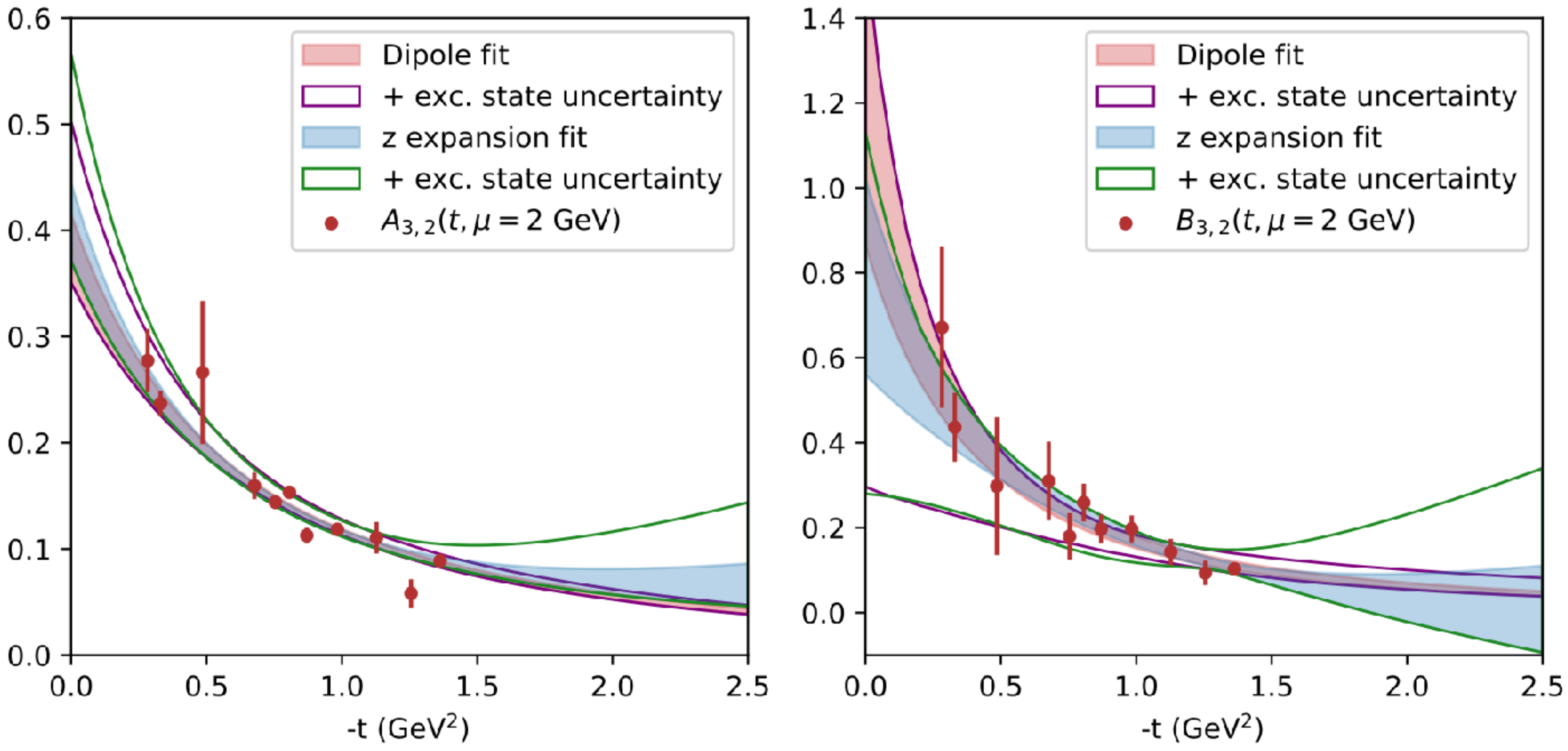
- Effect of matching on gravitational form factor (GFF)

GPDs on the lattice

D. Hackett, D. Pefkou, P. Shanahan (MIT) arXiv:2301.08484
S. Bhattacharya et al (ETMC/BNL/ANL) arXiv:2305.11117
H. Dutrieux et al (HadStruc) arXiv:2405.10304



GPDs on the lattice



- A novel GPD moment!

Conclusion

- In GPD physics, whereas much work remains to be done towards a complete error budget, we are already able to bring new quantities with a high-quality framework
- Future work will characterize singlet contributions and allow comparisons with experimental data with full skewness dependence.
- For unpolarized PDFs, the quest of precision is still on and presents beautiful challenges