Compton amplitude and the moments of nucleon structure functions from a lattice QCD calculation

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based on:
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PRD107 (2023) 5, 054503
and preliminary work

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in Collaboration with

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  - A. Hannaford-Gunn (current PhD)
  - E. Sankey (past Honours)
  - R. Smail (current PhD)
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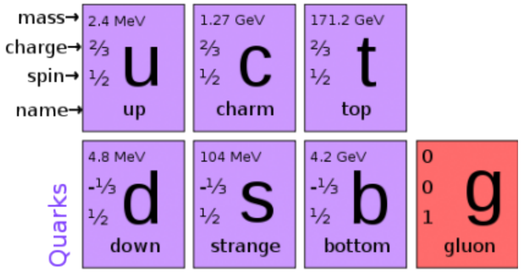
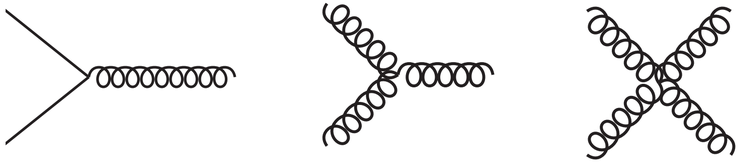
  - H. Perlt (Leipzig)
  - G. Schierholz (DESY, Hamburg)
  - H. Stüben (Hamburg)
- Japan:
  - Y. Nakamura (RIKEN, Kobe)

Introduction

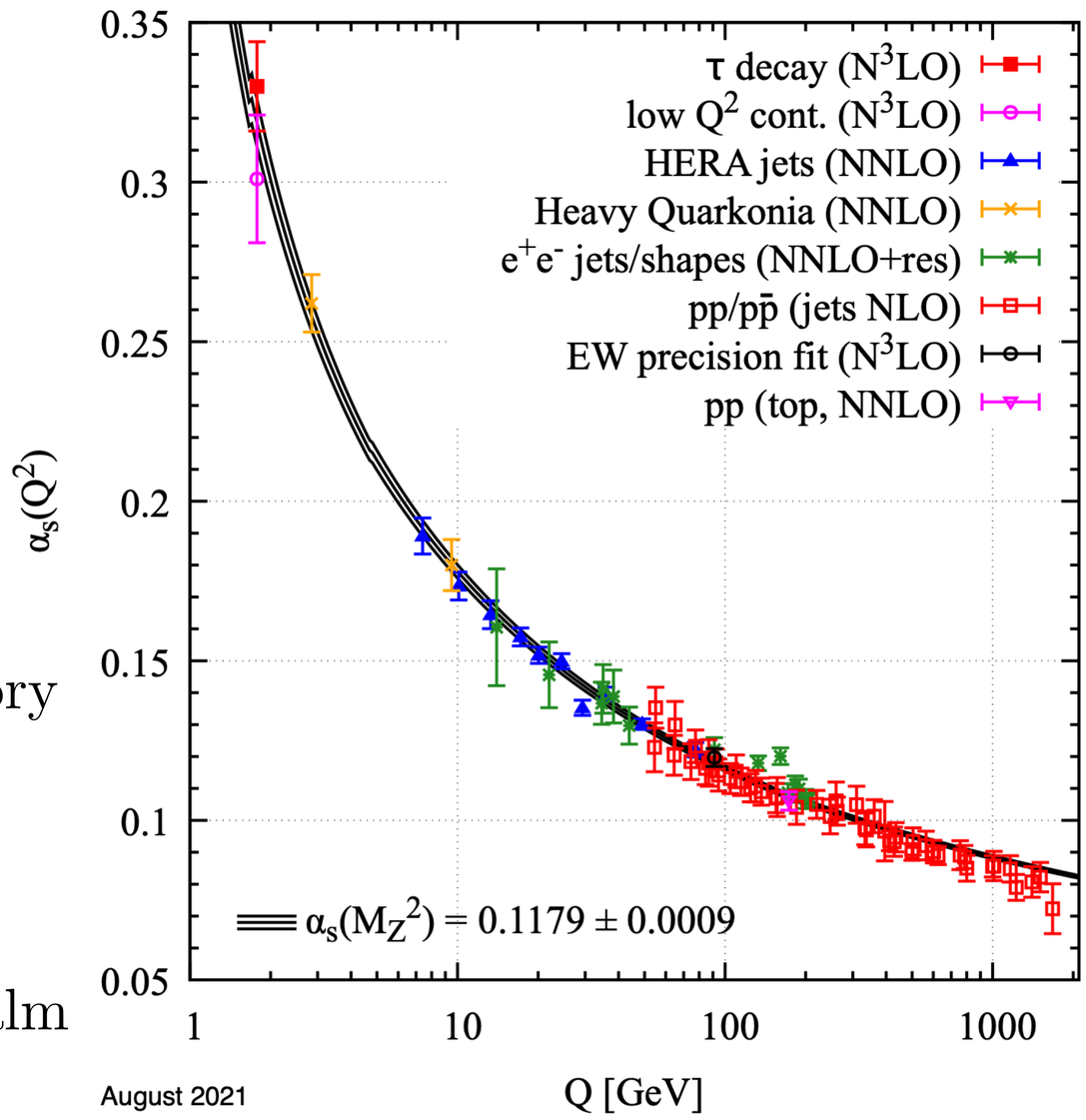
$$\mathscr{L} = \sum_q \bar{\psi}_q (i\not{D} - m_q)\psi_q - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$



$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_s [A_\mu, A_\nu]$$

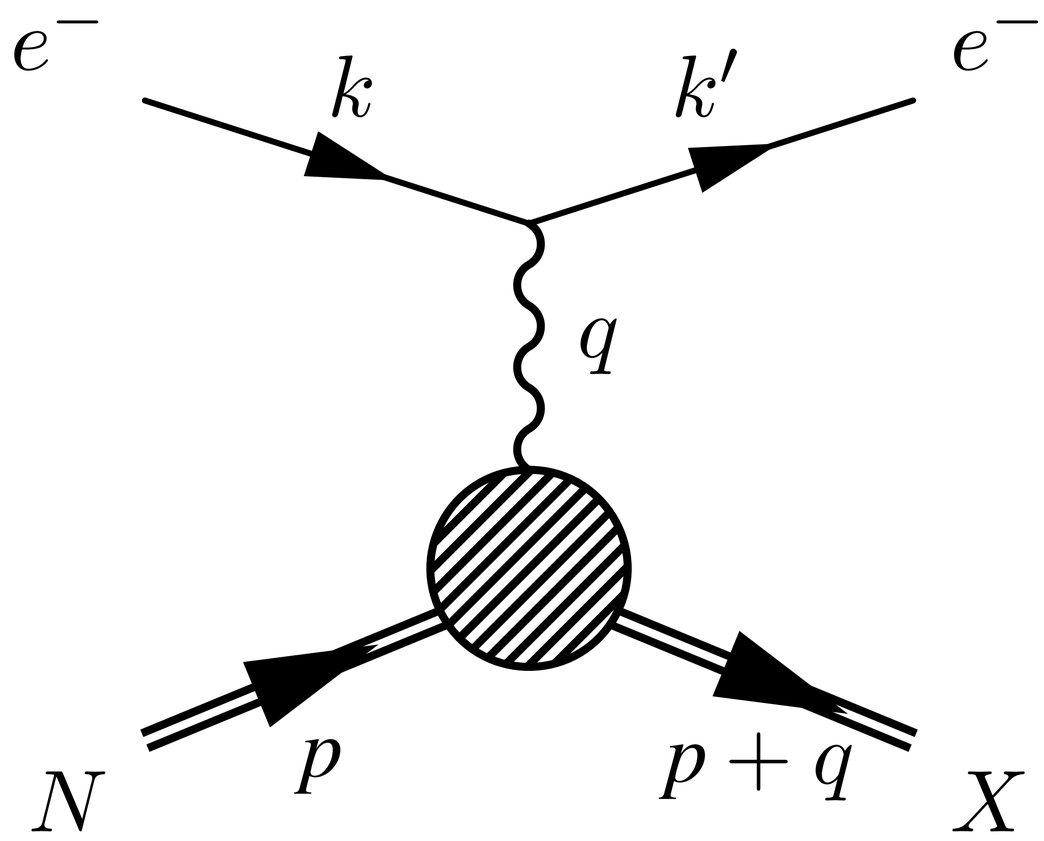


- Renormalisable non-Abelian $SU(3)$ gauge field theory
- Asymptotically free theory, perturbative methods applicable at $Q \gg \Lambda_{QCD}$
- Confinement leads to hadrons: nonperturbative realm



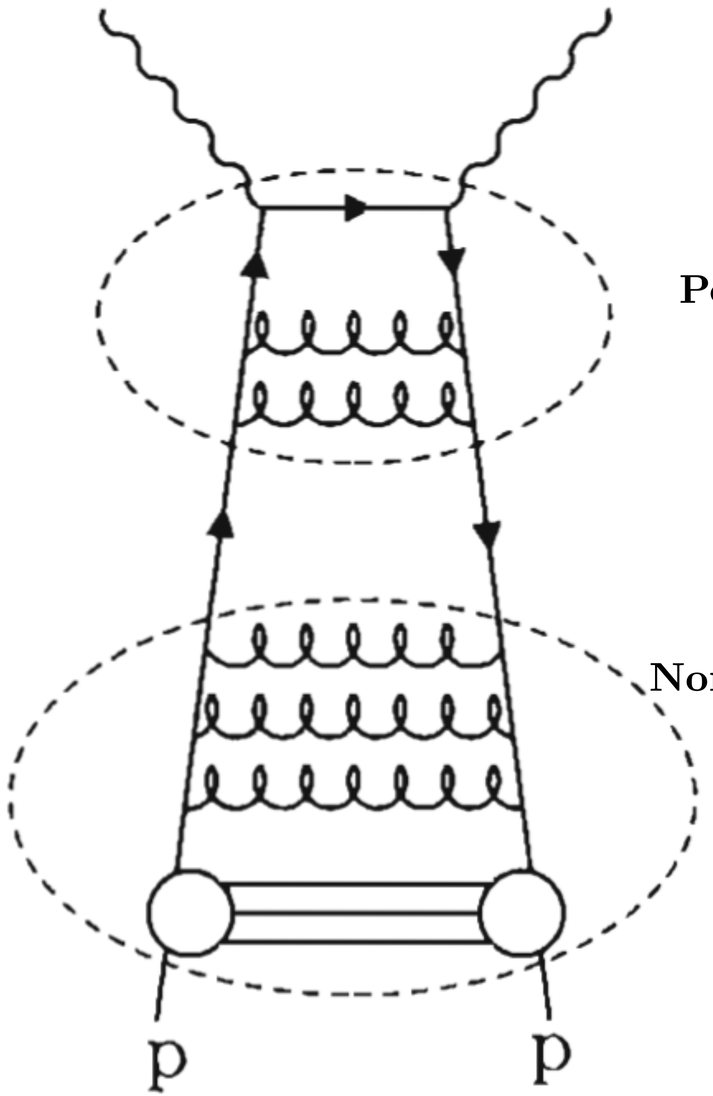
Introduction

Deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering



- Inclusive process, $\sum_X |X\rangle\langle X|$
- Crucial for understanding the hadron structure
- Theoretical formulation relies on Parton Distribution Functions (PDFs)

Introduction



Hard scale: Perturbatively calculable

Soft scale: Non-perturbative PDFs

- Operator Product Expansion (OPE) and the Factorisation theorem are key concepts

$$d^2\sigma \propto \sum_q C_q \otimes q + \mathcal{O}(M^2/Q^2)$$

factorisation theorem

hard scale coeff. **PDF**

OPE: $Q^2 \gg M^2$**, the leading twist term dominates the cross section**

Introduction

- Nucleon structure (leading twist)
  - Structure functions from first principles
  - Understanding the behaviour in the high- and low-x regions

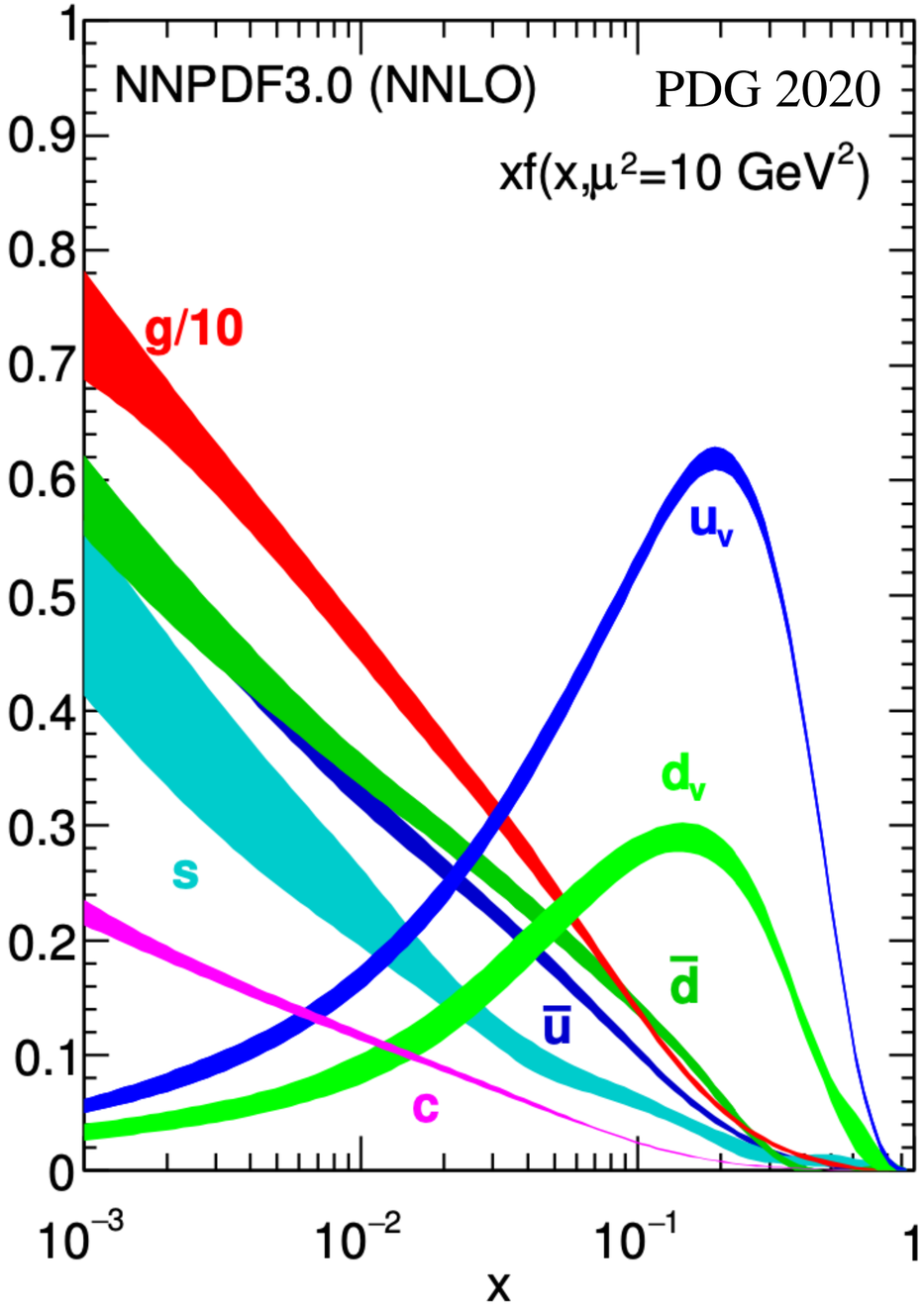
- In the parton model

$$F_2 \propto (q + \bar{q})$$

$$F_3^{\gamma Z} \propto (q - \bar{q})$$

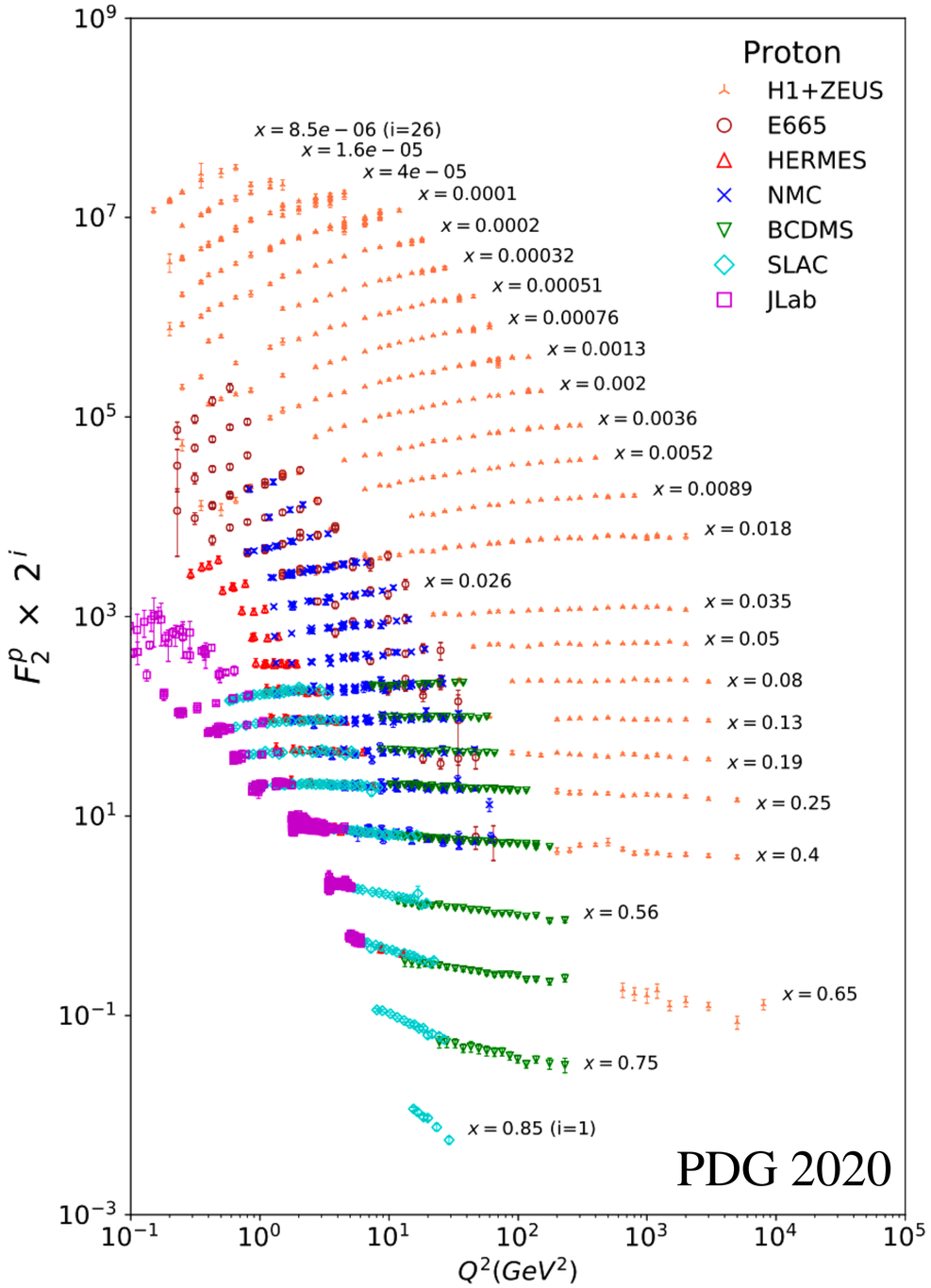
$$F_2^{W-} \propto u + \bar{d} + \bar{s} + c \ldots$$

$$F_3^{W-} \propto u - \bar{d} - \bar{s} + c \ldots$$



Introduction

- Scaling
  - Q^2 cuts of global QCD analyses
- Power corrections / Higher twist effects
  - Target mass corrections
  - Twist-4 contributions



Lattice QCD

ab initio
non-perturbative method

Two key equations:

$$\lim_{T\to\infty} \left\langle \hat{O}_2(t)\hat{O}_1(0)\right\rangle_T = \sum_h \langle 0|\hat{O}_2|h\rangle\langle h|\hat{O}_1|0\rangle e^{-E_h t}$$

Hadron d.o.f.

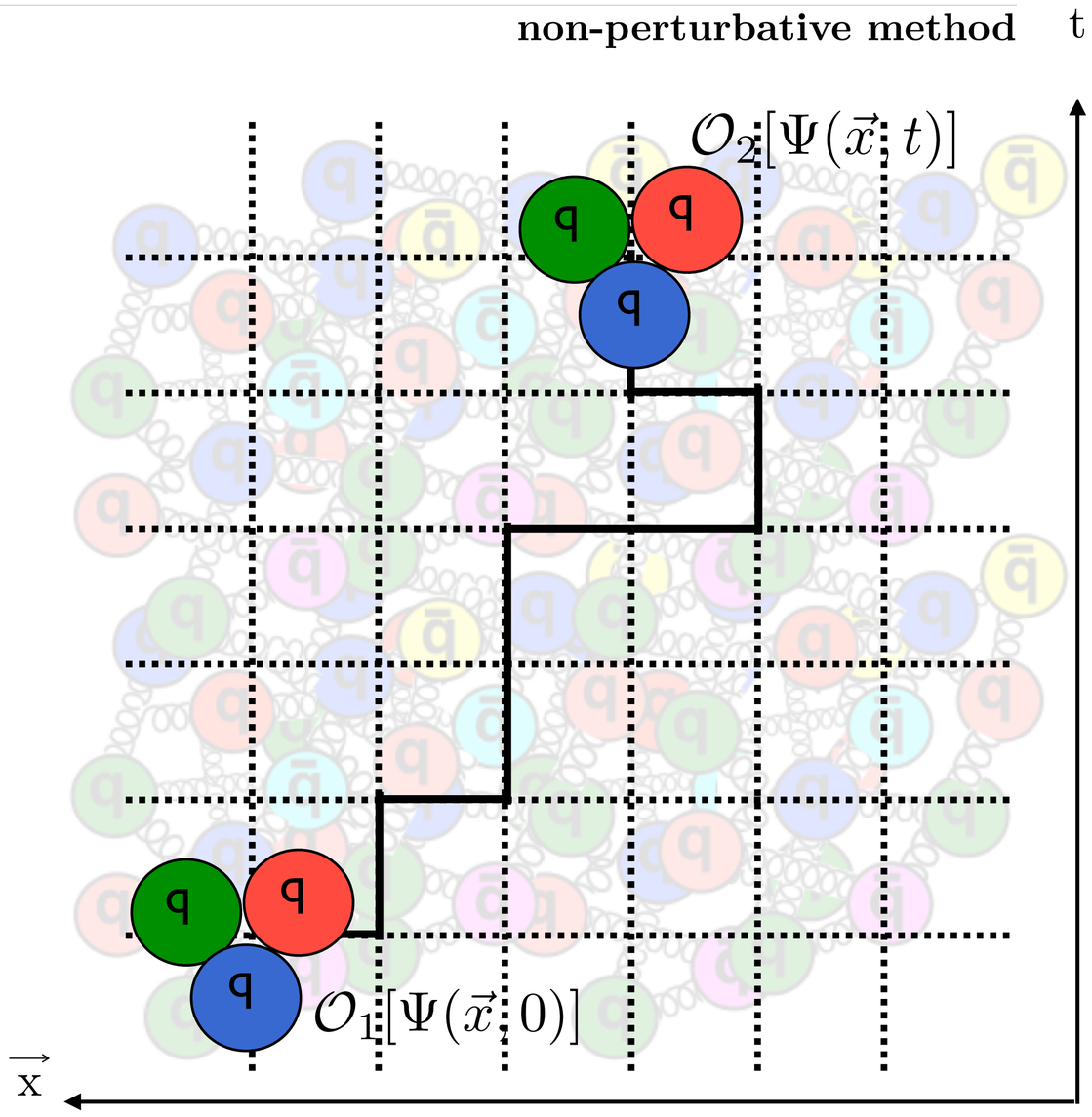
$$\left\langle \hat{O}_2(t)\hat{O}_1(0)\right\rangle = \frac{\int \mathcal{D}[\Psi] e^{-S_E[\Psi]} O_2[\Psi(\vec{x},t)] O_1[\Psi(\vec{x},0)]}{\int \mathcal{D}[\Psi] e^{-S_E[\Psi]}}$$

Quark-gluon d.o.f.

- Discretise the space-time continuum
- "Measure" quantities by computers
- Path integral has infinite dimensions
- Use tools of the stat. physics: *Importance Sampling*

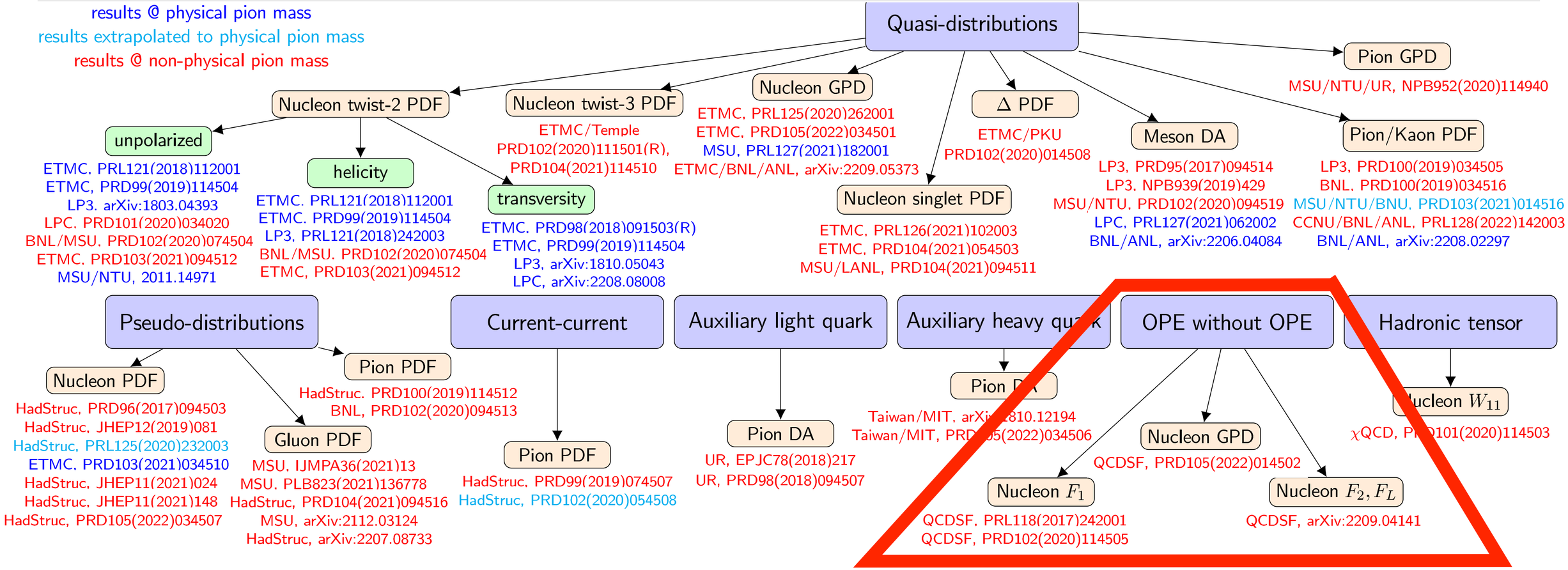
$$\langle \mathcal{O}\rangle = \frac{\int \mathcal{D}[\Psi] e^{-S_E[\Psi]} \mathcal{O}[\Psi]}{\int \mathcal{D}[\Psi] e^{-S_E[\Psi]}} = \lim_{N\to\infty} \sum_{n=1}^{N} \mathcal{O}[U_n]$$

- Take the continuum limit, $a \to 0$, $V \to \infty$, $m_\pi^{lat.} \to m_\pi^{phys.}$



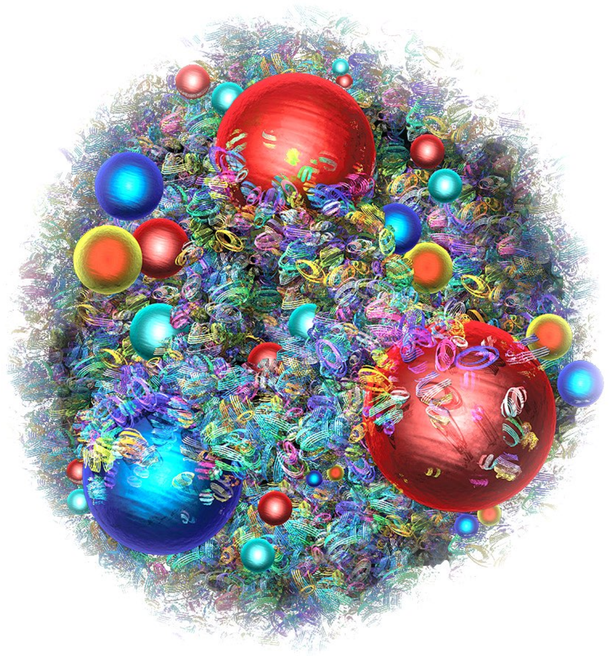
LQCD landscape

© K. Cichy, INT-22-83W



- **QCDSF-UKQCD-CSSM Collaboration**
- **Extended to nucleon F_3, and g_1, g_2**
- **Study of power corrections**

Outline

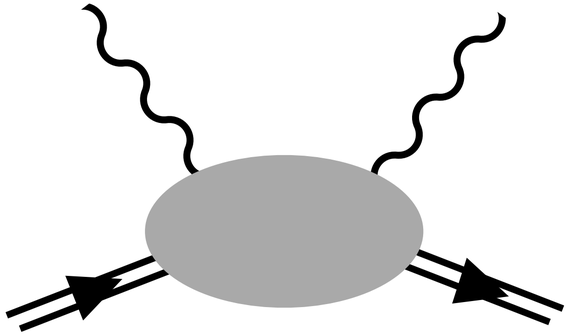


Credit: D Dominguez / CERN

- Forward Compton Amplitude & the Nucleon Structure Functions
  - Application of the Feynman-Hellmann Theorem
  - Moments of the Nucleon Structure Functions
- Outlook: Polarised, parity violating ...

Unpolarised

Forward Compton Amplitude



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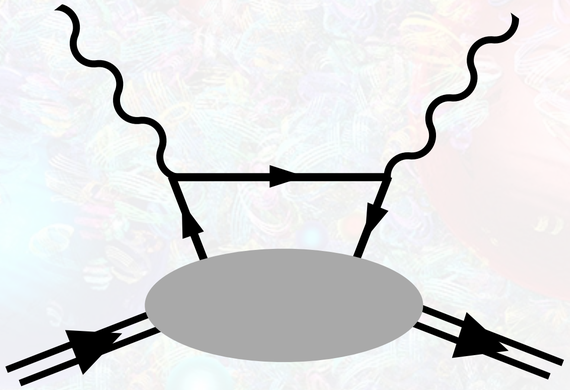


$+ \quad \mathcal{O}\left(\frac{M_N^2}{Q^2}, \frac{1}{Q^2}\right)$

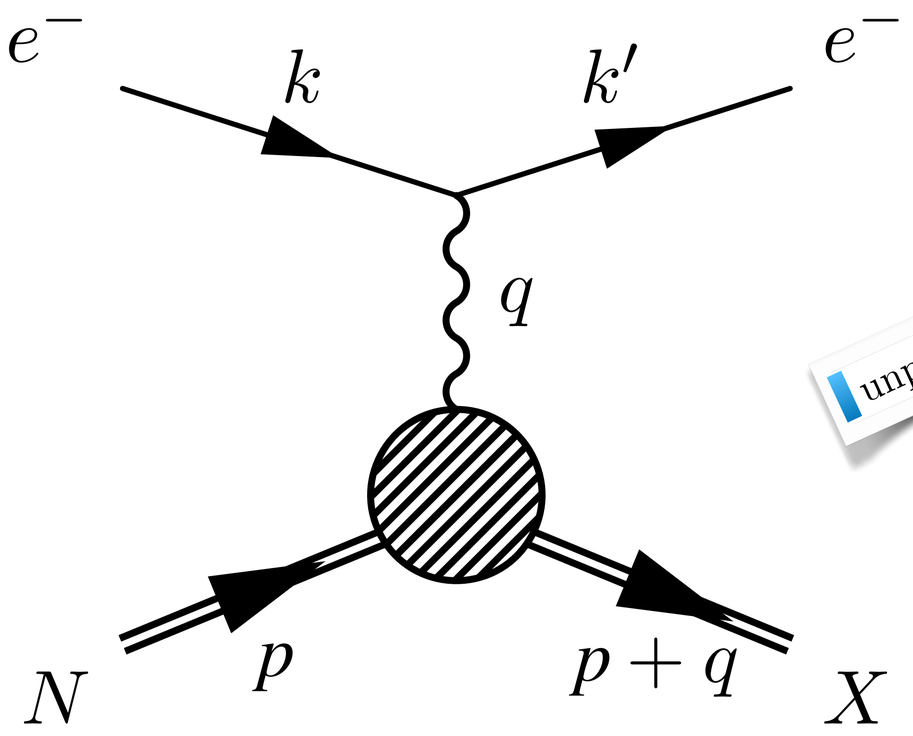
DIS and the Hadronic Tensor

Deep ($Q^2 \gg M^2$) inelastic ($W^2 \gg M^2$) scattering (DIS)

$$d\sigma \sim L_j^{\mu\nu} W_{\mu\nu}^j$$

$j = \gamma, Z,$ and γZ (neutral) or W (charged)

leptonic tensor ***hadronic tensor***



$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4 z e^{iq\cdot z} \rho_{ss'} \langle p, s' | [J_\mu(z), J_\nu(0)] | p, s \rangle$$

$$\rho_{ss'} = \frac{1}{2}\delta_{ss'}$$

unpolarised

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) \frac{F_2(x, Q^2)}{p\cdot q}$$

Structure Functions

Forward Compton Amplitude

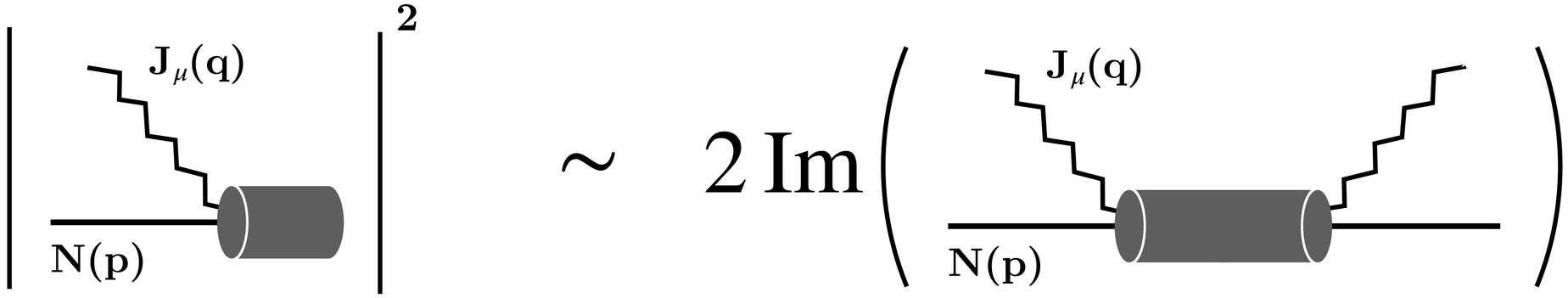
$$T_{\mu\nu}(p,q) = i\int d^4z\, e^{iq\cdot z}\rho_{ss'}\langle p,s'|\mathcal{T}\{J_\mu(z)J_\nu(0)\}|p,s\rangle \quad \text{, spin avg. } \rho_{ss'} = \frac{1}{2}\delta_{ss'}$$

$$\omega = \frac{2p\cdot q}{Q^2}$$

Same Lorentz decomposition as the Hadronic Tensor

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\mathcal{F}_1(\omega,Q^2) + \left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\frac{\mathcal{F}_2(\omega,Q^2)}{p\cdot q}$$

Compton Structure Functions (SF)



DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

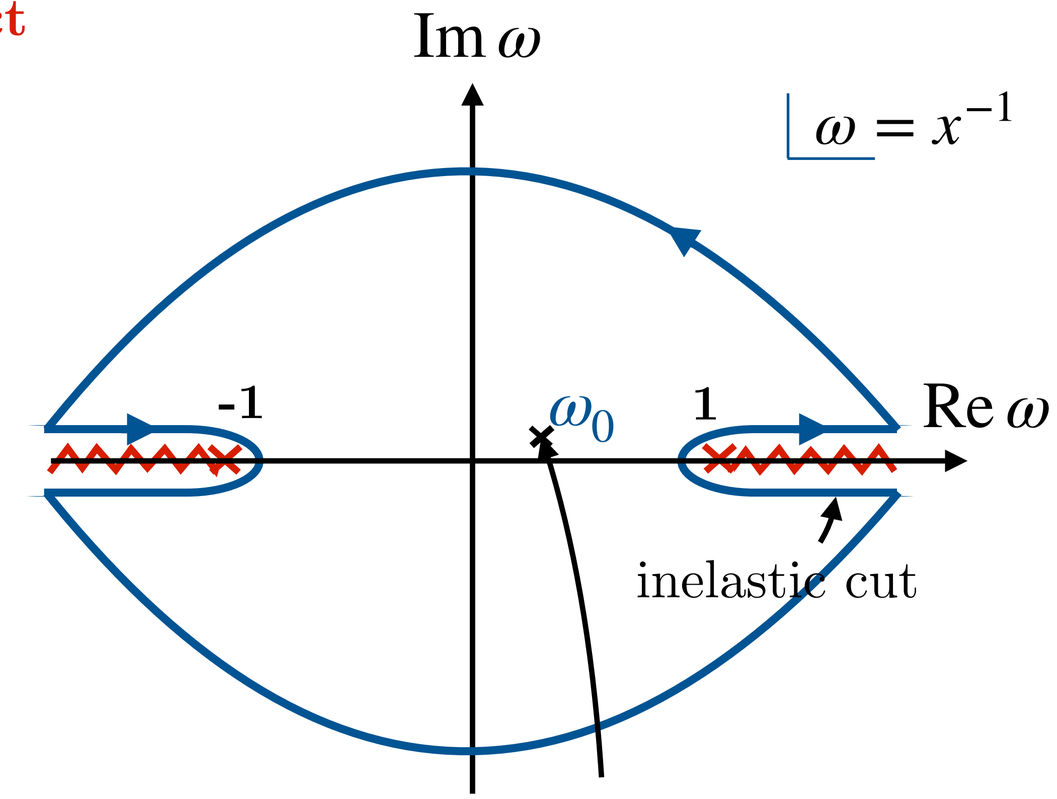
Nucleon Structure Functions

- **we can write down dispersion relations and connect Compton SFs to DIS SFs:**

$$\underbrace{\mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0,Q^2)}_{\equiv \overline{\mathcal{F}}_1(\omega, Q^2)} = 2\omega^2 \int_0^1 dx \frac{2x\, F_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

$$\mathcal{F}_2(\omega, Q^2) = 4\omega \int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

$$\underbrace{\mathcal{F}_L(\omega, Q^2) + \mathcal{F}_1(0,Q^2)}_{\equiv \overline{\mathcal{F}}_L(\omega, Q^2)} = \frac{8M_N^2}{Q^2} \int_0^1 dx F_2(x, Q^2) + 2\omega^2 \int_0^1 dx \frac{F_L(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$



Compton Amplitude is an analytic function in the unphysical region $|\omega_0| < 1$

Nucleon Structure Functions

- using the Taylor expansion, $\dfrac{1}{1-(x\omega)^2} = \sum_{n=1}^{\infty} (x\omega)^{2n-2}$

$\omega = \dfrac{2p\cdot q}{Q^2} \equiv x^{-1}$

$$\overline{\mathcal{F}}_{1,L}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1,L)}(Q^2)\ , \text{ where } M_{2n}^{(1)}(Q^2) = 2\int_0^1 dx\, x^{2n-1} F_1(x, Q^2)\ , \text{ and } \quad M_0^{(1)}(Q^2) = 0$$

$$\mathcal{F}_2(\omega, Q^2) = \sum_{n=1}^{\infty} 4\omega^{2n-1} M_{2n}^{(2)}(Q^2), \text{ where } M_{2n}^{(2,L)}(Q^2) = \int_0^1 dx\, x^{2n-2} F_{2,L}(x, Q^2)\ , \text{ and } \quad M_0^{(L)}(Q^2) = \frac{4M_N^2}{Q^2} M_2^{(2)}(Q^2)$$

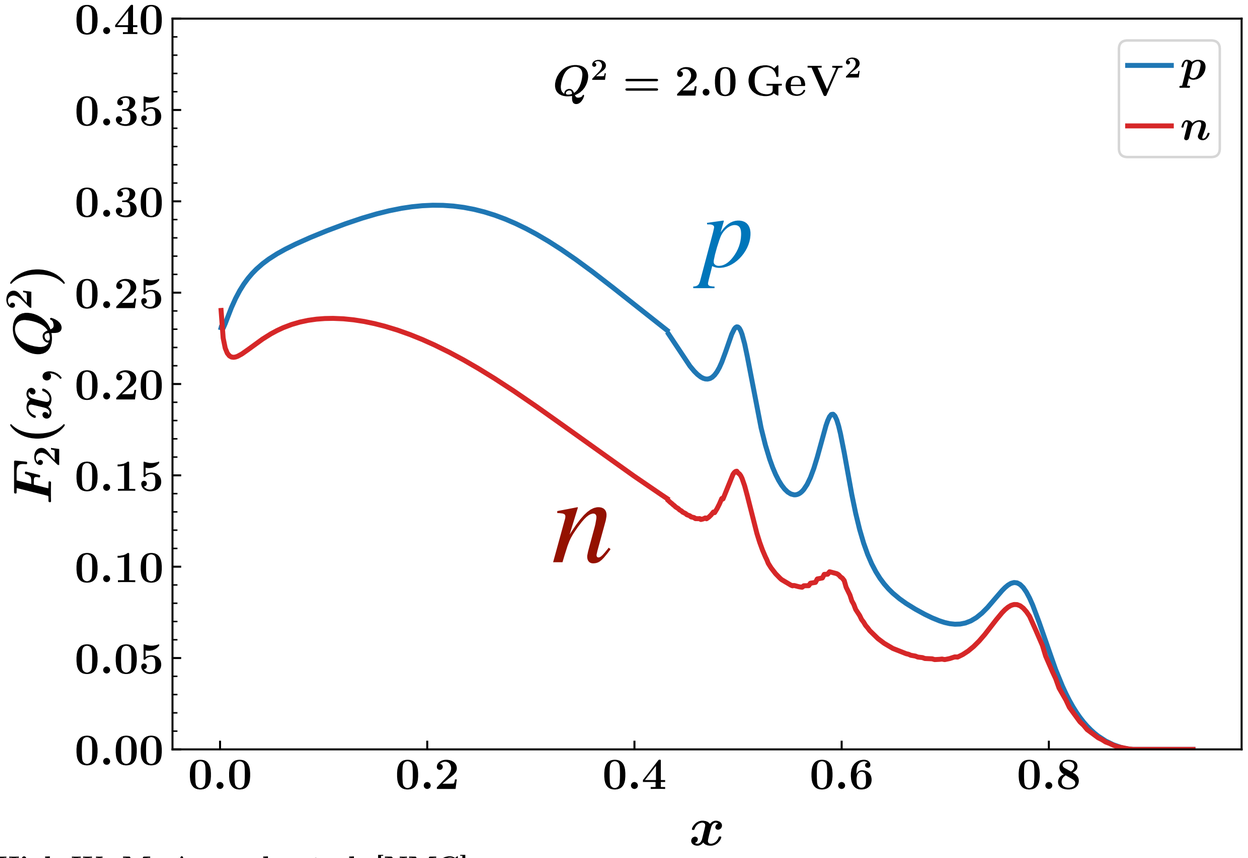
- $\mu = \nu = 3$ and $p_3 = q_3 = 0 \quad \Longrightarrow \quad \mathcal{F}_1(\omega, Q^2) = T_{33}(p, q)$

- $\mu = \nu = 0$ and $p_3 = q_3 = q_0 = 0 \quad \Longrightarrow \quad \dfrac{\mathcal{F}_2(\omega, Q^2)}{\omega} = \left[T_{00}(p, q) + T_{33}(p, q)\right] \dfrac{Q^2}{2E_N^2}$

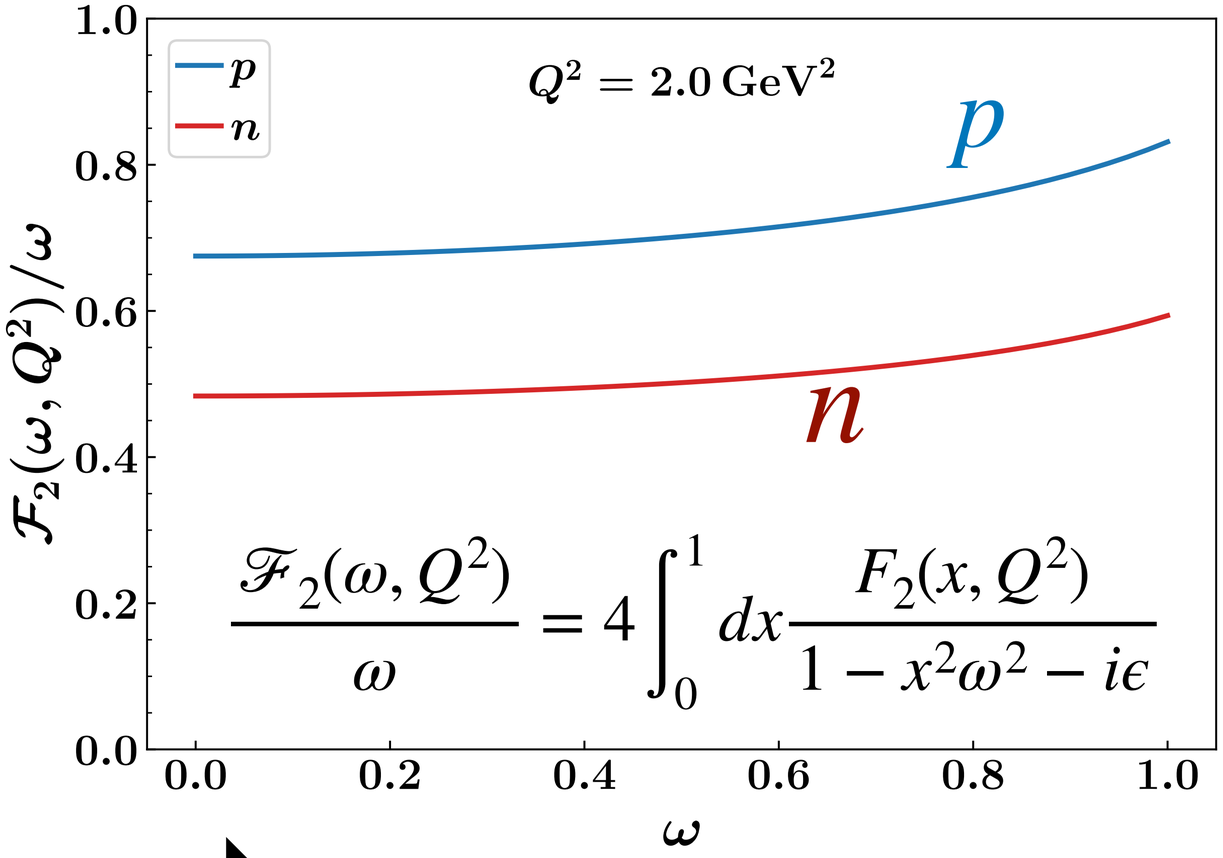
$$\mathcal{F}_L(\omega, Q^2) = -\mathcal{F}_1(\omega, Q^2) + \left(\frac{\omega}{2} + \frac{2M_N^2}{\omega Q^2}\right) \mathcal{F}_2(\omega, Q^2)$$

Shape of the Compton Amplitude

Structure functions



Compton Amplitudes



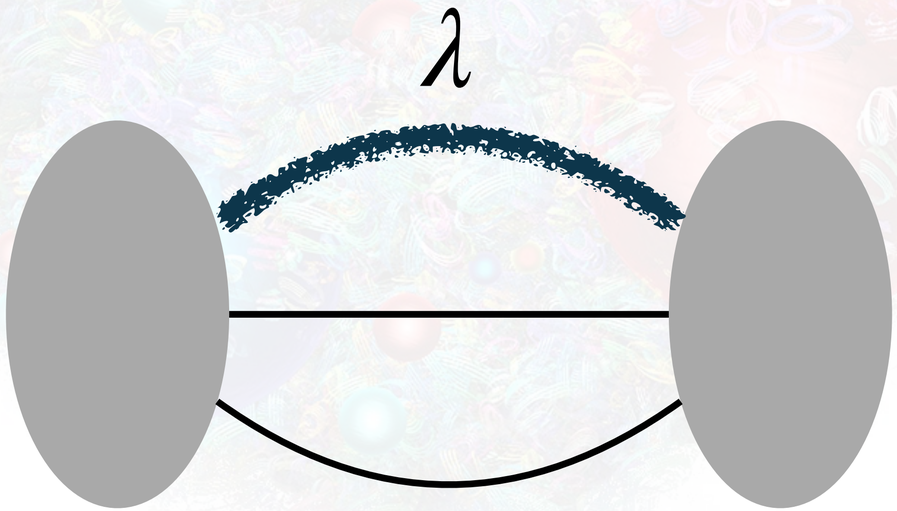
$$\frac{\mathcal{F}_2(\omega, Q^2)}{\omega} = 4\int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

High-W: M. Arneodo et al. [NMC], PLB364, 107-115 (1995), [hep-ph/9509406]
Low-W: M.E. Christy and P.E. Bosted, PRC81, 055213 (2010), [0712.3731]

dispersion relation

Feynman-Hellmann Theorem

@ 2nd order



FH Theorem at 1st order

in Quantum Mechanics:

$$\frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle$$

H_λ: perturbed Hamiltonian of the system

E_λ: energy eigenvalue of the perturbed system

ϕ_λ: eigenfunction of the perturbed system

- **expectation value of the perturbed system is related to the shift in the energy eigenvalue**

in Lattice QCD: energy shifts in the presence of a weak external field

$$S \to S(\lambda) = S + \lambda \int d^4x \, \mathcal{O}(x) \quad \longrightarrow \quad \bar{q}(x)\Gamma_\mu q(x) \qquad , \Gamma_\mu \in \{\mathbf{1}, \gamma_\mu, \gamma_5\gamma_\mu, \ldots\}$$

real parameter

e.g. local bilinear operator

@ 1st order

$$\frac{\partial E_\lambda}{\partial \lambda} = \frac{1}{2E_\lambda} \langle 0 | \mathcal{O} | 0 \rangle$$

$\mathbf{E}_\lambda \to$ **spectroscopy, 2-pt function**

$\langle 0 | \mathcal{O} | 0 \rangle \to$ **determine 3-pt**

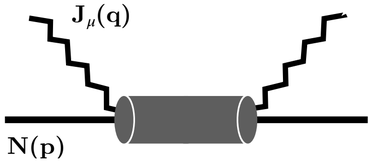
Applications:

- **σ - terms**
- **Form factors**

Compton amplitude via the FH relation at 2nd order

- **unpolarised Compton Amplitude**

$$T_{\mu\mu}(p,q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p)\,|\,\mathcal{T}\{J_\mu(z)J_\mu(0)\}\,|\,N(p)\rangle$$



- **Action modification**

local EM current

$J_\mu(z) = \sum_q e_q \bar{q}(z)\gamma_\mu q(z)$

$$S \to S(\lambda) = S + \lambda \int d^4z\,(e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}})\,J_\mu(z)$$

- **2nd order derivatives of the 2-pt correlator, $G_\lambda^{(2)}(\mathbf{p};t)$, in the presence of the external field**

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \left(\frac{\partial^2 A_\lambda(\mathbf{p})}{\partial\lambda^2} - tA(\mathbf{p})\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left.\frac{\partial^2 G_\lambda^{(2)}(\mathbf{p};t)}{\partial\lambda^2}\right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p})|\mathcal{T}\{\mathcal{J}(z)\mathcal{J}(0)\}|N(\mathbf{p})\rangle$$

from path integral

- **equate the time-enhanced terms:**

$$\left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p})|\mathcal{J}(z)\mathcal{J}(0)|N(\mathbf{p})\rangle}^{T_{\mu\mu}(p,q)} + (q \to -q)$$

Compton amplitude is related to the second-order energy shift

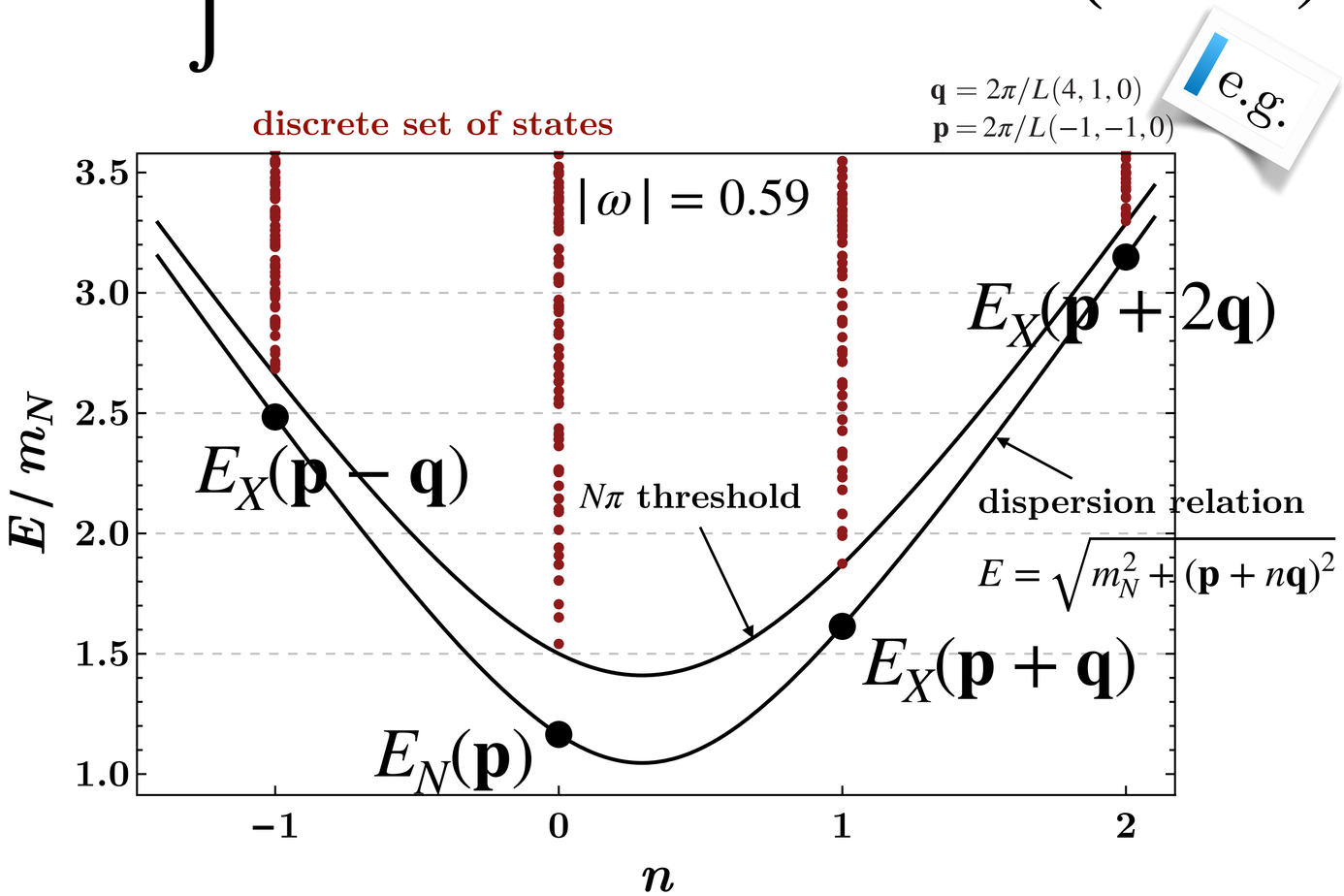
Compton amplitude via the FH relation at 2nd order

- **relevant contribution comes from the ordering where the currents are sandwiched**

$$\chi(t) \quad \mathcal{J}(z_4) \quad \mathcal{J}(y_4) \quad \bar{\chi}(0) \sim e^{-E_N(\mathbf{p})t} \int^t d\Delta\, e^{-\left(E_X(\mathbf{p}+\mathbf{q}) - E_N(\mathbf{p})\right)\Delta} (t-\Delta)$$

$\Delta = z_4 - y_4$

- under the condition $|\omega| < 1$, $E_X(\mathbf{p}+n\mathbf{q}) \gtrsim E_N(\mathbf{p})$, so the intermediate states <u>cannot go on-shell</u>
- ground state dominance is ensured in the large time limit



Moments of the Nucleon Structure Functions

Simulation Details

QCDSF/UKQCD configurations

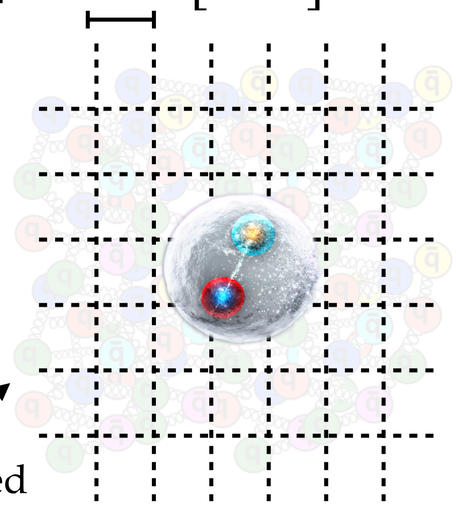
$\begin{pmatrix} 32^3 \times 64 \\ 48^3 \times 96 \end{pmatrix}$, 2+1 flavor (u/d+s)

$\beta = \begin{pmatrix} 5.50 \\ 5.65 \end{pmatrix}$, NP-improved Clover action

Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]

$m_\pi \sim \begin{bmatrix} 470 \\ 420 \end{bmatrix}$ MeV, ~SU(3) sym.

$m_\pi L \sim \begin{bmatrix} 5.6 \\ 6.9 \end{bmatrix}$ $\quad a = \begin{bmatrix} 0.074 \\ 0.068 \end{bmatrix}$ fm

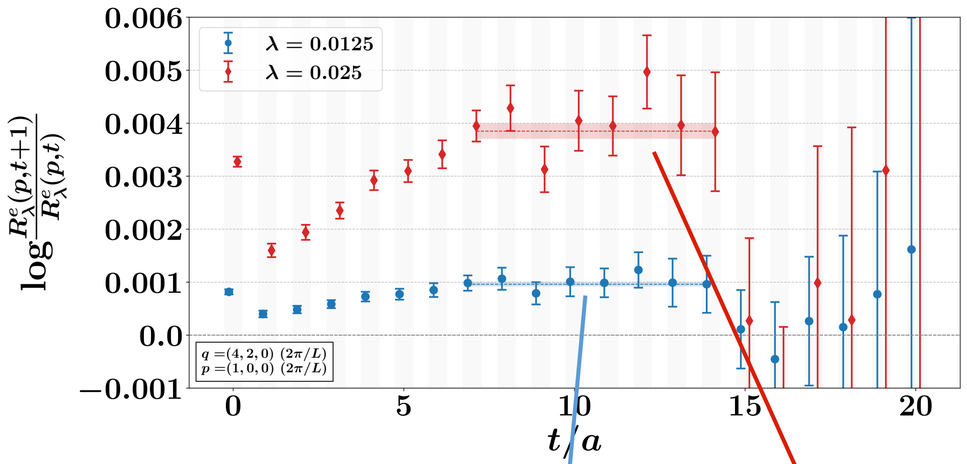


Unmodified QCD background

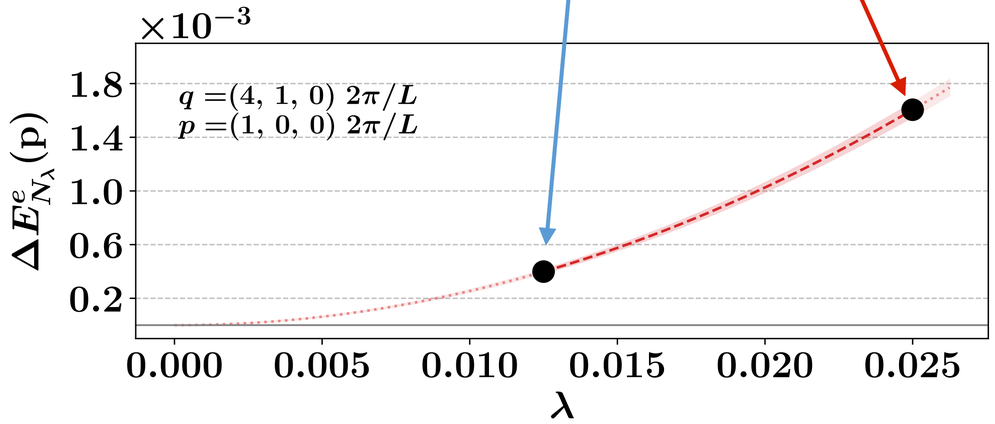
- FH implementation at the valence quark level
  - Valence u/d quark props with modified action, $S(\lambda)$
  - Local EM current insertion, $J_\mu(x) = Z_V \bar{q}(x)\gamma_\mu q(x)$
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- Several current momenta in the range, $1.5 \lesssim Q^2 \lesssim 7\, GeV^2$
- Up to $\mathcal{O}(10^4)$ measurements for each pair of Q^2 and λ
- Access to a range of $\omega = 2\, p \cdot q / Q^2$ values for several (p, q) pairs
  - An inversion for each q and λ, varying p is relatively cheap
- Connected 2-pt correlators calculated only, <u>no disconnected</u>

Strategy | Energy shifts

- **Extract energy shifts for each λ**



- **Get the 2nd order derivative**



Ratio of perturbed to unperturbed 2-pt functions

$$R^e_\lambda(\mathbf{p}, t) \equiv \frac{G^{(2)}_{+\lambda}(\mathbf{p}, t) G^{(2)}_{-\lambda}(\mathbf{p}, t)}{\left(G^{(2)}(\mathbf{p}, t)\right)^2} \xrightarrow{t \gg 0} A_\lambda(\mathbf{p}) e^{-2\Delta E^e_{N_\lambda}(\mathbf{p}) t}$$

Isolates 2nd-order energy shift by construct considering,

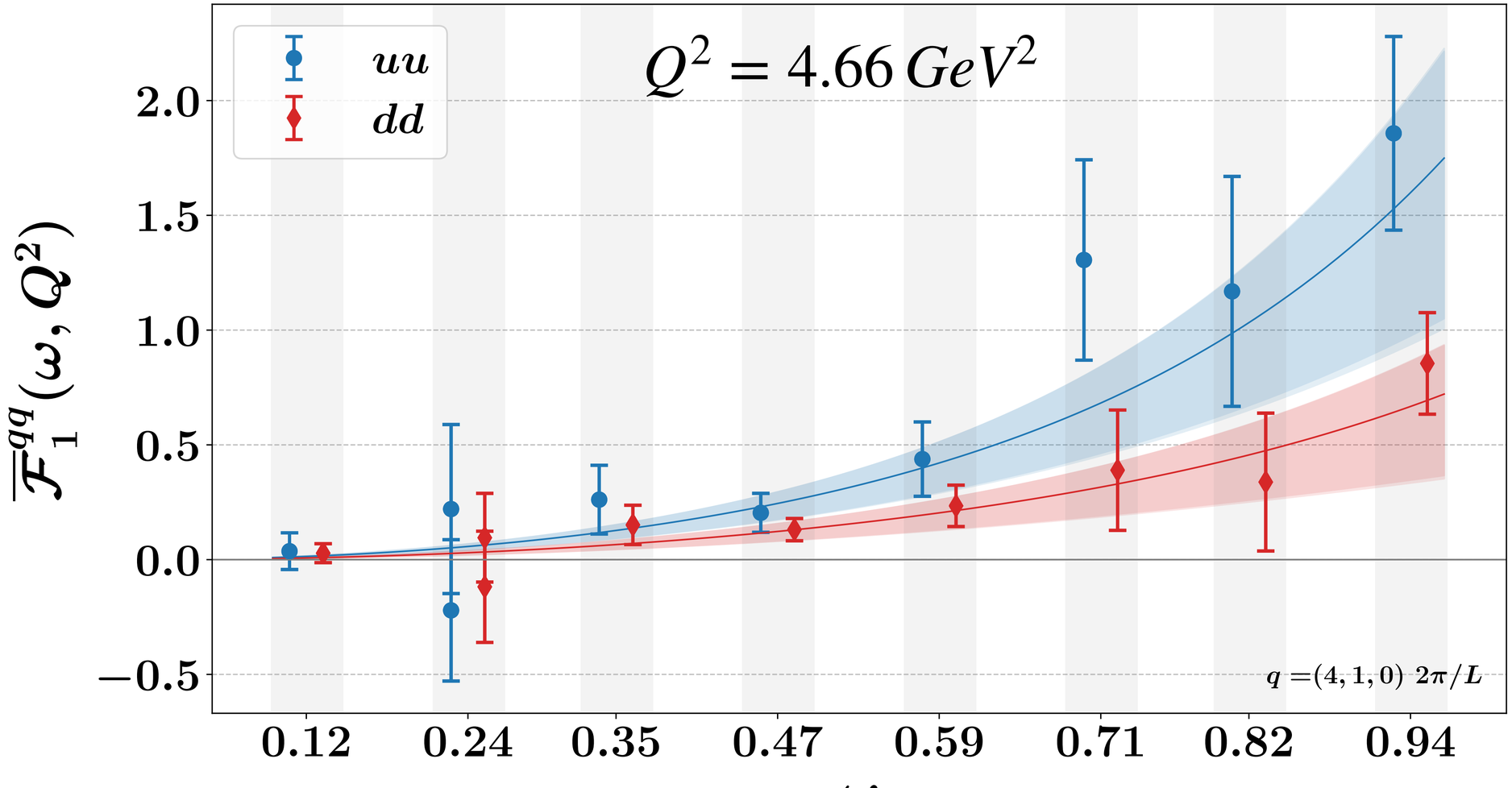
$$G^{(2)}_\lambda(\mathbf{p}; t) \sim A_\lambda(\mathbf{p}) e^{-E_{N_\lambda}(\mathbf{p}) t}$$

$$E_{N_\lambda}(\mathbf{p}) = E_N(\mathbf{p}) + \lambda \left.\frac{\partial E_{N_\lambda}(\mathbf{p})}{\partial \lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2!} \left.\frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial^2 \lambda}\right|_{\lambda=0} + \mathcal{O}(\lambda^3)$$

$$= E_N(\mathbf{p}) + \Delta E^o_N(\mathbf{p}) + \Delta E^e_N(\mathbf{p})$$

$\mathcal{F}_1$ Compton amplitude

$a = 0.074$ fm
$m_\pi \sim 470$ MeV
32^3x64, 2+1 flavour



uu

dd

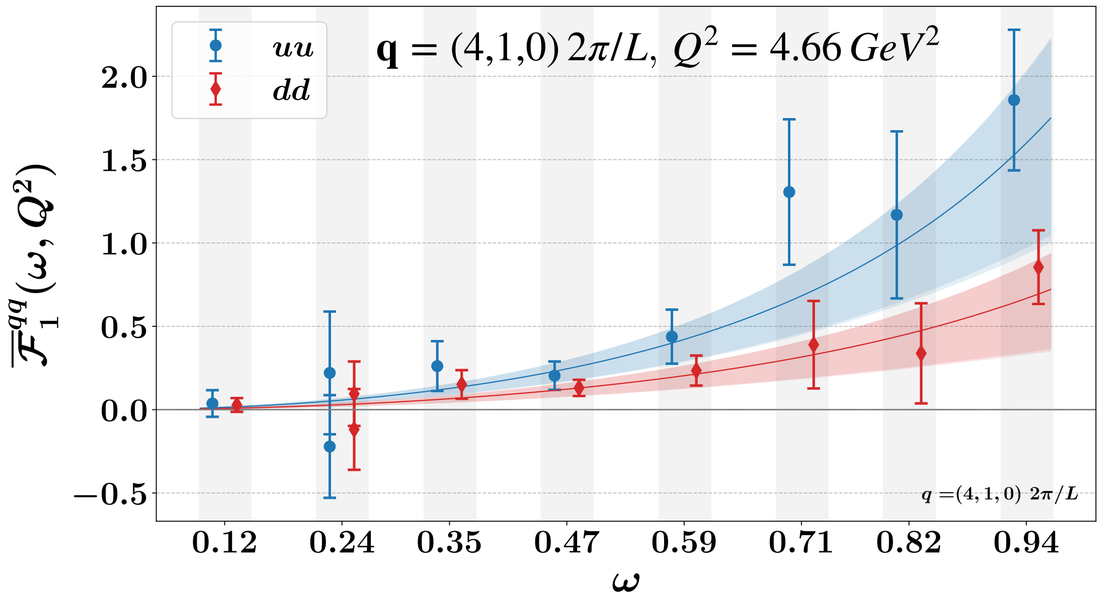
fixed **q** varying **p** → range of ω values

$$\omega = 1/x = \frac{2p \cdot q}{Q^2}$$

Moments | Fit

$a = 0.074$ fm
$m_\pi \sim 470$ MeV
32^3x64, 2+1 flavour

Remember:
$T_{33}(p,q) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$
$T_{33}(p,q) = \mathcal{F}_1(\omega, Q^2)$



$$\overline{\mathcal{F}}_1(\omega, Q^2) = 4(\omega^2 M_2^{(1)}(Q^2) + \omega^4 M_4^{(1)}(Q^2) + \cdots + \omega^{2n} M_{2n}^{(1)}(Q^2) + \cdots)$$

- **Enforce monotonic decreasing of moments for u and d only, not necessarily true for $u-d$**

$$M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \cdots \geq M_{2n}^{(1)}(Q^2) \geq \cdots \geq 0$$

We truncate at $n = 6$
No dependence to truncation order for $3 \leq n \leq 10$

- **Bayesian approach by MCMC method** — least-squares fluctuates, tricky to impose monotonic deceasing and positivity bound

Sample the moments from Uniform priors
individually for u- and d-quark

$$M_2^{(1)}(Q^2) \sim \mathcal{U}(0,1)$$

$$M_{2n}^{(1)}(Q^2) \sim \mathcal{U}\left(0, M_{2n-2}^{(1)}(Q^2)\right)$$

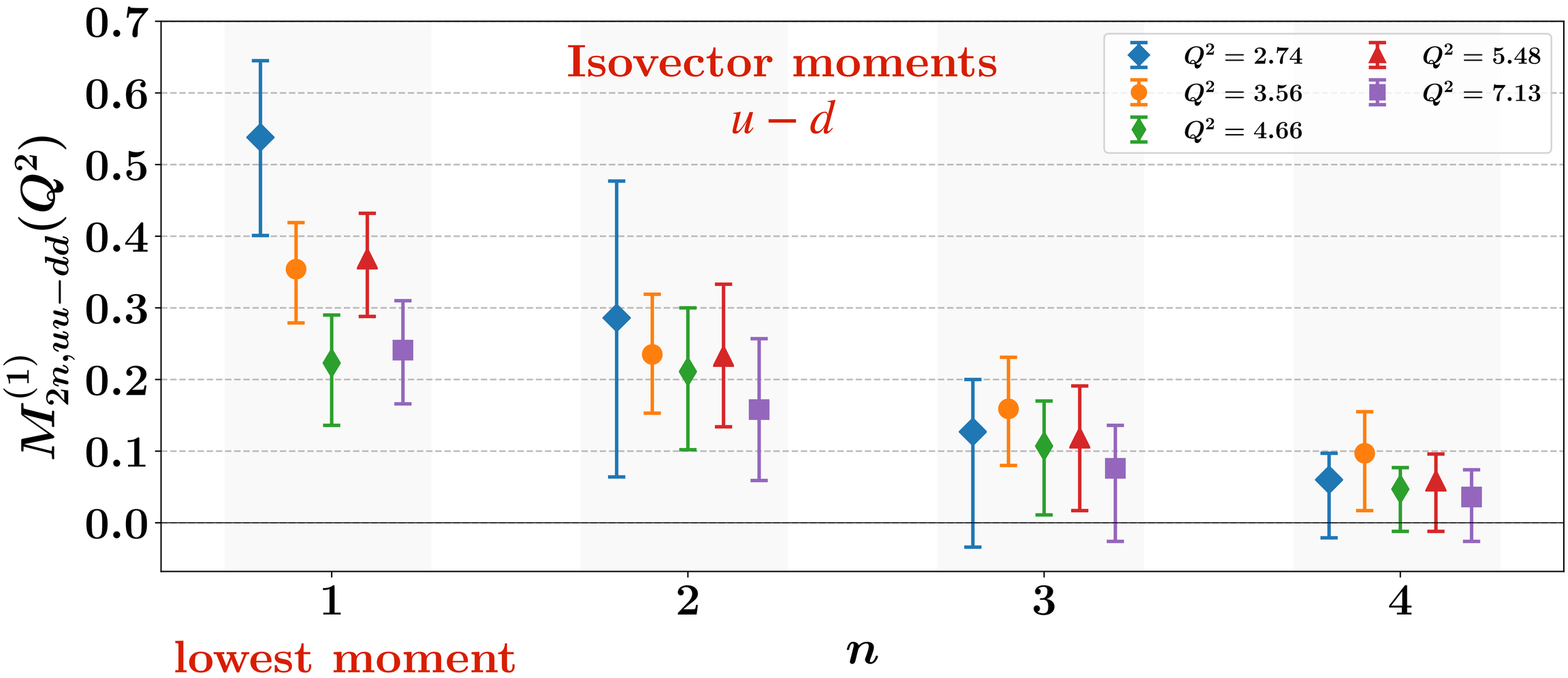
Multivariate Likelihood function, $exp(-\chi^2/2)$

$$\chi^2 = \sum_{i,j} \left[\overline{\mathcal{F}}_{1,i} - \overline{\mathcal{F}}_1^{obs}(\omega_i)\right] C_{ij}^{-1} \left[\overline{\mathcal{F}}_{1,j} - \overline{\mathcal{F}}_1^{obs}(\omega_j)\right]$$

covariance matrix

Moments

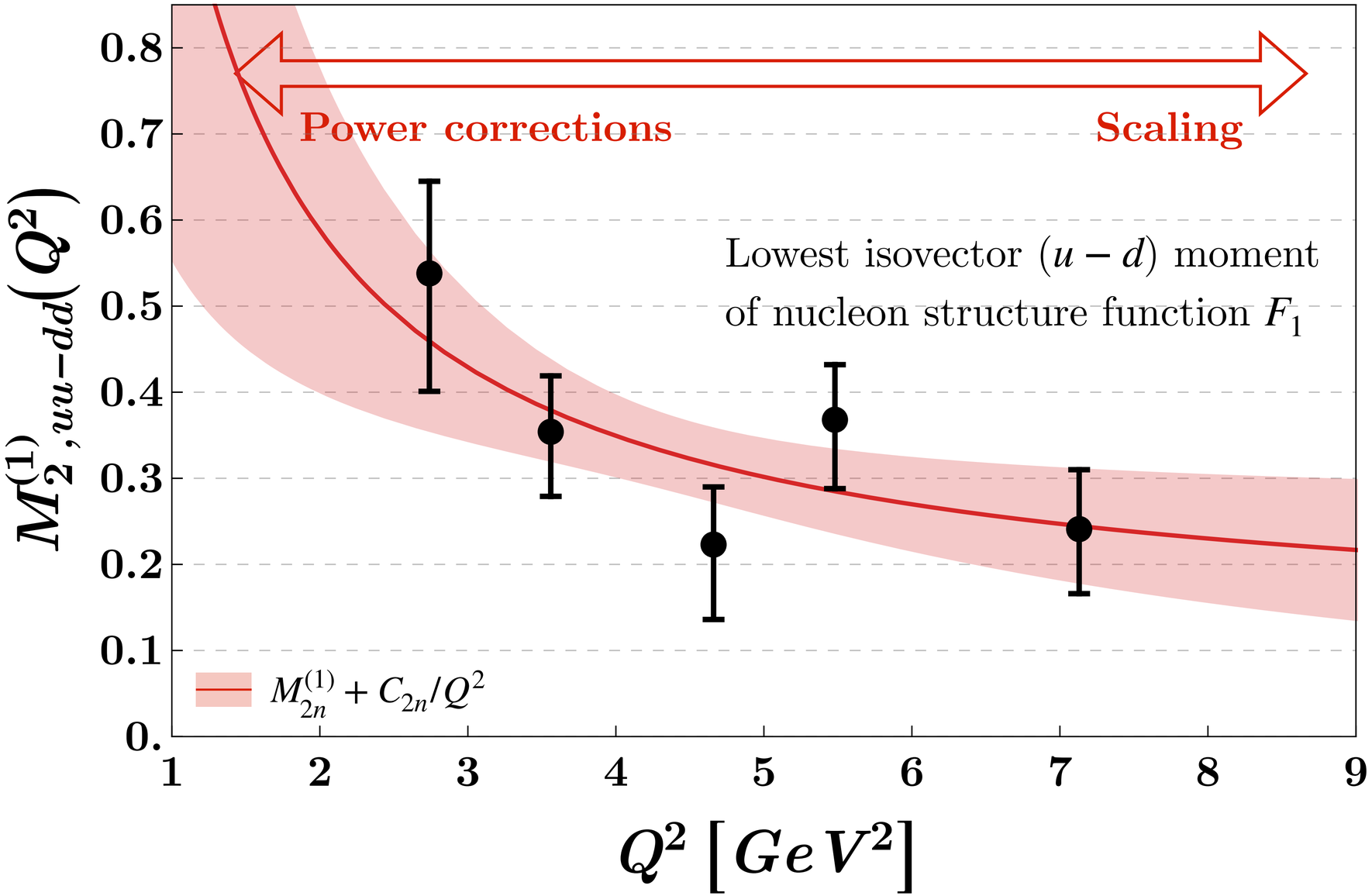
$a = 0.074\,\text{fm}$
$m_\pi \sim 470\,\text{MeV}$
32^3x64, 2+1 flavour



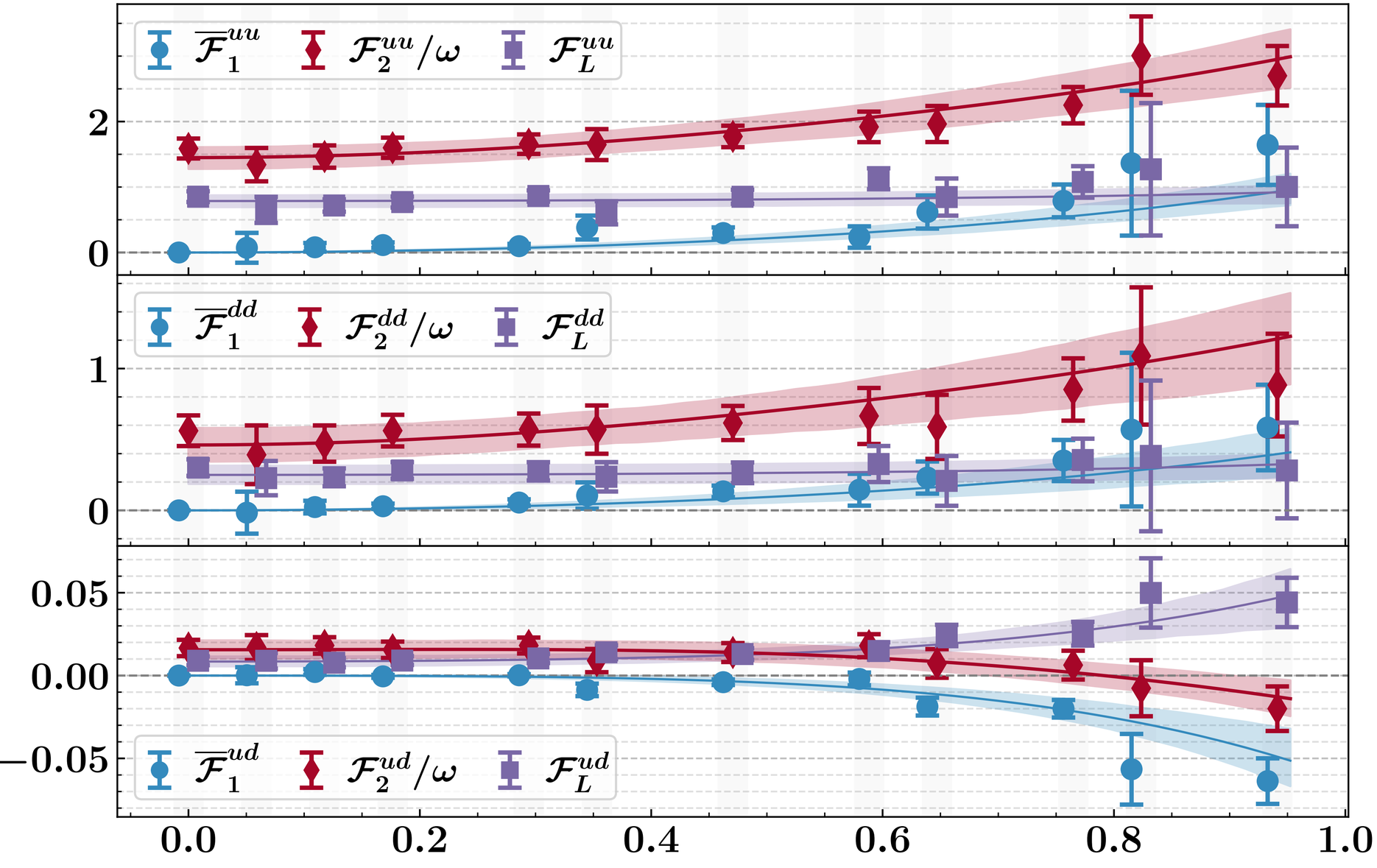
lowest moment

Power corrections

$a = 0.074$ fm
$m_\pi \sim 470$ MeV
32^3x64, 2+1 flavour



$\mathscr{F}_{2,L}$ Compton amplitude

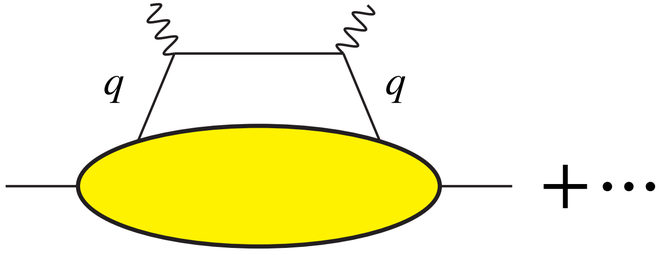


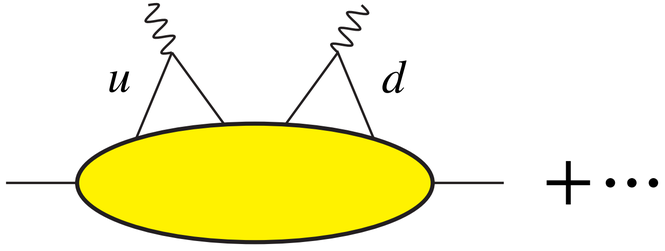
48^3x96, 2+1 flavour

$a = 0.068$ fm

$m_\pi \sim 420$ MeV

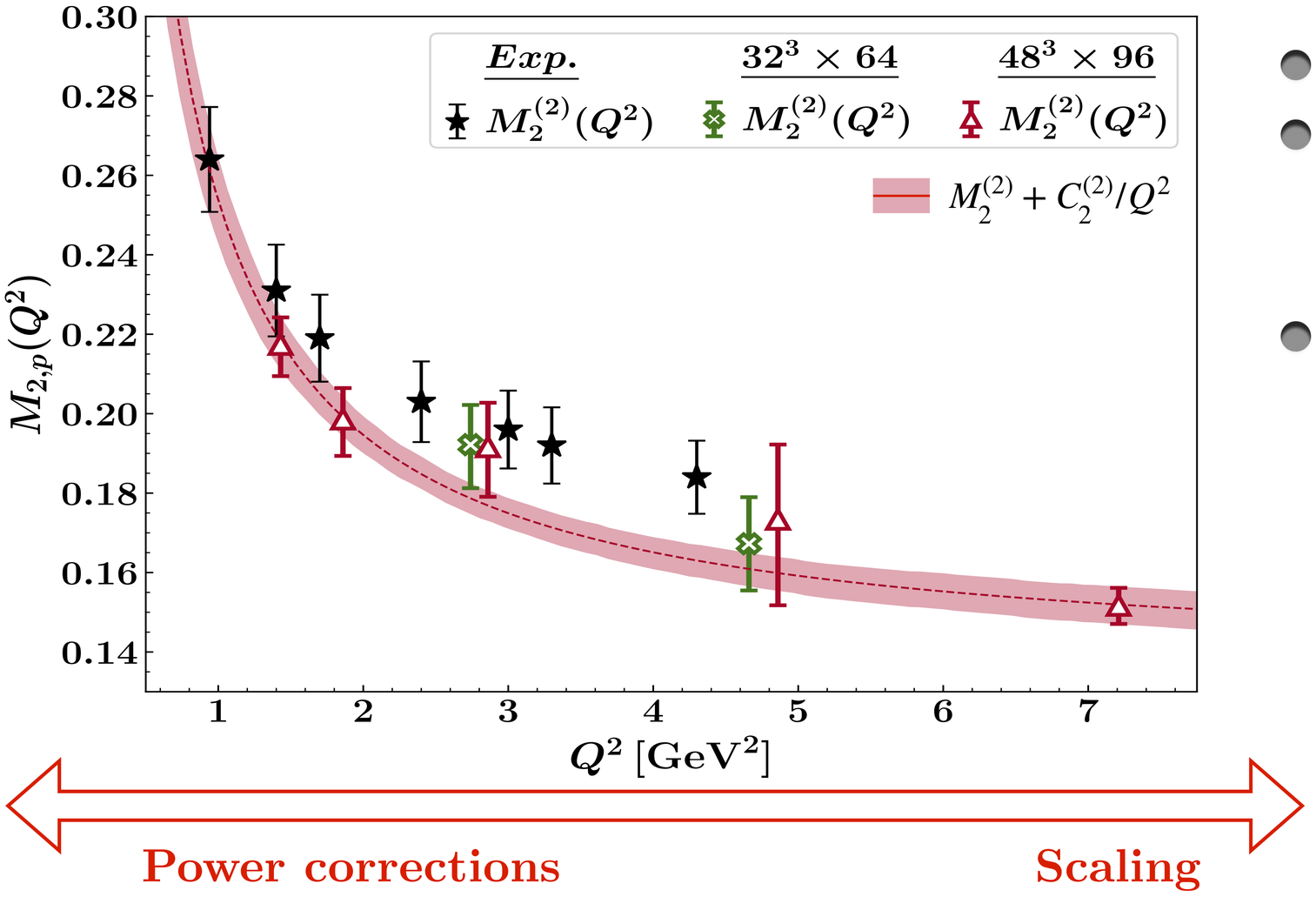
$Q^2 \sim 4.9\, GeV^2$





Moments | proton F_2

- **Unique ability to study the Q^2 dependence of the moments!**



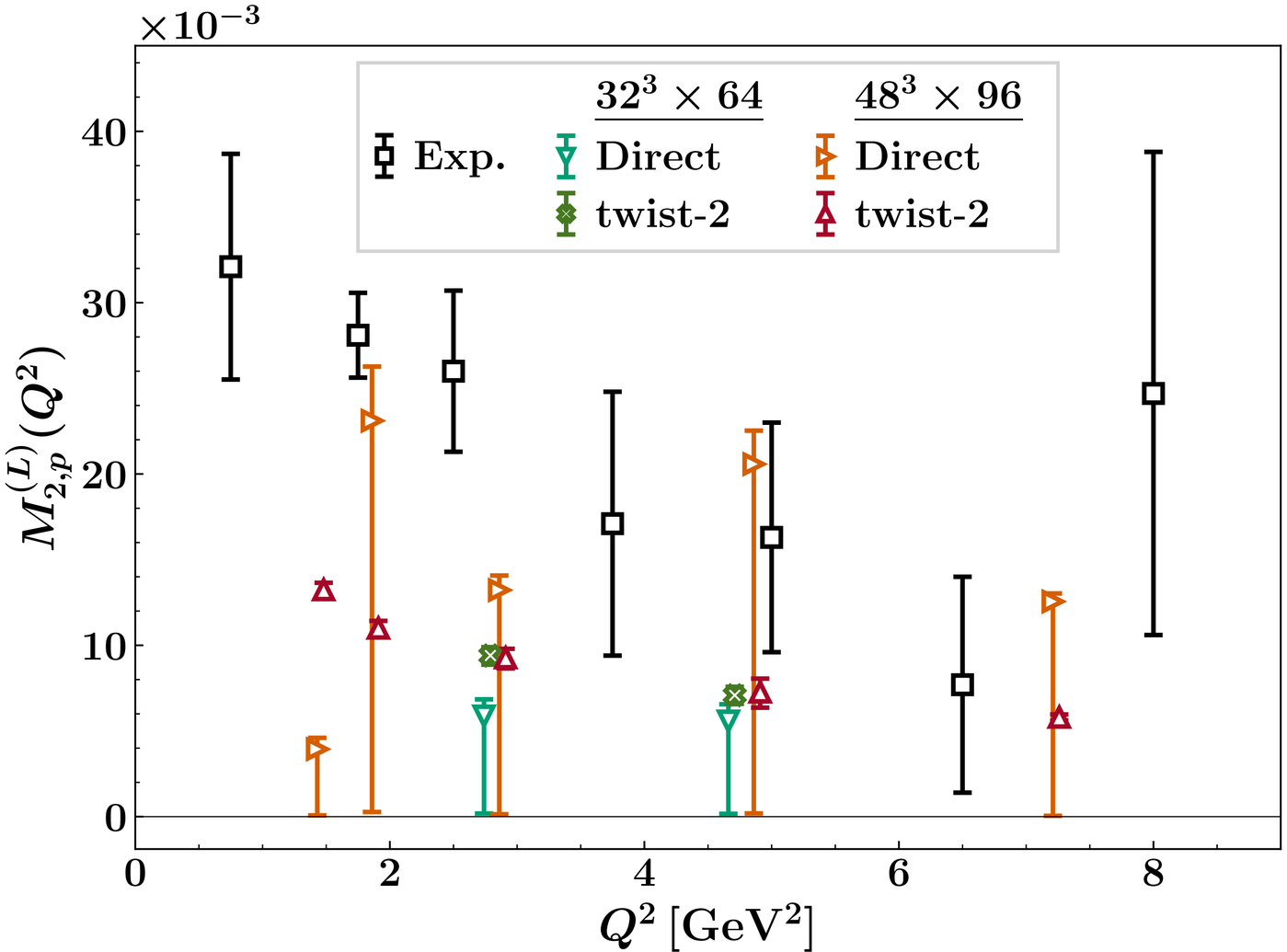
- Global PDF-fit cuts $\sim 1 - 10$ GeV^2
- Need $Q^2 > 10$ GeV^2 data to reliably extract partonic moments

- Power corrections below ~ 3 GeV^2 ?
  - Modelling via
  - $M_2^{(2)}(Q^2) = M_2^{(2)} + C_2^{(2)}/Q^2$

Exp $M_2^{(2)}$: C. S. Armstrong, R. Ent, C. E. Keppel, S. Liuti, G. Niculescu, and I. Niculescu, Phys. Rev. D 63, 094008 (2001), arXiv:hep-ph/0104055.

Moments | proton F_L

- **Unique ability to study the moments of F_L!**



Possible for the first time in a lattice QCD simulation!

- **Direct:** Fit to data points
  - Determines upper bounds
- **Twist-2:** Use the moments of F_2:
  - $M_2^{(L),QCD}(Q^2) = \frac{4}{9\pi}\alpha_s(Q^2)\, M_2^{(2)}(Q^2)$
  - Better precision, good agreement with exp. behaviour

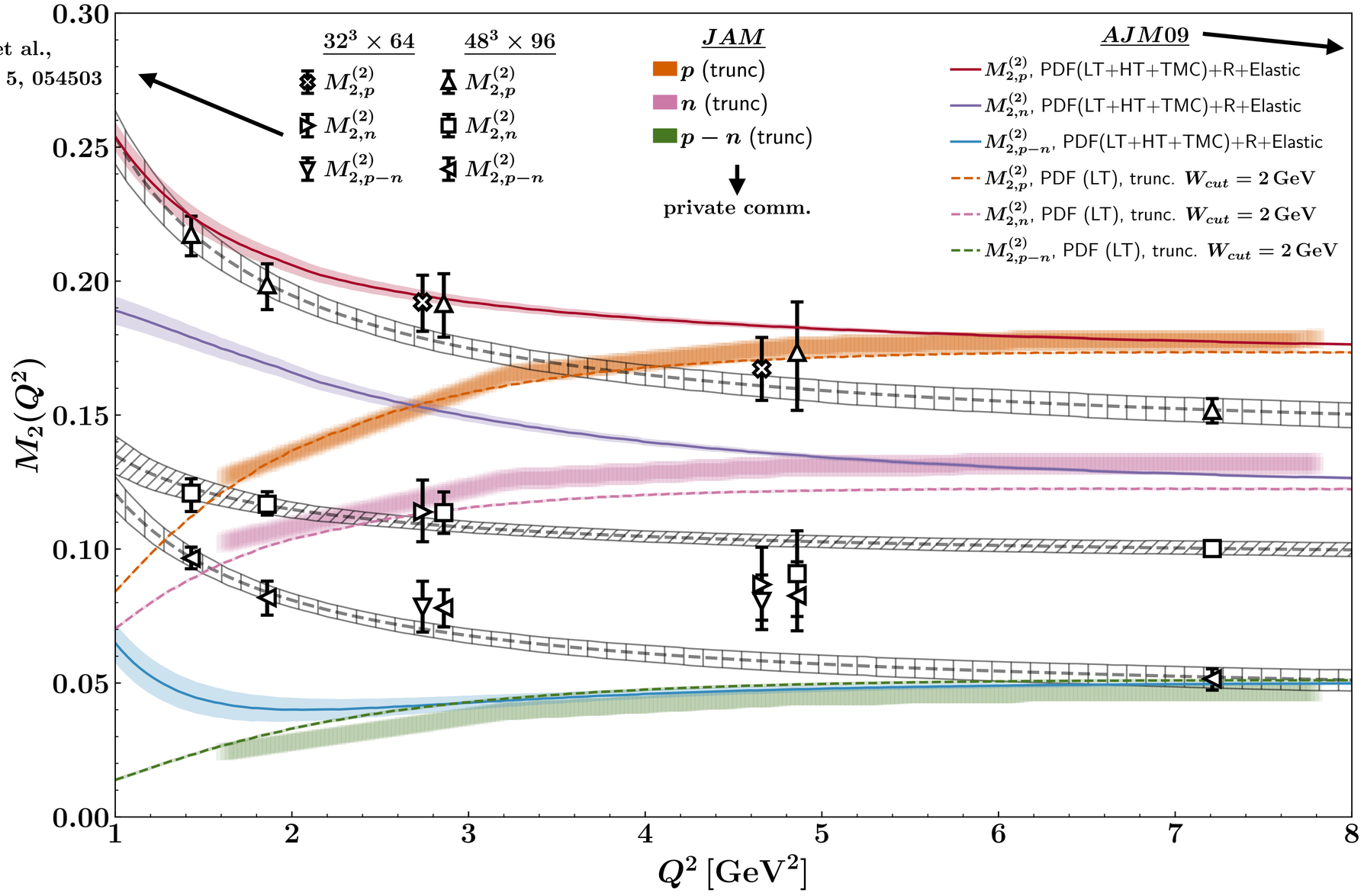
Exp Nachtmann $M_2^{(L)}$: P. Monaghan, A. Accardi, M. E. Christy, C. E. Keppel, W. Melnitchouk, and L. Zhu, Phys. Rev. Lett. 110, 152002 (2013), arXiv:1209.4542 [nucl-ex].

Outlook

- Confronting F_2 with pheno.
- Polarised g_1, g_2
- parity-violating unpolarised F_3

F_2 moments vs. (JAM, AJM)

Batelaan, kuc, et al., PRD107 (2023) 5, 054503



N.L. Hall et al., PRD83 (2013) 1, 013011

Outlook

- Confronting F_2 with pheno.
- Polarised g_1, g_2
- parity-violating unpolarised F_3

Polarised Structure Functions

$$T_{[\mu\nu]}(p,q,s) = i\varepsilon^{\mu\nu\alpha\beta}\frac{q_\alpha}{p\cdot q}\left[s_\beta\,\tilde{g}_1(\omega,Q^2) + \left(s_\beta - \frac{s\cdot q}{p\cdot q}p_\beta\right)\tilde{g}_2(\omega,Q^2)\right]$$

- Similar to the unpolarised case, we can extract $\tilde{g}_1$ and $\tilde{g}_2$
- via an OPE analysis: the first moment of $g_1(x)$ is related to axial current matrix elements

$$\Gamma_1(Q^2) = \int_0^1 g_1^{(u-d)}(x,Q^2)\,dx = \underbrace{\left(\Delta u - \Delta d\right)}_{\equiv g_A} C_1(\alpha_s(Q^2))$$

where, $C_1(\alpha_s(Q^2)) = 1 - \frac{\alpha_s(Q^2)}{\pi} - \mathcal{O}(\alpha_s^2)$

- $g_2(x)$ is twist-3, holds information on quark-gluon correlations
- Wandzura-Wilczek decomposition

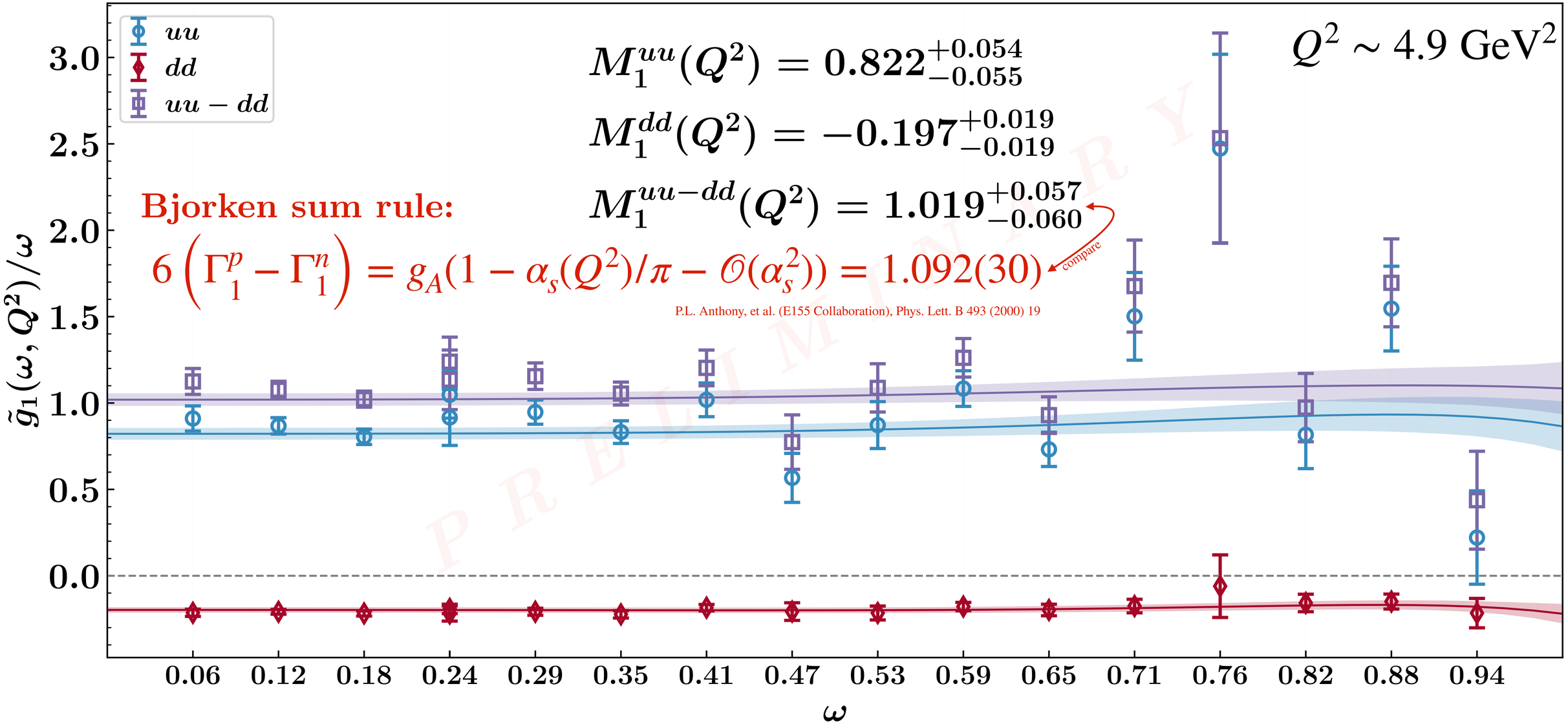
$$g_2(x,Q^2) = -\,g_1(x,Q^2) + \int_x^1 g_1(y,Q^2)\,dy + \bar{g}_2(x,Q^2)$$

- The Buckhardt — Cottingham sum rule

$$\int_0^1 g_2(x,Q^2)\,dx = 0$$

Polarised Structure Functions

48^3x96, 2+1 flavour
$a = 0.068$ fm
$m_\pi \sim 420$ MeV



Polarised Structure Functions

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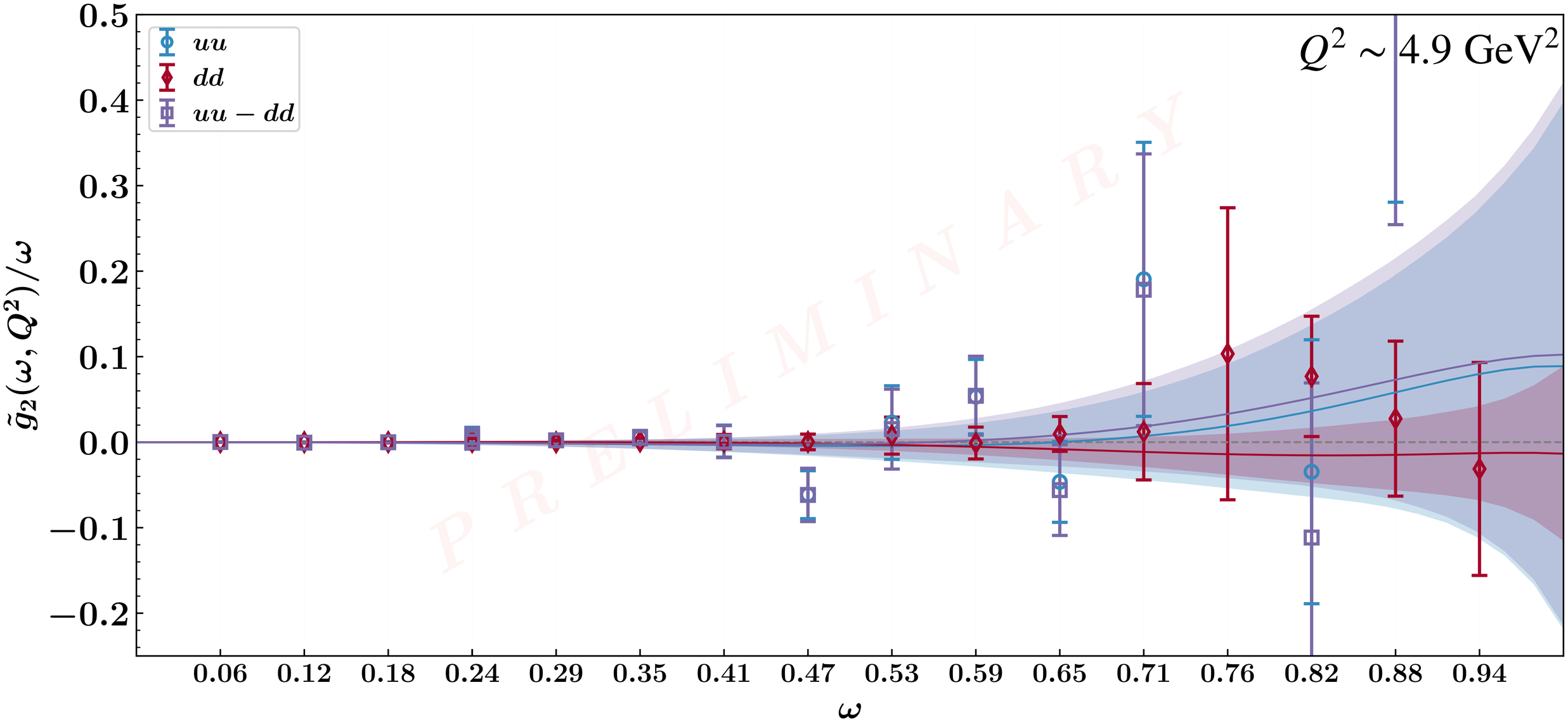


Outlook

- Confronting F_2 with pheno.
- Polarised g_1, g_2
- parity-violating unpolarised F_3

Motivation

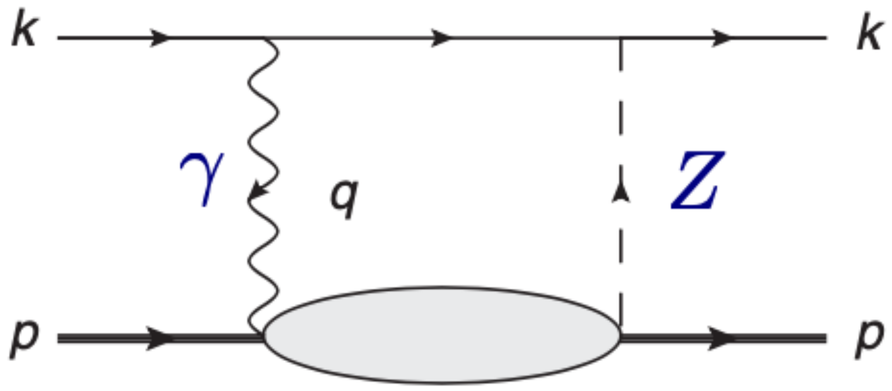
- Leading theoretical uncertainty in:
- Weak charge of the proton,

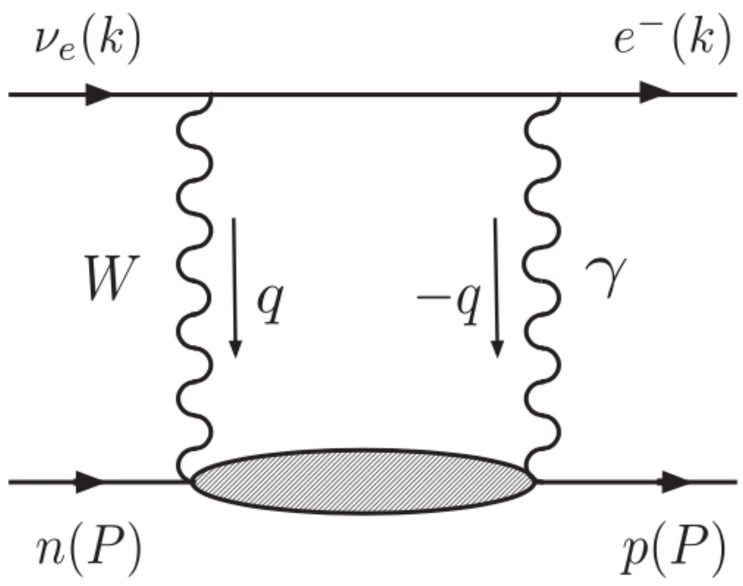
$$Q_W = (1 + \Delta_\rho + \Delta_e)(1 - 4\sin^2\theta_W(0) + \Delta'_e) + \Box_{AA}^{WW} + \Box_{AA}^{ZZ} + \Box_{VA}^{\gamma Z}$$

- CKM matrix element extracted from superallowed neutron β decays,

$$|V_{ud}|^2 = \frac{0.97148(20)}{1 + \Delta_R^V} \rightarrow (\ldots) + 2\Box_{VA}^{\gamma W}$$

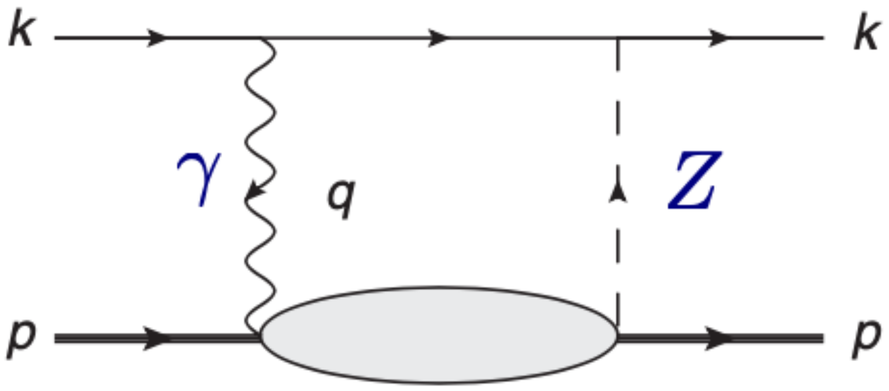
K. Shiells, P.G. Blunden, W. Melnitchouk, PRD104, 033003 (2021) [2012.01580]





Motivation

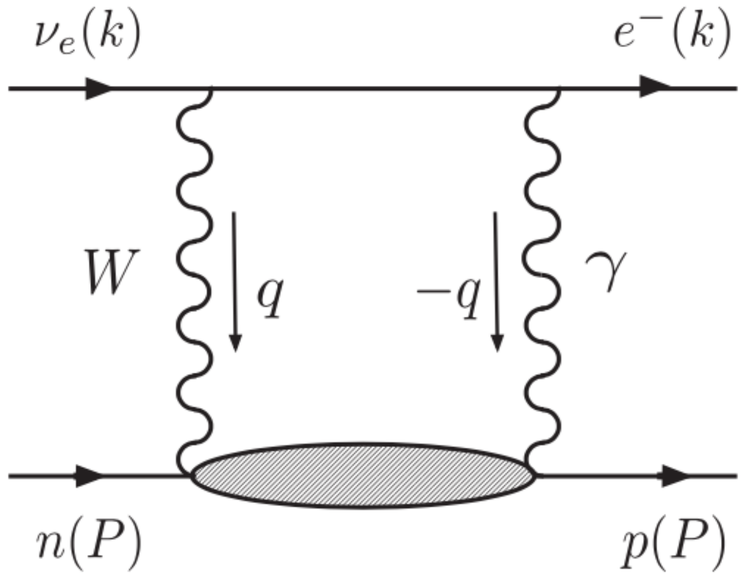
$$\Box_A^{\gamma Z} = \nu_e \frac{3\alpha_{EM}}{2\pi} \int_0^\infty \frac{dQ^2}{Q^2} \frac{M_Z^2}{M_Z^2 + Q^2} \underbrace{\int_0^1 dx F_3^{\gamma Z}(x, Q^2)}$$



First moment of F_3

$$\Box_{VA}^{\gamma W} = \frac{3\alpha_{EM}}{2\pi} \int_0^\infty \frac{dQ^2}{Q^2} \frac{M_W^2}{M_W^2 + Q^2} \overbrace{\int_0^1 dx F_3^{(0)}(x, Q^2)}$$

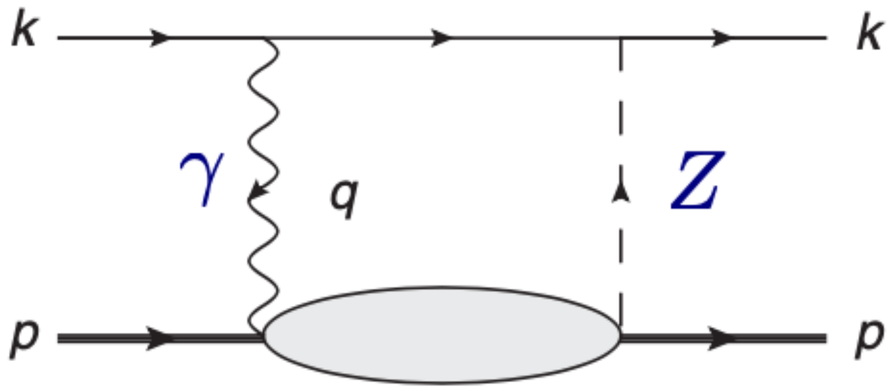
$$F_3^{(0)} = F_3^{\gamma Z,p} - F_3^{\gamma Z,n}$$



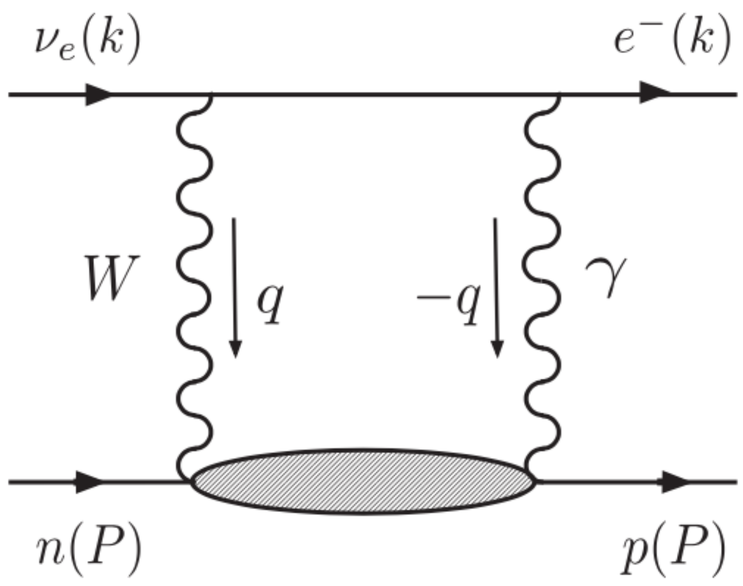
Motivation

- Box diagrams proportional to an integral over the whole Q^2 range

$$\square_A^{\gamma Z/W} \propto \int_0^\infty \frac{dQ^2}{Q^2} M_1^{(3)}(Q^2)\,(\ldots)$$



- Low-Q^2 (non-perturbative) regime dominates the integral
- F_3 is experimentally poorly determined in low Q^2
- Lattice approach is ideal for a high-precision determination of $M_1^{(3)}(Q^2)$



Parity Violating

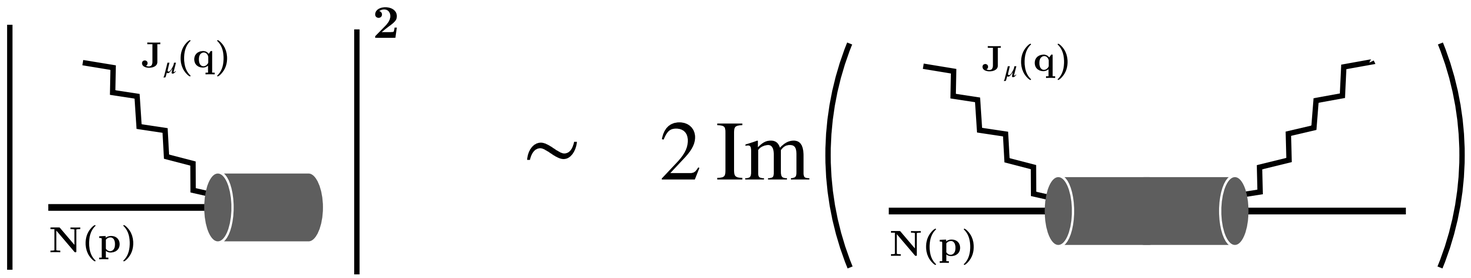
Forward Compton Amplitude

$$T_{\mu\nu}(p,q) = i\int d^4z\, e^{iq\cdot z}\rho_{ss'}\langle p,s'|\mathcal{T}\{J^V_\mu(z)J^A_\nu(0)\}|p,s\rangle \text{ , spin avg. } \rho_{ss'} = \frac{1}{2}\delta_{ss'}$$

$$= -g_{\mu\nu}\mathcal{F}_1(\omega,Q^2) + \frac{p_\mu p_\nu}{p\cdot q}\mathcal{F}_2(\omega,Q^2) + i\,\varepsilon^{\mu\nu\alpha\beta}\frac{p_\alpha q_\beta}{2p\cdot q}\mathcal{F}_3(\omega,Q^2)$$

$$+\frac{q_\mu q_\nu}{p\cdot q}\mathcal{F}_4(\omega,Q^2) + \frac{p_{\{\mu}q_{\nu\}}}{p\cdot q}\mathcal{F}_5(\omega,Q^2) + \frac{p_{[\mu}q_{\nu]}}{p\cdot q}\mathcal{F}_6(\omega,Q^2)$$

allowed terms because parity is violated



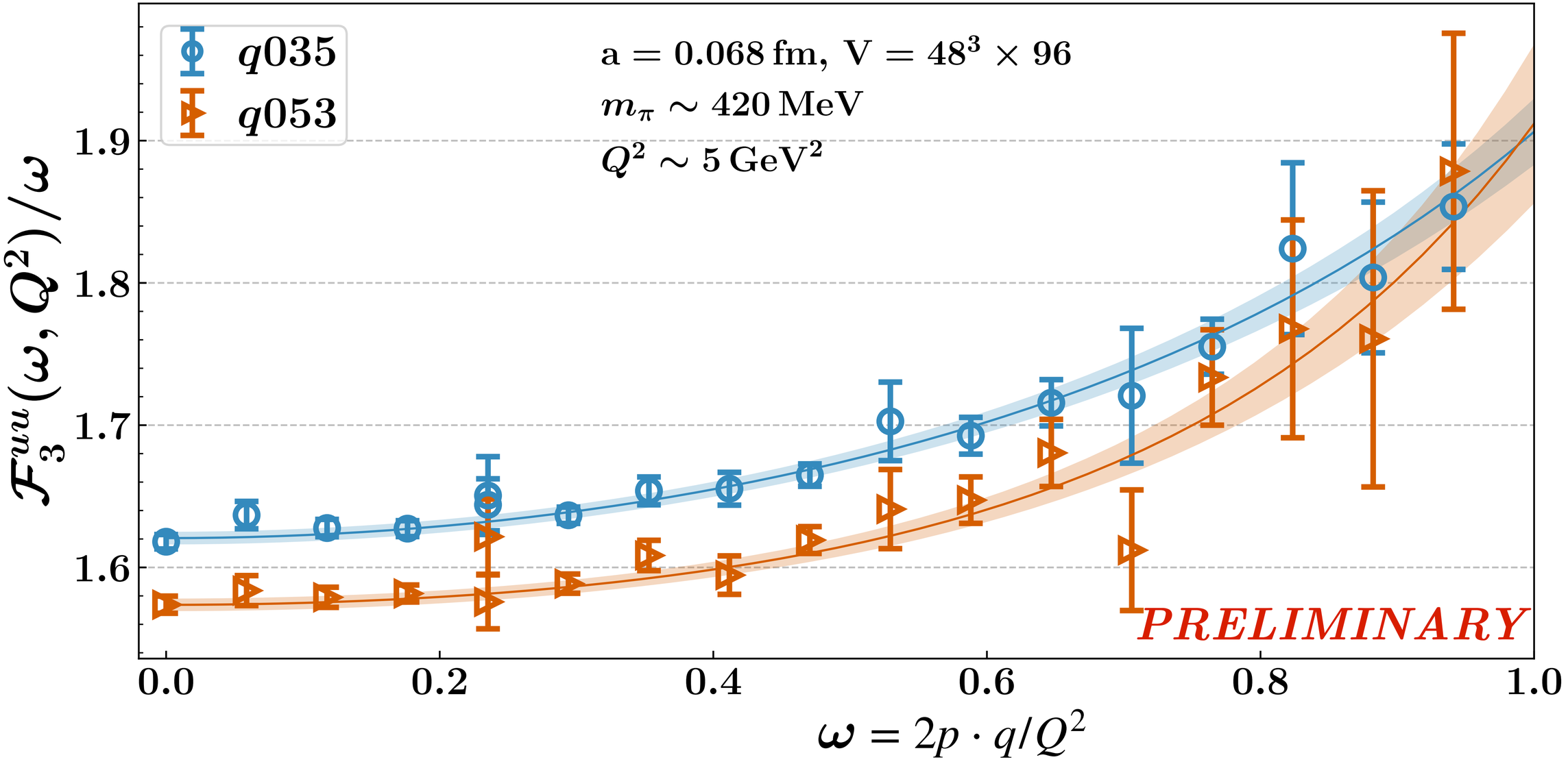
$$\omega = \frac{2p\cdot q}{Q^2}$$

$$\varepsilon^{0123} = 1$$

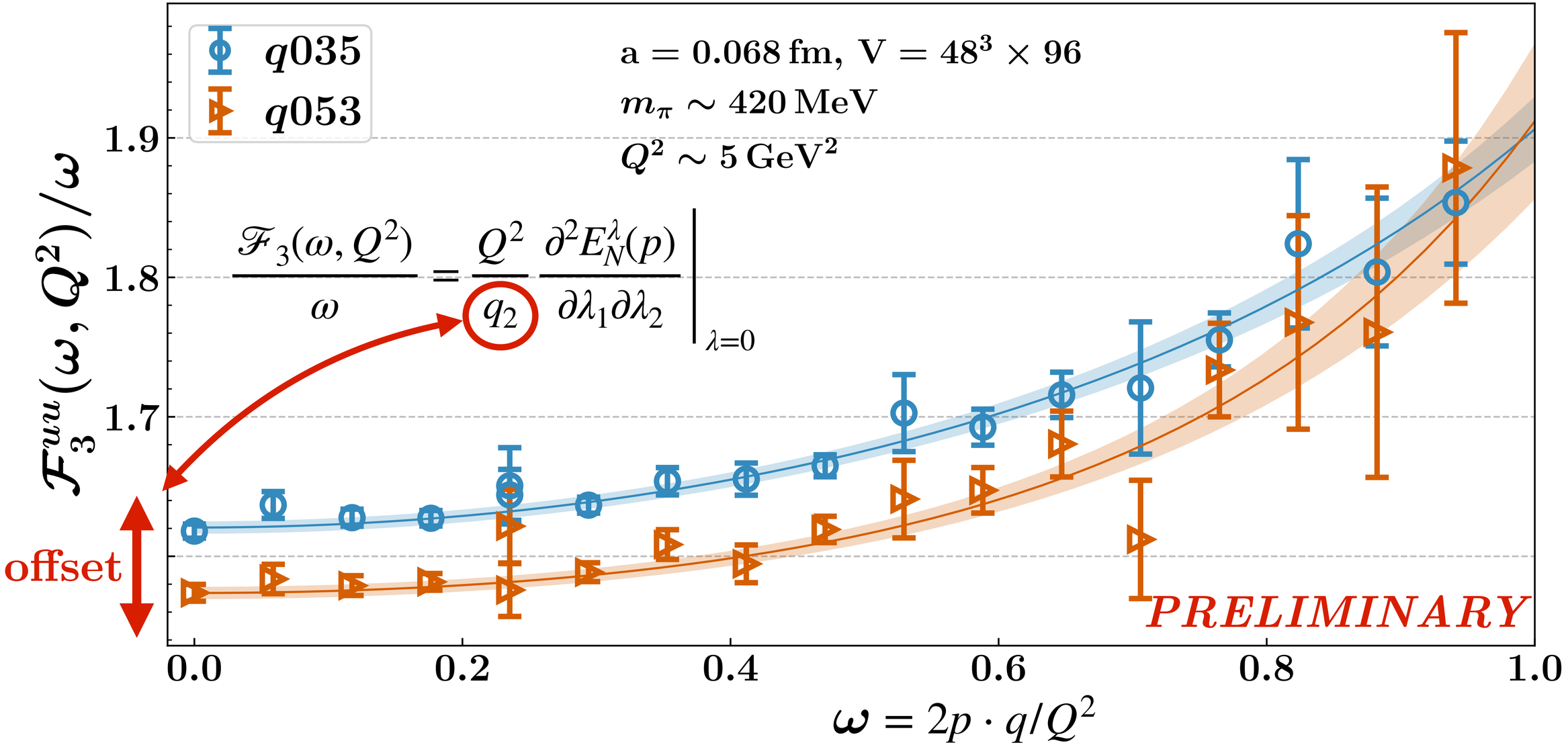
DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

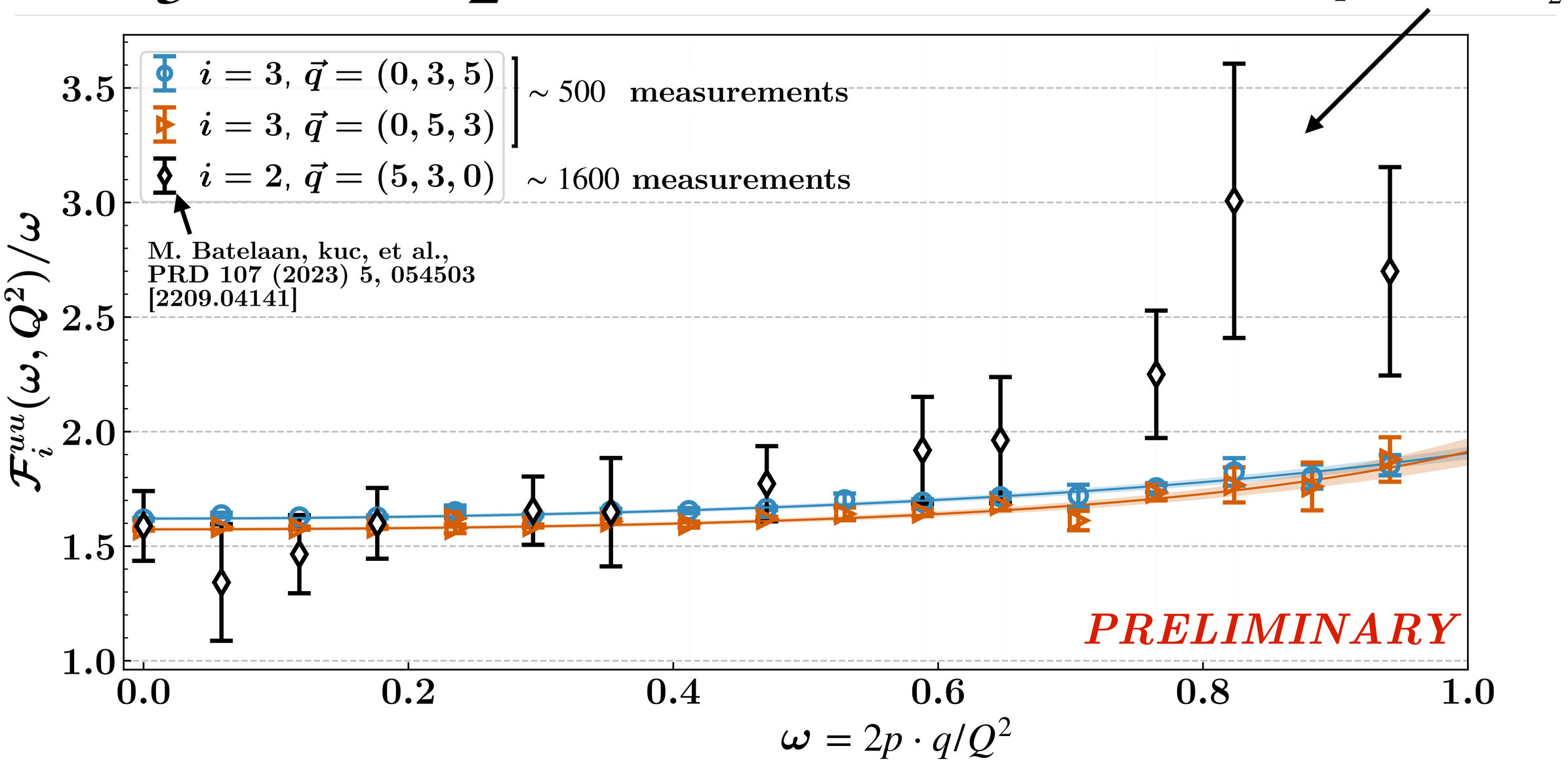
$\mathscr{F}_3$ Compton Amplitude



$\mathscr{F}_3$ Compton Amplitude



$\mathscr{F}_3$ vs. $\mathscr{F}_2$



Summary

- A versatile approach: F_1, F_2, F_L, F_3, and g_1 and g_2
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Exploratory calculation of $\mathscr{F}_3(\omega, Q^2)$
  - A good chance to study the discretisation errors

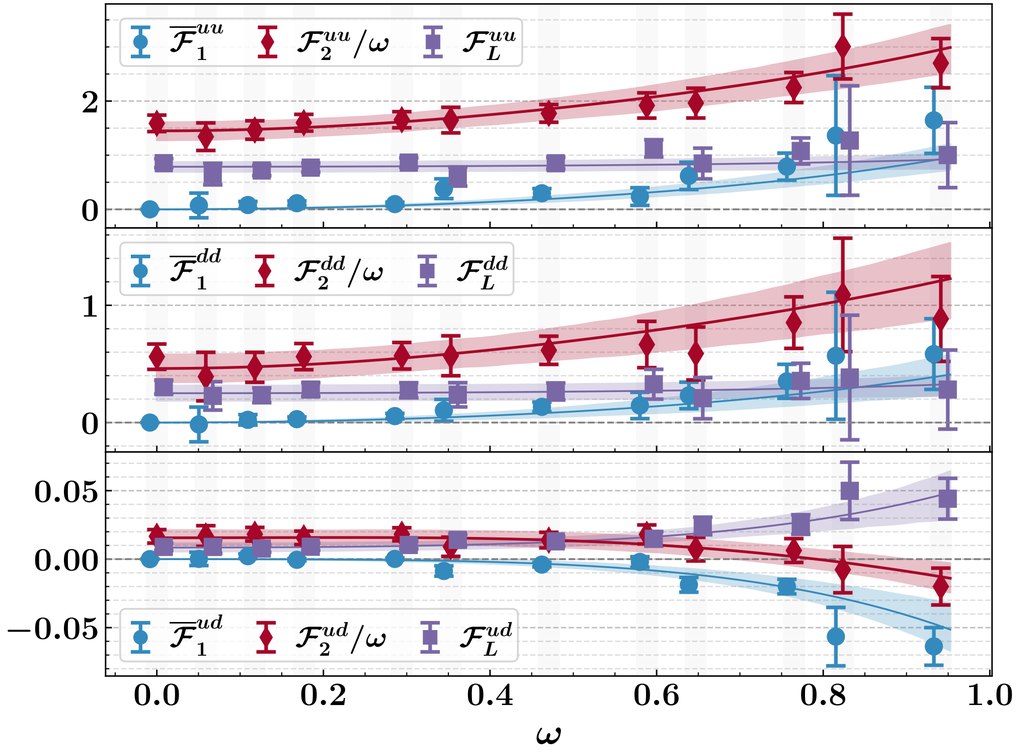
Summary

- In the long run:
  - Make contact with phenomenology: incorporate lattice Compton amplitude in global QCD fits
  - High-precision box-diagram estimates
  - Recover the x-dependence of PDFs

Backup

Moments | Fit details

48^3x96, 2+1 flavour
$a = 0.068$ fm
$m_\pi \sim 420$ MeV



$$\overline{\mathcal{F}}_1^{qq}(\omega, Q^2) = 2\sum_{n=1}^{\infty} M_{2n}^{(1)}(Q^2)\,\omega^{2n}$$

$$\frac{\mathcal{F}_2^{qq}(\omega, Q^2)}{\omega} = \frac{\tau}{1+\tau\omega^2}\sum_{n=0}^{\infty} 4\omega^{2n}\left[M_{2n}^{(1)} + M_{2n}^{(L)}\right](Q^2), \text{ where } \tau = \frac{Q^2}{4M_N^2}$$

- **Enforce monotonic decreasing of moments for uu and dd only, $|ud|^2 \leq 4uu * dd$**

$$M_2(Q^2) \geq M_4(Q^2) \geq \cdots \geq M_{2n}(Q^2) \geq \cdots \geq 0$$

We truncate at $n = 6$
No dependence to truncation order for $3 \leq n \leq 10$

- **Bayesian approach by MCMC method**

Sample the moments from Uniform priors
individually for u- and d-quark

$$M_2(Q^2) \sim \mathcal{U}(0, 1)$$
$$M_{2n}(Q^2) \sim \mathcal{U}\left(0, M_{2n-2}(Q^2)\right)$$

Normal Likelihood function, $exp(-\chi^2/2)$

$$\chi^2 = \sum_i \frac{\left(\overline{\mathcal{F}}_i - \overline{\mathcal{F}}^{obs}(\omega_i)\right)^2}{\sigma_i^2}$$

stat. errors via a bootstrap analysis

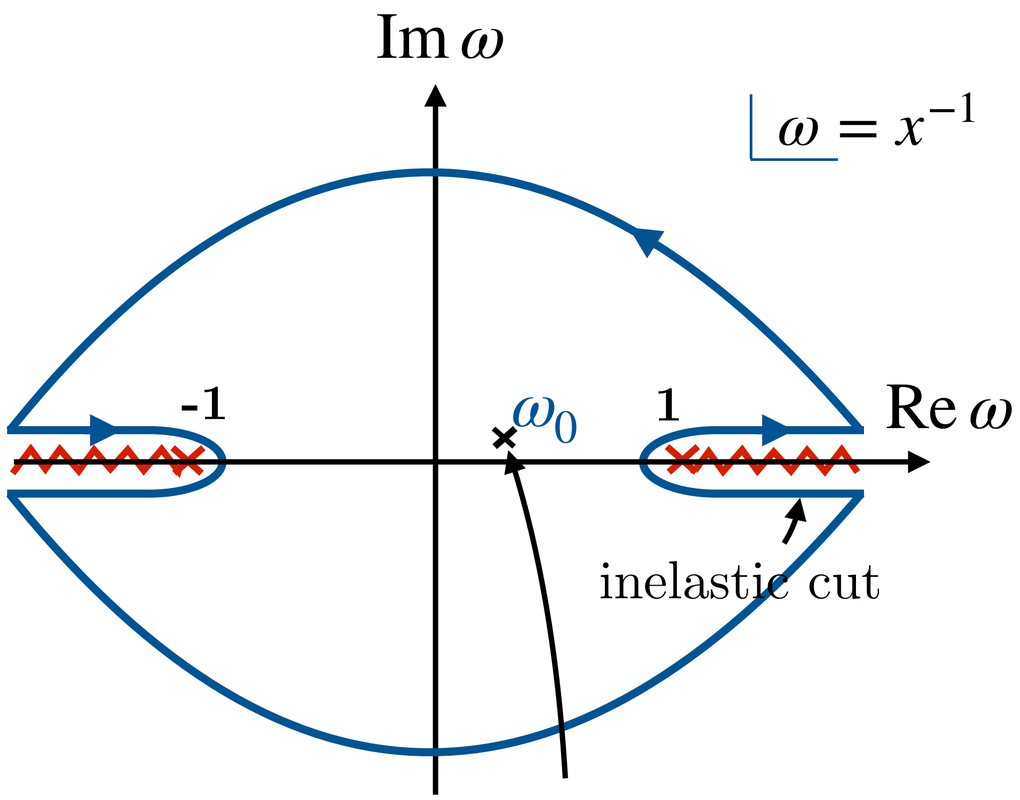
Nucleon Structure Functions | F_3

- for $\mu \neq \nu$ and $p_\mu = q_\mu = 0$, and $\beta \neq 0$, we isolate,

$$T_{\mu\nu}(p,q) = i\,\varepsilon^{\mu\nu\alpha\beta}\frac{p_\alpha q_\beta}{2p\cdot q}\mathscr{F}_3(\omega, Q^2)$$

- we can write down dispersion relations and connect Compton SFs to DIS SFs:

$$\mathscr{F}_3(\omega, Q^2) = 4\omega\int dx\frac{F_3(x,Q^2)}{1-x^2\omega^2}$$



Compton Amplitude is an analytic function in the unphysical region $|\omega_0| < 1$

Nucleon Structure Functions | F_3

$\omega = \frac{2p \cdot q}{Q^2} \equiv x^{-1}$

- using the Taylor expansion, $\frac{1}{1-(x\omega)^2} = \sum_{n=1}^{\infty} (x\omega)^{2n-2}$

$$\mathscr{F}_3(\omega, Q^2) = 4 \sum_{n=1,2,\ldots} \omega^{2n-1} M^{(3)}_{2n-1}(Q^2)$$

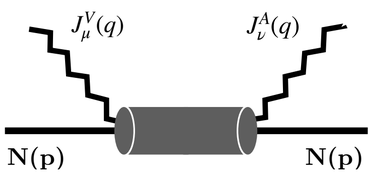
<u>Mellin moments</u>

$$M^{(3)}_{2n-1}(Q^2) = \int_0^1 dx\, x^{2n-2} F_3(x, Q^2), \quad \text{for } n = 1,2,3,\ldots$$

Compton amplitude via the FH relation at 2nd order

- **unpolarised Compton Amplitude**

$$T_{\mu\nu}(p,q) = \int d^4z e^{iq\cdot z} \langle N(p) | \mathcal{T}\{J^V_\mu(z) J^A_\nu(0)\} | N(p)\rangle$$



- **Action modification**

$$S \to S(\lambda) = S + \lambda_1 \int d^4z \cos(q\cdot z)\, J^V_\mu(z) + \lambda_2 \int d^4y \sin(q\cdot y)\, J^A_\nu(z)$$

local V, A currents

$$J^V_\mu(z) = Z_V \sum_q e_q \bar{q}(z)\gamma_\mu q(z)$$

$$J^A_\nu(z) = Z_A \sum_q \bar{q}(z)\gamma_\nu\gamma_5 q(z)$$

- **2nd order mixed derivatives of the 2-pt correlator, $G^{(2)}_\lambda(\mathbf{p};t)$, in the presence of the external field**

$$\left.\frac{\partial^2 G^{(2)}_\lambda(p;t)}{\partial\lambda_1\partial\lambda_2}\right|_{\lambda=0} = \left[\frac{\partial^2 A_\lambda(p)}{\partial\lambda_1\partial\lambda_2} - tA(p)\frac{\partial^2 E_{N_\lambda}(p)}{\partial\lambda_1\partial\lambda_2}\right] e^{-E_N(p)t}$$ from spectral decomposition

from path integral

$$\left.\frac{\partial^2 G^{(2)}_\lambda(p;t)}{\partial\lambda_1\partial\lambda_2}\right|_{\lambda=0} = -t\sum_{s,s'} iA_{ss'}(p)\frac{e^{-E_N(p)t}}{E_N(p)}\left[\int d^4z\, e^{iq\cdot z}\left\langle N_s(p)\,|\,J_\mu(z)J_\nu(0)\,|\,N_{s'}(p)\right\rangle - (q\to -q) + \ldots\right]$$

- **equate the time-enhanced terms:**

$$\left.\frac{\partial^2 E^\lambda_N(p)}{\partial\lambda_1\partial\lambda_2}\right|_{\lambda=0} = \frac{i}{2E_N(p)}\left[\overbrace{\int d^4z\, e^{iq\cdot z}\left\langle N_s(p)\,|\,J_\mu(z)J_\nu(0)\,|\,N_{s'}(p)\right\rangle}^{T_{\mu\nu}(p,q)} - (q\to -q)\right]$$

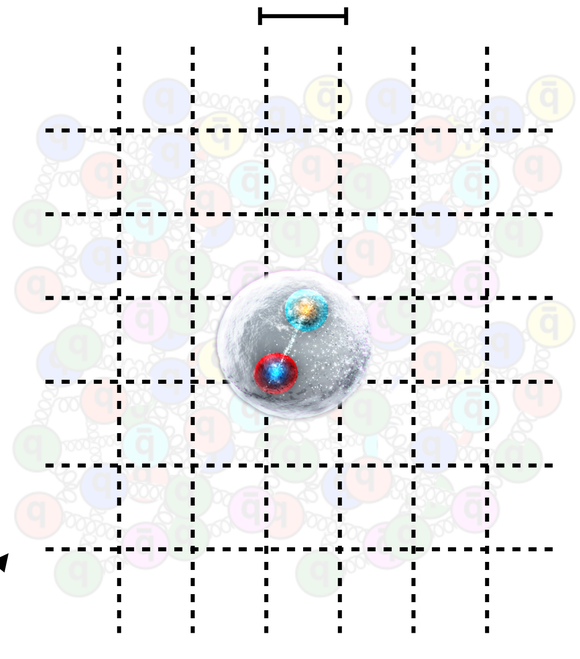
Compton amplitude is related to the second-order energy shift

Simulation Details | F_3

QCDSF/UKQCD configurations
$48^3 \times 96$, 2+1 flavor (u/d+s)
$\beta = 5.65$ Symanzik improved gauge
NP-improved Clover action
Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]

$m_\pi \sim 420$ MeV, SU(3) sym.

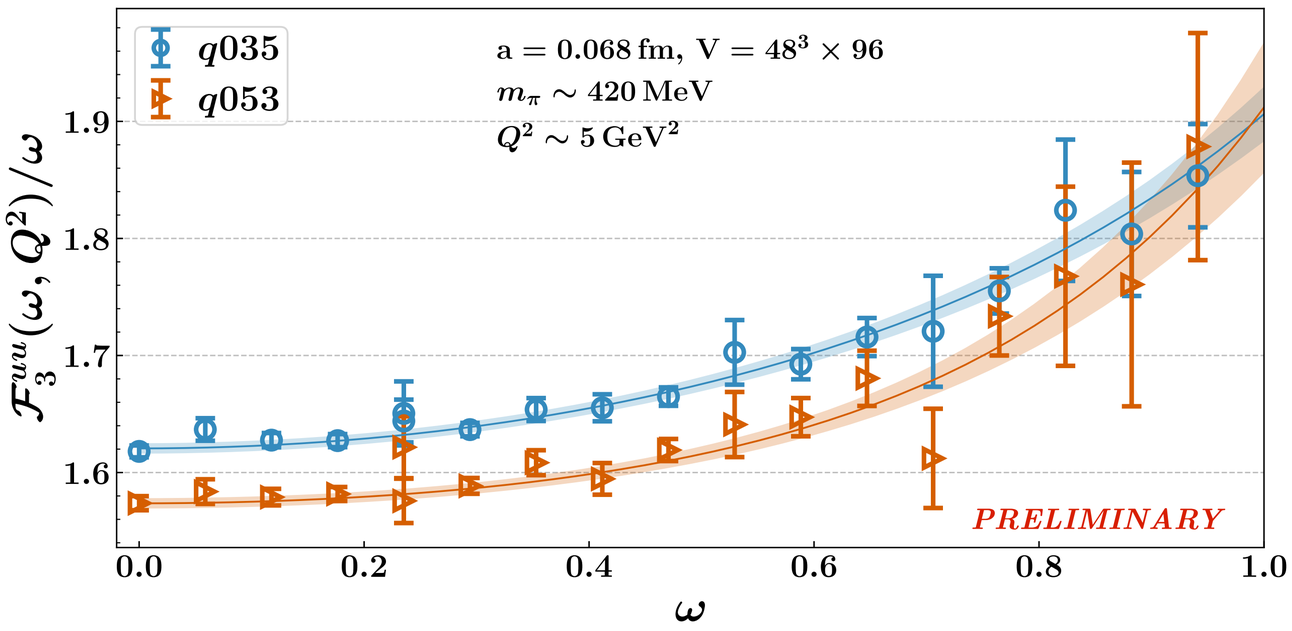
$m_\pi L \sim 6.9$ $\quad a = 0.068$ fm



Unmodified QCD background

- FH implementation at the valence quark level
  - Valence u/d quark props with modified action, $S(\lambda)$
  - Local V, A current insertions, $J_\mu^{V[A]}(x) = Z_{V[A]}\bar{q}(x)\gamma_\mu[\gamma_5]q(x)$
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- Presently, 1 current momenta $Q^2 \sim 5\,GeV^2$
- Roughly 500 measurements
- Access to a range of $\omega = 2\,p \cdot q/Q^2$ values for several (p, q) pairs
  - An inversion for each q and λ, varying p is relatively cheap
- Connected 2-pt correlators calculated only, <u>no disconnected</u>

Moments | Fit details | F_3



$$\frac{\mathcal{F}_3(\omega, Q^2)}{\omega} = \sum_{n=1,2,\ldots} 4\omega^{2n-2} M^{(3)}_{2n-1}(Q^2)$$

- **Enforce monotonic decreasing of moments for uu and dd only, $|ud|^2 \leq 4uu * dd$**

$$M_1(Q^2) \geq M_3(Q^2) \geq \cdots \geq M_{2n-1}(Q^2) \geq \cdots \geq 0$$

We truncate at $n = 6$

- **Bayesian approach by MCMC method**

Sample the moments from Uniform priors

individually for u- and d-quark

$M_1(Q^2) \sim |\mathcal{N}(0, 5)|$ positive half, long tail uninformative prior

$M_{2n+1}(Q^2) \sim \mathcal{U}(0, M_{2n-1}(Q^2))$ positive, bounded from above by the previous moment

Maximise the multivariate Likelihood function, $exp(-\chi^2/2)$

$$\chi^2 = \sum_{i,j} \left[\mathcal{F}_{3,i} - \mathcal{F}_3^{obs}(\omega_i)\right] C^{-1}_{ij} \left[\mathcal{F}_{3,j} - \mathcal{F}_3^{obs}(\omega_j)\right]$$

i, j runs through all the ω values of all flavour contributions