Excited J- - resonances from meson-meson scattering at the SU(3) flavor point in lattice QCD.
The lightest vector ($J^{PC} = 1^{--}$) mesons are the $\rho(770)$, $\omega(782)$, $\phi(1020)$

States are well understood in $e^+e^-$ annihilation due to their narrow widths and little background into decay into simple states like $\pi\pi$, $\pi\pi\pi$, $K\bar{K}$.

$\omega$ and $\phi$ states separated via decay channels $\pi\pi\pi$ vs $K\bar{K}$ (OZI)

Excited vector states picture:

$I=1$: There appears to be two states $\rho(1450)$, $\rho(1700)$

$I=0$: Three states $\omega(1420)$, $\omega(1650)$, $\phi(1680)$
Fig. 2


Fig. 5. $e^+e^- \rightarrow \pi^+\pi^-$ cross section versus $\sqrt{s}$. The Novosibirsk points are from ref. [2]. D. Bisello et al. (DM2), Phys. Lett. B 220, 321 (1989).

Presence of two states in $1^{--}$ from quark model it is natural to interpret these states as a radial excitation in S-wave $[2^3S_1]$, and an orbital excitation in D-wave $[^3D_1]$ (or some linear combination of the two).

One would then expect three nearly degenerate D-wave states $[^3D_{1,2,3}]$.

What has been claimed in the PDG:

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$J^P$</th>
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<tbody>
<tr>
<td>0</td>
<td>1$^-$</td>
</tr>
<tr>
<td>1</td>
<td>(0, 1, 2)$^+$</td>
</tr>
<tr>
<td>2</td>
<td>(1, 2, 3)$^-$</td>
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</table>

Isovector: $\rho(1400), \rho(1700), \rho_3(1690)$

Isoscalar: $\omega(1420), \omega(1650), \omega_3(1670) / \phi(1680), \phi_3(1850)$. 

Experimental Status

$J^P = 0^-, 1^-$, $2^-, 3^-$
A Place to start

Energy spectrum calculation by *hadpsec* of states with basis of single meson operators

Great picture of states we expect

These states actually feature as resonances.

We can do better.

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These $J^{--}$ states are resonances which appear as poles in the scattering amplitude

⇒ need a reliable description of the scattering amplitude.

⇒ we make use of a variety of parameterizations that satisfy basic properties (i.e. unitarity, analyticity) and fit to the finite volume spectrum via Luscher’s Quantization Condition.

⇒ calculate large basis of correlation functions in LQCD to produce finite volume spectrum

Finite-volume spectrum ↔ scattering amplitude.
Introduces three fundamental changes:

- Lattice spacing → does not likely play a big role
- Lattice volume → tool we need for scattering
- Quark mass → feature we make use of increasing pion mass

Compute correlation functions  \( C_{ij}(t) = \langle 0 | O_i(t)O_j(0) | 0 \rangle \) to extract the finite volume spectrum

\[
\Rightarrow C_{ij}(t) = \sum_{\alpha} \langle 0 | O_i | \alpha \rangle \langle \alpha | O_j | 0 \rangle e^{-E_{\alpha}t}
\]
Lattice QCD

Optimized operator constructed from applying the eigenvectors extracted from applying the variational method $h^\dagger = \sum_i v_i O_i$

Finite volume spectrum $\Rightarrow C_{ij}(t) = \sum_\alpha \langle 0 \mid O_i \mid \alpha \rangle \langle \alpha \mid O_j \mid 0 \rangle e^{-E_\alpha t}$

Single meson operators: $\sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \bar{\psi} \not{D} \not{D} \ldots \not{D} \psi$

Meson-meson operators: $\sum_{\vec{p}_1 + \vec{p}_2 = \vec{P}} C(\vec{p}_1, \vec{p}_2; \vec{P}) h_1^\dagger(\vec{p}_1) h_2^\dagger(\vec{p}_2)$

Momentum is quantized $\vec{p} = \frac{2\pi}{L} \vec{n}$

No interactions

$E = \sqrt{m_1^2 + \left(\frac{2\pi \vec{n}_1}{L}\right)^2} + \sqrt{m_2^2 + \left(\frac{2\pi \vec{n}_2}{L}\right)^2}$
Scattering in a finite volume

$2 \rightarrow 2$ scattering amplitudes are related to the finite volume spectrum via Luscher’s quantization condition: 
$$
\det \left[ 1 + i \rho \cdot t \cdot (1 + iM) \right] = 0
$$

$$
\rho_i(E) = \frac{2k_i}{E} \text{ is diagonal matrix of the phase space}
$$

$$
t_{ij}(E) \text{ is the symmetric scattering matrix satisfying unitarity } Im(t_{ij}^{-1}) = -i \delta_{ij} \rho_i
$$

$M_{ij}(E, L)$ contains the finite volume pieces

Calculations have been done for elastic, coupled channel, and coupled channels with spinning particles.
Elastic scattering in a finite volume

\[
\det \left[ 1 + i \rho \cdot t \cdot (1 + iM) \right] = 0
\]

For a single elastic channel (one partial wave) \( \Rightarrow t(E) = \frac{1}{\rho (\cot \delta(E) - i)} \)

Reduces to a single equation \( \cot \delta(E) = M(E, L) \)

Has been done in numerous examples

Can describe the amplitude through a phase shift.


More challenging, can no longer just write in terms of a single phase shift.

Solutions follow from K-matrix parameterizations of the amplitude: \( t^{-1} = K^{-1} + I \)

K-matrix real and symmetric \( K_{ij}(s) = \sum_{\alpha} g_i^{(\alpha)} g_j^{(\alpha)} \frac{1}{m_\alpha^2 - s} + \sum_{\beta} s^{\beta} \gamma_{ij} \)

⇒ guarantees unitarity

\[ \text{det} \left[ 1 + i\rho \cdot t \cdot (1 + iM) \right] = 0 \]

\[ I(s) = I(s_0) - \frac{s - s_0}{\pi} \int_{s_{\text{thr}}}^{\infty} \frac{\rho(s')}{(s' - s_0)(s' - s - i\epsilon)} ds' \]

Im \( I = -\rho \)
Coupled channel with nonzero spin

Orbital and angular momentum couple $\ell \otimes S \to J$

Can use K-matrix to handle this (ex. $0^{-+}, 1^{--}$ scattering in $J^P = 1^+$)

$$K_{1+} = \begin{pmatrix} \{^3 S_1 | ^3 S_1\} & \{^3 S_1 | ^3 D_1\} \\ \{^3 S_1 | ^3 D_1\} & \{^3 D_1 | ^3 D_1\} \end{pmatrix}$$

Done in both non-resonant and resonant systems:

"Dynamically-coupled partial-waves in $\rho \pi$ isospin-2 scattering from lattice QCD" - A. Woss, C. Thomas, J. Dudek

"The $b_1$ resonance in coupled $\pi \omega, \pi \phi$ scattering from lattice QCD" - A. Woss, C. Thomas, J. Dudek
This project studies the isoscalar $J^{--}$ excited mesons at the SU(3) flavor point.

Advantages:

$\Rightarrow$ Heavier light quark masses allow us to probe higher energy regions:
- first three-particle threshold gets moved higher up
- resonant states at lighter quark masses feature as stable particles
$\Rightarrow$ Fewer channels (ex. $\pi, K, \bar{K}, \eta$ are all just $\eta^8$)
This work (excited $J^{--}$ resonances...)

Old:

- Elastic scattering
- Coupled channel
- Spinning hadrons

New:

Multiple resonances in the same partial waves and irreps $\Rightarrow$ a proper test of the finite volume formalism
SU(3) Flavor

Two neutral members basis states $I = I_z = Y = 0$

$|1\rangle = \frac{1}{\sqrt{3}} (|\bar{u}u\rangle + |\bar{d}d\rangle + |\bar{s}s\rangle)$

$|8\rangle = \frac{1}{\sqrt{6}} (|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle)$

Pseudoscalar have small mixing angle from SU(3) states $\sim -10^\circ$

$|\eta\rangle \sim |\eta^8\rangle \quad |\eta'\rangle \sim |\eta^1\rangle$

Mixing splits into light and strange quarks (OZI)

$|\omega\rangle \sim \frac{1}{\sqrt{2}} (|\bar{u}u\rangle + |\bar{d}d\rangle) \quad |\phi\rangle \sim |\bar{s}s\rangle$
SU(3) Flavor

\begin{align*}
\eta^8 & \sim f_0^1 \\
\omega^8 & \sim \psi_0^1 \\
\eta^8 & \sim \psi_1^1 \\
\eta^1 & \sim f_0^1 \\
\omega^1 & \sim \psi_0^1 \\
0^- & \sim f_0^1 \\
1^- & \sim \psi_0^1 \\
0^+ & \sim f_0^1 \\
2^+ & \sim \psi_0^1 \\
1^+ & \sim f_0^1 \\
1^+ & \sim \psi_0^1 \\
\end{align*}
Channels in SU(3) Flavor

Conventional $\bar{q}q$ mesons live in either a singlet ($\bar{3} \otimes 3 \rightarrow 1$) or octet ($\bar{3} \otimes 3 \rightarrow 8$) representations.

Two ways to project to flavor singlet $8 \otimes 8 \rightarrow 1$, and trivially $1 \otimes 1 \rightarrow 1$.

Charge conjugation in neutral member of the octet

$|I = I_z = Y = 0\rangle$ for $8 \otimes 8 \rightarrow 1$:

$$\hat{C}(|8_1, C_1\rangle \otimes |8_2, C_2\rangle) \rightarrow C_1 C_2 (|8_1, C_1\rangle \otimes |8_2, C_2\rangle)$$

$\Rightarrow$ channels with $C=-$:

$\eta^8(0^{-+})\omega^8(1^{--}), f^i_0(0^{++})\omega^1(1^{--}), \eta^1(0^{-+})\omega^1(1^{--})$

$\Rightarrow$ can’t have identical particles with $C=-$
Channels

J=1: $\eta^8 \omega^8 \{3P_1\}, f_0^1 \omega^1 \{3S_1, 3D_1\}, \eta^1 \omega^1 \{3P_1\}$

J=2: $\eta^8 \omega^8 \{3P_2, 3F_2\}, f_0^1 \omega^1 \{3D_2\}, \eta^1 \omega^1 \{3P_2, 3F_2\}$

J=3: $\eta^8 \omega^8 \{3F_3\}, f_0^1 \omega^1 \{3D_3, 3G_3\}, \eta^1 \omega^1 \{3F_3\}$

Threshold behavior $t_\ell(s) \sim k^{2\ell}$ will suppress some channels in this region

$\Rightarrow$ J=1: $\eta^8 \omega^8 \{3P_1\}, f_0^1 \omega^1 \{3S_1\}, \eta^1 \omega^1 \{3P_1\}$

$\Rightarrow$ J=2: $\eta^8 \omega^8 \{3P_2, 3F_2\}, \eta^1 \omega^1 \{3P_2\}$

$\Rightarrow$ J=3: $\eta^8 \omega^8 \{3F_3\}$
**Lattice QCD**

Finite volume spectrum \( \Rightarrow C_{ij}(t) = \sum_\alpha \langle 0 | O_i | \alpha \rangle \langle \alpha | O_j | 0 \rangle e^{-E_{\alpha} t} \)

Single meson operators: \( \sum \frac{e^{i \vec{p} \cdot \vec{x}}}{\vec{x}} \bar{\psi} \vec{D} \vec{D} \ldots \vec{D} \psi \)

Meson-meson operators: \( \sum_{\vec{p}_1 + \vec{p}_2 = \vec{P}} C(\vec{p}_1, \vec{p}_2; \vec{P}) h^i_1(\vec{p}_1) h^i_2(\vec{p}_2) \)

Will include \( \eta^8(\vec{p}_1) \omega^8(\vec{p}_2), \eta^1(\vec{p}_1) \omega^1(\vec{p}_2), f_0^1(\vec{p}_1) \omega^8(\vec{p}_2) \)

\( \eta^1 \omega^1 / f_0^1 \omega^1 \)
Three resonances in a single irrep.

\[ J^P = (1,3,...)^- \]

Much messier than what we are typically used to working with.

Appears to be a decoupling within the heavier channels \( f_0^1 \eta^1, \eta^1 \omega^1 \).
\[ \langle 0|O_i|n \rangle = \eta^1(p_1^{[001]}) \omega^1(p_2^{[001]}) \]

\[ \langle 0|O_i|n \rangle = f_0^1(p_1^{[001]}) \omega^1(p_2^{[001]}) \]
$J^P = (1, 3, \ldots)^-$  $J^P = (2, \ldots)^-$  $J^P = (2, 3, \ldots)^-$  $J^P = (3, \ldots)^-$
$J^P = 0^+, 1^-, 2^+, 3^-, \ldots$ $J^P = 2^\pm, 3^\pm, \ldots$

$J^P = 2^\pm, 3^\pm, \ldots$ $J^P = 0^+, 1^-, 2^+, 3^-, \ldots$ $J^P = 0^+, 1^-, 2^+, 3^\pm, \ldots$
Plan of attack

Carry forward with elastic scattering in $\eta^8\omega^8$

$\Rightarrow$ fit to amplitudes of $J=2,3$ simultaneously ($T^-_2[000], E^-[000], A^-_2[000], B_1[001], B_2[001]$)

$\Rightarrow$ fix $J=3$ amplitude and fit for the $J=1$ amplitude ($T^-_1[000], A_1[001], A_1[111]$)

Perform analysis of $\eta^1\omega^1, f^1_0\omega^1$ as if channels were non-resonant and totally decoupled

Later relax the elastic assumption and allow $\eta^1\omega^1$ to couple

$\Rightarrow$ we find the change is rather insignificant
Parameterizations, J=2,3

J=2 dynamically coupled in P- and F-waves

Can handle this with the K-matrix $t^{-1} = K^{-1} + I$

$K_{J=2} = \begin{pmatrix} (3P_2|3P_2) & (3P_2|3F_2) \\ (3P_2|3F_2) & (3F_2|3F_2) \end{pmatrix}$

J=3 Breit-Wigner parameterization

$K_{J=3} = \frac{g_F^2}{m^2_R - s}$

$J^P$

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<thead>
<tr>
<th>$\ell$</th>
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<tbody>
<tr>
<td>$\ell = 0$</td>
<td>$(0, 1, 2)^-$</td>
</tr>
<tr>
<td>$\ell = 1$</td>
<td>$(1, 2, 3)^+$</td>
</tr>
<tr>
<td>$\ell = 2$</td>
<td>$(2, 3, 4)^-$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
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</tbody>
</table>

$I(s) = I(s_0) - \frac{s - s_0}{\pi} \int_{s_{thr}}^{\infty} \frac{\rho(s')}{(s' - s)(s' - s - i\epsilon)} ds'$

$\text{Im} I = -\rho$
\[ K_{J=2} = \frac{1}{m_R^2 - s} \begin{bmatrix} g_P^2 & g_P g_F & g_F^2 \\ g_P g_F & g_F^2 & \gamma_{PP}^2 \\ g_P g_F & g_F^2 & \gamma_{PF}^2 \end{bmatrix} \]

\[ K_{J=3} = \frac{g_F^2}{m_R^2 - s} \]

\[
\begin{align*}
J = 2 & \quad \left\{ \begin{array}{l}
m = 0.4322(15) \cdot a_t^{-1} \\
g_P = 0.753(37) \\
g_F = -4.13(29) \cdot a_t^2 \\
\gamma_{PP} = 0.1(33) \cdot a_t^2 \\
\gamma_{PF} = -110(17) \cdot a_t^2 \\
\gamma_{FF} = 143(322) \cdot a_t^6 \\
m = 0.4341(9) \cdot a_t^{-1} \\
g = 4.85(28) \cdot a_t^2
\end{array} \right. \\
J = 3 & \quad \left\{ \begin{array}{l}
m = 0.4319(39) \cdot a_t^{-1} \\
g = 4.85(28) \cdot a_t^2
\end{array} \right.
\]

\[ \chi^2/N_{\text{dof}} = \frac{120.3}{31-9} = 1.45 \]
1.42 < \chi^2 / N_{dof} < 1.46

1.66 < \chi^2 / N_{dof} < 1.86
$$K_{J=1} = \frac{g_a^2}{m_a^2 - s} + \frac{g_b^2}{m_b^2 - s} + \gamma$$

\[
m_a = 0.3881(14) \cdot a_t^{-1}
\]
\[
g_a = 1.46(10)
\]
\[
m_b = 0.4242(17) \cdot a_t^{-1}
\]
\[
g_b = -0.36(13)
\]
\[
\gamma = 20.9(86) \cdot a_t^2
\]

\[
\chi^2/N_{dof} = \frac{91.3}{72-5} = 1.36
\]
\[ K_{J=1} = \frac{g_a^2}{m_a^2 - s} + \frac{g_b^2}{m_b^2 - s} + \gamma s^n \]
Elasticity

Zero is a feature of elastic unitarity

\[ t = \frac{1}{\rho (\cot \delta - i)} \]

Cannot generate with an effective range

\[ k^3 \cot \delta = \frac{1}{a} + \frac{1}{2} rk^2 + \ldots \]
In N.R. scattering, the interactions are generated by a potential.

Lighter resonance has further to tunnel through the barrier to decay.

⇒ should be narrow compared to heavier resonance

⇒ must explain through relativistic treatment
Resonance interpretation

\[ t(s) = \frac{N(s)}{D(s)} \]

Write dispersively
\[
\frac{1}{2\pi i} \oint \frac{D(s')}{s' - s} = D(s_0) - \frac{s - s_0}{\pi} \int_{s_{thr}}^{\infty} \frac{N(s')\rho(s')}{(s' - s)(s' - s_0)} ds'
\]

⇒ can add poles to D(s) that feature as zeros in t(s)

⇒ create nearby poles in t(s)

⇒ these “CDD” poles have an interpretation that they would be stable particles if there were not lighter mesons for which it to decay
Add the [011]A\textsubscript{1} irreps and fit all simultaneously

Very good constraint $N_{\text{dof}} = 180$

\begin{align*}
\chi^2 / N_{\text{dof}} &= 258.3 / (192 - 12) = 1.43 \\
K_{J=1} &= \frac{g_a^2}{m_a^2 - s} + \frac{g_b^2}{m_b^2 - s} + \gamma \\
K_{J=2} &= \frac{1}{m_R^2 - s} \begin{bmatrix} g_P^2 & g_P g_F \\ g_P g_F & g_F^2 \end{bmatrix} + \begin{bmatrix} \gamma_{PP} & \gamma_{PF} \\ \gamma_{PF} & 0 \end{bmatrix} \\
K_{J=3} &= \frac{g_F^2}{m_R^2 - s}
\end{align*}
Comparing to the $\omega^*_J$, $\phi^*_J$

Relationship to familiar meson states:

$$
1 = \frac{1}{2\sqrt{2}} \left( K^+ \bar{K}^- + K^- \bar{K}^+ - K^0 \bar{K}^*0 - \bar{K}^0 K^*0 + \pi^+ \rho^- + \pi^- \rho^+ - \pi^0 \rho^0 - \eta_8 \omega_8 \right)
$$

Pseudoscalar states have little mixing from SU(3) eigenstates $\eta \sim \eta_8$, $\eta' \sim \eta_1$

Vector states are nearly ideally flavor mixed:

$$
\omega = \sqrt{\frac{2}{3}} \omega_1 + \sqrt{\frac{1}{3}} \omega_8 ; \phi = \sqrt{\frac{1}{3}} \omega_1 - \sqrt{\frac{2}{3}} \omega_8 \quad \Gamma = g^2 \frac{\rho}{M}
$$

Isoscalar states are admixtures of octet and singlet, but we haven’t computed the octet

$$
|I = I_z = Y = 0,8\rangle = \sqrt{\frac{1}{20}} \left( K^+ \bar{K}^- + K^- \bar{K}^+ - K^0 \bar{K}^*0 - \bar{K}^0 K^*0 \right) - \sqrt{\frac{1}{5}} \left( \pi^+ \rho^- + \pi^- \rho^+ - \pi^0 \rho^0 - \eta_8 \omega_8 \right)
$$

$8 = \eta_8 \omega_1$
Comparing to the $\omega_J^*, \phi_J^*$

Can enforce OZI to find octet contributions $g^1 = c_{\eta^8\omega^1}; g^8 = c_{\eta^8\omega^8}; h^8 = c_{\eta^8\omega^1}$

$\phi^* \to \pi\rho, \eta\omega$ give the constraints $g^8 = -\frac{\sqrt{5}}{4}g^1; h^8 = -\frac{1}{2\sqrt{2}}g^1$

$$\Gamma(\omega^* \to \pi\rho) = 3 \frac{\rho}{M} \frac{3}{16} (g^1)^2$$
$$\Gamma(\omega^* \to K\bar{K}^*) = 4 \frac{\rho}{M} \frac{3}{64} (g^1)^2$$
$$\Gamma(\omega^* \to \eta\omega) = 1 \frac{\rho}{M} \frac{1}{16} (g^1)^2$$

$\Gamma(\phi^* \to K\bar{K}^*) = 4 \frac{\rho}{M} \frac{3}{32} (g^1)^2$

$\Gamma(\phi^* \to \eta\phi) = 1 \frac{\rho}{M} \frac{1}{4} (g^1)^2$

$\Gamma(\rho^* \to \pi\omega) = 1 \frac{\rho}{M} \frac{3}{16} (g^1)^2$

$\Gamma(\rho^* \to K\bar{K}^*) = 2 \frac{\rho}{M} \frac{3}{32} (g^1)^2$

$\Gamma = g^2 \frac{\rho}{M}$

Can predict $\rho_J^*$ decays from OZI

$$|I = I_z = 1, 8\rangle = -\sqrt{\frac{3}{10}} (K^+\bar{K}^{*0} + \bar{K}^{*0}K^+) + \frac{1}{\sqrt{5}}\pi^+\omega_8 + \frac{1}{\sqrt{5}}\eta_8\rho^+$$
Comparing to the $\omega_J^*$, $\phi_J^*$

$J=3$: 
\[ \Gamma(\rho_3 \rightarrow \pi \omega, K\overline{K}^*) = 22, 2 \text{ MeV} \]
\[ \Gamma(\omega_3 \rightarrow \pi \rho, KK^*, \eta \omega) = 62, 2, 1 \text{ MeV} \]
\[ \Gamma(\phi_3 \rightarrow K\overline{K}^*, \eta \phi) = 20, 3 \text{ MeV} , \]
\[ \Gamma_{\omega_3}^{tot} \sim 168(10) \text{ MeV} \]
\[ \Gamma_{\phi_3}^{tot} \sim 87(25) \text{ MeV} \]
\[ \Gamma_{\rho_3}^{\pi \omega} \sim 30(10) \text{ MeV} \]
\[ \Gamma_{\rho_3}^{KK\pi} \sim 7 \text{ MeV} \]

$J=2$: 
\[ \Gamma(\rho_2 \rightarrow \pi \omega, K\overline{K}^*) = 125, 36 \text{ MeV} \]
\[ \Gamma(\omega_2 \rightarrow \pi \rho, KK^*, \eta \omega) = 365, 36, 17 \text{ MeV} \]
\[ \Gamma(\phi_2 \rightarrow K\overline{K}^*, \eta \phi) = 148, 44 \text{ MeV} , \]
Comparing to the $\omega_J^*, \phi_J^*$

$J=1$: 
\[ \Gamma(\rho_b \to \pi\omega, K\bar{K}^*) = 9.3 \text{ MeV} \]
\[ \Gamma(\omega_b \to \pi\rho, K\bar{K}^*, \eta\omega) = 25.3, 1 \text{ MeV} \]
\[ \Gamma(\phi_b \to K\bar{K}^*, \eta\phi) = 13, 5 \text{ MeV} \]

\[ \omega^*(1606) \quad \Gamma_{\pi\rho} \sim 84 \text{ MeV} \]
\[ \rho^*(1730) \quad \Gamma_{\pi\omega} \sim 0 \]

\[ \Gamma(\rho_a \to \pi\omega, K\bar{K}^*) = 133, 9 \text{ MeV} \]
\[ \Gamma(\omega_a \to \pi\rho, K\bar{K}^*, \eta\omega) = 384, 4, 5 \text{ MeV} \]
\[ \Gamma(\phi_a \to K\bar{K}^*, \eta\phi) = 154, 25 \text{ MeV} \]

\[ \omega^*(1440) \quad \Gamma_{\pi\rho} \sim 240 \text{ MeV} \]
\[ \rho^*(1463) \quad \Gamma_{\pi\omega} \sim 52 - 78 \text{ MeV} \]
Only 4 levels with large $\eta^1 \omega^1$ overlap.

Only real difference in fit-1 which features two $\eta^1 \omega^1$ parameters.

Potentially a small coupling $c_{\eta^1 \omega^1} \lesssim 0.04$ does not change overall width.

Statistical uncertainties on $f_{\eta^1}^\omega^1$ energy levels prevent a proper C.C. analysis with this channel.
Mild changes in the amplitude.

\[ a_t |c_{\eta^1\omega^1}) | \sim 0.07(2) \] is small and comparable to F-wave coupling.
Additional singularities

Unphysical sheet real axis pole $a_t \sqrt{s} \sim 0.23$ on many parameterizations

$\Rightarrow$ wanders a bit and remains far from physical scattering

Additional real axis pole $a_t \sqrt{s} \sim 0.24$ for simple phase space parameterization

$\Rightarrow$ not surprising this parameterization has poorer analytic properties

$\Rightarrow$ residue is real, a true p-wave bound state has imaginary coupling
Amplitude analytic structure

The full scattering amplitude $T(s,t)$ relates all scattering channels $s,t,u$- through an analytic continuation.

$s$-channel unitarity constrains the “right hand cut” to form $2^{N_{\text{chan}}}$ Riemann sheets

⇒ built into our parameterizations

Analyticity requires poles off axis real valued poles be on unphysical sheets.

⇒ reject parameterizations that have these

$t,u$-channel unitarity manifests themselves in the form of a “left hand cut”

⇒ not described but we know where they are

⇒ hope is we remain far enough away
Cross Channels

S-Channel

\[ \eta^8(p_1) \quad \omega^8(p_2) \quad \eta^8(p_3) \quad \omega^8(p_4) \]

T-Channel

\[ \eta^8(p_1) \quad \eta^8(p_3) \quad \omega^8(p_2) \quad \omega^8(p_4) \]

U-Channel

\[ \eta^8(p_1) \quad \eta^8(p_3) \quad \omega^8(p_2) \quad \omega^8(p_4) \]
Cuts

\[ R = m_{\omega s}^2 - m_{\eta s}^2 \]

\(-\infty < s \leq (m_{\omega s} - m_{\eta s})^2\)

S-channel

U-channel
Stable particles in cross-channels add additional singularities

- Stable $\omega^1$ in U-channel
- Stable $f_0^1$ in T-channel

Right-most part of additional cuts at $a_t\sqrt{s} = 0.299$ compared to threshold of $a_t\sqrt{s} = 0.3632$
Additional Singularities

Physical sheet pole at $a_t\sqrt{s} = 0.278(26)$ wrong residue.

⇒ asses this as a “ghost” occurring from improper treatment of the LHC

Noisy third unphysical sheet pole lies beyond region of constraint $a_tE \sim 0.46$.

⇒ artifact not present in all parameterizations

⇒ could be feeling presence of a hybrid $1^{--}$ meson we expect in that region
We extract 4 resonances consistent with the quark model prediction.

1−−: broader lighter first resonance and heavier narrower second resonance

2−−: broad resonance coupled mostly to P-wave

3−−: narrow F-wave resonance

Finite volume formalism can handle multiple resonances in same partial wave and nearly degenerate resonances in same irrep.
Future

Calculation of the octet is underway:

⇒ more channels
⇒ identical particles $\eta^8\eta^8, \omega^8\omega^8$
⇒ nearly degenerate thresholds in $\eta^8\omega^8, \eta^8\omega^1$

Would like to be able to study the hybrid candidate that lies slightly above in $1^{--}$
⇒ likely requires three-particle formalism